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Isogonal cevians and inequalities for quotients of cevians

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1 Isogonal cevians

In a triangle ABC with sides a, b, c, semiperimeter s and angle bisector $w = AD$, let $d = AM$ be a cevian. Let $e = AN$ be its isogonal cevian, which is defined as its reflection in the angle bisector; see Figure 1.

Our first result is a formula for the length of the isogonal cevian.

Theorem 1. *Let $d = AM$ be a cevian in a triangle ABC and let $BM/MC = x/y$. Then, for the length of its isogonal cevian $e = AN$, it holds*

$$e = \frac{(x+y)bc}{xb^2 + yc^2} d.$$

Eine Cevane oder Ecktransversale ist eine Strecke, die einen Eckpunkt eines Dreiecks mit der gegenüberliegenden Seite verbindet. Cevane spielen eine zentrale Rolle in der Dreiecksgeometrie. Die isogonale Cevane einer Cevane ist deren Spiegelung an der entsprechenden Winkelhalbierenden. Die Autoren der vorliegenden Arbeit berechnen die Länge der isogonalen Cevane in Abhängigkeit der anliegenden Dreiecksseiten, der Länge der Cevane und dem Verhältnis, unter dem die Cevane die Dreiecksseite teilt. Daraus ergibt sich dann eine Formel für die Länge einer Cevane in Abhängigkeit der angrenzenden Dreiecksseiten, dem entsprechenden Eckwinkel und dem Winkel zwischen der Cevane und der Winkelhalbierenden. Als Anwendung werden die Länge der Symmediane und mehrere Ungleichungen für Quotienten einiger spezieller Cevane angegeben.

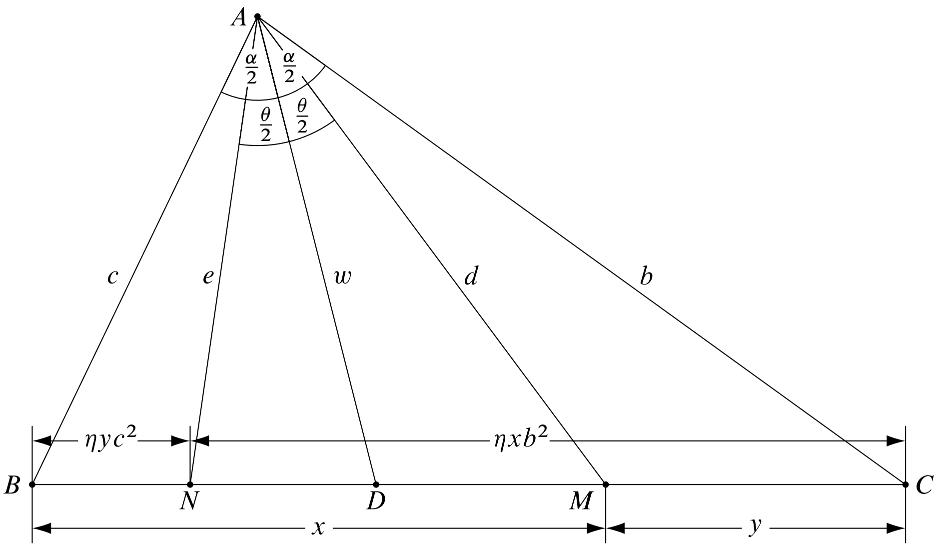

Figure 1. Isogonal cevians AM and AN

Proof. We will use the following relation for the segments determined by the feet of the isogonal cevians, descriptively named "Whisker-Lemma" [3, p. 161]:

$$\frac{BM}{MC} \cdot \frac{BN}{NC} = \frac{c^2}{b^2}.$$

Another result that we will use is Stewart's theorem [1, 5] which says that, in a triangle ABC with sides $BC = a$, $CA = b$, $AB = c$, cevian $AD = d$ and segments $BD = m$, $DC = n$, the relationship

$$b^2 m + c^2 n = a(d^2 + mn)$$

holds. Without loss of generality, we can take $a = 1$. Then we have that x and y are simply the lengths BM and MC. By the Whisker-Lemma, we have $BN = \eta y c^2$, $NC = \eta x b^2$, where η is to be determined from $\eta y c^2 + \eta x b^2 = 1$. Hence $\eta = 1/(xb^2 + yc^2)$. Stewart's theorem for the cevians d and e yields

$$d^2 + xy = xb^2 + yc^2 = \frac{1}{\eta},$$
$$e^2 + \eta^2 c^2 b^2 xy = \eta(yc^2 b^2 + xb^2 c^2) = \eta b^2 c^2.$$

Dividing the second equation by $\eta^2 c^2 b^2$ and subtracting from the first gives

$$e = d \cdot bc\eta,$$

and the proof is complete. ∎

2 The formula for the cevian

Knowing how much one cevian AM deviates from the angle bisector AD and how it divides the corresponding side is enough information to calculate its length. We have the following remarkable result; see Figure 1.

Theorem 2. *Let AM be a cevian in a triangle ABC, let $BM/MC = x/y$, and let $\theta/2 = \angle DAM$. Then we have*

$$AM \cdot \cos\frac{\theta}{2} = \Big(\frac{x}{x+y}b + \frac{y}{x+y}c\Big)\cos\frac{\alpha}{2}. \tag{1}$$

Proof. Let AN be the isogonal cevian of AM. From Theorem 1, using AM in the nominator of $(AN + AM)/(AN \cdot AM)$, we have

$$\frac{AN + AM}{AN \cdot AM} = \frac{AN \cdot \big[1 + \frac{xb^2+yc^2}{(x+y)bc}\big]}{AN \cdot AM} = \frac{(xb+yc)(b+c)}{(x+y)bcAM}. \tag{2}$$

By Pappus, for the angle bisector $w_a = AD$ in triangle ABC, we have

$$w_a = AD = \frac{2bc}{b+c}\cos\frac{\alpha}{2}. \tag{3}$$

On the other hand, AD is also angle bisector in triangle ANM. Hence

$$AD = \frac{2AN \cdot AM}{AN + AM}\cos\frac{\theta}{2}.$$

Dividing the former by the latter equation and substituting (2), we arrive at formula (1). ■

3 Applications

As a first application, we have new proof of [4, Theorem 1], which is an inequality for the general cevian AM,

$$AM \geqslant \Big(\frac{\lambda}{\lambda+1}b + \frac{1}{\lambda+1}c\Big)\cos\frac{\alpha}{2},$$

where $BM/MC = \lambda$. It is obtained simply by putting $BM/MC = \lambda = x/y$ in (1) and noting that

$$AM \geqslant AM \cdot \cos\theta = \Big(\frac{x}{x+y}b + \frac{y}{x+y}c\Big)\cos\frac{\alpha}{2}. \tag{4}$$

Next we give an easy derivation of the length of a symmedian s_a of a triangle ABC – the cevian which is a reflection of the median m_a in the corresponding angle bisector. We will show that

$$s_a = \frac{bc\sqrt{2(b^2+c^2)-a^2}}{b^2+c^2}. \tag{5}$$

Let, in Theorem 1, $AM = m_a$ be the median. Then $x = y = 1$, and we have, for its isogonal cevian, the symmedian $AN = s_a$,

$$s_a = \frac{2bc}{b^2 + c^2} m_a.$$

The length of the median is well known, $m_a = \sqrt{2(b^2 + c^2) - a^2}/2$. Hence, for the symmedian s_a, we obtain (5).

Remark 1. The standard way to derive the length of the symmedian (5) is to apply Stewart's theorem knowing that the symmedian $s_a = AD$ divides the side BC in the ratio of the squares of the adjacent sides, that is $BD/DC = c^2/b^2$. The last fact is a consequence of the Whisker-Lemma.

4 Inequalities for quotients of cevians

Further applications concern inequalities for quotients of some cevians. Let h_a, g_a and n_a denote the lengths of the altitude, Gergonne cevian and Nagel cevian, respectively, in triangle ABC, from the vertex A. For completeness, we will give the definitions in the sequel. We have that [2] (see Figure 2)

$$h_a \leqslant g_a \leqslant w_a \leqslant m_a \leqslant n_a.$$

Our aim is to obtain improvements of $m_a/h_a \geqslant 1$, $m_a/w_a \geqslant 1$, $n_a/h_a \geqslant 1$ and $n_a/w_a \geqslant 1$.

Since any cevian AN in a triangle ABC is at least as long as the corresponding altitude h_a, we have, from Theorem 1,

$$\frac{AM}{h_a} \geqslant \frac{AM}{AN} = \frac{xb^2 + yc^2}{(x + y)bc}. \tag{6}$$

Taking $AM = m_a$ to be the median, we have $x = y = 1$, and we get, from (6),

$$\frac{m_a}{h_a} \geqslant \frac{b^2 + c^2}{2bc},$$

which is the well-known first Tsintsifas inequality from [6].

A *Nagel point* is the point of concurrency of the three segments of a triangle connecting each vertex to the point of contact of the corresponding excircle and the opposite side. It is named after the German geometer Christian Heinrich von Nagel (1803–1882).

Let $AM = n_a$ be the Nagel cevian. Then $BM/MC = (s - c)/(s - b) = x/y$, and we have, from (6),

$$\frac{n_a}{h_a} \geqslant \frac{(s - c)b^2 + (s - b)c^2}{abc} \geqslant 1.$$

The last inequality is easily proved using the substitution

$$x = s - a, \quad y = s - b, \quad z = s - c,$$

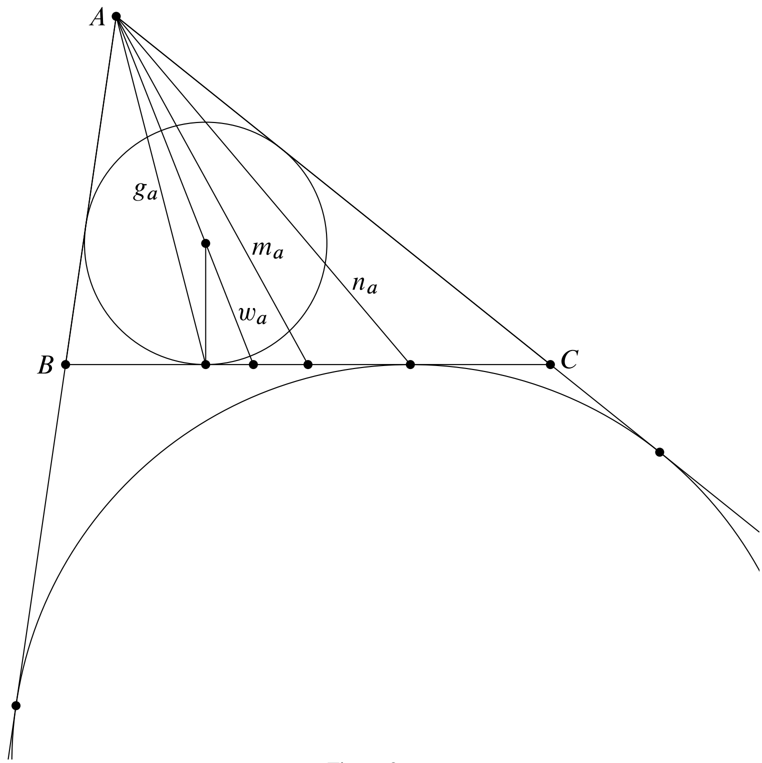

Figure 2

for three positive $x, y, z > 0$. Then $(s-c)b^2 + (s-b)c^2 \geqslant abc$ is equivalent to the simple inequality

$$y^3 + z^3 + x(y^2 + z^2) \geqslant 2xyz + y^2 z + yz^2.$$

From (4) and (3), we have

$$\frac{AM}{w_a} \geqslant \left(\frac{x}{x+y}b + \frac{y}{x+y}c\right)\frac{b+c}{2bc}. \tag{7}$$

Taking $AM = m_a$ to be the median, $x = y = 1$ and (7) gives us the second Tsintsifas inequality from [6],

$$\frac{m_a}{w_a} \geqslant \frac{(b+c)^2}{4bc}.$$

For the Nagel cevian $AM = n_a$, it holds $x/y = (s-c)/(s-b)$, and from (7), it follows

$$\frac{n_a}{w_a} \geqslant \frac{(s-c)b + (s-b)c}{2abc}(b+c) \geqslant 1.$$

The last inequality can be rewritten as the above inequality $(s-c)b^2 + (s-b)c^2 \geqslant abc$.

A *Gergonne point*, named after the French geometer Joseph Diaz Gergonne (1771–1859) is the point of concurrency of the three segments of a triangle connecting each vertex to the point of contact of the incircle and the opposite side.

For the end, let $AM = g_a$ be the Gergonne cevian. Then

$$\frac{BM}{MC} = \frac{s-b}{s-c} = \frac{x}{y}.$$

Inequality (7) is

$$\frac{g_a}{w_a} \geqslant \frac{(s-b)b + (s-c)c}{2abc}(b+c).$$

Since $1 \geqslant g_a/w_a$, we have a curious proof of the inequality

$$(s-b)b^2 + (s-c)c^2 \leqslant abc.$$

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