
Short note **The Milne-Thomson formula for the
harmonic conjugate and its associated
holomorphic function**

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Abstract. In this note, we justify the formula by Milne-Thomson giving a direct determination of the holomorphic function from its real part on disks.

The goal of this note is to justify a very nice formula given by Milne-Thomson in [3] (see also [4, p. 132], [1, p. 21], [2] and [6, p. 144] e.g.) on the construction of the holomorphic function (modulo a purely imaginary additive constant) whose real part is a given harmonic function in an open disk. This formula, given in item (3) below (and always appreciated by my students), was just stated in these books/papers, but without an explanation of how to interpret $u(\frac{z}{2}, -i\frac{z}{2})$ in case $u(x, y)$ is harmonic in the real variables x, y . Here we close this gap, hoping that this formula finally finds its way into the curriculum of every introductory course in complex analysis.

Proposition 1. *Let D be the disk $\{\xi \in \mathbb{C} : |\xi| < r\}$ or $D = \mathbb{C}$. Suppose that $u: D \rightarrow \mathbb{R}$ is harmonic, and for $\xi = x + iy \in D$ with $x, y \in \mathbb{R}$, let $\tilde{D} := \{(x, y) \in \mathbb{R}^2 : x + iy \in D\}$ and $\tilde{u}(x, y) := u(x + iy)$. The following assertions hold.*

- (1) *There is $f \in H(D)$ with $\operatorname{Re} f = u$.*
- (2) *$\tilde{u}(x, y)$ can be extended to a function*

$$U: \begin{cases} B \rightarrow \mathbb{C}, \\ (z, w) \mapsto U(z, w) \end{cases}$$

holomorphic in a neighborhood B of the origin in \mathbb{C}^2 containing \tilde{D} (when \tilde{D} is viewed as a subset of \mathbb{C}^2). The extension is understood in the sense that $U(x, y) = \tilde{u}(x, y)$ for $(x, y) \in \tilde{D} \cap B = \tilde{D}$. In case $r < \infty$, B contains $\frac{r}{\sqrt{2}}\mathbb{B}_2$, where \mathbb{B}_2 is the unit ball in \mathbb{C}^2 , and $B = \mathbb{C}^2$ if $D = \mathbb{C}$.

- (3) *$f(z) + \overline{f(0)} = 2U(\frac{z}{2}, -i\frac{z}{2})$ for $z \in D$.*

In other words, we may formally replace the real arguments x, y of u by $\frac{z}{2}, -i\frac{z}{2}$ to directly obtain f (modulo a constant).

Proof. (1) This is well known (see e.g. [5, Theorem 12.42]). The idea is that the C^1 -function $h := u_x - iu_y$ satisfies the Cauchy–Riemann equations due to $u_{xx} + u_{yy} = 0$. Hence h is holomorphic and admits a primitive $f(z) := \int_0^z h(\xi) d\xi$, say $f = a + ib$. Now

$$h = f' = f_x = a_x + ib_x = a_x - ia_y.$$

Consequently, $u_x = a_x$ and $u_y = a_y$. We conclude that $u(x, y) = a(x, y) + r$, $r \in \mathbb{R}$. Now we define v by $v := b$.

(2) Let $f \in H(D)$ satisfy $\operatorname{Re} f = u$ in D . In particular, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for some $a_n \in \mathbb{C}$, the series converging locally uniformly. Thus, for $(x, y) \in \tilde{D}$,

$$2\tilde{u}(x, y) = \sum_{n=0}^{\infty} a_n (x + iy)^n + \sum_{n=0}^{\infty} \bar{a}_n (x - iy)^n.$$

Now, for $(z, w) \in \mathbb{C}^2$, put

$$2U(z, w) := \sum_{n=0}^{\infty} a_n (z + iw)^n + \sum_{n=0}^{\infty} \bar{a}_n (z - iw)^n.$$

These series converge absolutely for those $(z, w) \in \mathbb{C}^2$ satisfying $|z \pm iw| < r$ (if $r < \infty$), or in \mathbb{C}^2 if $D = \mathbb{C}$, to the function

$$F(z, w) := f(z + iw) + \overline{f(\bar{z} + i\bar{w})}.$$

Hence U is holomorphic in

$$B := \{(z, w) \in \mathbb{C}^2 : |z \pm iw| < r\}.$$

Note that if $\sqrt{|z|^2 + |w|^2} < \frac{r}{\sqrt{2}}$, then by Cauchy–Schwarz,

$$|z \pm iw| \leq |z| + |w| \leq \sqrt{2(|z|^2 + |w|^2)} < \sqrt{2} \frac{r}{\sqrt{2}} = r,$$

implying that $\frac{r}{\sqrt{2}}\mathbb{B}_2 \subseteq B$. Also, if $(x, y) \in \tilde{D} \subseteq \mathbb{C}^2$, then $|x \pm iy| < r$, hence $(x, y) \in B$.

(3) First we note that, for $z \in D$, we have $(\frac{z}{2}, -i\frac{z}{2}) \in B$ since

$$\sqrt{\left|\frac{z}{2}\right|^2 + \left|-\frac{iz}{2}\right|^2} = \sqrt{\frac{|z|^2}{2}} < \frac{r}{\sqrt{2}}.$$

Hence, by the mere definition of U , we get

$$2U\left(\frac{z}{2}, -i\frac{z}{2}\right) = f(z) + \bar{a}_0 = f(z) + \overline{f(0)}. \quad \blacksquare$$

Shifting the center of the disk yields the following (somewhat surprising) formula (3), stated in [6] without a proof.

Proposition 2. Let $a \in \mathbb{C}$, and let D be the disk $\{\xi \in \mathbb{C} : |\xi - a| < r\}$ or $D = \mathbb{C}$. Suppose that $u: D \rightarrow \mathbb{R}$ is harmonic, and for $\xi = x + iy \in D$ with $x, y \in \mathbb{R}$, let

$$\tilde{D} := \{(x, y) \in \mathbb{R}^2 : x + iy \in D\} \quad \text{and} \quad \tilde{u}(x, y) := u(x + iy).$$

The following assertions hold.

- (1) There is $f \in H(D)$ with $\operatorname{Re} f = u$.
- (2) $\tilde{u}(x, y)$ can be extended to a function

$$U: \begin{cases} B \rightarrow \mathbb{C}, \\ (z, w) \mapsto U(z, w) \end{cases}$$

holomorphic in a neighborhood B of the point $(\operatorname{Re} a, \operatorname{Im} a)$ in \mathbb{C}^2 containing \tilde{D} .

- (3) $f(z) + \overline{f(a)} = 2U\left(\frac{z+\bar{a}}{2}, \frac{z-\bar{a}}{2i}\right)$ for $z \in D$.
- (4) A harmonic conjugate v of u (that is a function for which $u + iv$ is holomorphic in D) is given by

$$v(z) = 2 \operatorname{Im} U\left(\frac{z + \bar{a}}{2}, \frac{z - \bar{a}}{2i}\right).$$

- (5) The set of all holomorphic functions h in D with $\operatorname{Re} h = u$ is given by

$$h(z) = 2U\left(\frac{z + \bar{a}}{2}, \frac{z - \bar{a}}{2i}\right) - \alpha + i\sigma,$$

where $\alpha = u(a) = U(\operatorname{Re} a, \operatorname{Im} a)$ and $\sigma \in \mathbb{R}$.

Proof. The proof of (1)–(3) works as in the case $a = 0$. For $f(z) = \sum_{n=0}^{\infty} a_n(z - a)^n$, just take

$$B := \{(z, w) \in \mathbb{C}^2 : |z + iw - a| < r\} \cap \{(z, w) \in \mathbb{C}^2 : |z - iw - \bar{a}| < r\},$$

which is a neighborhood of $(\operatorname{Re} a, \operatorname{Im} a) \in \mathbb{C}^2$, and for $(z, w) \in B$,

$$2U(z, w) := \sum_{n=0}^{\infty} a_n(z + iw - a)^n + \sum_{n=0}^{\infty} \bar{a}_n(z - iw - \bar{a})^n.$$

Note that if z is close to a , then $\frac{z+\bar{a}}{2}$ is close to $\operatorname{Re} a$ and $\frac{z-\bar{a}}{2i}$ close to $\operatorname{Im} a$. Moreover, if $(x, y) \in \tilde{D} \subseteq \mathbb{C}^2$, then $(x, y) \in B$, too.

For (4), it suffices to take the imaginary part of $f = u + iv$. Note that with v any other function of the form $v + \beta$ with $\beta \in \mathbb{R}$ is a harmonic conjugate to u , too.

Assertion (5) immediately follows from (4). ■

Acknowledgments. I thank Ronen Peretz for some discussions. I also thank Peter Pflug for telling me that the extension of harmonic functions in \mathbb{R}^n to holomorphic functions in \mathbb{C}^n is well known in the theory of complex analysis of several variables (under the headword of Lie ball).

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