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Daniel Bernoulli's "immensely fertile equation": Notes on relations between a nonlinear and a linear second order equation

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1 Overview

The interrelation between

$$-u'' + Vu = 0 \tag{1}$$

(to which the general linear second order equation can easily be reduced) and the nonlinear first order Riccati equation is very well known. Little known is the observation that (1) and

$$\frac{r''}{r} - \frac{1}{r^4} = V \tag{2}$$

Bekannt ist die Äquivalenz der linearen Differentialgleichung (Dgl) 2. Ordnung mit der einer nichtlinearen 1. Ordnung, der Riccatischen Gleichung. Daniel Bernoullis Untersuchungen der Schwingungen inhomogener Saiten (1767) beruhen auf ihrer Äquivalenz mit einer nichtlinearen Dgl 2. Ordnung. Letztere wurde bislang aufgrund späterer Wiederentdeckungen nach Pinney (1950), Milne (1930), Ermakov (1880) oder Steen (1874) bzw. Kombinationen dieser Namen benannt. Zwei Anwendungen von Daniel Bernoullis Dgl werden hier vorgestellt. Erstens läßt sich mit minimalem Rechenaufwand zeigen, daß das Produkt zweier Lösungen der linearen Dgl 2. Ordnung Lösung einer linearen Dgl 3. Ordnung ist (Steen); die Beobachtung als solche stammt von Liouville (1839). Zweitens, motiviert durch den Sturmschen Vergleichssatz, der die Oszillationen der Lösungen zweier linearer Dglen 2. Ordnung zueinander in Beziehung setzt, können mit Hilfe einer geeigneten Umkehrung der Schwarzschen Ungleichung auch deren L^2-Normen verglichen werden (Johann Walter 1968).

are equivalent via

$$r = (u_1^2 + u_2^2)^{1/2} \tag{3}$$

in the sense that a positive solution of (2) yields a fundamental system of solutions u_1, u_2 of (1), and conversely, given such a system with Wronski determinant $u_1 u_2' - u_1' u_2 = 1$ (which is no loss of generality), (3) defines a positive solution of (2). (See also the generalisation in Theorem 1 below.)

As far as we can see, equation (2) and its connection to (1) first occurred in a paper of 1767 by Daniel Bernoulli (1700–1782) on the vibration of strings of non-uniform thickness [4]. In Section 2, we discuss this paper and parallel investigations by Euler (1707–1783) [E287, E442, E567]. Here we profited greatly from Truesdell's outstanding treatise [29, pp. 307–309, 315]. For him, however, equation (2) could not be an object of independent interest, and in his analysis of a 1773 paper by Lagrange (1736–1813) (we shall comment upon briefly in Section 3), the equation is not even written out explicitly [29, pp. 353–355]. In fact, he does not share Daniel Bernoulli's enthusiasm about the examples of non-uniform strings he could deal with, and his final judgement is decidedly negative [29, p. 316]. At the end of Section 2, we also point out two contributions towards a proof of a conjecture of Euler's concerning the inhomogeneous string (see [24] and [6, p. 86 f.]).

Differential equation (2) was frequently rediscovered – most notably by A. Steen (1874) and V. P. Ermakov (1880) and as late as 1950 by E. Pinney. Their principal motive was to enlarge the stock of functions V for which (1) is explicitly solvable[1]. Many mathematicians today would presumably find such an aspect less interesting if not dubious. Adolphe Steen (1816–1886), however, whose paper of 1874 remained basically unknown until the late 1990s, also made an unexpected use of (2) [26, p. 182 f.] (citations will always be from the English translation which Ray Redheffer and his wife provided and where printing errors of the original were deliberately not corrected). He rewrote (2) as a nonlinear second order equation in r^2 and, after differentiating again, found a linear third order equation for r^2 (see (20) below). Section 4 centres around this idea to prove that products of solutions of (1) satisfy a (uniquely determined) linear third order equation, a result that is rarely emphasised in the general ODE literature although it has a variety of applications. Since the Green function that belongs to the inhomogeneous form of (1) is proportional to $u_1(x)u_2(y)$, its diagonal satisfies this equation (this is used, e.g., in [14]). Quite a different application is given in [15]. We also try to sort out the conflicting statements in the literature as to who was the first to make the observation above. In all likelihood, it was Liouville (1809–1882) in a paper of 1839 on the (im)possibility of solving (1) in finite terms [20, p. 431].

Relation (3) suggests that equation (2) can also be helpful when studying (1) in a Hilbert space setting. Section 5 discusses a paper by Johann Walter [30] who noted that pairs of solutions of (2) enjoy certain common properties that can be used to infer the self-adjointness of a Sturm–Liouville operator associated with the left-hand side of (1) in $L^2(x_0, \infty)$ from that of a "smaller" one (Theorem 6 below).

[1] References to Ermakov and Pinney can be found in [26]. Meanwhile, the former's paper is available in an English translation [9], followed by a commentary on his impact on a Lie-type view of nonlinear ODEs [18].

2 Daniel Bernoulli and Euler on inhomogeneous strings

Daniel Bernoulli's aim in [4] is to study isochronal vibrations of a string with variable density σ and constant tension T, which we would write as $u(x,t) := y(x)\cos\omega t$, where y is a solution of

$$y'' + \frac{\omega^2}{c^2} y = 0 \tag{4}$$

with $c := (T/\sigma)^{1/2}$. Following Truesdell [29, p. 307 ff.], we use this notation as it facilitates comparison with Euler and later developments although Bernoulli himself does not display ω in this paper. He starts with the observation (presumably the result of many trials with other phase functions) that

$$y(x) := q(x)\sin\Big(\int_0^x \frac{\mathrm{d}t}{aq^2(t)}\Big) \quad (x \in [0,l]) \tag{5}$$

(l is the length of the string and $a > 0$ a constant) has the property

$$-\frac{y''}{y} = \frac{1}{a^2q^4} - \frac{q''}{q} = \frac{\omega^2}{c^2}. \tag{6}$$

He calls (6) "une équation infiniment féconde" [4, p. 286] because every q will give rise to a string with density σ and vibration y. Indeed, given $q > 0$ and $k \in \mathbb{N}$, we can achieve $y(l) = 0$ by choosing a such that

$$\frac{1}{a}\cdot\int_0^l \frac{\mathrm{d}t}{q^2(t)} = k\pi, \tag{7}$$

and we obtain a string with density σ and a vibration with $k-1$ nodes [4, p. 287]. Of course, the string may be capable of other vibrations as well. (He also mentions that, with suitable constants α, β,

$$y(x) := \alpha\, q(x)\sin\Big(\int_0^x \frac{\mathrm{d}t}{aq^2(t)} + \beta\Big)$$

is the general solution of (4). There is a printing error in his indefinite integral on [4, p. 285].)

Next he concludes from (6) that infinitely many modes for one and the same string can be found when q is taken to be linear, e.g.,

$$q(x) := 1 + \frac{x}{b} \tag{8}$$

(see [4, p. 290 f.]), in which case $q'' = 0$ and σ becomes

$$\sigma(x) := \sigma_0[q(x)]^{-4} = \sigma_0\Big(1 + \frac{x}{b}\Big)^{-4}. \tag{9}$$

As an aside, using the density (9), the differential equation (4) takes the form

$$s^2 z'' + d^2 s^m z = 0, \quad d > 0 \text{ a constant}, \tag{10}$$

with solutions $z = s^{1/2} Z_\gamma(\frac{2d}{m} s^{m/2})$, $\gamma = \pm 1/m$ (see [31, p. 97 (4)]). The Bessel functions Z_γ of the first or second kind are elementary functions if and only if γ is a half-odd integer. In our case, $s = 1 + x/b$, $m = -2$ and $d = b/a$. With the representations of $Z_{\pm 1/2}$ (see [31, pp. 54 (3), 55 (6)]), we obtain

$$z(s) = s\Big(c_1 \sin\frac{d}{s} + c_2 \cos\frac{d}{s}\Big)$$

as the general solution of (10). Since we wish to have $y(0) = 0$, $y'(0) = 1/a$, i.e., $z(1) = 0$, $z'(1) = d$, we find $c_1 = -\cos d$, $c_2 = \sin d$. Thus $z(s) = \sin(d - \frac{d}{s})$, which is (5) once the integration with (8) has been performed.

Studies of the corresponding Riccati equation by Nicolaus II, Daniel's elder brother, and by Daniel himself had already revealed in the early 1720s that (10), after elementary transformations, can be integrated if $1/m$ is of the form $(2n - 1)/2$, $n \in \mathbb{Z}$ (see the profound presentation of the involved history of Riccati's equation by U. Bottazini in [5, pp. 142–165]; cf. also [31, pp. 85–87]). The "only if" part had to await proof by Liouville in 1841 [20, pp. 117–123].

With (8), the left-hand side of (7) can immediately be computed and inserted into (6). The frequencies γ_k in $\omega = 2\pi\gamma_k$ are then given by

$$\gamma_k = \frac{k}{2l}\Big(1 + \frac{l}{b}\Big)\Big(\frac{T}{\sigma_0}\Big)^{1/2} = k\gamma_1 \quad (k \in \mathbb{N}). \tag{11}$$

Thus the overtones of such a string are all harmonic, and in the limit $b \to \infty$, one obtains the frequencies of the homogeneous string. He mentions this [4, p. 291], but does not write down (11) explicitly because, as we said before, ω does not occur. (Prior to this paper, however, he had already calculated frequencies for this case [29, p. 303].)

On [4, p. 300], he mentions that other examples of q can be found by connecting q''/q and $1/q^4$, i.e., guessing a solution of

$$q'' = cq^{-3}, \quad c > 0 \text{ a constant.} \tag{12}$$

The function

$$q(x) := \Big[1 + \Big(\frac{x}{b}\Big)^2\Big]^{1/2} \tag{13}$$

satisfies (12) with $c = b^{-2}$ and therefore leads to

$$\sigma(x) := \sigma_0\Big[1 + \Big(\frac{x}{b}\Big)^2\Big]^{-2}.$$

In fact, he could have chosen

$$q(x) := (\lambda x^2 + \mu x + \gamma)^{1/2}, \tag{14}$$

in which case $c = \lambda\gamma - \mu^2$ with $\lambda > 0$.

Example (9) had also been considered by Euler in [E287], a paper published in 1764, which Daniel Bernoulli had not seen when he completed [4], as he mentions in a footnote at the very beginning. Euler treats this case in two ways, in [E287, §§ 15–19] by a special ansatz into the wave equation, then in [E287, §§ 68–71] by looking at the corresponding Riccati equation[2]. (A more systematic presentation [E442] appeared in 1773; see [29, pp. 302–307, 313–315] for a critical assessment of these two papers.) In [E287, § 37], Euler writes that the homogeneous string and the one with density (9) are probably ["verisimile"] the only ones for which all overtones are harmonic. In a letter to his nephew Johann III, however, Daniel Bernoulli claimed to have counterexamples to this conjecture [29, p. 303]. Euler's final reflections on this subject are in [E567], presented in 1774 and published 10 years later, where he calls (5) a general formula that pleased him wonderfully ["mirifice"]. In [E567, § 3], he derives (5) in detail, noting that the ansatz $y = q \sin S$, S a primitive of a function $s > 0$, leads to the term $\cos S$ in y'' unless $2q's + qs' = 0$, i.e., $s = \text{const.} \cdot q^{-2}$ (his notation is slightly different). It is only in this paper that he uses (6) to treat (14) (cf. also [29, p. 315]).

It appears that the first attempt to prove Euler's conjecture was published by Wilma Mothwurf [24]. Ironically, she took her knowledge of the differential equation (6) from a then recent paper by Madelung [21], as her study departed from Euler's papers [E287, 442] where (6) does not appear. Madelung recommended (6) in particular for numerical calculations as had been done only months earlier by W. E. Milne [22, 23]. (As soon as Schrödinger had proposed his famous equation in 1926, Madelung showed that it was equivalent to a nonlinear system of two partial differential equations, one a continuity equation, the other having a hydrodynamical flavour.)

Wilma Mothwurf looks for a string with density σ which has the property that the q in (6) satisfies

$$\int_0^l \frac{1}{q^2} = C(l)\gamma_k \quad (l \geq 0). \tag{15}$$

Here C is a smooth function and $\gamma_k = k\gamma_1$ ($k \in \mathbb{N}$). Differentiating (15), (6) with $a = 1$ becomes

$$(2\pi\gamma_k)^2 \frac{\sigma(x)}{T} = \gamma_k^2 [C'(x)]^2 - [C'(x)]^{1/2}\{[C'(x)]^{-1/2}\}'' \tag{16}$$

for $x \in [0, l]$ and all $l \geq 0$. Dividing by γ_k^2, we obtain

$$\frac{\sigma(x)}{T} = [C'(x)]^2$$

in the limit $k \to \infty$. With this information, we see from (16) that $[C'(x)]^{-1/2}$ is a linear function. Hence

$$\sigma(x) = \sigma_0(\alpha x + \beta)^{-4} \tag{17}$$

and $C(l) = l/[\beta(\alpha l + \beta)]$. The vibration modes therefore vanish at l if

$$\frac{2\pi\gamma_k l}{\beta(\alpha l + \beta)} = k\pi,$$

and we are back to (11). Although equation (15) may not be a convincing point of departure, her paper is a remarkably early investigation of an inverse spectral problem.

[2] The general equation (10) is considered in §§ 76–78 (cf. also [31, p. 87 f.]).

Almost a decade later, Borg in a pioneering memoir proved, among many other things, the following result [6, pp. 83–86]. If a string of constant length, with fixed endpoints and a smooth density σ (and another property too technical to mention here) has solely harmonic overtones, then (17) holds. Here smoothness means that σ''' is absolutely continuous on $[0, l]$. The proof uses the asymptotic behaviour of the eigenvalues of a regular Sturm–Liouville operator and the completeness of products of certain functions in a Hilbert space. It is important to notice that, in general, it is not sufficient to prescribe just a single spectrum to determine a potential or a density.

3 Lagrange on the shape of columns

In a slightly different form, equation (2) reappears in a paper by Lagrange [17], written three years after the publication of [4], where he aims at finding the shape of a (rotationally symmetric) hinged column that supports the greatest possible weight without bending. He inserts $y = r \sin\varphi$ into the equation $Xy'' + Py = 0$ which Euler had used to study the strength of columns under a constant load P (X is the function which describes the elasticity of the column) and obtains

$$0 = (Pr + Xr'' - r\varphi'^2)\sin\varphi + (2r'\varphi' + r\varphi'')\cos\varphi.$$

Since r and φ are arbitrary functions, the bracketed terms have to be zero. Integration of $2r'\varphi' + r\varphi'' = 0$ shows $r^2\varphi' = h = \text{const.}$, which gives y the structure of (5), although r now solves

$$\frac{h^2}{r^4} - X\frac{r''}{r} = P.$$

He then tries to maximise the "relative strength" of a column over a class of functions X, but conceptual as well as computational errors lead him to the mistaken result that the cylinder is the strongest hinged column (see the analysis in [29, pp. 352–355]). The problem was not solved until around 1990 as the 1960 paper Truesdell mentions did not clinch the matter. We refer the reader to the perceptive article [8].

It is peculiar, at least to a present-day reader, that Lagrange does not mention [4]; given the fact that, at the time, the vibrating string was a topic of frequent and sometimes heated discussions, it seems unlikely that he did not know Daniel Bernoulli's paper. Indeed, Truesdell gives a number of examples where his references are sparse [29, pp. 265, 350, 352].

4 Adolphe Steen on products of solutions of (1)

Steen stated the equivalence of equations

$$-u'' + Vu = 0 \tag{1}$$

and

$$\frac{r''}{r} - \frac{1}{r^4} = V \tag{2}$$

via $r = (u_1^2 + u_2^2)^{1/2}$ in a clear way, and by starting with a positive linear function r rediscovered Daniel Bernoulli's function (9) with the corresponding fundamental system of (1) [26, p. 142(12)]. The latter's example (13) was obtained by subjecting (2) to what would now be called a Liouville–Green transformation [26, p. 144f. (17)]. He also noticed the following interrelation between (1) and

$$\frac{r''}{r} + \frac{1}{r^4} = V. \tag{18}$$

If u_1, u_2 is a fundamental system of (1) with Wronskian 1 and $u_1 u_2 > 0$ on some interval, then $r = (2u_1 u_2)^{1/2}$ is a positive solution of (18) on this interval (top line in [26, p. 142]; note the printing error), and conversely, given a positive solution of (18), $r \exp \circ s$ and $r \exp \circ (-s)$, s as in (21) below, form a fundamental system of (1).

Next, he observes that (2) and (18) can be written as

$$\begin{aligned} \pm 1 &= r^2(rr'' + r'^2) - (rr')^2 - Vr^4 \\ &= r^2\Big[\frac{1}{2}(r^2)'' - Vr^2\Big] - \frac{1}{4}[(r^2)']^2 \end{aligned} \tag{19}$$

(top line in [26, p. 149]; note the misprint in the last line on [26, p. 148]). Differentiating again, one obtains

$$0 = \frac{1}{2}(r^2)'[(r^2)'' - 2Vr^2] + \frac{1}{2}r^2[(r^2)''' - 2(Vr^2)'] - \frac{1}{2}(r^2)'(r^2)''.$$

Since the first term cancels the last, division by $\frac{1}{2}r^2$ shows that both $u_1^2 + u_2^2$ and $2u_1u_2$ satisfy the linear third order equation

$$y''' - 4Vy' - 2V'y = 0. \tag{20}$$

The linearity of (20) makes the assumption $u_1 u_2 > 0$ irrelevant, and so the product of *any* two solutions of (1) satisfies (20). Steen does not say so, however, and concludes his paper by prescribing r^2 and using (20) as a linear first order equation for V to produce new but somewhat artificial examples where (1) can be solved explicitly.

Astonishingly, [26, p. 136] credit a 1927 paper by Gambier with this elegant derivation of (20). Moreover, on [26, p. 137], they attribute to a fairly involved 1898 paper by Chini [7] the more general observation that the nonlinear superposition (22) below satisfies (23), but we now show that this result can be obtained by very elementary modifications of Steen's proof.

Theorem 1. *Let $I := [x_0, X)$, where $X \le \infty$ and $V \in C^0(I)$.*

(a) *If $0 < r \in C^2(I)$ is a solution of (2), then*

$$u_1 := r \cos \circ s, \quad u_2 := r \sin \circ s; \quad s(x) := \int_{x_0}^{x} \frac{1}{r^2} \quad (x \in I) \tag{21}$$

is a fundamental system of (1) with Wronskian

$$W := W(u_1, u_2) := u_1 u_2' - u_1' u_2 = 1.$$

(b) *Let A, B, C be numbers with $A > 0$ and $d := AC - B^2 > 0$. If $u_1, u_2 \in C^2(I)$ are solutions of* (1) *with Wronskian $W = 1$, then*

$$r := (Au_1^2 + 2Bu_1u_2 + Cu_2^2)^{1/2} \tag{22}$$

is a positive solution of

$$\frac{r''}{r} - \frac{d}{r^4} = V. \tag{23}$$

Proof. (a) *presents no problem, but* (b) *does.* Guided by Steen, we proceed from

$$\frac{1}{2}(r^2)' = Au_1u_1' + B(u_1'u_2 + u_1u_2') + Cu_2u_2' \tag{24}$$

by differentiation to

$$r'^2 + rr'' = Au_1'^2 + 2Bu_1'u_2' + Cu_2'^2 + Vr^2$$

or

$$r^3r'' - Vr^4 = r^2(Au_1'^2 + 2Bu_1'u_2' + Cu_2'^2) - (rr')^2. \tag{25}$$

With $a_1 := A^{1/2}u_1 + BA^{-1/2}u_2$, $a_2 := (d/A)^{1/2}u_2$, $b_1 := a_1'$, $b_2 := a_2'$, (22), (24) can be expressed as

$$r^2 = a_1^2 + a_2^2, \quad rr' = a_1b_1 + a_2b_2.$$

In view of Lagrange's identity, the right-hand side of (25) therefore equals

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1b_1 + a_2b_2)^2 = (a_1b_2 - a_2b_1)^2 = (d^{1/2}W)^2 = d. \qquad \blacksquare$$

Remark 2. (a) For $V \in C^1(I)$, we note that r^2, and so any linear combination of u_1^2, u_1u_2 and u_2^2 satisfies (20). The linear independence of these functions results from the identity

$$W(u_1^2, u_1u_2, u_2^2) = 2[W(u_1, u_2)]^3$$

for the Wronskians of these functions.

(b) The reason why there is no term y'' in (20) is that a term u' is missing in (1). To generalise (1), it is convenient to start with the symmetric form

$$-(pu')' + Vu = 0 \tag{26}$$

($0 < p \in C^1(I)$ for simplicity). If $r > 0$ is a solution of

$$\frac{(pr')'}{r} - \frac{1}{pr^4} = V, \tag{27}$$

then (21) is a fundamental system of (26) with $pW = 1$ when s is replaced by

$$s(x) := \int_{x_0}^{x} \frac{1}{pr^2}.$$

Conversely, if u_1, u_2 form a fundamental system of (26) with Wronskian $pW = 1$, then (22) is a solution of

$$\frac{(pr')'}{r} - \frac{d}{pr^4} = V$$

since $r^3(pr')' - Vr^4$ is p times the right-hand side of (25).

(c) Let $0 < p \in C^2(I)$ and $V \in C^1(I)$. Then $y := r^2$ solves

$$p(py')'' - 4pVy' - 2(pV)'y = 0. \tag{28}$$

Again this follows by differentiating (cf. (19))

$$d = p[r^3(pr')' - Vr^4] = \frac{1}{2}py[(py')' - 2Vy] - \frac{1}{4}(py')^2$$

and dividing by $\frac{1}{2}y$. When (26) is replaced by the slightly more general equation

$$w'' + fw' + gw = 0 \tag{29}$$

with $f, g \in C^1(I)$, we have to perform the differentiations in (28) and to identify p'/p with f and $(-V)/p$ with g and then obtain

$$y''' + 3fy'' + (2f^2 + f' + 4g)y' + 2(2fg + g')y = 0. \tag{30}$$

Equation (30) is indeed uniquely determined, for the difference of (30) and any other linear third order equation solved by u_1^2, u_1u_2 and u_2^2 would be a linear second order equation with three linearly independent solutions. Of course, one could also derive (30) from (20) by using the equivalence of (1) and (29) via

$$w = u\exp\Big(-\frac{1}{2}\int f\Big), \quad V = \frac{1}{2}f' + \frac{1}{4}f^2 - g,$$

however, at the expense of longer calculations.

The impression forced upon the reader of Steen's paper is that it was he who found (20) as the linear equation of smallest order which is satisfied by products of solutions of (1) (alas, this happened to [15, 26]), but this is not the case. In [26, p. 143], Steen mentions that a result of Liouville (no specific reference is given) can be used to show that solutions of (2) cannot be algebraic functions when V is a polynomial. The notation used points to Liouville's paper [20], and it is this paper to which his subsequent study [27] is devoted. There is a footnote on [20, p. 431], impossible to overlook, where Liouville derives (20) by triple differentiation of u_1u_2. Steen has deduced this equation in a motivated and elegant way, but it is astounding that he fails to mention Liouville at this particular point.

Whittaker–Watson [32, p. 298] refer to an 1880 paper by Appell [1] for (30), which is somewhat misleading as Appell uses (30) only to illustrate his study of rational functions of linearly independent solutions of linear n-th order equations [2, pp. 411–414]. (We believe that Chini's above-mentioned paper [7] also has its proper place in this wider context). Ince [13, p. 395] gives the following exercise: find the linear equation whose solutions are products of solutions of [our equation (1)] and explain why it is of the third order, adding "[Lindemann]" either as a reference or a help for the reader. Since the context of the exercise concerns the Lamé equation, the hint eventually leads to [19]. On [19, pp. 118, 120], there is indeed a proof with two references, Hermite's 1872/1873 Cours d'analyse and an 1878 paper of his [11, pp. 118–122, 475–478].

5 A comparison of pairs of solutions of Daniel Bernoulli's equation

We now describe a more intricate use of

$$\frac{(pr')'}{r} - \frac{1}{pr^4} = V \tag{27}$$

to study

$$-(pu')' + Vu = 0 \tag{26}$$

by means of the fundamental system

$$u_1 := r\cos\circ s, \quad u_2 := r\sin\circ s; \quad s(x) := \int_{x_0}^{x} \frac{1}{pr^2} \quad (x \in I) \tag{31}$$

on $I := [x_0, X)$, $X \leq \infty$. The theory created by Hermann Weyl in 1909 implies that the left-hand side of (26) gives rise to a family of self-adjoint operators in $L^2(I)$, solely parameterised by boundary conditions at x_0, if and only if (26) has a solution which is not of integrable square in a neighbourhood of X (see, e.g., [3, § 4]).

Johann Walter [30] noted that any two solutions r, $R > 0$ of (27) had common properties (see Corollary 4 and Remark 5 below) that can be exploited to infer the existence of a solution of (26) which is not in L^2 at infinity from that of an equation which is in a certain sense "smaller". In a footnote, he indicated that r and R can explicitly be related using the fact that all solutions of a Riccati equation can be represented in terms of a fundamental system of the associated linear second order equation (cf., e.g., [12, p. 108 f.]). Redheffer was so surprised by the conclusion that R/r was bounded that he gave in [25, Corollary 3] a proof which was independent of any such connecting formula, but also less explicit. In later years, Walter realised that the formula he had arrived at in [30, equation (4)] could not be correct as stated, but it was not granted to him to mend it. The proof of the formula given below avoids the Riccati equation and so any discussion of possible zeros of solutions of (26).

We stick to the assumptions on p and V made in Theorem 1 and Remark 2 (b), but one might as well assume that p is positive a.e. on I and that $1/p$, V are locally integrable on I. In this case, a solution of (26) or (27) is a function w which, together with pw', is locally absolutely continuous on I and satisfies the corresponding equation a.e. on I.

Theorem 3. *Let r and R be positive solutions of* (27) *and*

$$\alpha := (R/r)(x_0), \quad \beta := [p(rR' - r'R)](x_0),$$
$$f(t) := \cos^2 t + (\beta/\alpha)\sin 2t + [(\beta/\alpha)^2 + \alpha^{-4}]\sin^2 t, \quad (t \geq 0).$$

Then

$$R^2 = \alpha^2 (f \circ s) r^2.$$

Proof. In addition to (31), we have a second fundamental system of (26), namely the one generated by R, which we denote by capital letters. As a consequence, there exist numbers α, β, γ such that

$$U_1 = \alpha u_1 + \beta u_2, \quad U_2 := \gamma u_2.$$

Note that $U_2(x_0) = 0$, while $u_1(x_0) \neq 0$. Hence $R(x_0) = U_1(x_0) = \alpha r(x_0)$. From

$$u_1' = r' \cos \circ s - \frac{\sin \circ s}{pr}, \quad u_2' = r' \sin \circ s + \frac{\cos \circ s}{pr},$$

we conclude $\gamma = 1/\alpha$ since

$$(1/pR)(x_0) = U_2'(x_0) = \gamma(1/pr)(x_0).$$

Moreover,

$$R'(x_0) = U_1'(x_0) = (R/r)(x_0)r'(x_0) + \beta(1/pr)(x_0),$$

and the assertion follows since $R^2 = U_1^2 + U_2^2$. ■

Corollary 4. *The following statements hold.*

(a) $r \in L^2(I) \Leftrightarrow R \in L^2(I)$.

(b) $\sup|(r^2 p^{1/2})' p^{1/2}| < \infty \Leftrightarrow \sup|(R^2 p^{1/2})' p^{1/2}| < \infty$.

Proof. "⇒" in (a) is a consequence of

$$R^2 \leq \text{const.}\, r^2, \tag{32}$$

and "⇐" follows by changing the roles of r and R in (32). Part (b) results from

$$\alpha^{-2}(R^2 p^{1/2})' p^{1/2} = (r^2 p^{1/2})' p^{1/2}(f \circ s) + r^2(f' \circ s)p/pr^2$$

and the boundedness of $f \circ s$ and $f' \circ s$. ■

Remark 5. Suppose (26) is oscillatory on $I := [x_0, \infty)$, i.e., (26) has a non-trivial solution with infinitely many zeros. Then Sturm's oscillation theorem [13, 224 ff.] guarantees that all solutions, in particular (31) and the pair generated by R, have this property. Hence

$$\int_{x_0}^{\infty} \frac{1}{pR^2} = \infty = \int_{x_0}^{\infty} \frac{1}{pr^2}. \tag{33}$$

Thus there exist intervals $[X_1, X_2]$, $[x_1, x_2]$ with

$$\int_{X_1}^{X_2} \frac{1}{pR^2} = \pi = \int_{x_1}^{x_2} \frac{1}{pr^2}. \tag{34}$$

Equation (34) is listed as an additional common property of solutions of (27). However, if (26) is non-oscillatory, again by Sturm's theorem, every non-trivial solution of (26) has at most finitely many zeros. Consequently, both integrals in (33) are finite and (34) may be impossible for one or both integrals, notwithstanding the possibility that (26) has a solution with one or more zeros.

Returning to (33), one observes that there exist a number $a \geq x_0$ and a function $\varphi(\cdot, r)$ with

$$\int_x^{x+\varphi(x,r)} \frac{1}{pr^2} = \pi$$

for $x \geq a$. Every linear combination of (31) with a zero at x has $x + \varphi(x, r)$ as the next zero, and the same applies when r is replaced by R. Hence $\varphi(\cdot, r) = \varphi(\cdot, R)$. J. Walter's comparison theorem can now be formulated as follows. We emphasise that his sophisticated proof, also given below, is in essence unaffected.

Theorem 6 ([30, Satz 3]). *Let $I := [x_0, \infty)$ and $p \in C^1(I)$. Assume that equation (26) with $V_1, V \in C^0(I)$ is oscillatory on I. Let r_1, r be positive solutions of (27) corresponding to V_1, V, respectively. Suppose $\alpha := \sup|(r_1^2 p^{1/2})' p^{1/2}| < \infty$ and $r_1 \notin L^2(I)$. If*

$$\varphi(\cdot, r) \geq \varphi(\cdot, r_1), \tag{35}$$

then $r \notin L^2(I)$.

Proof. Let $x, y \in I$, $x < y$ and

$$s(x, y; r_1) := \int_x^y \frac{1}{pr_1^2}.$$

The idea is to prove a sort of converse of the Schwarz inequality

$$\left(\int_x^y p^{-1/2}\right)^2 \leq s(x, y; r_1) \int_x^y r_1^2,$$

viz.,

$$\left(\int_x^y p^{-1/2}\right)^2 \geq \left\{\sup_{[x,y]} |(r_1^2 p^{1/2})' p^{1/2}| + \frac{1}{s(x, y; r_1)}\right\}^{-1} \int_x^y r_1^2. \tag{36}$$

Inequality (36) will be established below.

From the definition of $\varphi(\cdot, r)$ and (35), we have

$$\pi = \int_x^{x+\varphi(x,r)} \frac{1}{pr^2} \geq \int_x^{x+\varphi(x,r_1)} \frac{1}{pr^2}$$

for $x \geq a$. Moreover, we can define an unbounded sequence by

$$x_1 \geq a, \quad x_{j+1} = x_j + \varphi(x_j, r_1)$$

with the property $s(x_j, x_{j+1}; r_1) = \pi$ $(j \in \mathbb{N})$. Inserting this sequence into (36), we obtain

$$\begin{aligned}
\int_{x_j}^{x_{j+1}} r^2 &\geq \left(\int_{x_j}^{x_{j+1}} p^{-1/2}\right)^2 \frac{1}{s(x_j, x_{j+1}; r)} \\
&\geq \left[\frac{1}{s(x_j, x_{j+1}; r)}\right]\left[\alpha + \frac{1}{s(x_j, x_{j+1}; r_1)}\right]^{-1} \int_{x_j}^{x_{j+1}} r_1^2 \\
&\geq \frac{1}{\pi}\left(\alpha + \frac{1}{\pi}\right)^{-1} \int_{x_j}^{x_{j+1}} r_1^2.
\end{aligned}$$

Hence $r \notin L^2(I)$.

In order to prove (36), we use the mean value theorem for integrals, which provides us with numbers $t_1, t_2 \in (x, y)$ such that

$$s(x, y; r_1) = \Big[\frac{1}{(r_1^2 p^{1/2})(t_1)}\Big] \int_x^y p^{-1/2}$$

and

$$\begin{aligned}\int_x^y r_1^2 &= (r_1^2 p^{1/2})(t_2) \int_x^y p^{-1/2} \\ &= \Big\{\int_{t_1}^{t_2} (r_1^2 p^{1/2})' + \Big[\frac{1}{s(x, y; r_1)}\Big] \int_x^y p^{-1/2}\Big\} \int_x^y p^{-1/2} \\ &\le \Big\{\sup_{[x,y]} |(r_1^2 p^{1/2})' p^{1/2}| + \frac{1}{s(x, y; r_1)}\Big\} \Big(\int_x^y p^{-1/2}\Big)^2. \qquad \blacksquare\end{aligned}$$

On account of Sturm's theorem, the distance of consecutive zeros of a solution of (26) increases with V. Thus (35) holds if $V \ge V_1$. A second sufficient condition for (35) is given in [30, Satz 5] (there is a printing error in "Beispiel zu Satz 5": $g_1 = x^{-1/2}$ is correct).

When (26) is non-oscillatory, the situation is completely different and in fact simpler as one then has a fundamental system of principle and non-principle solutions at hand [10, p. 355]. It is a remarkable coincidence that, just a little earlier, this case was conclusively treated in [16].

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Daniel Bernoulli's paper [4] is [St54] in [28] and planned to be contained in Vol. 6 of his Collected Works. Euler's papers, written in Latin of course, are cited by their Eneström number; the English translation of their titles is taken from [29]. The Euler Archive provides easy access to the originals.

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