
Cycles cross ratio: An invitation

Vladimir V. Kisil

Vladimir Kisil received his PhD degree from the Odessa University in 1992. After several inspiring PostDoc positions (Mexico City, Ghent, Cyprus), he landed at Leeds and became a Reader in Applied Analysis there. His main research interests are applications of symmetries in various fields of mathematics and physics. Vladimir often supports his research with various software, partially written by himself; the latter is shared with the community through the open source model.

1 Introduction

We introduce a new invariant in [Lie sphere geometry](#) ([3, Chapter 3], [7], [19]) which is analogous to the [cross ratio](#) of four points in projective ([30], [35, § 9.3], [37, Appendix A]) and hyperbolic geometries ([2, § 4.4], [32, III.5], [34, § I.5]). To avoid technicalities and stay visual, we will work in two dimensions; luckily, all main features are already illuminated in this case. Our purpose is not to give an exhausting presentation (in fact, we are hoping it is far from being possible now), but rather to draw attention to the new invariant and its benefits to the various geometrical settings. There are several far-reaching generalisations in higher dimensions and non-Euclidean metrics; see [Section 5](#) for a discussion.

Das Doppelverhältnis von vier Punkten ist eine zentrale Invariante in der projektiven und der hyperbolischen Geometrie. In der vorliegenden Arbeit wird der Begriff des Doppelverhältnisses ausgedehnt auf vier Zyklen. Mit Zyklen ist dabei der Oberbegriff von Kreisen, Geraden und Punkten gemeint. Zyklen bilden eine unter den Transformationen der Möbius-Geometrie und der Lie'schen Kugelgeometrie geschlossene Familie. Das erweiterte Doppelverhältnis ist nun invariant unter eben diesen Transformationen. Der Wert des Doppelverhältnisses gibt darüber hinaus Auskunft über die Lage der Zyklen, etwa ob sie orthogonal, tangential oder invers zueinander liegen. Der Autor lädt dazu ein, weitere Eigenschaften des Begriffs zu erforschen. Obwohl der neue Begriff weitreichende Verallgemeinerungen zulässt, vermeidet der Autor allzu technische Einzelheiten und stellt die Konzepte leserfreundlich und visuell in zwei Dimensionen dar. Er stellt zudem eine interaktive Version des Artikels zur Verfügung, die als Jupyter-Notebook implementiert ist und es ermöglicht, konkrete Berechnungen auf den Computer zu verlagern.

In abstract framework, the new invariant can be considered in any projective space with a bilinear pairing.

There is an [interactive version of this paper](#) implemented as a Jupyter notebook [29] based on the MoebInv software package [28].

2 Preliminaries: Projective space of cycles

We introduce our topic following its historic development. This conforms to our hands-on approach as an opposition to a start from the most general abstract construction.

Circles are the traditional subject of geometrical studies, and their numerous properties were already widely presented in Euclid's *Elements*. However, advances in their analytical presentations is not a so distant history.

A straightforward parametrisation of a circle equation

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

by a point (g, f, c) in some subset of the three-dimensional Euclidean space \mathbb{R}^3 was used in [32, Chapter II]. Abstractly, we can treat a *point* (x_0, y_0) of a plane as the zero radius circle with coefficients $(g, f, c) = (-x_0, -y_0, x_0^2 + y_0^2)$.

It is more advantageous to use the equation

$$k(x^2 + y^2) - 2lx - 2ny + m = 0, \quad (2)$$

which also includes *straight lines* for $k = 0$. This extension comes at a price: parameters (k, l, n, m) shall be treated as elements of the three-dimensional projective space $P\mathbb{R}^3$ rather than the Euclidean space \mathbb{R}^4 since (2) with (k, l, n, m) and $(k_1, l_1, n_1, m_1) = (\lambda k, \lambda l, \lambda n, \lambda m)$ defines the same set of points for any $\lambda \neq 0$. This parametrisation is known as tetracyclic/polyspheric coordinates, cf. [20, § 2.4.1], [4, § 20.7].

The next observation is that the linear structure of $P\mathbb{R}^3$ is relevant for circles geometry. For example, the traditional concept of *pencil of circles* [10, § 2.3] is nothing else but the linear span in $P\mathbb{R}^3$, cf. [34, § I.1.c]. Therefore, it will be convenient to accept all points $(k, l, n, m) \in P\mathbb{R}^3$ on equal ground even if they correspond to an empty set of solutions (x, y) in (2). The latter can be thought as “circles with imaginary radii”.

It is also appropriate to consider $(0, 0, 0, 1)$ as a representative of the point C_∞ at infinity, which complements \mathbb{R}^2 to the Riemann sphere. Following [37], we call circles (with real and imaginary radii), straight lines and points (including C_∞) on a plane by joint name *cycles*. Correspondingly, the space $P\mathbb{R}^3$ representing them – the *cycles space*.

Sometimes, we need to consider cycles with an *orientation*. This can be used, for example, to distinguish “inner” and “outer” tangency of cycles [12, 26, 28]. Furthermore, oriented cycles and their tangency relations are the starting point for the Lie sphere geometry. It is easy to encode cycles' orientations through polyspheric coordinates: cycles (k, l, n, m) and (k', l', n', m') have the opposite orientations if

$$(k, l, n, m) = (\lambda k', \lambda l', \lambda n', \lambda m') \quad \text{for some negative } \lambda,$$

and two points (k, l, n, m) and (k', l', n', m') are considered as representatives of the same oriented cycle if

$$(k, l, n, m) = (\lambda k', \lambda l', \lambda n', \lambda m') \quad \text{for a positive } \lambda.$$

An algebraic consideration often benefits from the introduction of complex numbers. For example, we can re-write (2) as

$$\begin{aligned} (-1 \quad \bar{z}) \begin{pmatrix} \bar{L} & -m \\ k & -L \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} &= kz\bar{z} - L\bar{z} - \bar{L}z + m \\ &= k(x^2 + y^2) - 2lx - 2ny + m, \end{aligned} \quad (3)$$

where $z = x + iy$ and $L = l + in$. We call *Fillmore–Springer–Cnops construction* (or *FSCc* for short) the association of the matrix

$$C = \begin{pmatrix} \bar{L} & -m \\ k & -L \end{pmatrix}$$

to a cycle with coefficients (k, l, n, m) , cf. [8], [9, § 4.1], [11].

A reader may expect a more straightforward realisation of the quadratic form (2), cf. [34, § I.1], [36, Theorem 9.2.11]:

$$(\bar{z} \quad 1) \begin{pmatrix} k & -L \\ -\bar{L} & m \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = kz\bar{z} - L\bar{z} - \bar{L}z + m. \quad (4)$$

However, FSCc will show its benefits in the next section. Meanwhile, we note that a point z corresponding to the zero radius cycle with the centre z is represented by the matrix

$$Z = \begin{pmatrix} \bar{z} & -z\bar{z} \\ 1 & -z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \bar{z} & \bar{z} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -z \\ 1 & -z \end{pmatrix} \quad \text{for } (k, l, n, m) = (1, x, y, x^2 + y^2), \quad (5)$$

where $\det Z = 0$. Also, the point at infinity can be represented by a zero radius cycle

$$C_\infty = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \quad \text{for } (k, l, n, m) = (0, 0, 0, 1). \quad (6)$$

More generally,

$$\det \begin{pmatrix} \bar{L} & -m \\ k & -L \end{pmatrix} = -k^2 r^2$$

for $k \neq 0$ and the cycle's radius r .

3 Fractional linear transformations and the invariant product

In the spirit of the [Erlangen programme](#) of Felix Klein (greatly influenced by Sophus Lie), a consideration of cycle geometry is based on a group of transformations preserving this family [23, 24]. Let

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

be an invertible 2×2 complex matrix. Then the *fractional linear transformation* (FLT for short) of the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is defined by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}: z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}. \quad (7)$$

It will be convenient to introduce the notation \overline{M} for the matrix

$$\begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ \overline{\gamma} & \overline{\delta} \end{pmatrix}$$

with complex conjugated entries of M . For a cycle C , the matrix \overline{C} corresponds to the reflection of C in the real axis $y = 0$. Also, due to special structure of FSCc matrix, we easily check that

$$\overline{C}C = C\overline{C} = -\det(C)I \quad \text{or} \quad \overline{C} \sim C^{-1} \text{ projectively if } \det(C) \neq 0. \quad (8)$$

If a cycle C is composed from a non-empty set of points in $(x, y) \in \mathbb{R}^2$ satisfying (2), then their images under transformation (7) form again a cycle C_1 with notable link to FSCc.

Lemma 1. *Transformation (7) maps a cycle*

$$C = \begin{pmatrix} \overline{L} & -m \\ k & -L \end{pmatrix} \text{ into a cycle } C_1 = \begin{pmatrix} \overline{L}_1 & -m_1 \\ k_1 & -L_1 \end{pmatrix}$$

such that $C_1 = \overline{M}CM^{-1}$.

Clearly, the map $C \rightarrow \overline{M}CM^{-1}$ is meaningful for any cycle, including imaginary ones; thus we regard it as FLT action on the cycle space. For the matrix form (4), the above identity $C_1 = \overline{M}CM^{-1}$ needs to be replaced by the matrix congruence $C_1 = M^*CM$, cf. [34, § II.6.e], [36, Theorem 9.2.13]. This difference is significant in view of the following definition.

Definition 2. For two cycles C and C_1 , define the *cycles product* by

$$\langle C, C_1 \rangle = -\text{tr}(C\overline{C}_1), \quad (9)$$

where tr denotes the trace of a matrix.

We call two cycles C and C_1 *orthogonal* if $\langle C, C_1 \rangle = 0$.

It is easy to find the explicit expression of the cycle product (9):

$$\langle C, C_1 \rangle = km_1 + k_1m - 2ll_1 - 2nn_1, \quad (10)$$

and observe that it is linear in coefficients of the cycle C (and C_1 as well). On the other hand, it is the initial definition (9) which allows us to use the invariance of trace under matrix similarity to conclude the following.

Corollary 3. *The cycles product is invariant under the transformation $C \mapsto \overline{MCM}^{-1}$. Therefore, FLT (7) preserves orthogonality of cycles.*

The cycle product is a rather recent addition to the cycle geometry, see independent works [8], [9, § 4.1], [11], [21, § 4.2]. Interestingly, expression (9) essentially repeats the GNS-construction in C^* -algebras [1] which is older by half a century at least.

Example 4 ([24, Chapter 6], [26], [27]). The cycles product and cycles orthogonality encode a great amount of geometrical characteristics. For example, for cycles represented by non-empty sets of points in \mathbb{R}^2 , we note the following.

- (i) A cycle is a straight line if it is orthogonal, $\langle C, C_\infty \rangle = 0$, to the zero radius cycle at infinity C_∞ , (6).
- (ii) A cycle Z represents a point if Z is self-orthogonal (*isotropic*): $\langle Z, Z \rangle = 0$. More generally, (8) implies $\langle C, C \rangle = 2 \det(C)$.
- (iii) A cycle C passes a point Z if they are orthogonal, $\langle C, Z \rangle = 0$.
- (iv) A cycle C represents a line in Lobachevsky geometry [35, Chapter 12] if it is orthogonal, $\langle C, C_{\mathbb{R}} \rangle = 0$, to the real line cycle

$$C_{\mathbb{R}} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}.$$

- (v) Two cycles are orthogonal as subsets of a plane (i.e. they have perpendicular tangents at an intersection point) if they are orthogonal in the sense of Definition 2.
- (vi) Two cycles C and C_1 are *tangent* (i.e. have a unique point in common) if

$$\langle C, C_1 \rangle^2 = \langle C, C \rangle \langle C_1, C_1 \rangle.$$

- (vii) *Inversive distance* [10, § 5.8] θ of two (non-isotropic) cycles is defined by the formula

$$\theta = \frac{\langle C, C_1 \rangle}{\sqrt{\langle C, C \rangle \langle C_1, C_1 \rangle}}. \quad (11)$$

In particular, the above discussed orthogonality corresponds to $\theta = 0$ and the tangency to $\theta = \pm 1$. For intersecting cycles, θ is the cosine of the intersecting angle.

- (viii) A generalisation of *Steiner power* $d(C, C_1)$ of two cycles is defined as, cf. [12, § 1.1],

$$d(C, C_1) = \langle C, C_1 \rangle + \sqrt{\langle C, C \rangle \langle C_1, C_1 \rangle}, \quad (12)$$

where both cycles C and C_1 are scaled to have $k = 1$ and $k_1 = 1$. Geometrically, the generalised Steiner power for spheres provides the square of *tangential distance*.

Remark 5. The cycles product is *indefinite*; see [13] for an account of the theory with some refreshing differences to the more familiar situation of inner product spaces. One illustration is the presence of self-orthogonal non-zero vectors; see Example 4 (ii) above. Another noteworthy observation is that the product (10) has the Lorentzian signature (1, 3) and \mathbb{R}^4 with this product is isomorphic to Minkowski space-time [21, § 4.2].

4 Cycles cross ratio

Due to the projective nature of the cycles space (i.e. matrices C and λC correspond to the same cycle), a *non-zero* value of the cycle product (9) is not directly meaningful. Of course, this does not affect the cycles orthogonality. A partial remedy in other cases is possible through various *normalisations* [24, § 5.2]. Usually, they are specified by either of the following conditions.

- (i) $k = 1$, which is convenient for metric properties of cycles. It brings us back to the initial equation (1) and is not possible for straight lines.
- (ii) $\langle C, C \rangle = \pm 1$, which was suggested in [21, § 4.2] and is useful, say, for tangency but is not possible for points.

Recall that the projective ambiguity is elegantly balanced in the *cross ratio* of four points ([2, § 4.4], [32, III.5], [34, § 1.5]):

$$(z_1, z_2; z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4}. \quad (13)$$

We use this classical pattern in the following definition.

Definition 6. A *cycles cross ratio* of four cycles C_1, C_2, C_3 and C_4 is

$$\langle C_1, C_2; C_3, C_4 \rangle = \frac{\langle C_1, C_3 \rangle}{\langle C_1, C_4 \rangle} : \frac{\langle C_2, C_3 \rangle}{\langle C_2, C_4 \rangle} \quad (14)$$

assuming $\langle C_1, C_4 \rangle \langle C_2, C_3 \rangle \neq 0$. If $\langle C_1, C_4 \rangle \langle C_2, C_3 \rangle = 0$ but $\langle C_1, C_3 \rangle \langle C_2, C_4 \rangle \neq 0$, we put $\langle C_1, C_2; C_3, C_4 \rangle = \infty$. The cycles cross ratio is generally undefined in the remaining case of an indeterminacy $\frac{0}{0}$.

Note that some additional geometrical reasons may help to resolve the last situation; see the consideration of orthogonality/tangency with zero radius cycle in Example 9.

As an initial justification of the definition, we list the following properties.

Proposition 7.

- (i) *The cycles cross ratio is a well-defined FLT-invariant of quadruples of cycles.*
- (ii) *The cycles cross ratio of four zero radius cycles is the squared modulus of the cross ratio for the respective points:*

$$\langle Z_1, Z_2; Z_3, Z_4 \rangle = |(z_1, z_2; z_3, z_4)|^2. \quad (15)$$

- (iii) *There is the cancellation formula*

$$\langle C_1, C; C_3, C_4 \rangle \langle C, C_2; C_3, C_4 \rangle = \langle C_1, C_2; C_3, C_4 \rangle. \quad (16)$$

Proof. The first statement follows from Corollary 3 and the construction of the cycles cross ratio. To show the second statement, we derive from (5) and (10) that

$$\langle Z_i, Z_j \rangle = |z_i - z_j|^2$$

if we use representations of zero radius cycles Z_i and Z_j by coefficients with $k_i = k_j = 1$. This implies (15). A demonstration of (16) is straightforward. ■

To demonstrate that there is more than just a formal similarity between the two, we briefly list some applications. First, we rephrase Example 4 (vii) in new terms.

Example 8. The *capacitance* $\text{cap}(C, C_1)$ of two cycles ([17, § 5.1], [18, Definition 3]) coincides with the following cycles cross product:

$$\text{cap}(C, C_1) = \langle C, C_1; C_1, C \rangle = \theta^2,$$

where θ is the inversive distance (11). Thereafter, FLT-invariance of the cycles cross ratio implies that the intersection angle of cycles is FLT-invariant. In particular, cycles are

- (i) orthogonal if $\langle C, C_1; C_1, C \rangle = 0$,
- (ii) tangent if $\langle C, C_1; C_1, C \rangle = \pm 1$, and
- (iii) disjoint if $|\langle C, C_1; C_1, C \rangle| > 1$.

Relation (i) is merely a consequence of the first-order orthogonality relation $\langle C, C_1 \rangle = 0$, which is fundamental to conformal and incidence geometries, cf. Example 4 (i)–(v). Meanwhile, the tangency condition (ii) is genuinely quadratic and shall be equally significant in Lie spheres geometry, the Steiner’s porism [17, 18], and other questions formulated purely in terms of cycles’ tangency.

Example 9. If a non-zero radius cycle passes a zero radius cycle (point), their cross ratio has an indeterminacy $\frac{0}{0}$. Geometrically, their relation can be seen in either ways: as orthogonality or tangency. Therefore, the indeterminacy of the cycle cross ratio can be geometrically resolved differently either to 0 (indicates orthogonality) or 1 (corresponds to tangency). More specifically (see the supporting symbolic computations in the notebook [29]):

- *Orthogonality.* Consider a cycle Z_t with a fixed centre and a variable squared radius t . Take a generic cycle C orthogonal to Z_t . To resolve an indeterminacy $\frac{0}{0}$, we use l’Hospital’s rule at the point $t = 0$, which corresponds to Z_t becoming a zero radius cycle. This produces $\langle C, Z_t; Z_t, C \rangle|_{t=0} = 0$.
- *Tangency.* For a zero radius cycle Z and passing it cycle C , consider a generic cycle $C_t = (1 - t) \cdot Z + t \cdot C$, $t \in [0, 1]$, from the pencil (linearly) spanned by Z and C . Since C_t touches C , Example 8 (ii) implies that $\langle C_t, C; C, C_t \rangle = 1$ for $t > 0$ and C_t coincides with Z for $t = 0$. Thus, we can extend the value 1 to $[Z, C; C, Z]$ by continuity.

The last technique makes the cycles cross ratio meaningful for Lie spheres geometry, which extends FLT by non-point Lie transformations, when a non-zero radius cycle is sent to a zero radius one.

Example 10. The Steiner power (12) can be written as

$$d(C, C_1) = \langle C, C_{\mathbb{R}}; C_1, C_{\mathbb{R}} \rangle + \sqrt{\langle C, C_{\mathbb{R}}; C, C_{\mathbb{R}} \rangle} \cdot \sqrt{\langle C_1, C_{\mathbb{R}}; C_1, C_{\mathbb{R}} \rangle}, \quad (17)$$

where $C_{\mathbb{R}}$ is the real line and cycles C and C_1 do not need to be normalised in any particular way. Thereafter, the Steiner power is an invariant of two cycles C and C_1 under Möbius transformations, since they fix the real line $C_{\mathbb{R}}$. The Möbius invariance is not so obvious from expression (12).

The next two applications will generalise the main features of the traditional cross ratio. Recall the other name of the cross ratio – the anharmonic ratio. The origin of the latter is as follows. Two points z_1 and z_2 on a line define a one-dimensional sphere with the centre $O = \frac{1}{2}(z_1 + z_2)$, which can be taken as the origin. Two points c_1 and c_2 are called *harmonically conjugated* (with respect to z_1 and z_2) if

$$c_1 \cdot c_2 = -z_1 \cdot z_2, \quad \text{cf.} \quad \begin{array}{ccccccc} & \bullet & & \bullet & & \bullet & \bullet & \bullet \\ & z_1 & & O & & c_1 & z_2 & c_2 \end{array}$$

It is easy to check that, in this case,

$$(c_1, c_2; z_1, z_2) = -1. \quad (18)$$

Thus, the cross ratio can be viewed as a measure how far four points are from harmonic conjugation, i.e. a measure of anharmonicity of a quadruple. To make a similar interpretation of the cycles cross ratio, recall that, for FSCc matrices of a cycle C_1 and its *reflection* C_2 in a cycle C , we have $C_2 = C\bar{C}_1C$, cf. [24, § 6.5]. That is, the reflection in a cycle C is the composition of FLT transform with FSCc matrix C and complex conjugation of matrix entries. It is easy to obtain the following.

Proposition 11. *If a cycle C_1 is a reflection of C_2 in a cycle C , then*

$$\langle C_1, C; C, C_1 \rangle = \langle C_2, C; C, C_2 \rangle.$$

More generally, the reflection in a cycle preserves the inversive distance, cf. Example 8.

The above condition is necessary; we describe a sufficient one as a *figure* in the sense of [26–28]. In short, a figure is an ensemble of cycles interrelated by cycles' relations. For the purpose of this paper, an FLT-invariant relation “to be orthogonal” between two cycles is enough. Software implementations of these figures can be found in [29].

Figure 12.

- (i) For two given cycles C and C_1 , construct the reflection $C_2 = C\bar{C}_1C$ of C_1 in C .
- (ii) Take any cycle C_o orthogonal to C and C_1 , see Illustration 1. All such cycles make a *pencil* – one-dimensional subspace of the projective space of cycles. That is because there are only two linear equations for orthogonality (10) to determine four projective coordinates (k_o, l_o, n_o, m_o) . By Proposition 11, C_o is also orthogonal to C_2 .
- (iii) Define cycles by orthogonality to C , C_o and itself (zero radius condition), that is the intersection points of C and C_o . Since self-orthogonality is a quadratic condition, there are two solutions: Z_1 and Z_2 .
- (iv) The harmonic conjugation of C_1 and C_2 (their reflection in C) implies, cf. (18),

$$\langle C_1, C_2; Z_1, Z_2 \rangle = 1. \quad (19)$$

This is demonstrated by symbolic computation in [29].

As the final illustration, we introduce Möbius-invariant distance between cycles. Recall that FLT with an $SL_2(\mathbb{R})$ matrix fixes the real line, and it is called a *Möbius transformation* ([23], [24, Chapter 1], [36, § 9.3]). The corresponding figure is as follows, see Illustration 2.

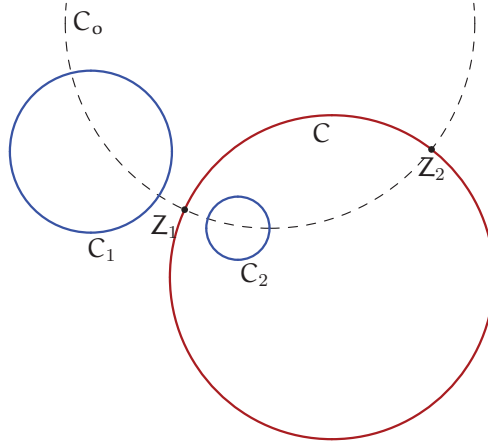


Illustration 1. Cycles cross ratio for two conjugated cycles

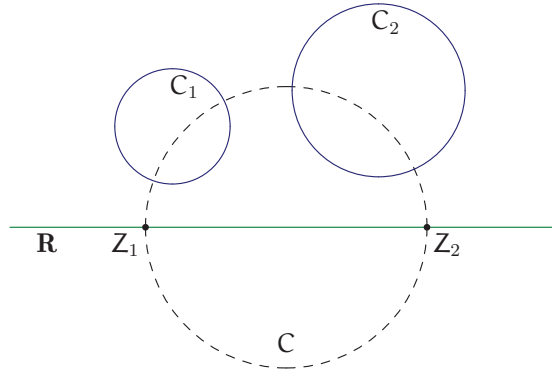


Illustration 2. Construction for Möbius invariant distance between two cycles

Figure 13.

- (i) Let two distinct cycles C_1 and C_2 be given, and they are different from the real line $C_{\mathbb{R}}$.
- (ii) Define a cycle C to be orthogonal to C_1 , C_2 and $C_{\mathbb{R}}$. It is specified by three linear equations for homogeneous coordinates (k, l, n, m) . In generic position, a solution is unique; however, it can be an imaginary cycle (with a negative square of the radius).
- (iii) Define cycles by orthogonality to C , $C_{\mathbb{R}}$ and itself, that is the intersection points of C and $C_{\mathbb{R}}$. In general position, there are two solutions which we denote by Z_1 and Z_2 . For an imaginary cycle C , first coordinates of Z_1 and Z_2 are conjugated complex numbers.

- (iv) Since the entire construction is completely determined by the given cycles C_1 and C_2 , we define the *distance between two cycles* by

$$d(C_1, C_2) = \frac{1}{2} \log \langle C_1, C_2; Z_1, Z_2 \rangle. \quad (20)$$

From Möbius invariance of the real line and cycles cross ratio, our construction implies the following.

Proposition 14.

- (i) *Distance (20) is Möbius invariant.*
- (ii) *For zero radius cycles C_1, C_2 , formula (20) coincides with the Lobachevsky metric on the upper-half plane.*
- (iii) *For any cycle C_3 orthogonal to C , the (signed) distance is additive: $d(C_1, C_3) = d(C_1, C_2) + d(C_2, C_3)$.*
- (iv) *If centres of C_1 and C_2 are on the imaginary axis (therefore Z_1 and Z_2 are zero and infinity), then $d(C_1, C_2) = \log(m_1/k_1) - \log(k_2/m_2)$.*

Obviously, Proposition 14 (ii) is the consequence of (15), and (iii) follows from the cancellation rule (16). On the other hand, the expression in (iv) can be obtained by a direct computation, see [29]. Note that, in this case, m_i/k_i is the square *tangential distance* (also known as the generalised Steiner power (12), (17)) from Z_1 (the origin) to the cycle C_i . Thus, for a zero radius C_i , it coincides with the usual distance between centres of Z_1 and C_i , and Proposition 14 (iv) recreates the classical result [2, (7.2.6)].

Comparing constructions on Illustration 1–2 and formulae (19)–(20), we can say that the invariant distance measures how far two cycles are from being reflections of each other in the real line.

5 Discussion and generalisations

We presented some evidence that the cycles cross ratio extends to Lie spheres geometry the concept of the cross ratio of four points. It is natural to expect that a majority of the classic theory ([2, § 4.4], [30], [32, III.5], [34, § I.5], [35, § 9.3], [37, Appendix A]) admits similar adaptation as well. However, we can expect even more than that.

One can lay down a general framework for the introduced invariant (14) in generic projective spaces as follows¹. Let V be a vector space over an arbitrary field \mathbb{F} with a bilinear pairing $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{F}$. Upon choosing any four points of the projective space on V , one may select arbitrary non-zero vectors, say c_1, c_2, c_3, c_4 , representing these points. Then the quotient

$$\frac{\langle c_1, c_3 \rangle}{\langle c_1, c_4 \rangle} : \frac{\langle c_2, c_3 \rangle}{\langle c_2, c_4 \rangle} \quad (21)$$

¹I am grateful to an anonymous referee for pointing this out in his/her report.

will in general be an element of \mathbb{F} . Whenever this scalar is well defined, it obviously will be an invariant under the natural action of the general orthogonal group $GO(V, \langle \cdot, \cdot \rangle)$ on the point set of the underlying projective space. More generally, a sort of projective space can be defined on a module V for a weaker structure than a field, say, algebras of dual and double numbers [31]. In such cases, numerous divisors of zero prompt a projective treatment ([6], [24, § 4.5]) of the new cross ratio (21). Another perspective direction to research are discrete Möbius geometries [17, 18].

The geometric applications of the new invariant (14), (21) are expected much beyond the currently presented situation of circles on a plane. Indeed, FSCc and cycles product based on Clifford algebras works in spaces of higher dimensions and with non-degenerate metrics of arbitrary signatures ([8], [9, § 4.1], [11], [24]). In a straightforward fashion, cycles cross ratio (14) remains a geometric FLT-invariant in higher dimensions as well.

A more challenging situation occurs if we have a degenerate metric and cycles are represented by parabolas [37] and respective FLT are based on dual numbers [5]. Besides theoretical interest, such spaces are meaningful physical models [15, 16, 20, 37]. The differential geometry loses its ground in the degenerate case and non-commutative/non-local effects appear [24, § 7.2]. The presence of zero divisors among dual numbers prompts a projective approach ([6], [24, § 4.5]) to the cross ratio of four dual numbers, which replaces (13). Variational methods do not produce FLT-invariant family of geodesics in the degenerate case; instead, geometry of cycles needs to be employed [22]. Our construction from Figure 13 shall be usable to define Möbius invariant distance between parabolic cycles as well, cf. [24, § 9.5].

Last but not least, if we restrict the group of transformations from FLT to Möbius maps (or any other subgroup of FLT which fixes a particular cycle), we will get a larger set of invariants. In particular, the FSCc matrix (3) and the corresponding cycles product (9) can use different number systems (complex, dual or double) independently from the geometry of the plane ([23], [24, § 5.3], [25]). Therefore, there will be three different cycles cross ratios (14) for each geometry of circles, parabolas and hyperbolas. This echoes the existence of nine Cayley–Klein geometries ([14], [33], [37, Appendix A]).

Various aspects of the cycles cross ratio appear to be a wide and fruitful field for further research.

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Vladimir V. Kisil
School of Mathematics
University of Leeds
Leeds LS2 9JT, UK
V.Kisil@leeds.ac.uk