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Unit squares with unit line segments

Manije Shahali and H. A. ShahAli

Manije Shahali studied computer sciences in Tehran. Her interests include sciences, mathematics and philosophy.
H. A. ShahAli studied math in Tehran and Hannover. His mathematical interests include number theory, combinatorics and geometry.

We begin with the following puzzle:

At most, how many unit squares can one make with 20 unit line segments?

Here a "unit square" is a "1×1-square" whose side length is 1 and is made by 4 unit line segments. Also, all shapes are considered in the plane.

For the above puzzle, we will first consider squares that are not connected; constructing them separately this way produces the fewest possible unit squares (here 5), as it requires the use of a maximum number of unit line segments per unit square (4 unit line segments), as shown in Figure 1.

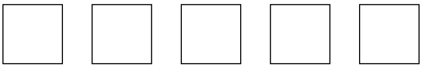
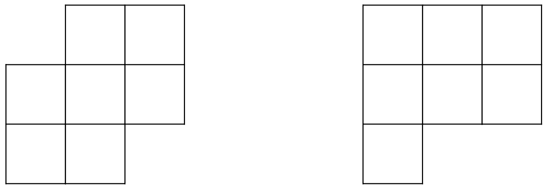


Figure 1. Five unconnected unit squares

Die Unterhaltungsmathematik umfasst verschiedenste Gebiete, von kuriosen Rätseln über unterhaltsame Spiele bis hin zu veritablen Forschungsfragen. Einer ihrer Zweige handelt von Konfigurationen aller Art, z. B. von Punkten, Strecken, Geraden usw. Oft sind diese Fragen mit einer Optimierung verknüpft. Die Autoren beginnen mit der Betrachtung eines solchen Rätsels: „Wie viele Einheitsquadrate kann man in der Ebene mit höchstens 20 Einheitsstrecken bilden?“ Bei der Untersuchung der entsprechenden allgemeinen Frage mit n Einheitsstrecken gelangen sie durch eine spiralförmige Konstruktion von Einheitsquadraten zu einer expliziten Formel. Diese liefert zum Beispiel, dass man maximal 478 Einheitsquadrate mit 1000 Einheitsstrecken konstruieren kann. Vielleicht wird ja die Leserschaft angeregt, die analoge Frage im Raum zu betrachten.



(a) (b)

Figure 2. Two solutions to the puzzle

To maximize the number of unit squares, we must construct them with as many shared unit line segments as possible. Figure 2 (a) and (b) show two such solutions to the puzzle, where we have achieved 7 unit squares in any answer, and each construction also contains two 2×2-squares. From Figure 2, it appears the maximum number of unit squares is 7 when the number of unit line segments is 20. Figure 2 (a) gives a symmetrical construction, but the asymmetric construction in Figure 2 (b) can help lead us to a more general solution, and proof, as shown in Figure 9 below.

Problems

At most, how many unit squares can one make with 1000 unit line segments? More broadly, what is the answer for a general positive integer n? And is there a unique answer?

This is an optimization problem: how to use the amount of material (unit line segments) to construct a maximum number of unit squares? Or equivalently, what is the minimum number of unit line segments needed to make a fixed number of unit squares?

We here present a theorem to provide the solution for any positive integer n, and as a special case, we will solve this problem for 1000 unit line segments.

When constructing new unit squares, we consider only squares chosen from a grid of aligned squares; we do not consider constructions such as shown in Figure 3 (a) and (b), where in Figure 3 (a), there are 3 unit squares built with only 9 unit line segments, and in Figure 3 (b), the two unit squares are not side-by-side. Throughout the figures, every new square is shown in gray, and the number cited in a square states the minimum number of unit line segments used for constructing the square.

(a) (b)

Figure 3. Unallowed cases in constructing squares

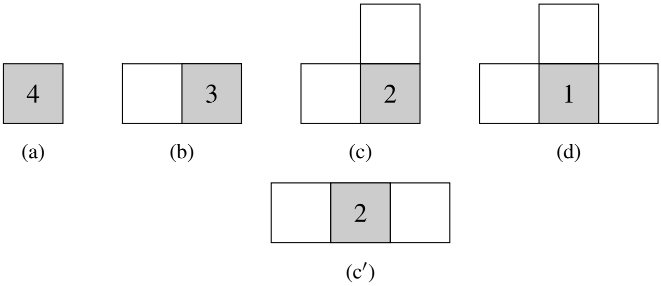


Figure 4. Various configurations in constructing a new square

If, in some step, a new unit square (shown in gray above) is built, we will have the following four cases.

(a) If the new square shares a side with no other unit squares, it needs 4 line segments, as shown in Figure 4 (a).

(b) If the new square shares a side with only another 1 unit square, it needs 3 new line segments, as shown in Figure 4 (b).

(c) If the new square shares sides with exactly 2 other unit squares, it needs 2 new line segments, as shown in Figure 4 (c) and (c′).

(d) If the new square shares sides with exactly 3 other unit squares, it needs 1 new line segment, as shown in Figure 4 (d).

Consider a completed construction of n unit squares; one can number the n squares in any arbitrary way by $1, 2, \dots, n$, and the k-th step in the construction process means the k-th square, and s_k is defined to be the minimum number of unit line segments needed to make the k-th square. Therefore, we have a numerical sequence $s_1, s_2, \dots, s_n$; also define $T_n = s_1 + s_2 + \cdots + s_n$, which is the total number of unit line segments making the construction. The sequence $s_1, s_2, \dots, s_n$ is not unique, but T_n for a completed construction is constant. The numbering and notation can be used also while proceeding with a construction process of unit squares. In a total construction of unit squares, it is clear that we need at least 4 unit line segments since, for the first square, we need at least 4 unit line segments.

Let $T \geq 4$ be the total constant number of unit line segments in a square construction; if in each step we use the least number of unit line segments to build a new square, then we will obtain the highest number of unit squares, and probably will be left with 0, 1 or 2 unit line segments in final since, for more than 2 unit segments, surely, we can build a new unit square because, for 3, one new square can be built on a pre-existing square, and for 4, one new square can be built solely using all the 4 segments or, on a pre-existing square, using 3 of the 4 segments.

Hence $T = s_1 + s_2 + \cdots + s_n + r$, where, for any k, $s_k \in \{1, 2, 3, 4\}$ is the least number of unit segments building the k-th square, and the remainder $r \in \{0, 1, 2\}$. As already said, since T is constant, when each s_k would take its least value, then n, the number of summands (of unit squares), would be the highest value.

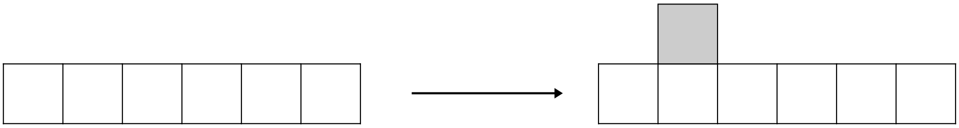


Figure 5. Building a new square on a k-row

If, for a construction of n unit squares, each term in the sequence $s_1, s_2, \ldots, s_n$ is minimal, then T_n will be minimal, and we call such a construction self-reliant so that its optimality is shown without comparison to any other construction.

We say one construction of unit squares is optimal if T_n is minimal. Also, we say one construction of unit squares is fully optimal if any partial sum $T_k = s_1 + s_2 + \cdots + s_k$ is minimal for all $k = 1, 2, \ldots, n$, for some numbering of the squares by $1, 2, \ldots, n$.

We call a row (or column) consisting of k unit squares a k-row (or k-column). If one new unit square is built on (and not in the direction of) a k-row (or k-column), it will ensure the possibility of constructing a total of $k - 1$ additional new unit squares on the k-row (or k-column), each one requiring 2 unit line segments to complete, in its 2 different directions on the k-row (or k-column) in the next following $k - 1$ steps. Thus if, on the outer side of a k-row (or k-column), there exist some squares, then exactly one of them is made by 3 unit line segments, and each of the possible $k - 1$ squares is made by 2 unit line segments. Figure 5 shows an example for $k = 6$.

We need the following lemma to prove the theorem.

Lemma. *The construction of unit squares is optimal if, in each step n, there is, in total,*

(i) *either one $k \times k$-square grid of unit squares with 0 to $k - 1$ additional successive unit squares on one of the sides,*

(ii) *or one $k \times (k + 1)$-rectangular grid of unit squares with 0 to k additional successive unit squares on one of the sides possessing $k + 1$ unit squares.*

Proof. We proceed by induction on steps of the constructions process.

- $n = 1$: Clearly, for the first unit square, at least 4 unit line segments are required, as shown on the left side of Figure 6 by a gray square ((i) for $k = 1$).
- $n = 2$: If the second unit square were to be made separately, we would need 4 more line segments. If it shares one side with a single other square, we need exactly 3 new line segments ((ii) for $k = 1$), like a 1×2-domino.
- $n = 3$: If the new gray square is built in the direction of the first two squares ((a) in Figure 6), the following square will also require 3 new line segments, whereas if the new gray square is built as per process (b), we can build the following square using only 2 new line segments. Hence we must proceed only with process (b) ((ii) for $k = 1$).

These prove the first 3 steps of the induction on the construction process.

Now, assuming that the n-th step holds, as above, we will now prove the $(n + 1)$-th step, in which we have the following two cases.

(1) In the n-th step, there may in total exist one $k \times k$-square grid of unit squares as shown in Figure 7 (a), where $n = k \times k$; in this case, the new unit square (in gray) should be

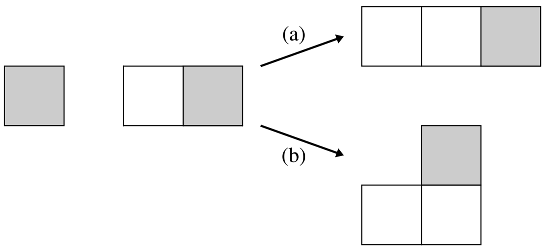


Figure 6. Three first steps in constructing squares

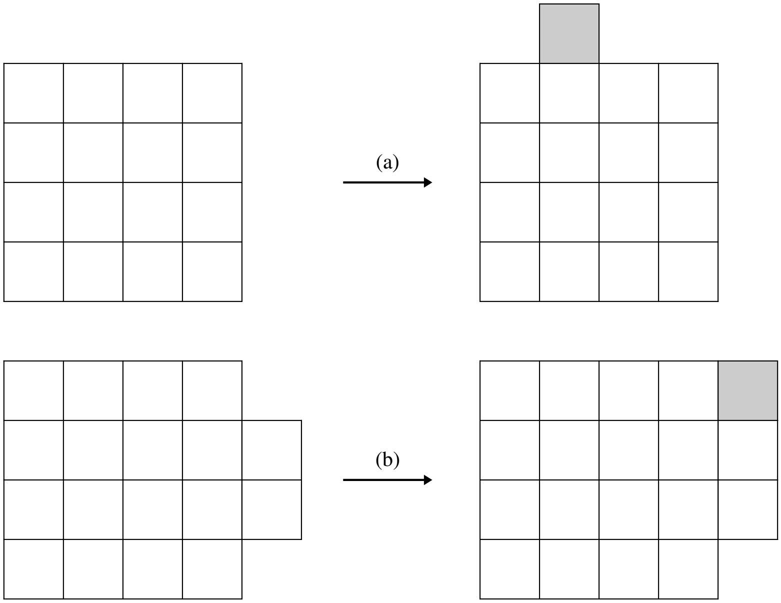


Figure 7. Adding new squares on a $k \times k$-square grid

built sharing a line segment with an outer unit square (no matter which one) in the grid, and it needs 3 new unit line segments. Also, there may exist 1 to $k - 1$ successive unit squares on one of the sides as shown in Figure 7 (b); in this case, the new subsequent square may be built with just 2 new unit line segments, as it shares 2 line segments with 2 pre-existing unit squares. Therefore, in the $(n + 1)$-th step, we will have one $k \times k$-square grid of unit squares with 1 to k additional unit squares on one of the sides so that, in the latter case, the construction will consist of one $k \times (k + 1)$-rectangular grid of unit squares.

(2) In the n-th step, there may in total exist one $k \times (k + 1)$-rectangular grid of unit squares as shown in Figure 8 (a); in this case, the new unit square should be constructed sharing a line segment with one of the unit squares on a side possessing $k + 1$ unit squares (no matter which one), as in Figure 8 (a), because this enables the construction of a subsequent square (in the example shown in Figure 8) which can be made with only 2 line

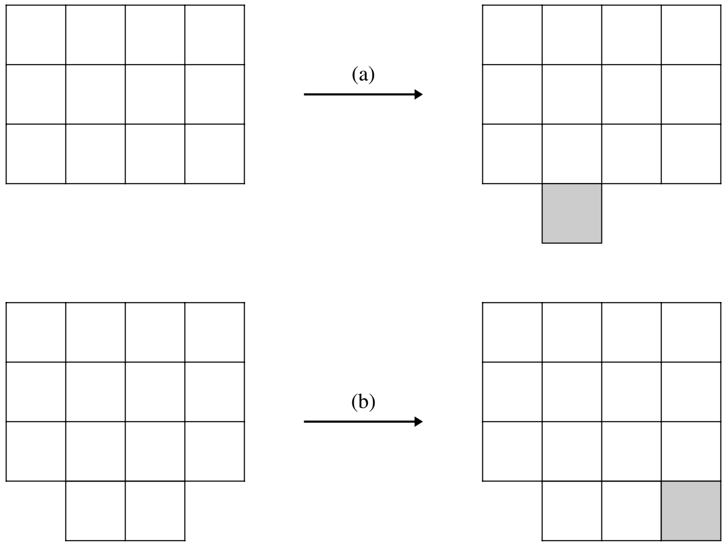


Figure 8. Adding new squares on a $k \times (k+1)$-rectangular grid

segments, which effectively delays by one step the need to make a new square using 3 new unit line segments. Also, there may exist 1 to k successive unit squares on a side possessing $k+1$ unit squares as shown in Figure 8 (b); in this case, the new subsequent square may be built with just 2 new unit line segments, as it shares 2 line segments with 2 pre-existing unit squares. Therefore, in any $(n+1)$-th step, we will have one $k \times (k+1)$-rectangular grid of unit squares with 1 to $k+1$ additional unit squares on one of the $(k+1)$-sides so that, in the latter case, the construction will consist of one $(k+1) \times (k+1)$-square grid of unit squares.

This completes the proof of the lemma. ■

Theorem. *Let T_n be the minimum number of unit line segments needed to construct n unit squares; then*

$$T_n = \begin{cases} 2n+1+\lfloor\sqrt{4n-4}\rfloor & \text{if } k^2 \le n+1 < k^2+k \text{ for some } k \in \mathbb{Z}_+, \\ 2n+1+\lfloor\sqrt{4n-3}\rfloor & \text{if } k^2+k \le n+1 < (k+1)^2 \text{ for some } k \in \mathbb{Z}_+, \end{cases}$$

where $\lfloor x \rfloor$ is the integer part of x.

Proof. We consider a counterclockwise spiral construction of unit squares $1, 2, \ldots, 7, \ldots$ around square 1 (like rolling a sheet of paper) as shown in Figure 9.

Now, we are ready to prove the theorem by expressing T_n in terms of n. If s_n is the minimum number of unit line segments needed to make the n-th unit square, then we obtain the following sequence:

$$s_n = 4, 3, 3, 2, 3, 2, 3, 2, 2, 3, 2, 2, 3, 2, 2, 2, 3, 2, 2, 2, \ldots$$

The lemma shows that this construction is optimal.

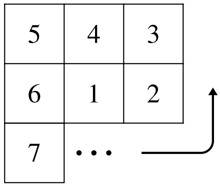


Figure 9. Spiral-like construction of unit squares

Starting with the third term in the sequence s_n, we observe the following patterned sequence:

$$3, 2, 3, 2, 3, 2, 2, 3, 2, 2, 3, 2, 2, 2, 3, 2, 2, 2, \ldots,$$

which is formed by the termwise sum of the constant sequence

$$2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, \ldots$$

and the following (0, 1)-sequence:

$$1, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, \ldots \tag{1}$$

consisting of successive powers of 10 (except 1) in 2 copies, dropping commas. We are going to locate 1's in sequence (1) and observe that the first 1's are put in the positions

$$1, 3, 5, 8, 11, 15, 19, \ldots \tag{2}$$

Now, we will determine sequence (2) denoted by u_m.

We get the following system of two linear recurrence relations:

$$\begin{cases} u_{2m} = u_{2m-1} + m + 1, \\ u_{2m+1} = u_{2m-1} + 2(m + 1), \end{cases}$$

which are easily solved; by the second relation, we get

$$\sum_{k=1}^{m} u_{2k+1} = \sum_{k=1}^{m} u_{2k-1} + 2 \sum_{k=1}^{m} (k + 1)$$
$$\implies u_{2m+1} = 1 + (m + 1)(m + 2) - 2,$$

and this gives $u_{2m+1} = m^2 + 3m + 1$; consequently, by the first relation,

$$u_{2m} = ((m - 1)^2 + 3(m - 1) + 1) + m + 1 = m^2 + 2m;$$

therefore,

$$\begin{cases} u_{2m} = m^2 + 2m, \\ u_{2m+1} = m^2 + 3m + 1. \end{cases} \tag{3}$$

And determining sequence (1) by (3), either $u_{2m} \le n < u_{2m+1}$ or $u_{2m+1} \le n < u_{2m+2}$:

$$\begin{cases} u_{2m} \le n < u_{2m+1} & \Longleftrightarrow\ m^2 + 2m \le n < m^2 + 3m + 1 \\ & \Longleftrightarrow\ (m+1)^2 \le n + 1 < (m+1)^2 + m + 1 \\ & \Longrightarrow\ 2m \le -2 + \sqrt{4n+4}, \\ u_{2m+1} \le n < u_{2m+2} & \Longleftrightarrow\ m^2 + 3m + 2 \le n + 1 < (m+1)^2 + 2(m+1) + 1 \\ & \Longleftrightarrow\ (m+1)^2 + m + 1 \le n + 1 < (m+2)^2 \\ & \Longrightarrow\ 2m + 1 \le -2 + \sqrt{4n+5}. \end{cases}$$

Then

$$T_n = \begin{cases} 4 + 3 + \sum_{k=1}^{n-2} s_k = 7 + 2(n-2) + \lfloor -2 + \sqrt{4n-4} \rfloor, \\ 4 + 3 + \sum_{k=1}^{n-2} s_k = 7 + 2(n-2) + \lfloor -2 + \sqrt{4n-3} \rfloor, \end{cases}$$

which gives

$$T_n = \begin{cases} 2n + 1 + \lfloor \sqrt{4n-4} \rfloor, \\ 2n + 1 + \lfloor \sqrt{4n-3} \rfloor, \end{cases}$$

and this completes the proof of the theorem. ■

Comment on the theorem. The reader can observe that the construction stated in the theorem, Figure 9, is an example of the general method discussed in the lemma and in fact is isomorphic with it; therefore, we can show that this construction is unique in some sense.

Let us consider the following two examples.

Examples. (1) Here are the first ten T_n's:

$$\begin{aligned} n &= 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \\ T_n &= 4, 7, 10, 12, 15, 17, 20, 22, 24, 27, \end{aligned}$$

and from the theorem, we easily observe that $\lim \frac{T_n}{n} = 2$ as $n \to \infty$.

(2) We have

$$n = k^2 \iff T_n = 2k(k+1).$$

Now, we solve the above problem for 1000 unit line segments.

Solution to the problem. By approximation and a few calculations in the theorem, we obtain

$$\begin{aligned} T_n &= 2 \times 478 + 1 + \lfloor \sqrt{4 \times 478 - 4} \rfloor \\ &= 2 \times 478 + 1 + \lfloor \sqrt{4 \times 478 - 3} \rfloor \\ &= 956 + 1 + 43 = 1000; \end{aligned}$$

hence $n = 478$. This states that one can construct 478 unit squares with 1000 unit line segments, and this answer is optimal. ∎

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Manije Shahali
Pacific Palisades, CA, USA
manija@addbits.com

H. A. ShahAli
Tehran, Iran
h_a_shahali@hotmail.com