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Short note An inequality for the ratio of polynomials

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Abstract. We show that the classical Chebyshev inequality for sums can be applied to obtain an inequality for the ratio of polynomials.

A classical result in the Theory of Inequalities is the following Chebyshev inequality for sums.

Let $p_k \geq 0$ $(k = 0, 1, \dots, n)$. If the sequences $(u_k)_{0\leq k\leq n}$ and $(v_k)_{0\leq k\leq n}$ are both decreasing or increasing, then

$$\sum_{k=0}^{n} p_k u_k \sum_{k=0}^{n} p_k v_k \leq \sum_{k=0}^{n} p_k \sum_{k=0}^{n} p_k u_k v_k. \tag{1}$$

If $p_k > 0$ $(k = 0, 1, \dots, n)$, then the sign of equality holds if and only if $u_0 = \dots = u_n$ or $v_0 = \dots = v_n$.

Inequality (1) is named after the Russian mathematician Pafnutii L. Chebyshev (1821–1894), who published an integral version of (1) in 1882. Interesting historical comments on (1) were recently given by Besenyei [1], who also provided a mechanical interpretation of Chebyshev's inequality due to Picard. Detailed information about the life and work of Chebyshev is given in a paper by Butzer and Jongmans [2].

An elegant proof of (1) can be found in the monograph "Inequalities" by Hardy, Littlewood and Pólya [4, Section 2.17]. Let

$$S_n = \sum_{k=0}^{n} p_k \sum_{k=0}^{n} p_k u_k v_k - \sum_{k=0}^{n} p_k u_k \sum_{k=0}^{n} p_k v_k. \tag{2}$$

Then

$$S_n = \frac{1}{2} \sum_{i=0}^{n} \sum_{j=0}^{n} p_i p_j (u_i - u_j)(v_i - v_j). \tag{3}$$

Moreover, we have the following identity which was given by Djoković [3] for the case of equal weights:

$$S_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} P_{\min(i,j)} \big(P_n - P_{\max(i,j)}\big)(u_i - u_{i+1})(v_j - v_{j+1}), \tag{4}$$

where $P_k = \sum_{i=0}^{k} p_i$. Since $(u_k)_{0\le k\le n}$ and $(v_k)_{0\le k\le n}$ are monotonic in the same sense, we conclude from (3) and (4) that $S_n \ge 0$, and if $p_k > 0$ $(k = 0, 1, \dots, n)$, then equality holds if and only if $u_0 = \cdots = u_n$ or $v_0 = \cdots = v_n$.

The aim of this note is to show that an application of Chebyshev's inequality leads to an inequality for the ratio of polynomials which we could not locate in the literature.

Theorem. *Let r, s and n be integers with $0 \le r < s \le n$. For all polynomials*

$$P(x) = \sum_{k=0}^{n} a_k x^k \quad \textit{and} \quad Q(x) = \sum_{k=0}^{n} b_k x^k$$

with

$$a_k > 0 \quad (k = 0, \dots, n), \quad b_k \ge 0 \quad (k = 0, \dots, n-1), \quad b_n > 0,$$
$$\frac{b_0}{a_0} \le \frac{b_1}{a_1} \le \cdots \le \frac{b_n}{a_n}$$

and all real numbers x, y with $0 < x \le y$, we have

$$x^{s-r} \frac{P^{(s)}(x)}{P^{(r)}(x)} \le y^{s-r} \frac{Q^{(s)}(y)}{Q^{(r)}(y)}. \tag{5}$$

The sign of equality holds in (5) if and only if

$$\frac{b_r}{a_r} = \cdots = \frac{b_n}{a_n} \quad \textit{and} \quad x = y. \tag{6}$$

Proof. We define

$$p_k = 0 \quad (0 \le k \le r-1), \quad p_k = a_k \frac{k!}{(k-r)!} x^k \quad (r \le k \le n),$$
$$u_k = 0 \quad (0 \le k \le s-1), \quad u_k = \frac{(k-r)!}{(k-s)!} \quad (s \le k \le n),$$

and

$$v_k = \frac{b_k}{a_k}\left(\frac{y}{x}\right)^k \quad (0 \le k \le n).$$

Since

$$u_0 = \cdots = u_{s-1} = 0 < u_k < \frac{k+1-r}{k+1-s} u_k = u_{k+1} \quad (s \le k \le n-1)$$

and

$$v_k = \frac{b_k}{a_k}\left(\frac{y}{x}\right)^k \le \frac{b_{k+1}}{a_{k+1}}\left(\frac{y}{x}\right)^{k+1} = v_{k+1} \quad (0 \le k \le n-1),$$

we conclude that $(u_k)_{0\le k\le n}$ and $(v_k)_{0\le k\le n}$ are increasing sequences. An application of (1) gives

$$\begin{aligned} x^s P^{(s)}(x) \cdot y^r Q^{(r)}(y) = \sum_{k=0}^{n} p_k u_k \sum_{k=0}^{n} p_k v_k &\le \sum_{k=0}^{n} p_k \sum_{k=0}^{n} p_k u_k v_k \\ &= x^r P^{(r)}(x) \cdot y^s Q^{(s)}(y). \end{aligned} \tag{7}$$

This leads to (5).

Next, we discuss the cases of equality. From (7), we obtain that equality holds in (5) if and only if $S_n = 0$, where S_n is defined in (2).

First, we assume that $S_n = 0$. Using $p_0 = \cdots = p_{r-1} = 0$, we conclude from (3) that

$$S_n = \frac{1}{2} \sum_{i=r}^{n} \sum_{j=r}^{n} p_i p_j (u_i - u_j)(v_i - v_j). \tag{8}$$

Since

$$p_k > 0 \quad (r \le k \le n) \quad \text{and} \quad (u_i - u_j)(v_i - v_j) \ge 0 \quad (r \le i, j \le n),$$

we obtain

$$S_n \ge \frac{1}{2} p_r p_n (u_r - u_n)(v_r - v_n).$$

We have $v_r \le v_n$. If $v_r < v_n$, then

$$(u_r - u_n)(v_r - v_n) = -\frac{(n-r)!}{(n-s)!}(v_r - v_n) > 0,$$

which implies $S_n > 0$, a contradiction. Thus,

$$v_r = \frac{b_r}{a_r}\left(\frac{y}{x}\right)^r = \frac{b_n}{a_n}\left(\frac{y}{x}\right)^n = v_n. \tag{9}$$

Using $0 < x \le y$ gives

$$\frac{b_r}{a_r} = \frac{b_n}{a_n}\left(\frac{y}{x}\right)^{n-r} \ge \frac{b_n}{a_n}.$$

Since $(b_k/a_k)_{0 \le k \le n}$ is increasing, we obtain

$$\frac{b_r}{a_r} = \frac{b_{r+1}}{a_{r+1}} = \cdots = \frac{b_n}{a_n}$$

so that (9) yields $x = y$.

Conversely, if (6) holds, then $v_r = v_n$. Since $(v_k)_{0 \le k \le n}$ is increasing, we conclude that $v_r = v_{r+1} = \cdots = v_n$ so that (8) yields $S_n = 0$. This means that equality holds in (5). ■

The following example offers a combinatorial inequality involving the product and the ratio of binomial coefficients. Let r, s and n be integers with $0 \le r < s \le n$, and let α, β be real numbers with $\alpha, \beta \ge 2n - 1$. We set

$$a_k = 1 \Big/ \binom{\alpha}{k} \quad \text{and} \quad b_k = \binom{\beta}{k} \quad (k = 0, 1, \ldots, n).$$

Then the assumptions of the theorem are satisfied so that we obtain, for all real numbers x, y with $0 < x \le y$,

$$\sum_{k=s}^{n} \frac{\binom{k}{s}}{\binom{\alpha}{k}} x^k \sum_{k=r}^{n} \binom{k}{r}\binom{\beta}{k} y^k \le \sum_{k=r}^{n} \frac{\binom{k}{r}}{\binom{\alpha}{k}} x^k \sum_{k=s}^{n} \binom{k}{s}\binom{\beta}{k} y^k.$$

Equality holds if and only if $r = n - 1$, $s = n$, $\alpha = \beta = 2n - 1$ and $x = y$.

As an immediate consequence of the theorem, we obtain the following inequality.

Corollary. *Let r, s and n be integers with $0 \le r < s \le n$, and let $c > 0$ be a real number. For all polynomials*

$$P(x) = \sum_{k=0}^{n} a_k x^k \quad \textit{with} \quad a_0 \ge a_1 \ge \cdots \ge a_n > 0$$

and for all $x \in (0, c]$, we have

$$x^{s-r} \frac{P^{(s)}(x)}{P^{(r)}(x)} \le \frac{\sum_{k=s}^{n} (k)_s c^k}{\sum_{k=r}^{n} (k)_r c^k}, \tag{10}$$

where $(k)_m = \prod_{j=0}^{m-1} (k - j)$. The upper bound is sharp.

Remarks. (i) Inequality (10) extends and refines a result of Soble [6] (see also [5, p. 123]), who presented an upper bound for $xP'(x)/P(x)$ on $(0, a]$, where $0 < a \le 1/e$.

(ii) A detailed collection of inequalities for polynomials can be found in Milovanović et al. [5, Chapter 2].

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