

Short note Angle sum of polygons in space

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Abstract. It is examined for which angles sums a polygon in space exists.

We consider polygons in the three-dimensional Euclidean space with n generally non-coplanar vertices ($n \geq 3$) and call them n -gons for short. An *angle* of an n -gon is defined as the angle between adjacent sides that is smaller than or equal to 180° . Intersecting sides, coinciding vertices, and even angles of 0° are permitted.

Theorem. *An n -gon in Euclidean space E^3 with angle sum S_n exists if and only if*

$$(n - 2) \cdot 180^\circ \geq S_n \geq \begin{cases} 0^\circ & \text{for even } n, \\ 180^\circ & \text{for odd } n. \end{cases} \quad (1)$$

Proof. First, we show by induction on n that the upper bound from (1) forms a necessary condition for the existence of an n -gon. Let $S_n = \alpha_1 + \alpha_2 + \dots + \alpha_n$ be the sum of the n -gon's consecutive angles. The base case $S_3 = 180^\circ$ is known. Adding to the n -gon a further vertex with angle α_{n+1} , as shown in Figure 1, we obtain the new vertex angles α'_1 and α'_n and the triangle angles β and γ . From $(n - 2) \cdot 180^\circ \geq S_n$ and using the spherical triangle inequality, it follows by the induction step that

$$\begin{aligned} ((n + 1) - 2) \cdot 180^\circ &\geq S_n + 180^\circ = S_n + \beta + \gamma + \alpha_{n+1} \\ &= S_n - \alpha_1 + (\alpha_1 + \beta) - \alpha_n + (\alpha_n + \gamma) + \alpha_{n+1} \\ &\geq S_n - \alpha_1 + \alpha'_1 - \alpha_n + \alpha'_n + \alpha_{n+1} = S_{n+1}. \end{aligned}$$

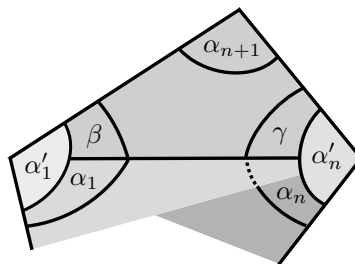


Figure 1

As regards the necessary conditions of the lower bound from (1), it suffices to show that $S_n \geq 180^\circ$ for odd n . To do this, we generalize an approach often used at school to prove that $S_3 = 180^\circ$: the angles α_i of an n -gon are translated such that their vertices come to lie in a common point O and, in addition, those with even index i are reflected at O . In this way, we obtain an *angle fan* with a common side of α_i and α_{i+1} for $1 \leq i \leq n - 1$, and an angle of 180° between the opposite sides of α_1 and α_n , as illustrated in Figure 2 for $n = 5$. Hence, again based on the spherical triangle inequality, it follows that $S_n \geq 180^\circ$.

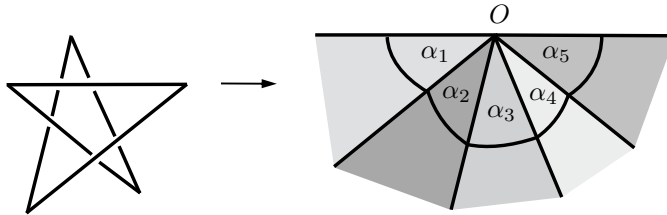


Figure 2

Next, we verify that (1) is sufficient for the existence of an n -gon by giving an example for each angle sum S_n .

For even n , consider an n -gon, as shown in Figure 3 for $n = 10$, but without point v . Its sides are diagonals of the lateral rectangles of a regular prism, and we choose their common length to be 1. This n -gon, which we call a *crown*, has equal angles. If the radius r of the circumscribed circle of the base area is continuously varied, the prism degenerates in two cases: for $r = 0$, it becomes a line segment with $S_n = 0^\circ$, and for $r = 1/(2 \sin \frac{\pi}{n})$, it results in a regular planar n -gon and thus $S_n = (n - 2) \cdot 180^\circ$. The continuity ensures that S_n assumes all values from (1) between these boundaries.

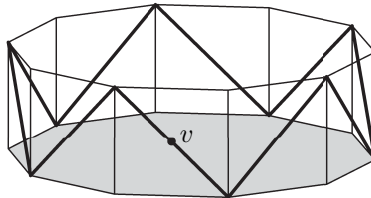


Figure 3

For odd n ($n \geq 5$), we add to a crown with $n - 1$ vertices a further vertex v which is the midpoint of a side, as in Figure 3 for $n = 11$. Since the angle at v is 180° , it follows for each r that $S_n = S_{n-1} + 180^\circ$, and thus S_n again assumes all values from (1). ■

Boundaries. The upper bound $S_n = (n - 2) \cdot 180^\circ$ can only be reached if, in the step of the above induction proof, it holds $\alpha'_1 = \alpha_1 + \beta$ and $\alpha'_n = \alpha_n + \gamma$, and consequently $\alpha'_1 \leq 180^\circ$ and $\alpha'_n \leq 180^\circ$. The two equations imply that a corresponding n -gon is planar and the two inequalities, which in addition exclude overlapping and concavity, that it is convex.

Concerning the lower bounds, an n -gon with even n and $S_n = 0^\circ$ is obviously linear. However, an n -gon with odd n and $S_n = 180^\circ$ is planar, which is due to the fact that the associated angle fan must be planar. If in such an n -gon all α_i are different from 0° , it can be characterized by having the largest turning number t , given by $t = (n - 1)/2$. Figure 4 shows a heptagon with $t = 3$ and thus $S_7 = 180^\circ$, together with the star (the great heptagram), which is the most symmetric version of the latter. An n -gon with $S_n = 180^\circ$ and one or more vanishing angles α_i is obtained by limiting processes. If $n - 1$ angles vanish and therefore the remaining one becomes 180° , we get again a linear n -gon.

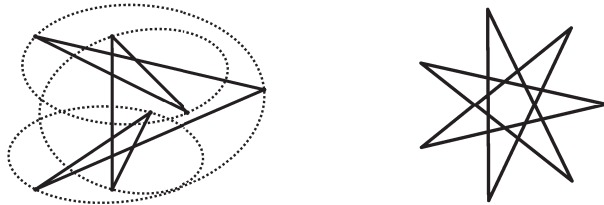


Figure 4

Summarizing the main point, we have that *an n -gon with a boundary angle sum S_n from (1) is planar.*

Generalization. The theorem holds for n -gons in any Euclidean space E^d with $d \geq 2$. For $d > 3$, the proof works in the same way as in E^3 . For $d = 2$, it remains to show that for each non-boundary angle sum S_n from (1) there exists a planar n -gon, which can easily be done by means of examples.

Remark. We could not find our result elsewhere in the present general form. However, for some classes of equilateral n -gons, it is implicitly contained in [1].

Reference

- [1] Y. Kamiyama, A filtration of the configuration space of spatial polygons, *Adv. Appl. Discrete Math.* **22** (2019), 67–74.

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