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Short note

A remark on an identity involving products of binomial coefficients

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1 Introduction

In their recent article [1] on reduction formulas for higher order derivations, B. Ebanks and A. Kézdy applied the identity

$$s(n) := \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+d+1-k}{n-j} \binom{d+j-k}{j} = \binom{n}{j} \tag{1.1}$$

for $n \in \mathbb{N}$ and each integer j satisfying $0 \le j \le n$ [1, Theorem 3.1]. The proof is based on hypergeometric summation using the WZ-method by Wilf and Zeilberger. It is worth noting that $s(n)$ is independent of $d \in \mathbb{R}$.

The purpose of this short note is to show that the seemingly complicated identity (1.1) has a very short elementary proof by taking advantage of the well-known identities

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^q = 0 \quad (q = 0, \dots, n-1). \tag{1.2}$$

The latter equation can easily be obtained by representing k^q as a linear combination of binomial coefficients, i.e., $k^q = \sum_{i=0}^{q} c_i \binom{k}{i}$ and observing that

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{k}{i} = \binom{n}{i} \sum_{k=i}^{n} (-1)^{n-k} \binom{n-i}{k-i} = 0 \quad (i = 0, \dots, n-1).$$

After giving an elementary proof in the next section, we present a proof by use of the residue calculus. Finally, we show a connection with the Karlsson–Minton formulas.

2 An elementary proof

Indeed, we show a more general identity than (1.1).

Proposition 1. *For $n \in \mathbb{N}$, arbitrary real numbers a_i, b_i and non-negative integers n_i $(i = 1, \ldots, r)$ satisfying $n_1 + \cdots + n_r = n$, it holds*

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{a_1 + b_1 k}{n_1} \cdots \binom{a_r + b_r k}{n_r} = \left(\prod_{i=1}^{r} b_i^{n_i} \right) \binom{n}{n_1, \ldots, n_r}. \tag{2.1}$$

Identity (1.1) is the special case $r = 2$, $a_1 = n + d + 1$, $a_2 = d + j$, $b_1 = b_2 = -1$, $n_1 = n - j$, $n_2 = j$.

Proof. If $n_i > 0$, we observe that $\binom{a_i + b_i k}{n_i} = (b_i k)^{n_i} / n_i! + P_i(k)$, where P_i is a certain polynomial of degree less than n_i. By (1.2), the left-hand side of equation (2.1) is equal to

$$\left(\prod_{i=1}^{r} \frac{b_i^{n_i}}{n_i!} \right) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^n.$$

Now the assertion follows since

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k(k-1) \cdots (k-n+1) = n!$$

by a further application of (1.2). ■

Remark 1. In the same way, one obtains, for $n \in \mathbb{N}$, $a_i, b_i \in \mathbb{N}$ and integers $n_i \geq 0$ $(i = 1, \ldots, r)$, that the left-hand side of equation (2.1) vanishes if $n_1 + \cdots + n_r < n$. In the case $n_1 + \cdots + n_r > n$, the left-hand side of equation (2.1) can be expressed as a linear combination of Stirling numbers of the second kind, given by

$$\sigma_{m,n} = \frac{1}{n!} \sum_{i=0}^{n} (-1)^{n-k} \binom{n}{k} k^m.$$

Remark 2. Taking advantage of equation (1.2), one can produce further identities of the above type.

3 Proof using residue calculus

Identity (2.1), i.e.,

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \prod_{j=1}^{r} \binom{a_j + b_j k}{n_j} = \left(\prod_{i=1}^{r} b_i^{n_i} \right) \binom{n}{n_1, \ldots, n_r},$$

provided that $n_1 + \cdots + n_r = n$, can be obtained by applying the residue calculus to the rational function

$$\frac{\prod_{j=1}^{r} \binom{a_j + b_j z}{n_j}}{\binom{n-z}{n+1}} =: \frac{g(z)}{h(z)}.$$

Note that $\binom{n-z}{n+1}$ is a polynomial in z of degree $n+1$ with simple zeros $z = 0, 1, \ldots, n$. Let $k \in \{0, 1, \ldots, n\}$. Expressing the denominator h of the rational function g/h as

$$h(z) = -\frac{z-k}{(n+1)!} \prod_{\ell=0,\, \ell \neq k}^{n} (\ell - z),$$

we obtain

$$\operatorname{Res}_{g/h}(k) = \lim_{z \to k} \frac{g(z)(n+1)!}{-\prod_{\ell=0,\, \ell\neq k}^{n} (\ell - z)} = (-1)^{k+1} \frac{(n+1)!}{k!\,(n-k)!} \prod_{j=1}^{r} \binom{a_j + b_j k}{n_j}. \tag{3.1}$$

This formula is valid also if k is a zero of the numerator g. Since

$$\begin{aligned} \frac{g(1/z)}{h(1/z)} &= (n+1)! \, \frac{\left(\prod_{j=1}^{r} \frac{z^{-n_j}}{n_j!}\right) \prod_{\ell=0}^{n_j - 1} (a_j z + b_j - \ell z)}{z^{-n-1} \prod_{\ell=0}^{n} (nz - 1 - \ell z)} \\ &= \frac{(n+1)!}{\prod_{j=1}^{r} n_j!} z^{n+1-(n_1+\cdots+n_r)} \frac{\prod_{j=1}^{r} \prod_{\ell=0}^{n_j-1} (a_j z + b_j - \ell z)}{\prod_{\ell=0}^{n} (nz - 1 - \ell z)}, \end{aligned}$$

the well-known formula

$$\operatorname{Res}_f(\infty) = -\operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right)$$

yields, in the case $n_1 + \cdots + n_r = n$,

$$\begin{aligned} \operatorname{Res}_{g/h}(\infty) &= -\frac{(n+1)!}{\prod_{j=1}^{r} n_j!} \frac{\prod_{j=1}^{r} \prod_{\ell=0}^{n_j-1} b_j}{\prod_{\ell=0}^{n} (-1)} \\ &= -(-1)^{n+1} (n+1) \binom{n}{n_1, \ldots, n_r} \prod_{j=1}^{r} b_j^{n_j}. \end{aligned} \tag{3.2}$$

Because the sum of the residues at all finite poles of g/h is equal to the minus residue at infinity, comparison of (3.1) and (3.2) proves identity (2.1).

4 Connection with the Karlsson–Minton formulas

The special case

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \prod_{j=1}^{r} \binom{a_j + k}{n_j} = \binom{n}{n_1, \ldots, n_r}, \tag{4.1}$$

provided that $n_1 + \cdots + n_r = n$, of Proposition 1 is a particular instance of the Karlsson–Minton formulas

$$ {}_{r+1}F_r\left({-(n_1+\cdots+n_r), a_1+n_1, \dots, a_r+n_r \atop a_1, \dots, a_r} ; 1 \right) = (-1)^{n_1+\cdots+n_r} \frac{(n_1+\cdots+n_r)!}{(a_1)_{n_1}\cdots(a_r)_{n_r}} $$

(see, e.g., [2, equation (1.9.3)]). The generalized hypergeometric function is defined by

$$ {}_pF_q\left({a_1, \dots, a_p \atop b_1, \dots, b_q} ; z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}, $$

where, for $k = 0, 1, 2, \dots$, $(c)_k = \Gamma(c+k)/\Gamma(c)$ denotes the Pochhammer symbol (rising factorial). Noting that $(-c)_k = (-1)^k k! \binom{c}{k}$, we have

$$ {}_{r+1}F_r\left({-n, a_1+n_1, \dots, a_r+n_r \atop a_1, \dots, a_r} ; 1 \right) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \prod_{j=1}^{r} \frac{\binom{a_j+n_j+k-1}{n_j}}{(a_j)_{n_j}/n_j!}. $$

Hence,

$$ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \prod_{j=1}^{r} \binom{a_j+n_j+k-1}{n_j} = \binom{n}{n_1, \dots, n_r}. $$

Since the right-hand side is independent of a_j, we can replace a_j with $a_j - n_j + 1$. This implies (4.1).

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References

[1] B. Ebanks and A. E. Kézdy, Reduction formulas for higher order derivations and a hypergeometric identity. *Aequationes Math.* **95** (2021), no. 6, 1053–1065

[2] G. Gasper and M. Rahman, *Basic hypergeometric series*. 2nd edn., Cambridge University Press, Cambridge, 2009

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