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Short note The Milne-Thomson formula for the harmonic conjugate and its associated holomorphic function

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Abstract. In this note, we justify the formula by Milne-Thomson giving a direct determination of the holomorphic function from its real part on disks.

The goal of this note is to justify a very nice formula given by Milne-Thomson in [3] (see also [4, p. 132], [1, p. 21], [2] and [6, p. 144] e.g.) on the construction of the holomorphic function (modulo a purely imaginary additive constant) whose real part is a given harmonic function in an open disk. This formula, given in item (3) below (and always appreciated by my students), was just stated in these books/papers, but without an explanation of how to interpret $u(\frac{z}{2}, -i\frac{z}{2})$ in case $u(x, y)$ is harmonic in the real variables x, y. Here we close this gap, hoping that this formula finally finds its way into the curriculum of every introductory course in complex analysis.

Proposition 1. *Let D be the disk $\{\xi \in \mathbb{C} : |\xi| < r\}$ or $D = \mathbb{C}$. Suppose that $u\colon D \to \mathbb{R}$ is harmonic, and for $\xi = x + iy \in D$ with $x, y \in \mathbb{R}$, let $\tilde{D} := \{(x, y) \in \mathbb{R}^2 : x + iy \in D\}$ and $\tilde{u}(x, y) := u(x + iy)$. The following assertions hold.*

(1) *There is $f \in H(D)$ with $\operatorname{Re} f = u$.*

(2) *$\tilde{u}(x, y)$ can be extended to a function*

$$U\colon \begin{cases} B \to \mathbb{C}, \\ (z, w) \mapsto U(z, w) \end{cases}$$

holomorphic in a neighborhood B of the origin in $\mathbb{C}^2$ containing $\tilde{D}$ (when $\tilde{D}$ is viewed as a subset of $\mathbb{C}^2$). The extension is understood in the sense that $U(x, y) = \tilde{u}(x, y)$ for $(x, y) \in \tilde{D} \cap B = \tilde{D}$. In case $r < \infty$, B contains $\frac{r}{\sqrt{2}}\mathbb{B}_2$, where $\mathbb{B}_2$ is the unit ball in $\mathbb{C}^2$, and $B = \mathbb{C}^2$ if $D = \mathbb{C}$.

(3) *$f(z) + \overline{f(0)} = 2U(\frac{z}{2}, -i\frac{z}{2})$ for $z \in D$.*

In other words, we may formally replace the real arguments x, y of u by $\frac{z}{2}, -i\frac{z}{2}$ to directly obtain f (modulo a constant).

Proof. (1) This is well known (see e.g. [5, Theorem 12.42]). The idea is that the C^1-function $h := u_x - iu_y$ satisfies the Cauchy–Riemann equations due to $u_{xx} + u_{yy} = 0$. Hence h is holomorphic and admits a primitive $f(z) := \int_0^z h(\xi)\,d\xi$, say $f = a + ib$. Now

$$h = f' = f_x = a_x + ib_x = a_x - ia_y.$$

Consequently, $u_x = a_x$ and $u_y = a_y$. We conclude that $u(x, y) = a(x, y) + r$, $r \in \mathbb{R}$. Now we define v by $v := b$.

(2) Let $f \in H(D)$ satisfy Re $f = u$ in D. In particular, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for some $a_n \in \mathbb{C}$, the series converging locally uniformly. Thus, for $(x, y) \in \tilde{D}$,

$$2\tilde{u}(x, y) = \sum_{n=0}^{\infty} a_n (x + iy)^n + \sum_{n=0}^{\infty} \overline{a}_n (x - iy)^n.$$

Now, for $(z, w) \in \mathbb{C}^2$, put

$$2U(z, w) := \sum_{n=0}^{\infty} a_n (z + iw)^n + \sum_{n=0}^{\infty} \overline{a}_n (z - iw)^n.$$

These series converge absolutely for those $(z, w) \in \mathbb{C}^2$ satisfying $|z \pm iw| < r$ (if $r < \infty$), or in $\mathbb{C}^2$ if $D = \mathbb{C}$, to the function

$$F(z, w) := f(z + iw) + \overline{f(\overline{z} + i\overline{w})}.$$

Hence U is holomorphic in

$$B := \{(z, w) \in \mathbb{C}^2 : |z \pm iw| < r\}.$$

Note that if $\sqrt{|z|^2 + |w|^2} < \frac{r}{\sqrt{2}}$, then by Cauchy–Schwarz,

$$|z \pm iw| \le |z| + |w| \le \sqrt{2(|z|^2 + |w|^2)} < \sqrt{2}\frac{r}{\sqrt{2}} = r,$$

implying that $\frac{r}{\sqrt{2}}\mathbb{B}_2 \subseteq B$. Also, if $(x, y) \in \tilde{D} \subseteq \mathbb{C}^2$, then $|x \pm iy| < r$, hence $(x, y) \in B$.

(3) First we note that, for $z \in D$, we have $(\frac{z}{2}, -i\frac{z}{2}) \in B$ since

$$\sqrt{\left|\frac{z}{2}\right|^2 + \left|\frac{-iz}{2}\right|^2} = \sqrt{\frac{|z|^2}{2}} < \frac{r}{\sqrt{2}}.$$

Hence, by the mere definition of U, we get

$$2U\left(\frac{z}{2}, -i\frac{z}{2}\right) = f(z) + \overline{a}_0 = f(z) + \overline{f(0)}. \qquad \blacksquare$$

Shifting the center of the disk yields the following (somewhat surprising) formula (3), stated in [6] without a proof.

Proposition 2. *Let $a \in \mathbb{C}$, and let D be the disk $\{\xi \in \mathbb{C} : |\xi - a| < r\}$ or $D = \mathbb{C}$. Suppose that $u\colon D \to \mathbb{R}$ is harmonic, and for $\xi = x + iy \in D$ with $x, y \in \mathbb{R}$, let*

$$\tilde{D} := \{(x, y) \in \mathbb{R}^2 : x + iy \in D\} \quad \text{and} \quad \tilde{u}(x, y) := u(x + iy).$$

The following assertions hold.

(1) *There is $f \in H(D)$ with $\operatorname{Re} f = u$.*

(2) *$\tilde{u}(x, y)$ can be extended to a function*

$$U\colon \begin{cases} B \to \mathbb{C}, \\ (z, w) \mapsto U(z, w) \end{cases}$$

holomorphic in a neighborhood B of the point $(\operatorname{Re} a, \operatorname{Im} a)$ in $\mathbb{C}^2$ containing $\tilde{D}$.

(3) *$f(z) + \overline{f(a)} = 2U\big(\frac{z+\overline{a}}{2}, \frac{z-\overline{a}}{2i}\big)$ for $z \in D$.*

(4) *A harmonic conjugate v of u (that is a function for which $u + iv$ is holomorphic in D) is given by*

$$v(z) = 2 \operatorname{Im} U\Big(\frac{z + \overline{a}}{2}, \frac{z - \overline{a}}{2i}\Big).$$

(5) *The set of all holomorphic functions h in D with $\operatorname{Re} h = u$ is given by*

$$h(z) = 2U\Big(\frac{z + \overline{a}}{2}, \frac{z - \overline{a}}{2i}\Big) - \alpha + i\sigma,$$

where $\alpha = u(a) = U(\operatorname{Re} a, \operatorname{Im} a)$ and $\sigma \in \mathbb{R}$.

Proof. The proof of (1)–(3) works as in the case $a = 0$. For $f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$, just take

$$B := \{(z, w) \in \mathbb{C}^2 : |z + iw - a| < r\} \cap \{(z, w) \in \mathbb{C}^2 : |z - iw - \overline{a}| < r\},$$

which is a neighborhood of $(\operatorname{Re} a, \operatorname{Im} a) \in \mathbb{C}^2$, and for $(z, w) \in B$,

$$2U(z, w) := \sum_{n=0}^{\infty} a_n (z + iw - a)^n + \sum_{n=0}^{\infty} \overline{a}_n (z - iw - \overline{a})^n.$$

Note that if z is close to a, then $\frac{z+\overline{a}}{2}$ is close to $\operatorname{Re} a$ and $\frac{z-\overline{a}}{2i}$ close to $\operatorname{Im} a$. Moreover, if $(x, y) \in \tilde{D} \subseteq \mathbb{C}^2$, then $(x, y) \in B$, too.

For (4), it suffices to take the imaginary part of $f = u + iv$. Note that with v any other function of the form $v + \beta$ with $\beta \in \mathbb{R}$ is a harmonic conjugate to u, too.

Assertion (5) immediately follows from (4). ∎

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