Elem. Math. (Online first)
DOI 10.4171/EM/459

Elemente der Mathematik

Two statements characterizing the Euclidean metric of a metric plane

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1 Introduction

E. E. Moise [8, p. 415] observed that the existence of an area function in an absolute plane such that the area of a triangle depends only on its base and its altitude is equivalent with the existence of a rectangle in that plane. This result, with continuity being taken as one of the axioms of absolute geometry, was re-proved in [13].

A like-minded result can be inferred from O. Bottema's analysis in [2] of the ratios in which the intersection point G of the medians of a triangle divides a given median. By means of computations in the hyperbolic plane, Bottema showed that, while it is possible for G to divide one median in the ratio $1:2$ (which is the case in Euclidean geometry), it cannot divide two medians in that $1:2$ ratio. One can thus conclude that, if there exists

Die euklidische Natur einer Ebene kann inzidenzgeometrisch in der Formulierung des klassischen Parallelenaxioms ausgedrückt werden, oder aber metrisch durch die Forderung der Existenz eines Rechtecks. Die inzidenzgeometrische Form ist, wie Max Dehn 1900 bewiesen hat, stärker als die metrische Form, die im vorliegenden Beitrag mit **R** bezeichnet wird. Der Autor zeigt nun die Gleichwertigkeit von **R** mit zwei Aussagen, nämlich *„es gibt zwei nicht-kongruente Dreiecke gleichen Flächeninhalts“* und *„es gibt ein Dreieck, dessen Seitenhalbierenden sich im Verhältnis* $1:2$ *schneiden“*. Die absolute Geometrie bezüglich der die Gleichwertigkeit der ersten Aussage gilt, ist diejenige der nicht-elliptischen metrischen Ebenen, in denen jedes Punktepaar einen Mittelpunkt hat, und für die zweite Aussage die der metrischen Ebenen, die eingehend von Friedrich Bachmann in seinem Buch *Aufbau der Geometrie aus dem Spiegelungsbegriff* untersucht wurden.

a triangle in an absolute plane in which the point of intersection of two medians divide each median in the ratio 1 : 2, then there exists a rectangle in that plane.

Our aim in this note is to prove these two results under the weakest imaginable assumptions regarding the "absolute plane".

In the case of the first result, we need to first elucidate what we mean by "area". While the notion of area can be made precise in Hilbert's plane absolute geometry $\mathcal{A}$ (the model of which will be referred to as *Hilbert planes*), whose axioms are the plane axioms of groups I, II, and III of Hilbert's *Grundlagen der Geometrie* (Foundations of Geometry) [5], as shown in [3], we will be looking for an even more general setting. We will be interested only in the area equality of two triangles with a common side, as we aim to show that, even if there exist two non-congruent triangles sharing a side and having congruent altitudes to that common side that have the same area, then there must exist a rectangle. Two triangles ABC and $AB'C$ in a Hilbert plane with Euclidean metric (that is, a Hilbert plane in which there is a rectangle) have the same area if and only if the altitude from B to AC and the altitude from B' to AC are congruent. In the case with non-Euclidean metric (that is, if there is no rectangle in the Hilbert plane), triangles ABC and $AB'C$ have the same area if and only if the sum of the angles is the same in both triangles (see [3] for more on area in the case of a non-Euclidean metric).

While in the form stated above the area equality of two triangles needs free mobility and order for its very expression, we can express this fact in a different manner in both the Euclidean and the non-Euclidean setting.

We start with the difficult case, expressing the notion of area equality in the case of a non-Euclidean metric. Given a triangle ABC, let U denote the midpoint of BC, let V denote the midpoint of AC, and let W denote the midpoint of AB. Let R denote the reflection of V in U, and let Q denote the reflection of V in W. Since $\widehat{VCU} \equiv \widehat{RBU}$ and $\widehat{VAW} \equiv \widehat{QBW}$, the angle $\widehat{QBR}$ represents the sum of the angles of triangle ABC (Figure 1). Since $BR \equiv BQ \equiv VC$, and VC is constant, given that A and C are fixed points, the isosceles triangle BQR has its two congruent sides of fixed length, and the angle between two sides is congruent to the sum of the angles of triangle ABC. The sum of the angles of triangles ABC and $AB'C$ are thus the same if and only if $QR \equiv Q'R'$, where by Q' and R' we have denoted the points obtained in the manner Q and R were, this time starting with triangle $AB'C$.

In the case of the Euclidean metric, the area equality of triangle ABC and ABC' amounts to the congruence of the altitudes from C and C'.

We thus need a geometry in which we can express segment congruence, and in which any pair of points has a unique midpoint.

The congruence core of plane absolute geometry was investigated in a series of papers by J. Hjelmslev [6, 7] and reached a particularly simple form in the axiomatics presented by F. Bachmann [1] for structures called *metric planes*. Although this is arguably the most important achievement in distance geometry since Euclid, the notion of a metric plane is not part of the active vocabulary of present-day mathematicians. It is for this reason that we will introduce them here, for the reader's convenience.

It will turn out that our theorems can be expressed and proved in the theory of non-elliptic metric planes in which every segment has a midpoint. Why "non-elliptic"? Metric planes are called *elliptic* if there are three line reflections in them whose composition is the

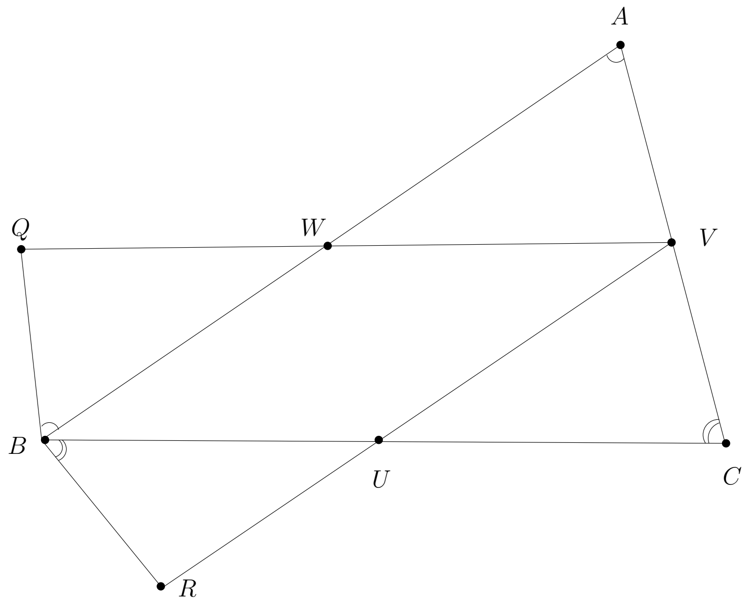

Figure 1. $\widehat{QBR}$ represents the sum of the angles of triangle ABC.

identity. Since this amounts to the existence of points which have several perpendiculars to a given line and the uniqueness of midpoints is no longer given, the basis for our reduction of the area problem to one about reflections vanishes. It also makes sense in the medians problem to restrict our attention to the non-elliptic case.

2 Metric planes

Metric planes, as presented in [1] (for different axiomatization, see [9, 10]), are defined in the language of groups. The idea behind the axiomatization is to focus on line reflections and on their properties when composed with other line reflections. One notices that if a and b are line reflections, and if $ab = ba$, then the lines defined by the reflections a and b must be perpendicular, and conversely. Since perpendicular lines intersect, and the composition of the reflections in them amounts to the reflection in their point of intersection, we can choose to refer to the product ab, in case $ab = ba$, as to a *point reflection*. Another familiar geometric notion, that of point-line incidence, can now be expressed in the language of line reflections. The point P determined by the point reflection ab, where a and b denote line reflections with $ab = ba$, is said to be incident with the line determined by the line reflection g if $Pg = gP$.

The axiom system consists of a *fundamental assumption*, that G be a group (written multiplicatively) generated by an invariant set S of involutory elements, as well as five axioms (see below). The elements of S will be denoted by lowercase Latin letters, and will be referred to as *line reflections* (or simply *lines*). Involutory products of two elements in S will be denoted by uppercase Latin letters, and will be referred to as *point reflections* (or simply *points*). For any two elements α and β in G, $\alpha \mid \beta$ denotes the fact that $\alpha \cdot \beta$ is involutory. The notation $\alpha_1, \dots, \alpha_n \mid \beta_1, \dots, \beta_m$ stands for the conjunction of all $\alpha_i \mid \beta_j$ with $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. The relation $P \mid g$ (or $g \mid P$) may be read as "point P is incident with line g", and $g \mid h$ as "line g is orthogonal to line h". The axioms for metric planes can be stated as follows.

A1. *For all points P and Q, there is a line g such that $P, Q \mid g$.*

A2. *If, for points P and Q and lines g and h, we have $P, Q \mid g, h$, then $P = Q$ or $g = h$.*

A3. *If the lines a, b, and c and the point P are such that $a, b, c \mid P$, then there is a line d such that $abc = d$.*

A4. *If the lines a, b, c, g are such that $a, b, c \mid g$, then there is a line d such that $abc = d$.*

A5. *There are lines g, h, j such that $g \mid h$, but none of $j \mid g$, $j \mid h$, or $j \mid gh$ holds.*

By **A1** and **A2**, there is a unique line joining two distinct points P and Q. We will denote it by $\langle P, Q \rangle$. It is, in general, not true that a midpoint exists for every pair of points. In this context, we say that M is a midpoint for A and B if $A^M = B$. Here we have denoted by α^β the element $\beta^{-1}\alpha\beta$ (so A^B denotes the reflection of A in B, P^g the reflection of P in g, and g^P the reflection of g in P). Nor is it in general true that a midpoint, should it exist, is unique. In fact, to ensure uniqueness (see [1, pp. 53–54]), it is necessary and sufficient to assume that the metric plane is not elliptic. This means that it satisfies the following axiom (here 1 stands for the neutral element of G).

∼P. *For all lines a, b, c, we have $abc \neq 1$.*

An alternate requirement, which is equivalent to **∼P**, is that, for any line g and any point P not incident with g, there exists a unique line h incident with P and perpendicular to g (if $P \mid g$, then the perpendicular through P to g is always unique, it is Pg). This requirement is obviously false in the case of plane elliptic geometry, in which, for every line, there is a point from which one can draw infinitely many perpendiculars to that line (that there are infinitely many perpendiculars from a *pole* of a line to that line follows from [1, Satz 20, § 6,12, p. 121], which states that there are no finite elliptic planes, so every line must be incident with infinitely many points). Double elliptic geometry is already excluded by the axioms for metric planes, as there is a unique line joining two distinct points in metric planes.

Since the theorem we want to prove depends for the definition of the area equivalence on the uniqueness of midpoints, we will be interested only in non-elliptic metric planes. Since we also need the unrestricted existence of midpoints, we also assume the following axiom.

M. *For all points A, B, there exists a point M such that $A^M = B$.*

We will denote the midpoint M of the segment defined by the pair (A, B) by $\mu(A, B)$. Metric planes can have a Euclidean metric, which means that there exists a rectangle.

R. *There are lines a, b, c, d such that $a, b \mid c, d$ and $a \neq b$, $c \neq d$.*

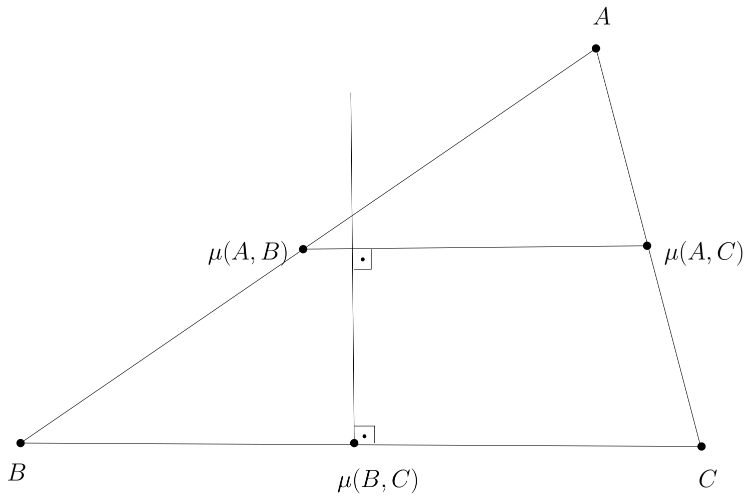

Figure 2. $\langle B, C\rangle\mu(B, C) \mid \langle\mu(A, B), \mu(A, C)\rangle$.

Or they can have a non-Euclidean metric, which means that there is no rectangle.

∼R. *If the lines a, b, c, d are such that $a, b \mid c, d$, then $a = b$ or $c = d$.*

Let $\mathcal{M}$ denote the theory axiomatized by {**A1–A5**} and $\mathcal{M}^+$ the theory axiomatized by {**A1–A5**} ∪ {∼**P**, **M**}.

The congruence of a pair of segments (P, Q) and (R, S) (denoted by $(P, Q) \equiv (R, S)$) can be defined, with $M = \mu(P, R)$ and $U = Q^M$, by the validity of one of

$$U^R = S, \quad U = S, \quad \text{and} \quad R \nmid \langle U, S\rangle \wedge U^{\langle R, R^{\langle U,S\rangle}\rangle} = S.$$

We will make repeated use of the following result [1, Satz 2 of § 4,1, p. 57]), which holds in $\mathcal{M}$, and which will be used only in case A, B, and C are three non-collinear points:

$$\langle B, C\rangle\mu(B, C) \mid \langle\mu(A, B), \mu(A, C)\rangle. \tag{2.1}$$

It states that the perpendicular bisector of side BC of the triangle ABC is perpendicular to the midline that connects the midpoints of AB and AC (Figure 2).

3 Two non-congruent triangles sharing a side and having congruent altitudes have the same area

The theorem we will prove states that, given $\mathcal{M}^+$, if two triangles with the same base and congruent altitudes are not congruent but have the same area, then **R** must hold.

To even express this result, we must define, in the language of non-elliptic metric planes with midpoints, what we mean by the phrase “triangles ABC and ABC' have the same area”. Whenever we refer to a “triangle” we mean three non-collinear points.

Definition 1. (i) In a metric plane with midpoints in which **R** holds, the triangles ABC and ABC' are said to have the same area if and only if one of the midpoints of the point pairs (C, C') and $(C^{\langle A,B\rangle}, C')$ lies on $\langle A, B\rangle$.

(ii) In a non-elliptic metric plane with midpoints in which $\sim$**R** holds, the triangles ABC and ABC' are said to have the same area if and only if, with U the midpoint of (B, C), U' the midpoint of (B, C'), V the midpoint of (A, C), V' the midpoint of (A, C'), W the midpoint of (A, B), $R = V^U$, $R' = V'^{U'}$, $Q = V^W$, and $Q' = V'^W$, we have $(Q, R) \equiv (Q', R')$.

We also need to explain what "non-congruent triangles sharing a side and having congruent altitudes" is supposed to mean.

Definition 2. We say that the triangles ABC and ABC' are non-congruent triangles and have congruent altitudes if $C \neq C'$, $C' \neq C^{\langle A,B\rangle}$, and

$$\mu(C, X) = \mu\big(\mu(C, C^{\langle A,B\rangle}), \mu(C', C'^{\langle A,B\rangle})\big) \quad \text{and} \quad \mu(C, X) \neq \mu(A, B)$$

holds for $X = C'$ or $X = C'^{\langle A,B\rangle}$.

Here $\mu(C, C^{\langle A,B\rangle})$, $\mu(C', C'^{\langle A,B\rangle})$ stand for the feet of the perpendiculars from C and C' to $\langle A, B\rangle$.

With these explanations, we are ready to state our first theorem.

Theorem 1. *The following holds in $\mathcal{M}^+$: if there exist two non-congruent triangles ABC and ABC' which have congruent altitudes and the same area, then* **R** *holds.*

Proof. Suppose ABC and ABC' are two non-congruent triangles which have congruent altitudes and the same area. We know from [11, Theorem 3.2] that, with $X = C'$ or $X = C'^{\langle A,B\rangle}$, we have $\langle M, N\rangle = \langle M', N'\rangle$, where M, N, M', and N' are such that $M = \mu(A, C)$, $N = \mu(B, C)$, $M' = \mu(A, X)$, and $N' = \mu(B, X)$ (Figure 3). Since ABX and ABC have congruent altitudes,

$$\mu(C^{\langle A,B\rangle}, X) = \mu\big(\mu(C, C^{\langle A,B\rangle}), \mu(X, X^{\langle A,B\rangle})\big), \tag{3.1}$$

and since the triangles are not congruent, we have $\mu(A, B) \neq \mu(C^{\langle A,B\rangle}, X)$. Since both $\mu(C, C^{\langle A,B\rangle})$ and $\mu(X, X^{\langle A,B\rangle})$ are incident with $\langle A, B\rangle$, we deduce from (3.1) that $\mu(C^{\langle A,B\rangle}, X)$ is incident with $\langle A, B\rangle$. By (2.1), we have

$$\langle C, X\rangle\mu(C, X) \mid \langle\mu(C, C^{\langle A,B\rangle}), \mu(X, C^{\langle A,B\rangle})\rangle,$$
$$\langle C, X\rangle\mu(C, X) \mid \langle M, M'\rangle,$$
$$\langle A, B\rangle\mu(A, B) \mid \langle M', N'\rangle.$$

Since $\langle\mu(C, C^{\langle A,B\rangle}), \mu(X, C^{\langle A,B\rangle})\rangle = \langle A, B\rangle$ and $\langle M, M'\rangle = \langle M', N'\rangle = \langle M, N\rangle$, axiom **R** is now seen to be satisfied with $a = \langle C, X\rangle\mu(C, X)$, $b = \langle A, B\rangle\mu(A, B)$, $c = \langle A, B\rangle$, and $d = \langle M, M'\rangle$. ■

That, if **R** holds, there exist two non-congruent triangles ABC and ABC' which have congruent altitudes is trivial, so the axioms **R** and "there exist two non-congruent triangles ABC and ABC' which have congruent altitudes and the same area" are equivalent with respect to $\mathcal{M}^+$.

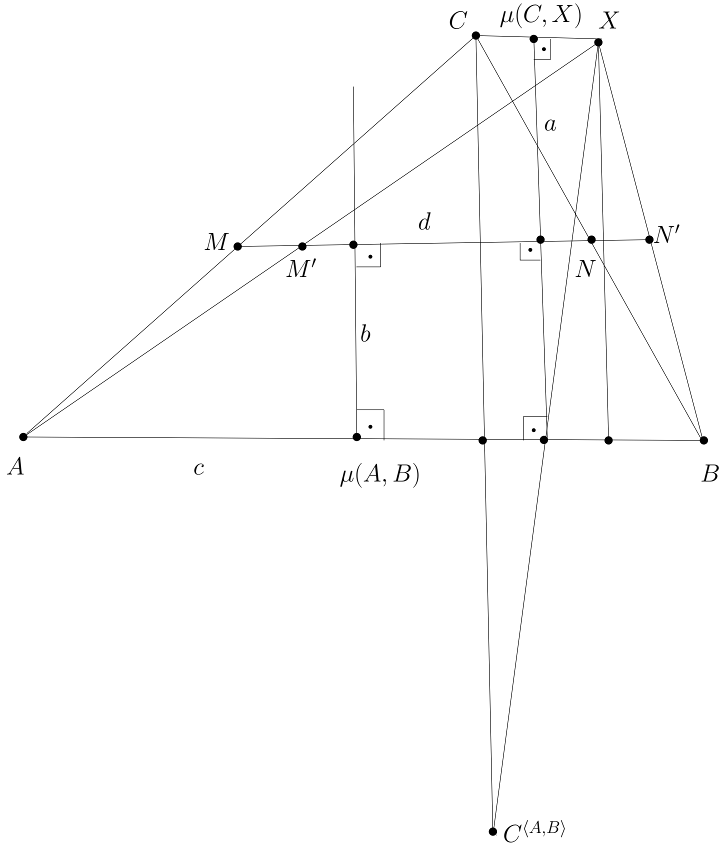

Figure 3. Lines a, b, c, and d form a rectangle.

4 Two medians intersect in a point which divides each median in the ratio 1 : 2

To express the fact that we are in the presence of a triangle with two medians intersecting in a point that divides each median in the ratio 1 : 2, we do not need to know that every segment has a midpoint, nor do we need to know that medians intersect in general. That is so because the existence of a triangle with that property is used in a conditional, if-then statement, for all we want to show is that the existence of such a triangle implies the existence of a rectangle. We will simply state that, if we are presented with three vertices of a triangle, with the midpoints for all sides, and with points on two medians witnessing the division in the 1 : 2 ratio of each median by a common point, then we can construct the vertices of a rectangle.

The base theory is now $\mathcal{M}$. We know from [1, Satz 1 of § 4,1, p. 56]) that,

$$\text{if } C^U = B \text{ and } B^V = A, \text{ then there exists a } W \text{ such that } C^W = A. \qquad (*)$$

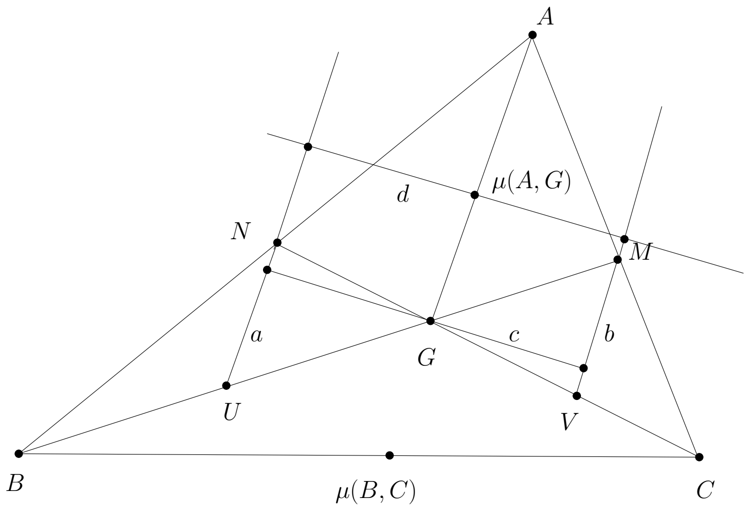

Figure 4. Lines a, b, c, and d form a rectangle.

Theorem 2. *The following holds in* $\mathcal{M}$*: if* A*,* B*, and* C *are three non-collinear points,* $A^N = B$*,* $A^M = C$*,* $B^U = G$*,* $U^G = M$*,* $C^V = G$*, and* $V^G = N$*, then* **R** *holds.*

Proof. (Figure 4) Since $G^U = B$ and $B^N = A$, there is, by ($*$), a point W such that $G^W = A$, which we denote by $\mu(A, G)$. By (2.1), we have

$$\langle A, G\rangle\mu(A, G) \mid \langle N, U\rangle, \quad \langle A, G\rangle\mu(A, G) \mid \langle M, V\rangle.$$

Reflections in lines preserve line-orthogonality, and reflections in points, being the product of two reflections in lines, also preserve line-orthogonality. We also have $P \mid g \to g^P = g$. Since $\langle G, G^{\langle N,U\rangle}\rangle \mid \langle N, U\rangle$ and $U^G = M$, $N^G = V$, and $\langle G, G^{\langle N,U\rangle}\rangle^G = \langle G, G^{\langle N,U\rangle}\rangle$, given that the reflection in G preserves orthogonality, we also have $\langle G, G^{\langle N,U\rangle}\rangle \mid \langle M, V\rangle$. We now notice that **R** holds with

$$a = \langle N, U\rangle, \quad b = \langle M, V\rangle, \quad c = \langle G, G^{\langle N,U\rangle}\rangle, \quad \text{and} \quad d = \langle A, G\rangle\mu(A, G). \qquad \blacksquare$$

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