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Short note The equation $x/y + y/z + z/x = 4$ revisited

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Abstract. We give an elementary argument to the fact that the equation $x/y + y/z + z/x = 4$ has no positive integer solutions.

Quoting from Spierpiński's classical book *250 Problems in Elementary Number Theory* [3, page 80], "we do not know whether the equation

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 4 \tag{1}$$

has positive integer solutions x, y, z". Erik Dofs [2] showed that the equation $a^3 + b^3 + c^3 = nabc$ has no *integer* solutions for $n = 4$ and many other values of n. By letting $x = a^2b$, $y = b^2c$, $z = c^2a$, equation (1) transforms into $a^3 + b^3 + c^3 = 4abc$. Hence equation (1) has no integer (positive integer) solutions. For the general equation $x/y + y/z + z/x = n$, elliptic curves are the natural setting; see [1]. So an elementary argument for Spierpiński's remark is desirable. In this note, we provide such an argument.

Theorem 1. *There do not exist positive integers x, y, z satisfying*

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 4. \tag{2}$$

We need the following lemma.

Lemma 1. *Let n be an odd positive integer such that every prime divisor p of n is equivalent to* $1 \pmod 4$. *Then there exist integers A, B such that $n = A^2 + B^2$.*

Proof. This is the special case of the Sum of Two Squares Theorem. For a proof, see Silverman [4, Theorem 25.1, page 196]. ■

For an odd prime number p, $(\cdot\,|\,p)$ denotes the Legendre symbol. Note that $(-1|p) = 1$ if and only if $p \equiv 1 \pmod 4$. We are now ready to prove Theorem 1. Assume that there exist positive integers x, y, z satisfying (2). If $xy - z^2 = 0$, $yz - x^2 = 0$, and $zx - y^2 = 0$, then $xyz = x^3 = y^3 = z^3$ so that $x = y = z$. Hence (2) is impossible. So at least one of $xy - z^2$, $yz - x^2$, $zx - y^2$ is nonzero. We can assume that $xz - y^2 \neq 0$. From (2), we have $x/y + y/z = 4 - z/x$. Hence $4x - z > 0$ and

$$\left(\frac{x}{y} - \frac{y}{z}\right)^2 = \left(4 - \frac{z}{x}\right)^2 - 4\frac{x}{z}.$$

Thus

$$\left(\frac{x(xz - y^2)}{y}\right)^2 = z\left(-4x^3 + z(z - 4x)^2\right). \tag{3}$$

Let $a = -x$ and $b = z$. Then $a < 0$, $b > 0$, and $4a + b = z - 4x < 0$. From (3), we have $b(4a^3 + b(4a + b)^2)$ is a rational square, hence an integral square. Let $d = \gcd(b, 4a^3 + b(4a + b)^2)$. Then $d \mid 4$. Therefore, $d \in \{1, 2, 4\}$.

Case 1: $d = 1$. Then $b = s^2$ and $4a^3 + b(4a + b)^2 = t^2$, where $s, t \in \mathbb{Z}^+$, $2 \nmid t$, $2 \nmid s$, and $\gcd(s, t) = 1$. Assume that p is a prime divisor of $4a + b$. Then p is odd. We have $t^2 \equiv 4a^3 \pmod{p}$. Therefore,

$$0 \equiv a^2(4a + b) \equiv t^2 + (as)^2 \pmod{p}. \tag{4}$$

Since $\gcd(a, b) = 1$ and $p \mid 4a + b$, we have $p \nmid a$ and $p \nmid b$. Hence $p \nmid t$ and $p \nmid as$. Thus, from (4), we have $(-1|p) = 1$. Hence $p \equiv 1 \pmod 4$. This holds for every prime divisor of $4a + b$. Since $4a + b < 0$, by Lemma 1, we have $-(4a + b) = A^2 + B^2$, where $A, B \in \mathbb{Z}$. Therefore, $-(4a + s^2) = A^2 + B^2$. Since $2 \nmid s$, we have

$$-1 \equiv -(4a + s^2) \equiv A^2 + B^2 \pmod 4,$$

which is impossible since $A^2 + B^2 \equiv 0, 1, 2 \pmod 4$.

Case 2: $d = 2$. Then $b = 2s^2$ and $4a^3 + b(4a + b)^2 = 2t^2$, where $s, t \in \mathbb{Z}^+$, and $\gcd(s, t) = 1$. Since $2 \mid b$, we have $2 \nmid a$. Since $t^2 = 2a^3 + 4s^2(2a + s^2)^2$, we have $2 \mid t$. So $4 \mid t^2$. Thus $4 \mid 2a^3$, which is impossible since $2 \nmid a$.

Case 3: $d = 4$. Then $b = 4s^2$ and $4a^3 + b(4a + b)^2 = 4t^2$, where $s, t \in \mathbb{Z}^+$, $\gcd(s, t) = 1$. We have

$$a^3 + 16s^2(a + s^2)^2 = t^2. \tag{5}$$

Since $2 \mid b$, we have $2 \nmid a$. From (5), we have $2 \nmid t$. Reducing (5) mod 8 gives $a \equiv 1 \pmod 8$. Since $a + s^2 = (4a + b)/4$, we have $a + s^2 < 0$. Let $a + s^2 = -2^r h$, where $r, h \in \mathbb{Z}_{\geq 0}$ and $2 \nmid h$. Assume that p is a prime divisor of h. Then p is odd. From (5), we have $a^3 \equiv t^2 \pmod{p}$. Therefore,

$$0 \equiv a^2(a + s^2) \equiv t^2 + (as)^2 \pmod{p}. \tag{6}$$

Since $p \mid a + s^2$, $b = 4s^2$, and $\gcd(a, b) = 1$, we have $p \nmid a$ and $p \nmid s$. From (6), we also have $p \nmid t$, hence $(-1|p) = 1$. Therefore, $p \equiv 1 \pmod 4$. This holds for every prime divisor of h. By Lemma 1, we have $h = A^2 + B^2$, where $A, B \in \mathbb{Z}$.

- $2 \mid s$. Since $a \equiv 1 \pmod 8$, we have $a + s^2 \equiv 1 \pmod 4$. Since $a + s^2 = -2^r h$, we have $r = 0$ and $h \equiv -1 \pmod 4$. Therefore,

  $$-1 \equiv h \equiv A^2 + B^2 \pmod 4,$$

  which is impossible since $A^2 + B^2 \equiv 0, 1, 2 \pmod 4$.

- $2 \nmid s$. Since $a \equiv 1 \pmod 8$, we have $a + s^2 \equiv 2 \pmod 8$. Since $a + s^2 = -2^r h$, we have $r = 1$ and $h \equiv -1 \pmod 4$. Therefore,

  $$-1 \equiv h \equiv A^2 + B^2 \pmod 4,$$

  which is impossible since $A^2 + B^2 \equiv 0, 1, 2 \pmod 4$.

References

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