Elemente der Mathematik (Online first)
DOI 10.4171/EM/481

Published by EMS Press

Approximation of polynomials by Hermite interpolation

Robert Kantrowitz and Michael M. Neumann

Robert Kantrowitz earned his Ph.D. from Syracuse University in 1990. Since then, he has served on the mathematics faculty at Hamilton College, his undergraduate alma mater. Much of his recent work focuses on analytic and geometric aspects of models of projectile motion. He has a large collection of mechanical pencils and fountain pens and dabbles in model railroading.

Michael Neumann received his doctoral and habilitation degrees from Saarland University in his native Germany and taught at Mississippi State University for more than thirty years. While his main work is in functional analysis and operator theory, his more recent research interests center around convex functions and projectile motion. He is passionate about the history of mathematics.

1 Discontinuous linear functionals

A theorem from elementary functional analysis states that a non-zero linear functional on a normed linear space E is discontinuous precisely when its kernel is dense in E; see, for instance, [3, Theorem 3.2 (a)]. In fact, if φ is a discontinuous linear functional on the normed space E, then there exists a bounded sequence $(q_n)_{n\in\mathbb{N}}$ of elements of E for which $|\varphi(q_n)| \to \infty$ as $n \to \infty$ and, for every such sequence and an arbitrary $p \in E$, the

Für ein beliebiges Polynom p auf dem Einheitsintervall garantiert ein Satz über diskontinuierliche lineare Funktionale auf normierten linearen Räumen die Existenz von Polynomen, deren Ableitungen an einem Endpunkt des Intervalls verschwinden und die p gleichmäßig approximieren. Im vorliegenden Artikel wird die Frage behandelt, ob solche Approximationen auch unter der weiteren Bedingung möglich sind, dass die approximierenden Polynome an äquidistanten Stützstellen mit p übereinstimmen. Es stellt sich heraus, dass die Hermite-Interpolationspolynome das Gewünschte leisten. Verlangt man hingegen, dass die Ableitung der Approximationspolynome im Mittelpunkt des Intervalls verschwindet, so tritt das Runge-Phänomen auf den Plan. In diesem Fall konvergieren die entsprechenden Hermite-Interpolationspolynome nämlich nicht, wenn $p'\left(\frac{1}{2}\right) \neq 0$ ist. Andererseits führt dann ein Wechsel von äquidistanten Stützstellen zu Tschebyscheff-Stützstellen wieder zu Interpolationspolynomen, die p gleichmäßig approximieren.

sequence $(p_n)_{n\in\mathbb{N}}$ given by

$$p_n = p - \frac{\varphi(p)}{\varphi(q_n)} q_n \quad \text{for all } n \in \mathbb{N} \tag{1}$$

satisfies $p_n \in \ker(\varphi)$ for all $n \in \mathbb{N}$ and $p_n \to p$ as $n \to \infty$; see, for example, the proof of part (ii) of [7, Proposition 8.6]. Of course, if $p \in \ker(\varphi)$, then $p_n = p$ for all $n \in \mathbb{N}$.

In the following, the relevant normed linear space will be the space $P[0,1]$ of polynomials on the compact interval $[0,1]$, endowed, as usual, with the supremum norm $\|\cdot\|_\infty$ given by $\|p\|_\infty = \sup\{|p(x)| : x \in [0,1]\}$ for all $p \in P[0,1]$. Also, for arbitrary $\alpha \in [0,1]$, the symbol φ_α will denote the point evaluation of the derivative at α given by $\varphi_\alpha(p) = p'(\alpha)$ for all $p \in P[0,1]$. Here we shall focus on the cases $\alpha = 0, \frac{1}{2}, 1$. The functional φ_0 is discontinuous since, for each $n \in \mathbb{N}$, the polynomial $q_n(x) = (x-1)^n$ satisfies $\|q_n\|_\infty = 1$ and $\varphi_0(q_n) = q_n'(0) = n(-1)^{n-1}$. Consequently, for given $p \in P[0,1]$, equation (1) becomes

$$p_n(x) = p(x) - \frac{p'(0)}{n(-1)^{n-1}}(x-1)^n,$$

which entails that $p_n'(0) = 0$ for all $n \in \mathbb{N}$ and $p_n \to p$ as $n \to \infty$ uniformly on $[0,1]$. A similar result may be obtained for φ_1 by using $q_n(x) = x^n$ for all $n \in \mathbb{N}$, but the case of φ_α for $0 < \alpha < 1$ is perhaps less obvious and will be addressed later.

It seems natural to wonder if the approximation result of the preceding example may be improved to accommodate certain conditions of interpolation type. For instance, given an arbitrary $p \in P[0,1]$ and a point $\alpha \in [0,1]$, the problem is to approximate p by a sequence $(p_n)_{n\in\mathbb{N}}$ of polynomials in the kernel of φ_α such that the condition of interpolation $p_n\left(\frac{j}{n}\right) = p\left(\frac{j}{n}\right)$ holds for $j = 0, 1, \ldots, n$. Such conditions are known from the theory of *Lagrange interpolation*, but the additional stipulation $p_n'(\alpha) = 0$ indicates that actually the more general theories of *Hermite* and *Birkhoff interpolation* are relevant here, as will be detailed in Section 3. We will see that the formulas provided by these theories are quite complicated and not really helpful for the convergence issues that we are interested in. We therefore pursue a different approach which is remarkably more elementary.

For the case of equally spaced interpolation points, the following dichotomy will be established. If α is one of the endpoints 0 or 1, then, for arbitrary $p \in P[0,1]$, the sequence of corresponding Hermite interpolation polynomials p_n turns out to converge uniformly to p, while, in the case of the midpoint $\alpha = \frac{1}{2}$, this sequence even fails to be bounded for every choice of p for which $p'\left(\frac{1}{2}\right) \neq 0$. By contrast, it will also be shown that convergence occurs even in the midpoint case $\alpha = \frac{1}{2}$ provided that equally spaced interpolation points are replaced by Chebyshev nodes of the first kind.

2 A workhorse lemma

The key is an analysis of the *node polynomials* $s_n \in P[0,1]$ defined by

$$s_n(x) = \prod_{j=0}^{n} \left(x - \frac{j}{n}\right) \quad \text{for all } x \in [0,1].$$

In the textbook [11], Trefethen discusses the historical context and the importance of node polynomials [11, equation (5.4)] in interpolation. For example, node polynomials allow for the alternative and advantageous Lagrange interpolation formula that is presented in [11, equation (5.9)]. In addition, they lead to a simple expression for the error that arises when applying Lagrange interpolation to approximate a function; see, for instance, the Polynomial Interpolation Error Theorem [4, Theorem 4.3]. Node polynomials also play a natural role in Hermite interpolation theory; see [9] or [4, Section 4.6], for example.

Evidently, the degree of the node polynomial s_n is $n+1$. We gather some facts about these polynomials in the following lemma which will turn out to be the workhorse for establishing our main theorems connected to Hermite interpolation.

Lemma 1. *For arbitrary $n \in \mathbb{N}$, the following assertions hold:*

(i) $s_n\left(\frac{k}{n}\right) = 0$ *for* $k = 0, 1, \dots, n$;

(ii) $s_n'(0) = (-1)^n \frac{n!}{n^n}$;

(iii) $s_n'(1) = \frac{n!}{n^n}$;

(iv) $|s_n(x)| \leq \frac{n!}{4n^{n+1}}$ *for all* $x \in [0,1]$;

(v) $s_n'\left(\frac{1}{2}\right) = \frac{(-1)^{n/2}(\frac{n}{2})!\,(\frac{n}{2})!}{n^n}$ *for even* $n \in \mathbb{N}$;

(vi) $\left|s_n\left(\frac{1}{2n}\right)\right| = \frac{(2n)!}{(2n)^{n+1} \cdot 2^n \cdot n!}$.

Proof. Assertion (i) is an immediate consequence of the definition of s_n. For part (ii), an application of the product rule to

$$s_n(x) = \prod_{j=0}^{n} \left(x - \frac{j}{n}\right) = x \prod_{j=1}^{n} \left(x - \frac{j}{n}\right)$$

yields

$$s_n'(x) = \prod_{j=1}^{n} \left(x - \frac{j}{n}\right) + x \left(\prod_{j=1}^{n} \left(x - \frac{j}{n}\right)\right)'.$$

Thus,

$$s_n'(0) = \prod_{j=1}^{n} \left(-\frac{j}{n}\right) = (-1)^n \frac{n!}{n^n}.$$

Similarly, for assertion (iii), we have

$$s_n'(1) = \prod_{j=0}^{n-1} \left(1 - \frac{j}{n}\right) = \prod_{j=0}^{n-1} \left(\frac{n-j}{n}\right) = \frac{n!}{n^n}.$$

For part (iv), let $N = \left\{\frac{j}{n} : j = 0, 1, \dots, n\right\}$, and note that, by (i), it suffices to consider a point $x \in [0,1] \setminus N$. For such x, we order the elements of N as a list $(x_0, x_1, \dots, x_n)$ such that

$$|x - x_0| \leq |x - x_1| \leq \cdots \leq |x - x_n|;$$

in particular, the point x lies between x_0 and x_1. It follows that

$$|s_n(x)| = |x - x_0||x - x_1| \prod_{j=2}^{n} |x - x_j| \leq \frac{1}{4n^2} \prod_{j=2}^{n} |x - x_j| \leq \frac{1}{4n^2} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdots \frac{n}{n},$$

which establishes estimate (iv).

Proceeding to assertion (v), for even $n \in \mathbb{N}$, the product rule applied to the formula

$$s_n(x) = \left(x - \frac{1}{2}\right) \prod_{\substack{j=0 \\ j \neq n/2}}^{n} \left(x - \frac{j}{n}\right)$$

leads to

$$s_n'(x) = \prod_{\substack{j=0 \\ j \neq n/2}}^{n} \left(x - \frac{j}{n}\right) + \left(x - \frac{1}{2}\right) \left(\prod_{\substack{j=0 \\ j \neq n/2}}^{n} \left(x - \frac{j}{n}\right) \right)'.$$

Thus,

$$\begin{aligned} s_n'\left(\frac{1}{2}\right) &= \prod_{j=0}^{\frac{n}{2}-1} \left(\frac{1}{2} - \frac{j}{n}\right) \prod_{j=\frac{n}{2}+1}^{n} \left(\frac{1}{2} - \frac{j}{n}\right) \\ &= \frac{1}{n^n} \prod_{j=0}^{\frac{n}{2}-1} \left(\frac{n}{2} - j\right) \prod_{j=\frac{n}{2}+1}^{n} \left(\frac{n}{2} - j\right) \\ &= \frac{1}{n^n} \cdot \frac{n}{2}\left(\frac{n}{2} - 1\right) \cdots 1 \cdot (-1)(-2) \cdots \left(-\frac{n}{2}\right) \\ &= \frac{(-1)^{n/2}}{n^n} \cdot \left(\frac{n}{2}\right)! \left(\frac{n}{2}\right)! \end{aligned}$$

Finally, for arbitrary $n \in \mathbb{N}$,

$$\begin{aligned} \left| s_n\left(\frac{1}{2n}\right) \right| &= \prod_{j=0}^{n} \left| \frac{1}{2n} - \frac{j}{n} \right| = \prod_{j=0}^{n} \left| \frac{1 - 2j}{2n} \right| = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)^{n+1}} \\ &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot 2 \cdot 4 \cdot 6 \cdots 2n}{(2n)^{n+1} \cdot 2^n \cdot n!} = \frac{(2n)!}{(2n)^{n+1} \cdot 2^n \cdot n!}, \end{aligned}$$

which establishes assertion (vi) and completes the proof. ■

3 Hermite interpolation polynomials

The following theorem is in the spirit of formula (1) and is an immediate consequence of assertions (i), (ii), and (iv) of Lemma 1. It provides simple formulas for polynomials p_n that agree with a given polynomial p at equally spaced points in the interval [0, 1], have derivatives that vanish at 0, and uniformly approximate p.

Theorem 2. *For arbitrary $p \in P[0,1]$ and $n \in \mathbb{N}$, the polynomial p_n given by*

$$p_n(x) = p(x) - \frac{p'(0)}{s_n'(0)} s_n(x) = p(x) - (-1)^n p'(0) \frac{n^n}{n!} s_n(x) \quad \textit{for all } x \in [0,1]$$

satisfies

(i) $p_n'(0) = 0$;

(ii) $p_n\left(\frac{k}{n}\right) = p\left(\frac{k}{n}\right)$ *for* $k = 0, 1, \ldots, n$;

(iii) $\|p_n - p\|_\infty \le |p'(0)|/(4n)$.

In particular, $\|p_n - p\|_\infty \to 0$ *as* $n \to \infty$.

We emphasize that no particular knowledge of the theory of interpolation polynomials is needed to establish Theorem 2, but it is gratifying to have a closer look at the connection.

Given an arbitrary function f on $[0,1]$ and intermediate points $0 = x_0 < x_1 < \cdots < x_n = 1$, there exists a unique polynomial h of degree at most $n+1$ for which $h'(0) = 0$ and $h(x_k) = f(x_k)$ for $k = 0, 1, \ldots, n$. This polynomial is called the *Hermite interpolation polynomial* for the given setting and may be computed by the formula

$$h(x) = f(0) + x^2 \sum_{k=1}^{n} \frac{f(x_k) - f(0)}{x_k^2} \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{x - x_j}{x_k - x_j} \quad \text{for all } x \in [0,1]. \tag{2}$$

Indeed, it is evident that formula (2) defines a polynomial of degree at most $n+1$ with the desired properties. Furthermore, it is easily seen that there is at most one polynomial of this type since, by Rolle's theorem, the derivative of the difference of two such polynomials has at least $n+1$ zeros and hence must be the zero function which, in turn, forces the difference to be identically zero.

To sketch the historical background and to explain our choice of terminology, we recall, for instance, from [1, Section 3.6] or [10, Section 14.1] that, in the classical theory of Hermite interpolation, one considers a collection of nodes $x_0 < x_1 < \cdots < x_n$ and pairs of real numbers $(f_{0,k}, f_{1,k})$ for $k = 0, 1, \ldots, n$. In this setting, there exists a unique polynomial h of degree at most $2n+1$ for which $h(x_k) = f_{0,k}$ and $h'(x_k) = f_{1,k}$ for $k = 0, 1, \ldots, n$. The polynomial h is sometimes referred to as the *classical Hermite interpolation polynomial*, and it is well known how to compute it.

While the classical theory does not apply to the interpolation problem of the present paper, it is covered by a more general interpolation formula due to Spitzbart [9]. In fact, [9, Theorem 1] provides a polynomial h of a suitable degree for which $h^{(j)}(x_k) = f_{j,k}$ for $j = 0, 1, \ldots, \alpha(k)$ and $k = 0, 1, \ldots, n$ with given integers $\alpha(k) \ge 0$ and real numbers $f_{j,k}$. As one might expect, the formula for this *generalized Hermite interpolation polynomial* h is a bit cumbersome, but it can be shown that it reduces to (2) in our case of interest and also to the classical formula from [10] in the case $\alpha(k) = 1$ for $k = 0, 1, \ldots, n$. It therefore seems justified to call all the polynomials from [9, Theorem 1] *Hermite interpolation polynomials* as, for instance, in [8]. Of course, one may also refer to them as *Birkhoff–Hermite polynomials*, but what is not needed here is the more sophisticated theory of *Birkhoff interpolation* for the case when $f_{j,k}$ is missing for at least one pair (j,k) with $j < \alpha(k)$; see [6, 8].

Now, if in the setting of Theorem 2, the degree of p is bounded by $n+1$, then so is the degree of p_n. This means that p_n *is* the Hermite interpolation polynomial for all sufficiently large n. We conclude that an arbitrary polynomial is the uniform limit on $[0,1]$ of its Hermite interpolation polynomials for point evaluation of the derivative at the origin provided that equally spaced intermediate points are chosen as the interpolation points. It seems to be a daunting task to derive this remarkable convergence result directly from representation formula (2).

The preceding results canonically transfer to an interval of the form $[0,a]$ for arbitrary $a>0$. Specifically, given a polynomial $\tilde{p}$ on $[0,a]$ and the nodes $0=x_0<x_1<\cdots<x_n=1$, we define $p(x)=\tilde{p}(ax)$ for all $x\in[0,1]$ and consider the Hermite polynomials h_n and $\tilde{h}_n$ corresponding to the conditions $h_n'(0)=\tilde{h}_n'(0)=0$, $h_n(x_k)=p(x_k)$, and $\tilde{h}_n(ax_k)=\tilde{p}(ax_k)$ for $k=0,1,\ldots,n$. From the definition and also from formula (2), it is then immediate that $h_n(x)=\tilde{h}_n(ax)$ for all $x\in[0,1]$. It turns out that a similar relationship holds for the polynomials suggested by Theorem 2.

Indeed, let s_n and $\tilde{s}_n$ denote the node polynomials for

$$(x_0,x_1,\ldots,x_n)\quad\text{and}\quad(ax_0,ax_1,\ldots,a\,x_n),$$

respectively. Then, clearly, $\tilde{s}_n(ax)=a^{n+1}s_n(x)$ for all $x\in[0,1]$, and therefore, $\tilde{s}_n'(0)=a^n s_n'(0)$. Moreover, the polynomials p_n and $\tilde{p}_n$ given by

$$p_n(x)=p(x)-\frac{p'(0)}{s_n'(0)}s_n(x)\quad\text{and}\quad\tilde{p}_n(ax)=\tilde{p}(ax)-\frac{\tilde{p}'(0)}{\tilde{s}_n'(0)}\tilde{s}_n(ax)$$

for all $x\in[0,1]$ satisfy

$$\tilde{p}_n(ax)=p(x)-\frac{p'(0)/a}{a^n s_n'(0)}a^{n+1}s_n(x)=p_n(x)\quad\text{for all } x\in[0,1].$$

In particular, the supremum norm of $\tilde{p}_n-\tilde{p}$ over $[0,a]$ coincides with that of p_n-p over $[0,1]$. Thus the conclusion of Theorem 2 remains valid for polynomials on $[0,a]$.

In view of Lemma 1 (iii), we obtain the following counterpart of Theorem 2.

Theorem 3. *For arbitrary $p\in P[0,1]$ and $n\in\mathbb{N}$, the polynomial p_n given by*

$$\begin{aligned}p_n(x)&=p(x)-\frac{p'(1)}{s_n'(1)}s_n(x)\\&=p(x)-p'(1)\frac{n^n}{n!}s_n(x)\quad\textit{for all } x\in[0,1]\end{aligned}$$

satisfies

(i) $p_n'(1)=0$;

(ii) $p_n\left(\frac{k}{n}\right)=p\left(\frac{k}{n}\right)$ *for* $k=0,1,\ldots,n$;

(iii) $\|p_n-p\|_\infty\le|p'(1)|/(4n)$.

In particular, $\|p_n-p\|_\infty\to0$ *as* $n\to\infty$.

4 Runge's phenomenon

While the interpolation polynomials associated with the functionals φ_0 and φ_1 behave very nicely, this is no longer true for $\varphi_{1/2}$. In fact, in this case, the polynomials exhibit the behavior that has come to be known as *Runge's phenomenon* (see [2, Section 4.3.4]). If a polynomial $p \in P[0,1]$ satisfies $p'\left(\frac{1}{2}\right) = 0$, then $p \in \ker(\varphi_{1/2})$, so the approximation of p by polynomials in $\ker(\varphi_{1/2})$ is trivial. We thus focus on polynomials p for which $p'\left(\frac{1}{2}\right) \neq 0$. The following result shows that, in this case, Runge's phenomenon always occurs.

Theorem 4. *Given a polynomial $p \in P[0,1]$ and an even integer $n \in \mathbb{N}$, let*

$$p_n(x) = p(x) - \frac{p'\left(\frac{1}{2}\right)}{s_n'\left(\frac{1}{2}\right)} s_n(x) = p(x) - \frac{p'\left(\frac{1}{2}\right) \cdot n^n}{(-1)^{n/2}\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!} s_n(x) \quad \text{for all } x \in [0,1].$$

Then the following assertions hold:

(i) $p_n'\left(\frac{1}{2}\right) = 0$;

(ii) $p_n\left(\frac{k}{n}\right) = p\left(\frac{k}{n}\right)$ *for* $k = 0, 1, \ldots, n$;

(iii) *if* $p'\left(\frac{1}{2}\right) \neq 0$, *then* $\left|p_n\left(\frac{1}{2n}\right) - p\left(\frac{1}{2n}\right)\right| > \left|p'\left(\frac{1}{2}\right)\right| \cdot \frac{n}{2}$ *for all* $n \geq 40$.

In particular, if $p'\left(\frac{1}{2}\right) \neq 0$, then $\|p_{2k} - p\|_\infty \to \infty$ and $\|p_{2k}\|_\infty \to \infty$ as $k \to \infty$ so that, in the present setting, the Hermite interpolation polynomials for p fail to approximate p.

Proof. It is clear that claim (i) holds, and (ii) is an immediate consequence of part (i) of Lemma 1. Turning to assertion (iii), the definition of p_n and parts (v) and (vi) of Lemma 1 yield

$$\begin{aligned}
\left|p_n\left(\frac{1}{2n}\right) - p\left(\frac{1}{2n}\right)\right| &= \left|p'\left(\frac{1}{2}\right)\right| \frac{\left|s_n\left(\frac{1}{2n}\right)\right|}{\left|s_n'\left(\frac{1}{2}\right)\right|} \\
&= \left|p'\left(\frac{1}{2}\right)\right| \frac{n^n (2n)!}{(2n)^{n+1} \cdot 2^n \cdot n! \cdot \left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!} \\
&= \left|p'\left(\frac{1}{2}\right)\right| \frac{(2n)!}{2n \cdot 2^{2n} \cdot n! \cdot \left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!} \\
&= \left|p'\left(\frac{1}{2}\right)\right| \frac{(n+1)(n+2)\cdots(2n)}{2n \cdot 2^{2n} \cdot \left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!} \\
&> \left|p'\left(\frac{1}{2}\right)\right| \frac{n^n}{2n \cdot 2^{2n} \cdot \left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!} \\
&= \left|p'\left(\frac{1}{2}\right)\right| \left(\frac{k^k}{2^{k+1} \cdot \sqrt{k} \cdot k!}\right)^2,
\end{aligned}$$

where, because n is even, we substituted $n = 2k$ for the appropriate $k \in \mathbb{N}$. The proof will thus be complete once we establish that $k^k > 2^{k+1} \cdot k \cdot k!$ whenever $k \in \mathbb{N}$ satisfies $k \geq 20$.

The base case for the inductive argument, namely that $20^{20} > 2^{21} \cdot 20 \cdot 20!$, is easily verified. So suppose that the inequality $k^k > 2^{k+1} \cdot k \cdot k!$ holds for some integer $k \geq 20$. Then

$$\begin{aligned}(k+1)^{k+1} &= (k+1)(k+1)^k = (k+1) \cdot k^k \cdot \left(1 + \frac{1}{k}\right)^k \\ &> 2^{k+1} \cdot k \cdot (k+1)! \left(1 + \frac{1}{k}\right)^k \\ &= 2^{k+1} \cdot (k+1) \cdot (k+1)! \cdot \frac{k}{k+1} \cdot \left(1 + \frac{1}{k}\right)^k \\ &> 2^{k+2} \cdot (k+1) \cdot (k+1)!\end{aligned}$$

because the increasing sequence $(e_k)_{k=1}^{\infty}$ defined by $e_k = \frac{k}{k+1}\left(1 + \frac{1}{k}\right)^k$ satisfies $e_k > 2$ for all $k \geq 5$. This completes the induction and hence the proof of the theorem. ■

Theorem 4 stands in remarkable contrast to the excellent convergence properties of the Lagrange interpolation polynomials corresponding to an analytic function; see [5, Section 2.2.3].

5 Polynomials of Chebyshev type

In the classical theory of Lagrange interpolation, it is well known that certain interpolation points that are not equally spaced mitigate Runge's phenomenon. It turns out that this approach also works nicely in the present setting of Hermite interpolation, as will be seen next.

For $n \in \mathbb{N}$, the *n-th Chebyshev polynomial* of the first kind is defined by

$$T_n(x) = \cos(n \arccos(x)) \quad \text{for } -1 \leq x \leq 1.$$

That this formula indeed produces a polynomial in x follows most easily from basic complex analysis. For arbitrary real θ, de Moivre's formula confirms that

$$\cos(n\theta) + i\,\sin(n\theta) = \mathrm{e}^{in\theta} = (\mathrm{e}^{i\theta})^n = (\cos(\theta) + i\,\sin(\theta))^n.$$

Equating the real parts and applying the binomial theorem yields

$$\begin{aligned}\cos(n\theta) &= \sum_{k=0}^{[n/2]} \binom{n}{2k} i^{2k} \sin^{2k}(\theta) \cos^{n-2k}(\theta) \\ &= \sum_{k=0}^{[n/2]} \binom{n}{2k} (\cos^2(\theta) - 1)^k \cos^{n-2k}(\theta).\end{aligned}$$

With the choice $\theta = \arccos(x)$, we conclude that

$$T_n(x) = \sum_{k=0}^{[n/2]} \binom{n}{2k} (x^2 - 1)^k x^{n-2k},$$

which implies that T_n is a polynomial of degree n. Moreover, by its trigonometric definition, T_n has the n zeros

$$x_k = \cos\left(\frac{2k-1}{2n}\pi\right) \quad \text{for } k = 1, 2, \ldots, n$$

in the interval $[-1, 1]$, and these are called the *Chebyshev nodes of the first kind* (see [2, Sections 4.4.1 and 4.4.2]). It is also clear that $|T_n(x)| \leq 1$ for all $x \in [-1, 1]$ and that $T_n'(0) = 0$ for even n, while $T_n'(0) = \pm n$ for odd n. The classical Chebyshev polynomials are naturally defined on the interval $[-1, 1]$, but a shift to the unit interval is straightforward. Indeed, let

$$q_n(x) = T_n(2x-1) \quad \text{for all } x \in [0, 1].$$

Then $\|q_n\|_\infty = 1$ and $q_n'\left(\frac{1}{2}\right) = 0$ for even n, while $q_n'\left(\frac{1}{2}\right) = \pm 2n$ for odd n. The shifted Chebyshev nodes are the roots of q_n in $[0, 1]$, namely the numbers

$$\hat{x}_k = \frac{1}{2} + \frac{1}{2}x_k = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2k-1}{2n}\pi\right) \quad \text{for } k = 1, 2, \ldots, n.$$

We are thus led to the following counterpoint to Theorem 4.

Theorem 5. *For arbitrary $p \in P[0, 1]$ and an odd $n \in \mathbb{N}$, let*

$$p_n(x) = p(x) - \frac{p'\left(\frac{1}{2}\right)}{q_n'\left(\frac{1}{2}\right)} q_n(x) \quad \textit{for all } x \in [0, 1].$$

Then we have

(i) $p_n'\left(\frac{1}{2}\right) = 0$;

(ii) $p_n(\hat{x}_k) = p(\hat{x}_k)$ *for* $k = 0, 1, \ldots, n$;

(iii) $\|p_n - p\|_\infty = |p'\left(\frac{1}{2}\right)|/(2n)$.

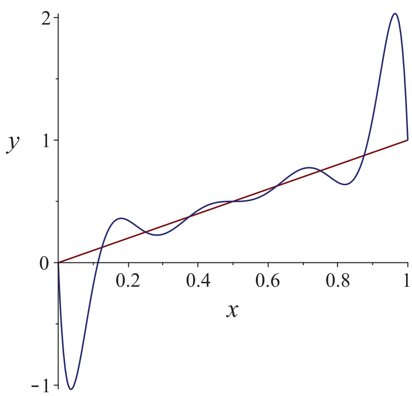
In particular, $\|p_{2k+1} - p\|_\infty \to 0$ *as* $k \to \infty$.

In the following graphics, the polynomial p defined by $p(x) = x$ on $[0, 1]$ is displayed along with its associated standard Hermite polynomial p_8 of Theorem 4 on the one hand, and its associated Chebyshev-type polynomial p_9 of Theorem 5 on the other hand.

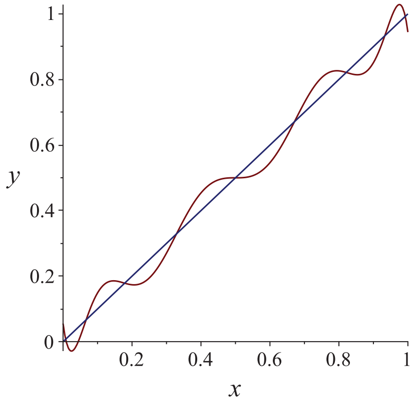
We leave it to the interested reader to explore the fates of Theorems 4 and 5 in the case of interpolation polynomials for which the derivative vanishes at α for an arbitrary interior point $0 < \alpha < 1$.

Finally, it is natural to wonder about the extent to which each of our four theorems remains valid for an arbitrary differentiable function replacing the polynomial p. Unfortunately, our formulas produce a sequence of polynomials only when applied to a polynomial p, so a different approach seems to be needed to handle the general case.

Acknowledgments. The authors thank the referee for careful reading of the manuscript and for helpful suggestions.



Standard Hermite polynomial p_8



Chebyshev-type polynomial p_9

References

[1] K. E. Atkinson, *An introduction to numerical analysis*. 2nd edn., John Wiley & Sons, New York, 1989

[2] A. Björck and G. Dahlquist, *Numerical methods*. Prentice-Hall Ser. Automatic Comput., Prentice-Hall, Englewood Cliffs, 1974

[3] B. Bollobás, *Linear analysis: An introductory course*. Cambridge Math. Textb., Cambridge University Press, Cambridge, 1990

[4] J. F. Epperson, *An introduction to numerical methods and analysis*. 2nd edn., John Wiley & Sons, Hoboken, 2013

[5] W. Gautschi, *Numerical analysis*. 2nd edn., Birkhäuser, Boston, 2012

[6] G. G. Lorentz and K. L. Zeller, Birkhoff interpolation. *SIAM J. Numer. Anal.* **8** (1971), 43–48

[7] J. D. Pryce, *Basic methods of linear functional analysis*. Dover Publications, Inc., Mineola, 2011

[8] I. J. Schoenberg, On Hermite–Birkhoff interpolation. *J. Math. Anal. Appl.* **16** (1966), 538–543

[9] A. Spitzbart, A generalization of Hermite's interpolation formula. *Amer. Math. Monthly* **67** (1960), 42–46

[10] G. Szegő, *Orthogonal polynomials*. 4th edn., Amer. Math. Soc. Colloq. Publ. 23, American Mathematical Society, Providence, 1975

[11] L. N. Trefethen, *Approximation theory and approximation practice*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2013

Robert Kantrowitz (corresponding author)
Mathematics and Statistics Department
Hamilton College
198 College Hill Road
Clinton, NY 13323, USA
rkantrow@hamilton.edu

Michael M. Neumann
Department of Mathematics and Statistics
Mississippi State University
Mississippi State, MS 39762, USA
neumann@math.msstate.edu