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Short note

The Garfunkel–Bankoff inequality and the Finsler–Hadwiger inequality

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Abstract. In this note, we give improvements of the Garfunkel–Bankoff inequality and the Finsler–Hadwiger inequality in a triangle.

In a triangle with angles A, B, C, the sides are a, b, c, and S is the area of the triangle. The semi-perimeter, circumradius and inradius are denoted by s, R and r, respectively. In [3], Garfunkel proposed the following inequality as an open problem:

$$\tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} \geq 2 - 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}. \tag{1}$$

This was first proved by Bankoff in [1], and is known as the Garfunkel–Bankoff inequality. It has received considerable attention from researchers in the field of geometrical inequalities and has motivated a number of papers providing various generalizations and analogues; see, e.g., [8] and the references therein.

In [4], the celebrated Finsler–Hadwiger inequality [7]

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S + \mathcal{Q},$$

where $\mathcal{Q} = (a-b)^2 + (b-c)^2 + (c-a)^2$, was improved to

$$a^2 + b^2 + c^2 \geq 4\sqrt{4 - \frac{2r}{R}}\,S + \mathcal{Q}. \tag{2}$$

In this note, we give a sharpened version of (1), and an improvement of (2). The proof of the theorems relies on the bound

$$s^2 \leq 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)}, \tag{3}$$

which is described in [2, Section 5.10] as "the fundamental inequality of a triangle".

Theorem 1. *In a triangle with angles A, B, C, the inequality*

$$\tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} \geq 2 - 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + \frac{r^2(R-2r)}{4R^2(R-r)}. \tag{4}$$

holds, with equality if and only if the triangle is equilateral.

Proof. Using the well-known identities

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{r}{4R}$$

and

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}=\frac{(4R+r)^2}{s^2}-2,$$

(4) is equivalent to

$$s^2\le\frac{4R^2(4R+r)^2(R-r)}{16R^3-24R^2r+9Rr^2-2r^3}. \tag{5}$$

By (3), it is sufficient to prove

$$2R^2+10Rr-r^2+2(R-2r)\sqrt{R(R-2r)}\le\frac{4R^2(4R+r)^2(R-r)}{16R^3-24R^2r+9Rr^2-2r^3}. \tag{6}$$

Putting $t=\frac{r}{R}$, we have $0<t\le\frac{1}{2}$, and (6) is equivalent to

$$2+10t-t^2+2(1-2t)\sqrt{1-2t}\le\frac{4(4+t)^2(1-t)}{16-24t+9t^2-2t^3}.$$

This is true since

$$\begin{aligned}&\left[\frac{4(4+t)^2(1-t)}{16-24t+9t^2-2t^3}-(2+10t-t^2)\right]^2-[2(1-2t)\sqrt{1-2t}]^2\\&=\frac{t^4(4t^6+12t^5-47t^4+36t^3+504t^2-512t+128)}{(2t^3-9t^2+24t-16)^2}\\&=\frac{t^4(t^2-4t+8)(t+4)^2(1-2t)^2}{(2t^3-9t^2+24t-16)^2}\ge 0,\end{aligned}$$

which is obviously correct for $0<t\le\frac{1}{2}$. ∎

By Euler's inequality $R\ge 2r$, (4) is stronger than (1), and equivalent if and only if the triangle is equilateral.

As an application of (5), we show that (2) can be improved to the following.

Theorem 2. *In a triangle with sides a,b,c, the inequality*

$$a^2+b^2+c^2\ge 4\sqrt{4-\frac{2r}{R}+\frac{r^2(R-2r)}{4R^2(R-r)}}S+\mathcal{Q}. \tag{7}$$

holds, with equality if and only if the triangle is equilateral.

Proof. Indeed, using the well-known identities $ab+bc+ca=s^2+4Rr+r^2$, $S=rs$, we get

$$a^2+b^2+c^2-[(a-b)^2+(b-c)^2+(c-a)^2]=4r(4R+r).$$

Then (7) is equivalent to (5). ∎

Kooi's inequality

$$s^2 \le \frac{R(4R+r)^2}{2(2R-r)}$$

is equivalent to the Garfunkel–Bankoff inequality (1) and to the refinement of the Finsler–Hadwiger inequality (2). Its geometric interpretation is $OM^2 \ge 0$, where O is the circumcenter and M is the Mittenpunkt of the triangle (see [6]). It can be derived directly from the fundamental triangle inequality (3), without using parameter t (see [5]).

In [6], M. Lukarevski and D. S. Marinescu gave a refinement of Kooi's inequality, namely

$$s^2 \le \frac{R(4R+r)^2}{2(2R-r)} - \frac{r^2(R-2r)}{4R}. \tag{8}$$

We point out that (5) is stronger than (8), since $R \ge 2r$ and

$$\begin{aligned} &\frac{R(4R+r)^2}{2(2R-r)} - \frac{r^2(R-2r)}{4R} - \frac{4R^2(4R+r)^2(R-r)}{16R^3-24R^2r+9Rr^2-2r^3} \\ &= \frac{r^3(R-2r)(4R-r)(20R^2-5Rr+2r^2)}{4R(2R-r)(16R^3-24R^2r+9Rr^2-2r^3)} \ge 0. \end{aligned}$$

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