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Short note A simple proof of the Fundamental Theorem of Algebra

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Many proofs of the Fundamental Theorem of Algebra, including various proofs based on the theory of analytic functions of a complex variable, are known. To the best of our knowledge, this proof is different from the existing ones.

Theorem. *Every polynomial* $p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n \in \mathbb{C}[z]$, *where* $n \in \mathbb{N}$, *has a zero in* $\mathbb{C}$.

Proof. For a contradiction, let us assume that $p(z)$ has no zero in $\mathbb{C}$. Thus the polynomials of the sequence $\{p_k(z)\}_{k=1}^{\infty}$, given by

$$p_k(z) := \frac{z \cdot p(kz)}{k^n} = z^{n+1} + \frac{a_1}{k} z^n + \frac{a_2}{k^2} z^{n-1} + \cdots + \frac{a_{n-1}}{k^{n-1}} z^2 + \frac{a_n}{k^n} z,$$

have only a simple zero at 0. Next we choose an arbitrary but fixed real number r (> 0). Since z^{n+1} is non-zero on $|z| = r$, there exists a $\delta > 0$ such that $|z^{n+1}| > \delta$ for every z, where $|z| = r$. Moreover, as $\{p_k(z)\}_{k=1}^{\infty}$ converges uniformly to z^{n+1} on $|z| = r$, there exists $N \in \mathbb{N}$ such that $|p_k(z)| > \frac{\delta}{2}$ for every $k \geq N$ and for every z, where $|z| = r$. Thus $\left\{\frac{1}{p_k(z)}\right\}_{k=1}^{\infty}$ converges uniformly to $\frac{1}{z^{n+1}}$ on $|z| = r$. Hence

$$\lim_{k \to \infty} \int_{|z|=r} \frac{1}{p_k(z)}\, dz = \int_{|z|=r} \frac{1}{z^{n+1}}\, dz.$$

Now, using Cauchy's integral formula, we get

$$\lim_{k \to \infty} \frac{2\pi i \cdot k^n}{p(0)} = \int_{|z|=r} \frac{1}{z^{n+1}}\, dz,$$

i.e.,

$$\lim_{k \to \infty} k^n = 0,$$

which is impossible. Thus $p(z)$ has a zero in $\mathbb{C}$. ∎

Note that

$$\int_{|z|=r} \frac{1}{z^{n+1}}\, dz = \frac{i}{r^n} \int_0^{2\pi} e^{-in\theta}\, d\theta = 0.$$

References

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