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One property of a planar curve whose convex hull covers a given convex figure

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1 Introduction

The authors of [1] (along with obtained interesting results) posed the following conjecture.

Conjecture 1 (A. Akopyan and V. Vysotsky [1]). *Let γ be a curve such that its convex hull covers a planar convex figure K. Then* $\operatorname{length}(\gamma) \geq \operatorname{per}(K) - \operatorname{diam}(K)$.

It should be noted that this conjecture has been proved in the case when γ is passing through all extreme points of K (see [1, Theorem 7]). This note is devoted to the proof of the above conjecture in the general case. Figures 1 and 2 show the difference between the general case and the special case mentioned above.

We identify the Euclidean plane with $\mathbb{R}^2$ supplied with the standard Euclidean metric d, where $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. For any subset $A \subset \mathbb{R}^2$, $\operatorname{co}(A)$ means the convex hull of A. For any points $B, C \in \mathbb{R}^2$, $[B, C]$ denotes the line segment between these points.

Arseniy Akopyan und Vladislav Vysotsky äusserten 2017 im American Mathematical Monthly folgende Vermutung: Wenn die konvexe Hülle einer ebenen Kurve γ eine ebene konvexe Figur K überdeckt, dann gilt $\operatorname{length}(\gamma) \geq \operatorname{per}(K) - \operatorname{diam}(K)$. Das heisst, die Länge der Kurve γ wird von unten durch den Umfang und den Durchmesser der Figur K abgeschätzt. Die Autoren der vorliegenden Arbeit beweisen diese Ungleichung. Sie identifizieren zudem alle Fälle, in denen Gleichheit auftritt. Die Abschätzung mag auf den ersten Blick harmlos erscheinen, dennoch mussten die Autoren für den Beweis recht tief in die mathematische Werkzeugkiste greifen. Es wäre interessant, Analogien dieser Ungleichung in euklidischen Räumen der Dimension drei und höher aufzustellen.

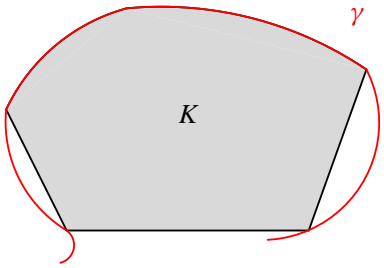

Figure 1. Illustration to the case when γ is passing through all extreme points of K.

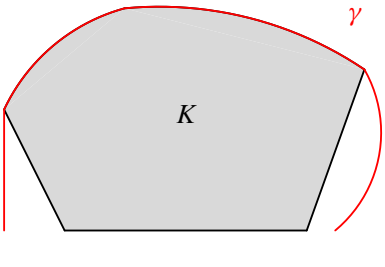

Figure 2. Illustration to the case when γ is not passing through some extreme points of K.

A *convex* (*planar*) *figure* is any compact convex subset of $\mathbb{R}^2$. We shall denote by $\mathrm{per}(K)$, $\mathrm{bd}(K)$ and $\mathrm{int}(K)$, respectively, the perimeter, the boundary, and the interior of a convex figure K. Note that the perimeter of any line segment (i.e. a degenerate convex figure) is assumed to be equal to its double length. Note also that the diameter

$$\mathrm{diam}(K) := \max\{d(x,y) \mid x,y \in K\}$$

of a convex figure K coincides with the maximal distance between two parallel support lines of K. Recall that an extreme point of K is a point in K which does not lie in any open line segment joining two points of K. The set of extreme points of K will be denoted by $\mathrm{ext}(K)$. It is well known that $\mathrm{ext}(K)$ is closed and $K = \mathrm{co}(\mathrm{ext}(K))$ for any convex figure $K \subset \mathbb{R}^2$.

A *planar curve* γ is the image of a continuous mapping $\varphi\colon [a,b] \subset \mathbb{R} \mapsto \mathbb{R}^2$. From now on, we will call planar curves simply *curves* for brevity, since no other curves are considered in this note. As usual, the length of γ is defined as

$$\mathrm{length}(\gamma) := \sup\left\{\sum_{i=1}^{m} d(\varphi(t_{i-1}),\varphi(t_i))\right\},$$

where the supremum is taken over all finite increasing sequences

$$a = i_0 < i_1 < \cdots < i_{m-1} < i_m = b$$

that lie in the interval $[a,b]$. A curve γ is called *rectifiable* if $\mathrm{length}(\gamma) < \infty$.

We call a curve $\gamma \subset \mathbb{R}^2$ *convex* (*closed convex*) if it is a closed connected subset of the boundary (respectively, the whole boundary) of the convex hull $\mathrm{co}(\gamma)$ of γ.

Let us consider the following example.

Example 1. Suppose that the boundary $\mathrm{bd}(K)$ of a convex figure K is the union of a line segment $[A,B]$ and a convex curve γ with the endpoints A and B. Then $K \subset \mathrm{co}(\gamma)$ and $\mathrm{length}(\gamma) = \mathrm{per}(K) - d(A,B)$. Moreover, $\mathrm{length}(\gamma) = \mathrm{per}(K) - \mathrm{diam}(K)$ if and only if $d(A,B) = \mathrm{diam}(K)$.

The main result of this note is the following.

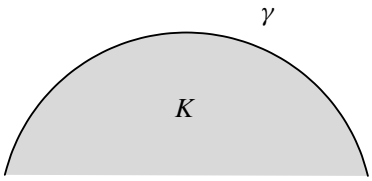

Figure 3. If K is a circular segment with the arc γ and the central angle subtending the arc is at most π, then the equality $\text{length}(\gamma) = \text{per}(K) - \text{diam}(K)$ holds.

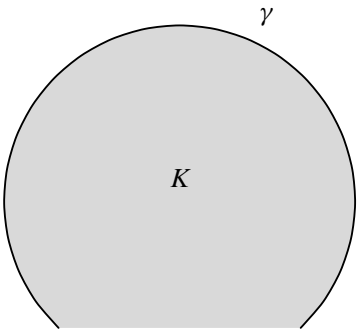

Figure 4. If K is a circular segment with the arc γ and the central angle subtending the arc is greater than π, then the inequality $\text{length}(\gamma) > \text{per}(K) - \text{diam}(K)$ holds.

Theorem 1. *For a given convex figure K and for any planar curve γ with the property $K \subset \text{co}(\gamma)$, the inequality*

$$\text{length}(\gamma) \geq \text{per}(K) - \text{diam}(K) \tag{1}$$

holds. Moreover, this inequality becomes an equality if and only if γ is a convex curve, $\text{bd}(K) = \gamma \cup [A, B]$, *and* $\text{diam}(K) = d(A, B)$, *where A and B are the endpoints of γ.*

Figures 3 and 4 illustrate the fulfillment of the equality in the inequality $\text{length}(\gamma) \geq \text{per}(K) - \text{diam}(K)$ for circular segments.

Remark 1. Since obviously $\text{per}(K) \geq 2\,\text{diam}(K)$, inequality (1) immediately implies the following widely known inequality: $\text{length}(\gamma) \geq \frac{1}{2}\,\text{per}(K)$; see e.g. [4].

The strategy of our proof is as follows. We fix a convex figure $K \subset \mathbb{R}^2$. Then we prove the existence of a curve γ_0 of minimal length among all curves γ satisfying the condition $K \subset \text{co}(\gamma)$ (Section 2). After that, we prove the inequality $\text{length}(\gamma_0) \geq \text{per}(K) - \text{diam}(K)$ and study all possible cases of the equality $\text{length}(\gamma_0) = \text{per}(K) - \text{diam}(K)$, where γ_0 is an arbitrary curve of minimal length among all curves γ satisfying the condition $K \subset \text{co}(\gamma)$ (Section 3). This allow us to get the proof of Theorem 1 in Section 4.

2 Some auxiliary results

To prove the desired results, we first recall some important properties of curves and convex figures.

Let us recall the following useful definition. A sequence of curves $\{\gamma_i\}_{i\in\mathbb{N}}$ converges uniformly to a curve γ if the curves γ_i admit parameterizations with the same domain that uniformly converges to some parameterization of γ. We will need the following result (see e.g. [3, Theorem 2.5.14]).

Proposition 1 (Arzela–Ascoli theorem for curves). *Given a compact metric space, any sequence of curves which have uniformly bounded lengths has a uniformly converging subsequence.*

We also note one important property (the lower semi-continuity of length) of the limit curve in the above assertion (see e.g. [3, Proposition 2.3.4]).

Proposition 2. *Suppose that a sequence of rectifiable curves* $\{\gamma_i\}_{i\in\mathbb{N}}$ *which converges pointwise to* γ *(with respect to parameterizations with the same domain) is given. Then the inequality* $\liminf_{i\to\infty} \mathrm{length}(\gamma_i) \geq \mathrm{length}(\gamma)$ *holds.*

The following property (of the monotonicity of perimeter) of convex figures is well known (see e.g. [2, § 7]).

Proposition 3. *If convex figures* K_1 *and* K_2 *in the Euclidean plane are such that* $K_1 \subset K_2$, *then* $\mathrm{per}(K_1) \leq \mathrm{per}(K_2)$, *and the equality holds if and only if* $K_1 = K_2$.

We need also the following well-known result (it could be proved using the Crofton formula; see e.g. [1, pp. 594–595]).

Proposition 4. *Let* $\varphi\colon [c,d] \to \mathbb{R}$ *be a parametric continuous curve with* $\varphi(c) = \varphi(d)$. *Then the length of the curve* $\gamma = \{\varphi(t) \mid t \in [c,d]\}$ *is greater than or equal to* $\mathrm{per}(\mathrm{co}(\gamma))$. *Moreover, the equality holds if and only if* γ *is a closed convex curve.*

Now, we are going to prove the following.

Proposition 5. *For a given convex figure* $K \subset \mathbb{R}^2$, *there is a curve* γ_0 *of minimal length among all curves* γ *satisfying the condition* $K \subset \mathrm{co}(\gamma)$.

Proof. If $\mathrm{int}(K) = \emptyset$, then the proposition is trivial. In what follows, we assume that $\mathrm{int}(K) \neq \emptyset$. Denote by $\Delta(K)$ the set of all planar curves γ such that $K \subset \mathrm{co}(\gamma)$. Let us consider $M = \inf\{\mathrm{length}(\gamma) \mid \gamma \in \Delta(K)\}$. It is clear that $M \leq \mathrm{per}(K)$ since $\mathrm{bd}(K)$ could be considered as a curve γ. Now, we consider the sequence of curves $\{\gamma_i\}_{i\in\mathbb{N}}$ from $\Delta(K)$ such that $\mathrm{length}(\gamma_i) \to M$ as $i \to \infty$. Without loss of generality, we may assume that $\mathrm{length}(\gamma_i) \leq M + 1$ for all $i = 1, 2, 3, \ldots$.

Let us take a point $O \in \mathrm{int}(K)$. There is $r > 0$ such that the ball with center O and radius r is inside K. For a fixed $i \in \mathbb{N}$, we consider the point $C_i \in \gamma_i$ such that $d(C_i, O) = \max\{d(x, O),\ x \in \gamma_i\}$ and the straight line l_i passing through O is perpendicular to the straight line OC_i. Since $K \subset \mathrm{co}(\gamma_i)$, there is a point $D_i \in \gamma_i$ such that the line segment $[C_i, D_i]$ intersects l_i. This means that

$$M + 1 \geq \mathrm{length}(\gamma_i) \geq d(C_i, D_i) \geq d(C_i, O) \geq r > 0.$$

It implies that $M \geq r > 0$ and

$$\gamma_i \subset B(O, M+1) := \{x \in \mathbb{R}^2,\ d(x, O) \leq M + 1\}.$$

Since the ball $B(O, M + 1)$ is compact and the lengths of the curves γ_i, $i = 1, 2, 3, \ldots$, are uniformly bounded, then the sequence $\{\gamma_i\}$ has a uniformly converging subsequence by Proposition 1. Passing to a subsequence if necessary, we can assume that the sequence

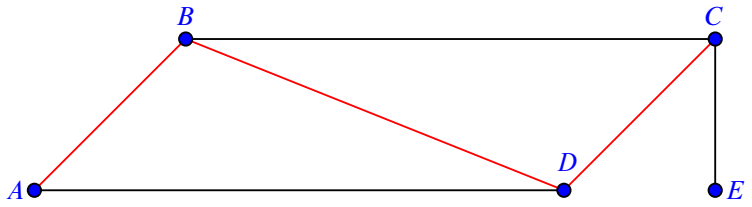

Figure 5. Illustration to Remark 3: a non-convex shortest curve γ.

$\{\gamma_i\}_{i\in\mathbb{N}}$ converges uniformly to some curve γ_0. Since $K \subset \mathrm{co}(\gamma_i)$ for $i = 1, 2, 3, \ldots$, then $K \subset \mathrm{co}(\gamma_0)$ too. The lower semi-continuity of length (see Proposition 2) implies $M = \lim_{i\to\infty} \mathrm{length}(\gamma_i) \geq \mathrm{length}(\gamma_0)$, and therefore, $\mathrm{length}(\gamma_0) = M$. This proves the proposition. ∎

Remark 2. Note that the curve γ_0 in Proposition 5 may not be unique. For instance, if K is an equilateral triangle, then the union of any two of its sides is such a curve.

Remark 3. Note also that the curve γ_0 in Proposition 5 could be non-convex. For instance, let K be the parallelogram $ABCD \subset \mathbb{R}^2$ with $A = (0, 0)$, $B = (1, 1)$, $C = (t + 1, 1)$ and $D = (t, 0)$, where $t \geq 1$. It is easy to see that the broken line $ABCE$ with $E = (t + 1, 0)$ is one of the shortest convex curves, whose convex hull covers K, and its length is $1 + \sqrt{2} + t$; see Figure 5. On the other hand, the length of the broken line $ABDC$ (whose convex hull is K) is equal to $2\sqrt{2} + \sqrt{2 - 2t + t^2}$. It is easy to check that

$$2\sqrt{2} + \sqrt{2 - 2t + t^2} < 1 + \sqrt{2} + t \quad \text{for } t > (3\sqrt{2} + 2)/4 \approx 1.5606.$$

The above discussion leads to the following natural problem.

Problem 1. Give a comprehensive description of the class of planar curves γ with the following property: there is a compact convex figure $K \subset \mathbb{R}^2$ such that γ is the shortest curves among all curves, whose convex hulls cover K.

In the next section, we give more detailed information about any curve of shortest length among all curves γ satisfying the condition $K \subset \mathrm{co}(\gamma)$ for a given K.

3 Some properties of shortest curves γ with $K \subset \mathrm{co}(\gamma)$

Let $U \subset \mathbb{R}^2$ be a convex figure. We say that a straight line $l \subset \mathbb{R}^2$ divides U into U_1 and U_2 if U_1 and U_2 are convex figures lying in different half-planes relative to l such that $U = U_1 \cup U_2$ and $U_1 \cap U_2 = U \cap l$.

We need the following two simple results.

Lemma 1. *Let $U \subset \mathbb{R}^2$ be a convex figure, and let us consider some points $E, F \in \mathrm{ext}(U)$. Then the straight line $l = EF$ divides U into convex figures U_1 and U_2 such that*

$$U_i = \mathrm{co}(\mathrm{ext}(U) \cap U_i), \quad i = 1, 2.$$

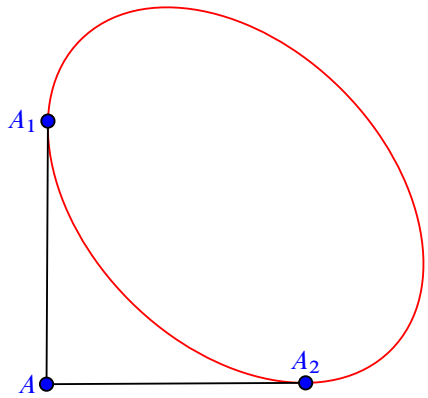

Figure 6. Illustration to Lemma 2: the convex figure U and $\triangle A_1AA_2$.

Proof. It is clear that $\mathrm{co}(\mathrm{ext}(U)\cap U_i)\subset U_i$. Let us suppose that $\mathrm{co}(\mathrm{ext}(U)\cap U_i)\neq U_i$. Then there is a point $C\in\mathrm{ext}(U_i)$ such that $C\notin\mathrm{co}(\mathrm{ext}(U)\cap U_i)$. On the other hand, $\mathrm{ext}(U_i)\subset\mathrm{ext}(U)$, and we obtain the contradiction. ∎

Lemma 2. *Let $U\subset\mathbb{R}^2$ be a convex figure. Let us suppose that a point $A\notin U$ and points $A_1, A_2\in U$ are such that the straight lines AA_1 and AA_2 are support lines for U and $AA_1\perp AA_2$. Then $d(A,A_1)+\mathrm{per}(U)>\mathrm{per}(\mathrm{co}(U\cup\{A\}))$.*

Proof. Let us consider the triangle A_1AA_2, and let γ^* be a part of $\mathrm{bd}(U)$ between the points A_1 and A_2 such that $U\subset\mathrm{co}(\gamma^*\cup\{A\})$ (Figure 6). It is clear that

$$\mathrm{bd}(\mathrm{co}(U\cup\{A\}))=\gamma^*\cup[A,A_1]\cup[A,A_2].$$

It is also clear that $\mathrm{per}(U)-\mathrm{length}(\gamma^*)$ is the length of the part of $\mathrm{bd}(U)$ complementary to γ^* between the points A_1 and A_2; hence, $\mathrm{per}(U)-\mathrm{length}(\gamma^*)\geq d(A_1,A_2)>d(A,A_2)$, and we get

$$d(A,A_1)+\mathrm{per}(U)>d(A,A_1)+\mathrm{length}(\gamma^*)+d(A,A_2)=\mathrm{per}(\mathrm{co}(U\cup\{A\})),$$

which proves the lemma. ∎

Let us fix a curve γ with an arc length parameterization $\varphi(t)$, $t\in[a,b]$, such that $K\subset\mathrm{co}(\gamma)$ and it has minimal possible length among all curves with this property. We put $A:=\varphi(a)$, $B:=\varphi(b)$ and $\widetilde{K}:=\mathrm{co}(\gamma)$.

Lemma 3. *In the above notation, $A,B\in\mathrm{ext}(\widetilde{K})$ and $A\neq B$. Moreover, $K\cap[A,B]\neq\emptyset$.*

Proof. Let us suppose that $A\notin\mathrm{ext}(\widetilde{K})$; then there is a sufficiently small $\varepsilon>0$ such that $\varphi([a,a+\varepsilon])\cap\mathrm{ext}(\widetilde{K})=\emptyset$ (recall that $\mathrm{ext}(\widetilde{K})$ is a closed set in $\mathbb{R}^2$). Hence, if we modify γ up to $\gamma_1:=\{\varphi(t)\mid t\in[a+\varepsilon,b]\}$, then we get a shorter curve with the same convex

hull. This contradictions shows that $A = \varphi(a) \in \operatorname{ext}(\widetilde{K})$. Similar arguments imply that $B = \varphi(\beta) \in \operatorname{ext}(\widetilde{K})$.

Suppose that $B = A$. Let us consider a support line l_1 for $\widetilde{K}$ through the point B. Since $B \in \operatorname{ext}(\widetilde{K})$, we may take a point $C \in l_1$ and a support line l_2 for $\widetilde{K}$ through C such that $C \notin \widetilde{K}$ and l_2 is perpendicular to l_1. Now, take a point $D \in \widetilde{K} \cap l_2$. Let γ^* be a part of $\operatorname{bd}(\widetilde{K})$ between the points B and D such that $\widetilde{K} \subset \operatorname{co}(\gamma^* \cup [C, D])$. Lemma 2 and Proposition 4 imply $d(C, D) + \operatorname{length}(\gamma^*) < \operatorname{per}(\widetilde{K}) \le \operatorname{length}(\gamma)$. Hence, the curve $\gamma^* \cup [C, D]$ is shorter than γ, and we get a contradiction due to $\widetilde{K} \subset \operatorname{co}(\gamma^* \cup [C, D])$. Therefore, $B \neq A$.

Let us suppose that $K \cap [A, B] = \emptyset$. Then the distance $\min\{d(x, y) \mid x \in K, y \in l\}$ between K and the straight line $AB =: l$ is positive (recall that $K \subset \widetilde{K}$ and A, B are extreme points of $\widetilde{K}$). Therefore, $K \subset \operatorname{co}\{\psi(t) \mid t \in [a + \varepsilon, b - \varepsilon]\} \subset \operatorname{co}(\gamma)$ for sufficiently small $\varepsilon > 0$. Since the curve $\gamma_2 := \{\psi(t) \mid t \in [a + \varepsilon, b - \varepsilon]\}$ is shorter than γ, we get a contradiction. This proves that $K \cap [A, B] \neq \emptyset$. ∎

Proposition 6. *Let us consider $\alpha, \beta \in [a, b]$ such that $\varphi(\alpha), \varphi(\beta) \in \operatorname{ext}(\widetilde{K})$. Then one of the following assertions holds:*

(1) $[\varphi(\alpha), \varphi(\beta)] \subset \operatorname{bd}(\widetilde{K})$;

(2) *the straight line l through the points $\varphi(\alpha)$ and $\varphi(\beta)$ divides $\widetilde{K}$ into $\widetilde{K}_1$ and $\widetilde{K}_2$ such that $(\widetilde{K}_i \setminus [\varphi(\alpha), \varphi(\beta)]) \cap K \neq \emptyset$, $i = 1, 2$.*

Proof. Let us suppose that $[\varphi(\alpha), \varphi(\beta)] \not\subset \operatorname{bd}(\widetilde{K})$; then every $\widetilde{K}_i$, $i = 1, 2$, has a point C_i from $\operatorname{ext}(\widetilde{K}) \setminus \{\varphi(\alpha), \varphi(\beta)\}$. It is clear that $C_i = \varphi(t_0)$ for some $t_0 \in [a, b]$.

If $(\widetilde{K}_i \setminus [\varphi(\alpha), \varphi(\beta)]) \cap K = \emptyset$, then $K \subset \operatorname{co}(\operatorname{ext}(\widetilde{K}_j))$, $j \in \{1, 2\} \setminus \{i\}$, by Lemma 3. Now, we will show how one can modify γ to a curve γ_1 which is shorter than γ, but $K \subset \operatorname{co}(\gamma_1)$.

If $t_0 = a$ ($t_0 = b$), then we can take a sufficiently small $\varepsilon > 0$ such that

$$\varphi([a, a + \varepsilon]) \cap l = \emptyset \quad (\text{respectively, } \varphi([b - \varepsilon, b]) \cap l = \emptyset).$$

Then we see that $K \subset \operatorname{co}(\operatorname{ext}(\widetilde{K}_j)) \subset \operatorname{co}(\gamma_1)$, where $\gamma_1 = \{\varphi(t) \mid t \in [a + \varepsilon, b]\}$ (respectively, $\gamma_1 = \{\varphi(t) \mid t \in [a, b - \varepsilon]\}$). Hence, we have found a curve that is shorter than γ and whose convex hull contains K, which is impossible.

If $t_0 \in (a, b)$, then we can take $t_1, t_2 \in [a, b]$, $t_1 < t_2$, such that $t_0 \in (t_1, t_2)$ and $\varphi([t_1, t_2]) \cap l = \emptyset$. Since $\varphi(t_0) \in \operatorname{ext}(\widetilde{K})$, then $\varphi([t_1, t_0]) \neq [\varphi(t_1), \varphi(t_2)]$. Now, we consider a curve $\gamma_2 = (\gamma \setminus \varphi([t_1, t_0])) \cup [\varphi(t_1), \varphi(t_2)]$. Obviously, γ_2 is shorter than γ, but its convex hull still contains K. This contradiction proves the proposition. ∎

Corollary 1. *Suppose that $\varphi(t_0)$ is an extreme point of $\widetilde{K}$ and it is not isolated in the set $\operatorname{ext}(\widetilde{K})$. Then $\varphi(t_0) \in K$.*

Proof. Let us take a sequence $\{t_n\}_{n \in \mathbb{N}}$, $t_n \in [a, b]$, such that all points $\varphi(t_n)$ are extreme for $\widetilde{K}$, $\varphi(t_n) \neq \varphi(t_0)$, $[\varphi(t_0), \varphi(t_n)] \not\subset \operatorname{bd}(\widetilde{K})$, and $\varphi(t_n) \to \varphi(t_0)$ as $n \to \infty$. By Proposition 6, the straight line l_n through the points $\varphi(t_n)$ and $\varphi(t_0)$ divides $\widetilde{K}$ into two convex figures; each of them contains some point of K. Let $\widetilde{K}_n$ be a one of these two figures, which has a smaller diameter. It is clear that $\operatorname{diam}(\widetilde{K}_n) \to 0$ as $n \to \infty$. If $C_n \in \widetilde{K}_n \cap K$, then $C_n \to \varphi(t_0)$ as $n \to \infty$. Since K is closed, we get $\varphi(t_0) \in K$. ∎

By Lemma 3, the points A and B are extreme points of $\widetilde{K}$. If A (respectively, B) is not an isolated point in the set $\mathrm{ext}(\widetilde{K})$, then $A \in K$ (respectively, $B \in K$). The following proposition deals with the case when A (or B) is isolated in $\mathrm{ext}(\widetilde{K})$.

Proposition 7. *If $A = \varphi(a)$ is isolated in $\mathrm{ext}(\widetilde{K})$, then there are $\tau_1, \tau_2 \in (a, b]$, $\tau_1 < \tau_2$, such that the following assertions hold:*

(1) $[A, \varphi(\tau_1)] \cup [A, \varphi(\tau_2)]$ *covers some neighborhood of A in* $\mathrm{bd}(\widetilde{K})$*;*

(2) $\varphi([a, \tau_1]) = [A, \varphi(\tau_1)]$*;*

(3) $\varphi([a, \tau_2]) \cup [A, \varphi(\tau_2)]$ *is a closed convex curve;*

(4) $[A, \varphi(\tau_1)] \cap K \neq \emptyset$*;*

(5) *the angle between the line segments $[A, \varphi(\tau_1)]$ and $[A, \varphi(\tau_2)]$ is equal to $\pi/2$.*

Similar results hold for $B = \varphi(b)$ if it is isolated in $\mathrm{ext}(\widetilde{K})$.

Proof. Since the point A is extreme in $\widetilde{K}$ and isolated in $\mathrm{ext}(\widetilde{K})$, then there are points $A_1, A_2 \in \mathrm{ext}(\widetilde{K}) \subset \mathrm{bd}(\widetilde{K})$ such that $[A, A_1], [A, A_2] \subset \mathrm{bd}(\widetilde{K})$ and $[A, A_1] \cup [A, A_2]$ covers some neighborhood of A in $\mathrm{bd}(\widetilde{K})$ (roughly speaking, A_1 and A_2 are closest extreme points to A with respect to different directions on $\mathrm{bd}(\widetilde{K})$). It is clear that $A_1 = \varphi(\tau_1)$ and $A_2 = \varphi(\tau_2)$ for some $\tau_1, \tau_2 \in (a, b]$. Without loss of generality, we may suppose that $0 < \tau_1 < \tau_2$.

Let us consider the following closed curves:

$$\gamma_1 = \varphi([a, \tau_1]) \cup [A, \varphi(\tau_1)], \quad \gamma_2 = \varphi([a, \tau_2]) \cup [A, \varphi(\tau_2)].$$

By Proposition 4, we get that $\mathrm{length}(\gamma_1) \geq \mathrm{per}(\mathrm{co}(\gamma_1))$ and $\mathrm{length}(\gamma_2) \geq \mathrm{per}(\mathrm{co}(\gamma_2))$. Since $[A, A_1], [A, A_2] \subset \mathrm{bd}(\widetilde{K})$, then $[A, A_1] \subset \mathrm{bd}(\mathrm{co}(\gamma_1))$ and $[A, A_2] \subset \mathrm{bd}(\mathrm{co}(\gamma_2))$. Due to the inclusion $\gamma_i \subset \mathrm{co}(\gamma_i)$, $i = 1, 2$, we may replace the curve γ with the curve

$$\gamma_i^- := \varphi([\tau_i, b]) \cup (\mathrm{bd}(\mathrm{co}(\gamma_i)) \setminus [A, A_i])$$

with the same convex hull $\widetilde{K}$. Since γ has minimal length among all curves whose convex hull covers K, we get $\mathrm{length}(\gamma_i) = \mathrm{per}(\mathrm{co}(\gamma_i))$ by Proposition 4. It means that γ_1 and γ_2 are closed convex curves by Proposition 4 (see Figure 7).

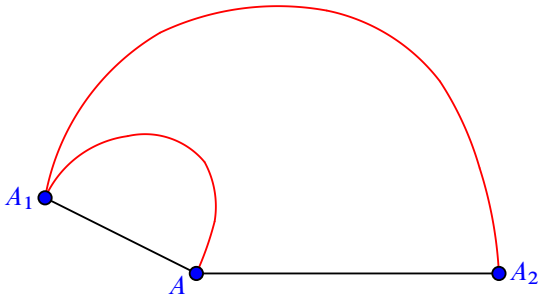

Figure 7. Illustration to the proof of Proposition 7: the curves γ_1 and γ_2.

Since $A, A_1 \in \mathrm{co}(\gamma_2)$, then $[A, A_1] \subset \mathrm{co}(\gamma_2)$. On the other hand, $[A, A_1] \subset \mathrm{bd}(\widetilde{K})$. Since $\mathrm{co}(\gamma_2) \subset \widetilde{K}$, we get $[A, A_1] \subset \mathrm{bd}(\mathrm{co}(\gamma_2))$. It implies that $[A, A_1] = \varphi([a, \tau_1])$ and $[A, A_2] \neq \varphi([a, \tau_2])$. Therefore, assertions (1)–(3) are proved.

Let us prove (4). If $[A, \varphi(\tau_1)] \cap K = \emptyset$, then there is $\varepsilon > 0$ such that $\mathrm{co}(\varphi([a, \tau_1 + \varepsilon]))$ and K are situated in different half-planes with respect to some straight line. Therefore, $K \subset \mathrm{co}(\gamma_3)$, where $\gamma_3 := \varphi([\tau_1 + \varepsilon, b]) \cup [A, \varphi(\tau_1 + \varepsilon)]$. On the other hand, γ_3 is shorter than γ (recall that $\varphi(\tau_1)$ is extreme in $\widetilde{K}$; hence, $\varphi([a, \tau_1 + \varepsilon]) \neq [A, \varphi(\tau_1 + \varepsilon)]$). This contradiction implies $[A, \varphi(\tau_1)] \cap K \neq \emptyset$.

Finally, let us prove (5). If $\angle A_1 A A_2 \neq \pi/2$, then we can take $A' \in [A, A_1]$ such that $A' \neq A$ and $d(A, A')$ is less than the distance from A to K. Then $A' = \varphi(\tau')$ for some $\tau' \in (a, \tau_1)$. Now, take a point $A'' \in [A, A_2]$ such that $[A', A'']$ is orthogonal to $[A, A_2]$. If we consider $\gamma_4 := \varphi([\tau', b]) \cup [A', A'']$, then $K \subset \mathrm{co}(\gamma_4)$ and $\mathrm{length}(\gamma_4) < \mathrm{length}(\gamma)$ (since the leg is shorter than the hypotenuse in any right triangle). This contradiction shows that $\angle A_1 A A_2 = \pi/2$.

Similar results for the point B we get automatically, reversing the parameterization of the curve γ. The proposition is completely proved. ∎

Proposition 8. *In the above notation, let η_1 be the smallest number in T, and let η_2 be the largest number in T, where $T = \{t \in [a, b] \mid \varphi(t) \in K\}$. Then the following inequality holds:*

$$\mathrm{length}(\gamma) + d(\varphi(\eta_1), \varphi(\eta_2)) \geq \mathrm{per}(\widetilde{K}) \geq \mathrm{per}(K).$$

Proof. Since $K \subset \widetilde{K}$, then the inequality $\mathrm{per}(\widetilde{K}) \geq \mathrm{per}(K)$ follows directly from Proposition 3. Therefore, it suffices to prove the inequality

$$\mathrm{length}(\gamma) + d(\varphi(\eta_1), \varphi(\eta_2)) \geq \mathrm{per}(\widetilde{K}). \tag{2}$$

We have $\varphi([a, \eta_1]) = [A, \varphi(\eta_1)] \subset \mathrm{bd}(\widetilde{K})$ and $\varphi([\eta_2, b]) = [\varphi(\eta_2), B] \subset \mathrm{bd}(\widetilde{K})$ by Proposition 7. Proposition 7 also implies that there is $\theta_1 \in (a, b]$ such that $[A, \varphi(\eta_1)] \cup [A, \varphi(\theta_1)]$ covers a neighborhood of A in $\mathrm{bd}(\widetilde{K})$ if $A \notin K$ and there is $\theta_2 \in [a, b)$ such that $[B, \varphi(\eta_2)] \cup [B, \varphi(\theta_2)]$ covers a neighborhood of B in $\mathrm{bd}(\widetilde{K})$ if $B \notin K$ (note that $\theta_1 = b$ if and only if $\theta_2 = a$).

Let us consider $\widehat{\gamma} = \varphi([\eta_1, \eta_2])$ and $\widehat{K} = \mathrm{co}(\widehat{\gamma})$. Note that $\widehat{K} \subset \widetilde{K}$ and $\widehat{K}$ contains all extreme points of $\widetilde{K}$ with the possible exception of points A and B (the latter is possible only if A or B is not in K). Therefore, $\widetilde{K} = \mathrm{co}(\widehat{K} \cup \{A, B\})$.

Since $\widehat{\gamma} \cup [\varphi(\eta_1), \varphi(\eta_2)]$ is a closed curve, Proposition 4 implies the inequality

$$\mathrm{length}(\widehat{\gamma}) + d(\varphi(\eta_1), \varphi(\eta_2)) \geq \mathrm{per}(\widehat{K}). \tag{3}$$

Let us consider the following four cases: (1) $A, B \in K$, (2) exactly one of the points A and B is in K, (3) $A, B \notin K$ and $\theta_1 < b$, (4) $A, B \notin K$ and $\theta_1 = b$.

In case (1), we have $\gamma = \widehat{\gamma}$ and $\widehat{K} = \widetilde{K}$; hence, (3) implies

$$\mathrm{length}(\gamma) + d(\varphi(\eta_1), \varphi(\eta_2)) \geq \mathrm{per}(\widetilde{K}),$$

and we got what we need.

Let us consider case (2). Without loss of generality, we may assume that $B \in K$ (hence, $\eta_2 = b$ and $\varphi(\eta_2) = B$) and $A \notin K$. Hence, $\widetilde{K} = \mathrm{co}(\widehat{K} \cup \{A\})$. Let us consider the triangle A_1AA_2, where $A_1 = \varphi(\eta_1)$ and $A_2 = \varphi(\theta_1)$. By Proposition 7, we have $\angle A_1AA_2 = \pi/2$. Since $A_1, A_2 \in \mathrm{bd}(\widetilde{K}) \cap \widehat{\gamma}$, we get that $A_1, A_2 \in \mathrm{bd}(\widehat{K})$. Then (3) and Lemma 2 imply

$$\begin{aligned}\mathrm{length}(\gamma) + d(\varphi(\eta_1), \varphi(\eta_2)) &= d(A, A_1) + \mathrm{length}(\widehat{\gamma}) + d(\varphi(\eta_1), \varphi(\eta_2)) \\ &\geq d(A, A_1) + \mathrm{per}(\widehat{K}) > \mathrm{per}(\widetilde{K}),\end{aligned}$$

which proves (2).

To deal with case (3), consider the triangles A_1AA_2 and B_1BB_2, where $A_1 = \varphi(\eta_1)$, $A_2 = \varphi(\theta_1)$, $B_1 = \varphi(\eta_2)$ and $B_2 = \varphi(\theta_2)$. By Proposition 7, $\angle A_1AA_2 = \angle B_1BB_2 = \pi/2$. Note that $\theta_1 < \eta_2$ and $\eta_1 < \theta_2$. Since $A_1, A_2, B_1, B_2 \in \mathrm{bd}(\widetilde{K}) \cap \widehat{\gamma}$, we get that

$$A_1, A_2, B_1, B_2 \in \mathrm{bd}(\widehat{K}).$$

Then (3) and Lemma 2 imply

$$\begin{aligned}\mathrm{length}(\gamma) + d(\varphi(\eta_1), \varphi(\eta_2)) &= d(A, A_1) + d(B, B_1) + \mathrm{length}(\widehat{\gamma}) + d(\varphi(\eta_1), \varphi(\eta_2)) \\ &\geq d(A, A_1) + d(B, B_1) + \mathrm{per}(\widehat{K}) \\ &> d(A, A_1) + \mathrm{per}(\mathrm{co}(\widehat{K} \cup \{B\})) \\ &> \mathrm{per}(\mathrm{co}(\mathrm{co}(\widehat{K} \cup \{B\}) \cup \{A\})) \\ &= \mathrm{per}(\mathrm{co}(\widehat{K} \cup \{A, B\})) = \mathrm{per}(\widetilde{K}),\end{aligned}$$

which proves (2).

Finally, let us consider case (4). In this case, we have $[A, B] \subset \mathrm{bd}(\widetilde{K})$, $A_2 = B$ and $A = B_2$. Let us consider the quadrangle AA_1B_1B, where $A_1 = \varphi(\eta_1)$ and $B_1 = \varphi(\eta_2)$. By Proposition 7, we have $\angle A_1AB = \angle B_1BA = \pi/2$. Since $A_1, B_1 \in \mathrm{bd}(\widetilde{K}) \cap \widehat{\gamma}$, we get that $A_1, B_1 \in \mathrm{bd}(\widehat{K})$.

We denote by γ_3 a part of $\mathrm{bd}(\widehat{K})$ between A_1 and A_2 such that $\widetilde{K} \subset \mathrm{co}(\gamma_3 \cup \{A, B\})$ (see Figure 8). It is clear that

$$\mathrm{bd}(\widetilde{K}) = \gamma_3 \cup [A, A_1] \cup [B, B_1] \cup [A, B].$$

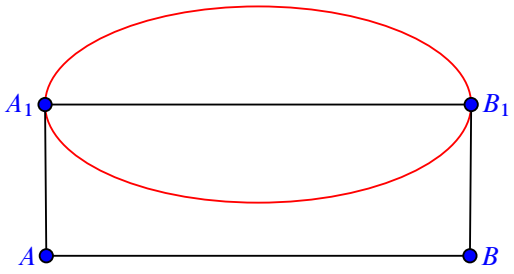

Figure 8. Illustration to case (4) in the proof of Proposition 8.

Note that $\mathrm{per}(\widehat{K}) - \mathrm{length}(\gamma_3)$ is the length of the curve $(\mathrm{bd}(\widehat{K}) \setminus \gamma_3) \cup \{A_1, B_1\}$, connecting the points A_1 and B_1. Hence, $\mathrm{per}(\widehat{K}) - \mathrm{length}(\gamma_3) \geq d(A_1, B_1) \geq d(A, B)$, and we get

$$\begin{aligned}\mathrm{length}(\gamma) + d(\varphi(\eta_1), \varphi(\eta_2)) &= d(A, A_1) + d(B, B_1) + \mathrm{length}(\widehat{\gamma}) + d(\varphi(\eta_1), \varphi(\eta_2)) \\ &\geq \mathrm{per}(\widehat{K}) + d(A, A_1) + d(B, B_1) \\ &\geq \mathrm{length}(\gamma_3) + d(A, B) + d(A, A_1) + d(B, B_1) \\ &= \mathrm{per}(\widetilde{K}).\end{aligned}$$

Hence, we have proved (2) for all possible cases. The proposition is proved. ∎

Remark 4. We see from the above proof that the equality

$$\mathrm{length}(\gamma) + d(\varphi(\eta_1), \varphi(\eta_2)) = \mathrm{per}(\widetilde{K})$$

is fulfilled if and only if $\varphi([a, b]) \cup [A, B]$ is a convex curve (that coincides with $\mathrm{bd}(\widetilde{K})$) and the quadrangle AA_1B_1B, where $A_1 = \varphi(\eta_1)$ and $A_2 = \varphi(\theta_1)$, is a rectangle (in particular, $A_1 = A$ and $B_1 = B$). Consequently, since $\mathrm{per}(\widetilde{K}) = \mathrm{per}(K)$ implies $\widetilde{K} = K$, the equality

$$\mathrm{length}(\gamma) + d(\varphi(\eta_1), \varphi(\eta_2)) = \mathrm{per}(K)$$

is fulfilled if and only if $\varphi([a, b]) \cup [A, B] = \mathrm{bd}(K)$.

Since $\mathrm{diam}(K) \geq d(\varphi(\eta_1), \varphi(\eta_2))$, then Proposition 8 and Remark 4 imply the following corollary.

Corollary 2. *If a curve γ has shortest length among all curves whose convex hulls cover a given compact convex figure K, then the following inequality holds:*

$$\mathrm{length}(\gamma) + \mathrm{diam}(K) \geq \mathrm{per}(K).$$

Moreover, the equality in this inequality is fulfilled if and only if γ is convex, $\mathrm{bd}(K) = \gamma \cup [A, B]$ *and* $\mathrm{diam}(K) = d(A, B)$*, where A and B are the endpoints of the curve γ.*

4 Proof of Theorem 1

Let us fix a convex figure $K \subset \mathbb{R}^2$. By Proposition 5, there is a curve γ_0 of minimal length among all curves γ satisfying the condition $K \subset \mathrm{co}(\gamma)$. By Corollary 2, we get

$$\mathrm{length}(\gamma) + \mathrm{diam}(K) \geq \mathrm{length}(\gamma_0) + \mathrm{diam}(K) \geq \mathrm{per}(K)$$

for any curve γ such that $K \subset \mathrm{co}(\gamma)$, which proves (1). We have the equality in (1) if and only if $\mathrm{length}(\gamma) = \mathrm{length}(\gamma_0)$ (hence, we may assume that $\gamma = \gamma_0$ without loss of generality), γ is convex, $\gamma \cup [A, B] = \mathrm{bd}(K)$ and $\mathrm{diam}(K) = d(A, B)$, where A and B are the endpoints of the curve γ. Therefore, we obtain just convex figures K and corresponding curves γ exactly as in Example 1.

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