Elemente der Mathematik 78 (2023), 123–126
DOI 10.4171/EM/475

Short note

Isosceles tetrahedrons with integer edges and volume in the $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ grid

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Abstract. By adding parallelogram identities, we prove an equation on the edge lengths of a tetrahedron and the diagonal lengths of its medial octahedron. By this and with the help of Euler bricks we find in the grid $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ a family of isosceles tetrahedrons with integer edge length and integer volume.

1 Notation

Let A, B, C and D be elements of $\mathbb{R}^n$, $n \geq 2$, such that there are no three points on a line; we call such a configuration an n-tetrahedron. E, F, G, H, I and K stand for the midpoints of the edges AD, BC, DC, AB, AC and BD respectively.

We write $h = (PQ)$ for the equivalence class of the arrow (PQ). The scalar product of two vectors h and g is denoted by $h^t \cdot g$, so the norm of the vector h is given by $\|h\| = \sqrt{h^t \cdot h}$; for the squared norm, we write $h^2 = h^t \cdot h$. Adding the two equations $(a \pm b)^t(a \pm b) = \|a\|^2 + \|b\|^2 \pm 2 \cdot a^t b$, we get the parallelogram identity $\|a+b\|^2 + \|a-b\|^2 = 2 \cdot (\|a\|^2 + \|b\|^2)$.

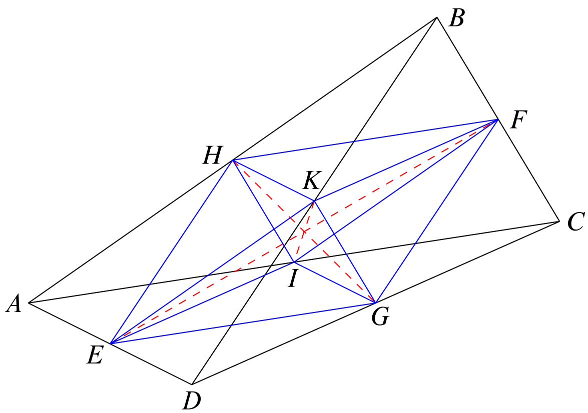

Figure 1

Furthermore, we use the vectors $a = (BC)$, $b = (AB)$, $c = (AD)$, $d = (DC)$, $e = (AC)$, $f = (DB)$ and the diagonal vectors of the n-octahedron $EFGHKI$ $u = (EF)$, $v = (HG)$ and $w = (KI)$.

In Figure 1, we can see different objects. First of all, we see a 3-tetrahedron $ABCD$ and its medial 3-octahedron $EFGHKI$ with its body diagonals EF, HG, KI. Secondly, we find a 2-tetrahedron or quadrilateral $ABCD$ with its diagonals AC and BD, and thirdly, we recognize the three parallelograms $EHFG$, $EKFI$ and $HKGI$.

2 A tetrahedron identity

An n-tetrahedron can be described with three vectors. We take these three as a basis, and so, without loss of generality, all we do here is in fact three or two dimensional Euclidean geometry.

Lemma 2.1. *For the opposite edges of an n-tetrahedron and the diagonals of its medial octahedron the following equations hold:*

(α) $e^2 + f^2 = 2 \cdot (u^2 + v^2)$,

(β) $b^2 + d^2 = 2 \cdot (u^2 + w^2)$,

(γ) $a^2 + c^2 = 2 \cdot (v^2 + w^2)$.

Proof. In $EHFG$, we have the diagonals u and v, the side vectors $(EH) = \frac{1}{2} f = (GH)$ and $(EG) = \frac{1}{2} e = (HF)$. (α) follows from the parallelogram identity. Analogously, we find (β) and (γ) with the parallelograms $EKFI$ and $HKGI$ respectively. ■

Theorem 2.2. *Let a, b, c, d, e and f be the side vectors of the n-tetrahedron $ABCD$, and u, v and w the diagonal vectors of the medial n-octahedron $EFGHKI$; then the following holds.*

(a) *The sum of the squared edges equals four times the sum of the squared diagonals of the medial n-octahedron:*

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 = 4 \cdot (u^2 + v^2 + w^2).$$

(b) *The Euler identity $a^2 + b^2 + c^2 + d^2 = e^2 + f^2 + 4 \cdot w^2$, see* [6].

Proof. (a) Add (α), (β) and (γ) of Lemma 2.1.

(b) Add (β) and (γ), rearrange the terms and finish with (α):

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 &= 2 \cdot (u^2 + w^2) + 2 \cdot (v^2 + w^2) \\ &= 2 \cdot (u^2 + v^2) + 4 \cdot w^2 \\ &= e^2 + f^2 + 4 \cdot w^2. \end{aligned}$$

■

For $n = 2$, Theorem 2.2 (b) becomes the classical Euler's quadrilaterals formula [2], in which we distinguish edges from diagonals.

3 Isosceles tetrahedrons with integer edges and integer volumes

An isosceles tetrahedron has congruent faces, and therefore opposite edges have the same length.

Lemma 3.1. *Let u, v, w be the body diagonal vectors of the medial octahedron of an isosceles tetrahedron with edge vectors b, c, e. The volume of the isosceles tetrahedron and the volume of its medial octahedron is denoted by V_{T} and V_{OC} respectively. Then the following holds.*

(a) *The vectors u, v and w are pairwise orthogonal, and the line segments EF, HG and KI bisect one another.*

(b) $V_{\mathrm{T}} = 2 \cdot V_{\mathrm{OC}} = \frac{\|u\| \cdot \|v\| \cdot \|w\|}{3}$.

Proof. (a) As $(EG) = (HF) = \frac{1}{2}(AC)$, $(HE) = (FG) = \frac{1}{2}(BD)$ and $(AC) = (BD)$, $EHFG$ is a rhombus, that is why $v = (HG)$ and $u = (EF)$ are orthogonal. Analogously, we find the same for $w = (KL)$, $u = (EF)$, as well as for $w = (KL)$, $v = (HG)$. Hence, in an isosceles tetrahedron, the vectors u, v and w are pairwise orthogonal and also by a rhombus argument the mentioned line segments intersect in a common point and bisect one another.

(b) The tetrahedrons $BHKF$, $CFGI$, $DGKE$ and $AEIH$ are similar to the tetrahedron $ABCD$, so we have $V_{\mathrm{OC}} = V_{\mathrm{T}} - 4 \cdot (\frac{1}{2})^3 \cdot V_{\mathrm{T}} = \frac{1}{2} \cdot V_{\mathrm{T}}$; with $V_{\mathrm{OC}} = \frac{\|u\| \cdot \|v\| \cdot \|w\|}{2 \cdot 3}$, the formula follows. ■

More on isosceles tetrahedrons can be found in [1,8,9].

Now we construct with the help of Euler bricks [7] isosceles tetrahedrons whose edges have integer length and integer volumes. An Euler brick is a rectangular cuboid whose edges u, v, w and face diagonals all have integer lengths.

Theorem 3.2. *If the body diagonals u, v, w of the medial octahedron of an isosceles tetrahedron form an Euler brick, then the lengths of the edges of the isosceles tetrahedron as well as its volume are integers.*

Proof. As in an isosceles tetrahedron opposite edges have the same length, by Lemma 2.1, we get $e^2 = u^2 + v^2$, $b^2 = u^2 + w^2$, $c^2 = v^2 + w^2$. These are the defining Diophantine equations for an Euler brick. Each integer solution of $e^2 = u^2 + v^2$ is a Pythagorean triple of the form $(m^2 + n^2, m^2 - n^2, 2mn)$. As $\|u\| \cdot \|v\| \equiv (m^2 - n^2) \cdot 2mn \equiv 0 \bmod (3)$, which is easily seen by inserting all elements of $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $V_{\mathrm{T}} = \frac{\|u\| \cdot \|v\| \cdot \|w\|}{3}$ is an integer. ■

Corollary 3.1. *Let (r, s, t) be an Euler brick triple.*

(a) *The vertices $A(0/0/0)$, $B(0/-s/t)$, $C(-r/0/t)$, $D(-r/-s/0)$ form in the $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ grid an isosceles tetrahedron $ABCD$ with integer lengths of its edges and integer volume $\frac{r \cdot s \cdot t}{3}$.*

(b) *Let $ABCD$ be such an isosceles tetrahedron T and A_{EKFI}, A_{EHFG}, A_{HKGI} the areas of the parallelograms $EKFI$, $EHFG$, $HKGI$; then the squared surface area of the isosceles tetrahedron is given by*

$$S_{\mathrm{T}}^2 = 16 \cdot (A_{EKFI}^2 + A_{EHFG}^2 + A_{HKGI}^2).$$

Proof. (a) A forward calculation shows $e^2 = f^2 = r^2 + t^2, c^2 = a^2 = r^2 + s^2$ and $b^2 = d^2 = s^2 + t^2$, so $ABCD$ is an isosceles tetrahedron, and $u = (0, 0, t)^t$, $v = (-r, 0, 0)^t$, $w = (0, s, 0)^t$; hence by Lemma 3.1 and Theorem 3.2, we get the volume.

(b) We have

$$\begin{aligned} S_{\mathrm{T}}^2 &= 4 \cdot \|(0, -s, t) \times (-r, 0, t)^t\|^2 \\ &= 4 \cdot \big((rs)^2 + (rt)^2 + (st)^2\big) \\ &= 16 \cdot (A_{EKFI}^2 + A_{EHFG}^2 + A_{HKGI}^2). \end{aligned}$$

■

Concerning the volume, the smallest Euler brick has sides $\|u\| = 44$, $\|v\| = 117$, $\|w\| = 240$ (see [3–5]). Because of this result, the smallest isosceles tetrahedron in the $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ grid has edge lengths $\|c\| = 267$, $\|b\| = 244$ and $\|e\| = 125$. Its volume equals 411,840.

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