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Another extension of Lobachevsky's formula

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1 Introduction

Let $f(x)$ be a continuous function such that $f(x \pm \pi) = f(x)$ for all $x \geq 0$. Lobachevsky's formula [7] states that

$$\int_0^\infty \frac{\sin^2 x}{x^2} f(x)\, dx = \int_0^\infty \frac{\sin x}{x} f(x)\, dx = \int_0^{\pi/2} f(x)\, dx. \tag{1}$$

This implies the classical Dirichlet integral formula

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}. \tag{2}$$

Lobatschewskis berühmte Formel

$$\int_0^\infty \operatorname{sinc}^2(x) f(x)\, dx = \int_0^\infty \operatorname{sinc}(x) f(x)\, dx = \int_0^{\pi/2} f(x)\, dx$$

für π-periodische Funktionen f enthält die Sinus cardinalis Funktion $\operatorname{sinc}(x) = \frac{\sin x}{x}$ respektive deren Quadrat. Seine ursprüngliche Arbeit stammt aus dem Jahr 1842. Seitdem hat sie die Aufmerksamkeit vieler Mathematikerinnen und Mathematiker erregt, darunter Alfred Cardew Dixon und Godfrey Harold Hardy. In dieser Arbeit wird Lobatschewskis Formel erweitert. Der Autor verwendet dazu die höheren Ableitungen der Partialbruchentwicklung von $\csc x$. Dies erlaubt es ihm, entsprechende explizite Formeln für alle ungeraden Potenzen der sinc-Funktion zu finden, wobei zusätzliche Korrekturterme auftauchen. Es stellt sich heraus, dass der Ansatz auch für alle geraden Potenzen funktioniert.

Various proofs of (2) are given by Hardy [4] and Dixon [2]. In general, as a byproduct of the solution to the American Mathematical Monthly Problem 11423 [8], Chen [1] established

$$\int_0^\infty \frac{\sin^{2n} x}{x^{2m}}\,dx = \frac{\pi}{2^{2n}(2m-1)!}\sum_{k=1}^{n}(-1)^{k+m}\binom{2n}{n-k}(2k)^{2m-1} \qquad \text{for } 1 \le m \le n,$$

$$\int_0^\infty \frac{\sin^{2n+1} x}{x^{2m+1}}\,dx = \frac{\pi}{2^{2n+1}(2m)!}\sum_{k=0}^{n}(-1)^{k+m}\binom{2n+1}{n-k}(2k+1)^{2m} \qquad \text{for } 0 \le m \le n.$$

Recently, Jolany [6] generalized formula (1) to

$$\int_0^\infty \frac{\sin^4 x}{x^4} f(x)\,dx = \int_0^{\pi/2} f(x)\,dx - \frac{2}{3}\int_0^{\pi/2} \sin^2 x f(x)\,dx$$

and also gave a method for computing

$$\int_0^\infty \frac{\sin^{2n} x}{x^{2n}} f(x)\,dx$$

for $n \in \mathbb{N}$ in general. But he did not reveal an explicit formula.

In this note, applying higher derivatives to the partial fraction expansion of $\csc x$, we extend Lobachevsky's formula to all odd powers. In particular, we find

$$\int_0^\infty \frac{\sin^3 x}{x^3} f(x)\,dx = \int_0^{\pi/2} f(x)\,dx - \frac{1}{2}\int_0^{\pi/2} \sin^2 x f(x)\,dx, \qquad (3)$$

$$\int_0^\infty \frac{\sin^5 x}{x^5} f(x)\,dx = \int_0^{\pi/2} f(x)\,dx - \frac{5}{6}\int_0^{\pi/2} \sin^2 x f(x)\,dx + \frac{1}{24}\int_0^{\pi/2} \sin^4 x f(x)\,dx.$$

Moreover, our approach also gives an explicit formula for all even powers.

2 Proof of (3)

We start with the proof of (3), which will serve as an illustration of the proof for general odd powers. To this end, we need two lemmas.

Lemma 1. *Let $x \ne n\pi$ for $n \in \mathbb{Z}$. Then (see [7, 1.422, p.43])*

$$\frac{1}{\sin x} = \frac{1}{x} + \sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{x-k\pi} + \frac{1}{x+k\pi}\right). \qquad (4)$$

Since

$$\frac{d^2}{dx^2}\left(\frac{1}{\sin x}\right) = \frac{2}{\sin^3 x} - \frac{1}{\sin x},$$

differentiating (4) twice term by term yields the following lemma.

Lemma 2. *Let $x \neq n\pi$ for $n \in \mathbb{Z}$. Then*

$$\frac{2}{\sin^3 x} - \frac{1}{\sin x} = \frac{2}{x^3} + \sum_{k=1}^{\infty} (-1)^k 2\Big(\frac{1}{(x-k\pi)^3} + \frac{1}{(x+k\pi)^3}\Big). \tag{5}$$

Now, we extend Lobachevsky's formula for the cubic power as follows.

Theorem 1. *Let $f(x)$ be a continuous function satisfying $f(x \pm \pi) = f(x)$ for all $x \geq 0$. Then identity (3) holds.*

Proof. Let

$$I = \int_0^{\infty} \frac{\sin^3 x}{x^3} f(x)\, dx.$$

We rewrite the integral as

$$I = \sum_{n=0}^{\infty} \int_{n\pi/2}^{(n+1)\pi/2} \frac{\sin^3 x}{x^3} f(x)\, dx.$$

Next, we split the sum into two parts by the parity of n to get

$$I = \sum_{k=0}^{\infty} \int_{k\pi}^{k\pi+\frac{\pi}{2}} \frac{\sin^3 x}{x^3} f(x)\, dx + \sum_{k=1}^{\infty} \int_{k\pi-\frac{\pi}{2}}^{k\pi} \frac{\sin^3 x}{x^3} f(x)\, dx. \tag{6}$$

Let $x = k\pi + t$. Then

$$\int_{k\pi}^{k\pi+\frac{\pi}{2}} \frac{\sin^3 x}{x^3} f(x)\, dx = (-1)^k \int_0^{\pi/2} \frac{\sin^3 t}{(k\pi+t)^3} f(t)\, dt.$$

Similarly, let $x = k\pi - t$. Then

$$\int_{k\pi-\frac{\pi}{2}}^{k\pi} \frac{\sin^3 x}{x^3} f(x)\, dx = (-1)^{k-1} \int_0^{\pi/2} \frac{\sin^3 t}{(k\pi-t)^3} f(t)\, dt.$$

Using these two integrals and (6), we get

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{\sin^3 t}{t^3} f(t)\, dt + \sum_{k=1}^{\infty} \int_0^{\pi/2} (-1)^k \Big(\frac{\sin^3 t}{(k\pi+t)^3} + \frac{\sin^3 t}{(t-k\pi)^3}\Big) f(t)\, dt \\ &= \int_0^{\pi/2} \sin^3 t \Big[\frac{1}{t^3} + \sum_{k=1}^{\infty} (-1)^k \Big(\frac{1}{(t+k\pi)^3} + \frac{1}{(t-k\pi)^3}\Big)\Big] f(t)\, dt. \end{aligned}$$

Here the interchange of the order of the summation and integration is justified by the uniform convergence of the series. Finally, applying identity (5), we find that

$$I = \int_0^{\pi/2} \sin^3 t \frac{1}{2}\Big(\frac{2}{\sin^3 t} - \frac{1}{\sin t}\Big) f(t)\, dt = \int_0^{\pi/2} f(t)\, dt - \frac{1}{2}\int_0^{\pi/2} \sin^2 t f(t)\, dt.$$

This proves (3) as claimed. ∎

3 Proof of the general odd powers

We now demonstrate how the proof of (3) sheds light to treat general odd powers. Let $f^{(n)}$ be the nth derivative of f. By (4), for every $n \in \mathbb{N}$, we have

$$\csc^{(2n)} x = (2n)!\left[\frac{1}{x^{2n+1}} + \sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{(x-k\pi)^{2n+1}} + \frac{1}{(x+k\pi)^{2n+1}}\right)\right]. \tag{7}$$

Based on Lemma 2, we need to prove that $\csc^{(2n)} x$ is a linear combination of $\csc^k x$ (so $1/\sin^k x$) with $1 \le k \le 2n+1$. For $n = 2, 3$, by direct calculation or with Mathematica, we have

$$\csc^{(4)} x = \csc x(\cot^4 x + 18\cot^2 x\csc^2 x + 5\csc^4 x) \tag{8}$$

$$= \csc x - 20\csc^3 x + 24\csc^5 x, \tag{9}$$

$$\csc^{(6)} x = \csc x(\cot^6 x + 179\cot^4 x\csc^2 x + 479\cot^2 x\csc^4 x + 61\csc^6 x) \tag{10}$$

$$= -\csc x + 182\csc^3 x - 840\csc^5 x + 720\csc^7 x. \tag{11}$$

In view of $\csc^2 x = 1 + \cot^2 x$, (8) and (10) suggest that $\csc^{(2n)} x$ is a product of $\csc x$ with a polynomial of degree n in $\cot^2 x$. Therefore, as (9) and (11) indicate, $\csc^{(2n)} x$ will be a polynomial of degree (2n+1) in $\csc x$ (so $1/\sin x$).

We now proceed to establish the above assertions. Following Hoffman [5], let $Q_n(x)$ be the derivative polynomials for $\sec x$ which are defined by

$$\sec^{(n)} x = \sec x\, Q_n(\tan x), \quad n \in \mathbb{N}_0. \tag{12}$$

For $k \in \mathbb{N}_0$ and a fixed x, since

$$\sec^{(k)}(x+t)|_{t=0} = \sec^{(k)} x,$$

the Taylor series expansion of $\sec(x+t)$ leads to the following exponential generating function for $\{Q_n\}$:

$$\frac{\sec(x+t)}{\sec x} = \sum_{n=0}^{\infty} Q_n(\tan x)\frac{t^n}{n!}. \tag{13}$$

Since

$$\frac{\sec(x+t)}{\sec x} = \frac{\cos x}{\cos x\cos t - \sin x\sin t} = \frac{\sec t}{1-\tan x\tan t},$$

applying $\tan x \to x$ enables us to rewrite (13) as

$$G(x,t) := \frac{\sec t}{1 - x\tan t} = \sum_{n=0}^{\infty} Q_n(x)\frac{t^n}{n!}. \tag{14}$$

Similarly, applying the chain rule to (12) yields

$$Q_n(-x) = (-1)^n Q_n(x), \quad Q_{n+1}(x) = (1+x^2)Q_n'(x) + xQ_n(x). \tag{15}$$

By $\tan(x+\pi/2) = -\cot x$ and $\sec(x+\pi/2) = -\csc x$, using (12) and (15), we find that

$$\csc^{(n)} x = (-1)^n \csc x \, Q_n(\cot x). \tag{16}$$

To find a closed formula for $Q_n(x)$, let $S(n,k)$ be the *secant numbers* of order k which are defined by

$$\sec x \tan^k x = \sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!} \quad (k \in \mathbb{N}_0). \tag{17}$$

We obtain the following lemma.

Lemma 3. *Let $Q_n(x)$ be defined by (14), and let $S(n,k)$ be defined by (17). Then*

$$Q_n(x) = \sum_{k=0}^{n} S(n,k) x^k, \tag{18}$$

and

$$\csc^{(n)} x = (-1)^n \csc x \sum_{k=0}^{n} S(n,k) \cot^k(x). \tag{19}$$

Proof. Using the geometric series expansion in (14), we have

$$\begin{aligned} G(x,t) &= \sum_{k=0}^{\infty} \sec t \tan^k t x^k = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} S(n,k) \frac{t^n}{n!} \right) x^k \quad \text{(use (17))} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} S(n,k) x^k \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} Q_n(x) \frac{t^n}{n!}. \end{aligned}$$

This proves (18). Consequently, (19) follows from (16) and (18). ■

By the parity of $\tan^k x$, we see that $S(n,k) \neq 0$ only if both n and k have the same parity. Thus,

$$\csc^{(2n)} x = \csc x \sum_{k=0}^{n} S(2n,2k) \cot^{2k}(x).$$

By the binomial theorem, we find

$$\begin{aligned} \csc^{(2n)} x &= \sum_{k=0}^{n} \sum_{i=0}^{k} (-1)^{k-i} S(2n,2k) \binom{k}{i} \csc^{2i+1} x \\ &= \sum_{i=0}^{n} (-1)^i \left(\sum_{k=i}^{n} (-1)^k S(2n,2k) \binom{k}{i} \right) \csc^{2i+1} x. \end{aligned} \tag{20}$$

On the other hand, for $n \in \mathbb{N}$, let

$$I_n = \int_0^{\infty} \frac{\sin^{2n+1} x}{x^{2n+1}} f(x)\,dx = \sum_{n=0}^{\infty} \int_{n\pi/2}^{(n+1)\pi/2} \frac{\sin^{2n+1} x}{x^{2n+1}} f(x)\,dx.$$

Similar to the proof of (3), we have

$$I_n = \int_0^{\pi/2} \sin^{2n+1} t \left[\frac{1}{t^{2n+1}} + \sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{(k\pi + t)^{2n+1}} + \frac{1}{(t - k\pi)^{2n+1}} \right) \right] f(t)\, dt.$$

This, together with (7) and (20), yields the main result.

Theorem 2. *Let $f(x)$ be a continuous function satisfying $f(x \pm \pi) = f(x)$ for all $x \geq 0$. For every $n \in \mathbb{N}$, we have*

$$\int_0^{\infty} \frac{\sin^{2n+1} x}{x^{2n+1}} f(x)\, dx = \frac{1}{(2n)!} \sum_{i=0}^{n} (-1)^i \left(\sum_{k=i}^{n} (-1)^k S(2n, 2k) \binom{k}{i} \right) \int_0^{\pi/2} \sin^{2(n-i)} x f(x)\, dx.$$

4 Computation of $Q_n(x)$

In this section, we show how to use (17) and (18) to find $Q_n(x)$ and $\csc^{(2n)} x$ practically. Differentiating (17) gives

$$\frac{1}{k!} (\sec x \tan^k x)' = \frac{\sec^3 x \tan^{k-1} x}{(k-1)!} + \frac{\sec x \tan^{k+1} x}{k!} = \sum_{n=k}^{\infty} S(n,k) \frac{x^{n-1}}{(n-1)!} = \sum_{n=k-1}^{\infty} S(n+1,k) \frac{x^n}{n!}. \tag{21}$$

Moreover, using $\sec^2 x = 1 + \tan^2 x$, we have

$$\frac{\sec x \tan^{k-1} x}{(k-1)!} + k(k+1) \frac{\sec x \tan^{k+1} x}{(k+1)!} = \frac{\sec^3 x \tan^{k-1} x}{(k-1)!}. \tag{22}$$

Equations (21) and (22) imply that

$$S(n+1,k) = S(n,k-1) + (k+1)^2 S(n,k+1) \quad (n \geq 0,\ k \geq 1). \tag{23}$$

In particular, we have $S(n,0) = S_n$, which are the classical secant numbers (see A000364 in OEIS®) defined by

$$\sec x = \sum_{n=0}^{\infty} S_n \frac{x^n}{n!}.$$

Using (23), together with the well-known S_n, we can calculate $S(n,k)$ and then $Q_n(x)$ from (18) as well. For example, we find $S(2n, 2k)$ for $0 \leq k \leq n \leq 10$ in Table 1.

	k					
n	0	2	4	6	8	10
0	1					
2	1	2				
4	5	28	24			
6	61	662	1320	720		
8	1385	24568	83664	100800	40320	
10	50521	1326122	6749040	13335840	11491200	3628800

Table 1. Some nonzero values of $S(n,k)$ for $0 \le k \le n \le 10$.

Thus, explicitly, we find that

$$\begin{aligned} Q_2(x) &= 1 + 2x^2, \\ Q_4(x) &= 5 + 28x^2 + 24x^4, \\ Q_6(x) &= 61 + 662x^2 + 1320x^4 + 720x^6, \\ Q_8(x) &= 1385 + 24568x^2 + 83664x^4 + 100800x^6 + 40320x^8, \\ Q_{10}(x) &= 50521 + 1326122x^2 + 6749040x^4 + 13335840x^6 \\ &\quad + 11491200x^8 + 3628800x^{10}. \end{aligned}$$

Using these derivative polynomials, beside recovering (9) and (11), we obtain

$$\begin{aligned} \csc^{(8)} x &= \csc x - 1640 \csc^3 x + 23184 \csc^5 x - 60480 \csc^7 x + 40320 \csc^9 x, \\ \csc^{(10)} x &= -\csc x + 14762 \csc^3 x - 599280 \csc^5 x + 3659040 \csc^7 x \\ &\quad - 6652800 \csc^9 x + 3628800 \csc^{11} x. \end{aligned}$$

5 Remarks

Identity (19) indeed enables us to find an explicit formula for all even powers too. Based on Jolany's proof [6] for the power of 4, for every even power $2n$, we need to compute the $2(n-1)$th derivative of $\csc^2 x$. Applying the Leibniz rule and (19), we have

$$\begin{aligned} (\csc^2 x)^{(m)} &= \sum_{k=0}^{m} \binom{m}{k} (\csc x)^{(k)} (\csc x)^{(m-k)} \\ &= \csc^2 x \sum_{k=0}^{m} \binom{m}{k} \left(\sum_{i=0}^{k} S(k,i) \cot^i x \right) \left(\sum_{j=0}^{m-k} S(m-k,j) \cot^j x \right) \\ &= \csc^2 x \sum_{k=0}^{m} \sum_{i=0}^{k} \sum_{j=0}^{m-k} \binom{m}{k} S(k,i) S(m-k,j) \cot^{i+j} x. \end{aligned} \tag{24}$$

It is easy to see that $i + j$ is always even when m is even. Therefore, the derivative $(\csc^2 x)^{(2(n-1))}$ is a polynomial of degree $2n$ in $\csc x$. Using Table 1 for $S(n,k)$ and (24), we have

$$\begin{aligned}
(\csc^2)^{(4)} x &= \csc^2 x(16 + 120\cot^2 x + 120\cot^4 x) \\
&= 16\csc^2 x - 120\csc^4 x + 120\csc^6 x, \\
(\csc^2)^{(6)} x &= \csc^2 x(272 + 3696\cot^2 x + 8400\cot^4 x + 5040\cot^6 x) \\
&= -64\csc^2 x + 2016\csc^4 x - 6720\csc^6 x + 5040\csc^8 x, \\
(\csc^2)^{(8)} x &= \csc^2 x(7936 + 168960\cot^2 x + 645120\cot^4 x \\
&\qquad + 846720\cot^6 x + 362880\cot^8 x) \\
&= 256\csc^2 x - 32640\csc^4 x + 282240\csc^6 x \\
&\qquad - 604800\csc^8 x + 362880\csc^{10} x.
\end{aligned}$$

These identities will yield Lobachevsky's formula for

$$\int_0^\infty \frac{\sin^6 x}{x^6} f(x)\,dx, \quad \int_0^\infty \frac{\sin^8 x}{x^8} f(x)\,dx, \quad \text{and} \quad \int_0^\infty \frac{\sin^{10} x}{x^{10}} f(x)\,dx$$

respectively.

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