Elemente der Mathematik 78 (2023), 174–177
DOI 10.4171/EM/486

Short note

Two proofs of Bondy's theorem on induced subsets and two related questions

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Let $A = \{a_1, \dots, a_n\}$ be a set of n elements, and let $\mathcal{F} = \{A_1, \dots, A_m\}$ be a family of m subsets of A. In the sequel, we will call a set of n elements an n-set, and we denote any matrix M of order $m \times n$ as $M = M_{m\times n} = (a_{ij})$ with $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$. Also, in a matrix M, any column whose deletion makes all the new rows distinct is called a *solution*. For A_i, let v_i be the incidence vector of A_i defined by $v_i = (\delta_{i1}, \dots, \delta_{in})$ with

$$\delta_{ij} = \begin{cases} 0 & \text{if } a_j \notin A_i, \\ 1 & \text{if } a_j \in A_i, \end{cases}$$

for $j = 1, \dots, n$. Then $M_{m\times n} = (v_1, \dots, v_m)^T$ is the incidence matrix of $\mathcal{F}$. Conversely, any (0, 1)-matrix can be interpreted as an incidence matrix. Now suppose that $m = n$, and $\{A_1, \dots, A_n\}$ is a family $\mathcal{F}$ of n distinct subsets of A.

A. Hajnal asked, in a private conversation with J. A. Bondy, the following question.

Is there necessarily an $(n-1)$-set $A' \subset A$ such that the sets $A_i \cap A'$ are all distinct?

Bondy proved this in [3] affirmatively with a simple graph theoretical argument.

Bondy's theorem. *If $\mathcal{F} = \{A_1, \dots, A_n\}$ is a family of n distinct subsets of an n-set A, then there is an $(n-1)$-set $A' \subset A$ such that the sets $A_i \cap A'$, where $1 \le i \le n$, are all distinct.*

In other words, there is always an element $a \in A$ such that $A_1 \setminus \{a\}, \dots, A_n \setminus \{a\}$ (or similarly $A_1 \cup \{a\}, \dots, A_n \cup \{a\}$) are yet distinct. The example $A, A \setminus \{a_1\}, \dots, A \setminus \{a_n\}$ of $n + 1$ subsets of A shows that n is the best bound for the cardinality of $\mathcal{F}$ in general (because $A \setminus \{a_k\} = A \setminus \{a_k\} \setminus \{a_k\}$ for every k).

B. Bollobás states in [2] three different proofs, I. Anderson expresses in [1] a graph theoretical proof, and A. Winter gives in [4] a linear algebraic proof.

For some theorems, the contraposition is also nice and interesting, especially if the proof of the contraposition is easier. Bondy's theorem is of this kind. Bondy's theorem and its contraposition may be formulated by (0, 1)-incidence matrices. The main purpose of this article is to state the contraposition of Bondy's theorem (as Theorem 1) and to give a neat and purely combinatorial proof which is a straightforward consequence of the simple Fact 1 below. Although a theorem and its contraposition are logically equivalent in that a proof of one is sufficient to establish the truth of the other, in this article, we will also give a matrix theoretical proof to Bondy's theorem as Theorem 2, where we give priority to determinants and minors to prove it, although this proof, in comparison to the proof of Theorem 1, is a bit trickier, as it is based on Fact 2 below.

Fact 1. *If a graph is on n vertices and has at least n edges, then it has at least one cycle.*

Fact 2. *The determinant of a square matrix vanishes only when its rows (or columns) are linearly dependent. (If the rows or columns are not distinct, then the determinant vanishes; this is a consequence of Fact 2, although it has also other proofs.)*

Fact 1 is a well-known simple theorem in graph theory whose proof is an easy exercise, and Fact 2 is one of the essential properties of determinants whose proof may be found in any textbook on matrix theory or linear algebra.

Theorem 1. *Let M be a square $(0, 1)$-matrix such that, after removing any column, there are at least two equal rows; then there are at least two equal rows in M.*

Proof (by graphs). Let M have n rows $R_1, \ldots, R_n$ and n columns $C_1, \ldots, C_n$. Overlooking any C_k, for $k = 1, \ldots, n$, let $R_1^{(k)}, \ldots, R_n^{(k)}$ be the same rows $R_1, \ldots, R_n$ unless their k-th entry is overlooked. Now, for any k, there are $i \neq j$ such that $R_i^{(k)} = R_j^{(k)}$. Then consider a graph on $R_1, \ldots, R_n$ (as n vertices) with n edges $\{R_i^{(k)}, R_j^{(k)}\}$, for $k = 1, \ldots, n$; hence by Fact 1, there exists at least one cycle $R_{i_1}, \ldots, R_{i_m}$, for some $1 \leq i_1 < \cdots < i_m \leq n$, and we therefore simply observe that $R_{i_1} = \cdots = R_{i_m}$. This completes the proof. ■

Theorem 2. *If M is a $(0, 1)$-matrix of order $n \times n$ with all the rows distinct, then we can substitute all entries of one proper column by 0 (or 1) such that the new rows are yet distinct. Or equivalently, there is at least one column whose deletion makes all rows of the new $n \times (n-1)$ matrix yet distinct.*

Proof (by matrices). First, let $\det(M) \neq 0$. Consider the number of 1's in each row of M, and let the number of 1's in the i-th row be minimal (such a row may not be unique; it is even possible that all the rows have the same number of 1's). Then we fix this i, and expand $\det(M)$ by the i-th row, and we get

$$\det(M) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij},$$

where M_{ij} is the (i, j)-th minor (or the minor of M corresponding to the entry a_{ij}); i.e. it is the determinant of the $(n-1) \times (n-1)$ submatrix formed from M by deleting the i-th row and the j-th column. Now, for some j, we have $a_{ij} M_{ij} \neq 0$ because otherwise $a_{ij} M_{ij} = 0$ for all $j \in \{1, \ldots, n\}$, and then $\det(M) = 0$, which is not this case. And from $a_{ij} M_{ij} \neq 0$, we achieve $a_{ij} = 1$ and $M_{ij} \neq 0$; therefore, all rows of the matrix under the minor M_{ij} are distinct; otherwise, $M_{ij} = 0$, which is a contradiction! Now, denote by N the $n \times (n-1)$ submatrix derived from M by deleting the j-th column of M; then this

i-th row of N is also different from any other k-th row of N, for all $k \in \{1, \dots, n\} \setminus \{i\}$, because if, for $i \neq k$, the i-th row and k-th row in N were equal, hence we would have either $a_{kj} = 1$, which means in this case that both the i-th and k-th rows in M would be equal, which is a contradiction since all the rows of M are distinct; or $a_{kj} = 0$, which would mean that the number of 1's in the k-th row would be less than the number of 1's in the i-th row in M, which was minimal: a contradiction! Hence, deleting this j-th column of M makes all the new rows distinct, too.

Next, let $\det(M) = 0$. Then, by Fact 2, the columns of M are linearly dependent; i.e. there are scalars $k_1, \dots, k_n$, not all zero, without loss of generality say $k_n \neq 0$, such that

$$k_1 C_1 + \cdots + k_n C_n = 0 \quad \text{or} \quad k_1 C_1 + \cdots + k_{n-1} C_{n-1} = -k_n C_n,$$

where $C_1, \dots, C_n$ are all the columns of M. Overlooking the column C_n, consider, in the reduced $n \times (n-1)$ matrix, any two rows R_s and R_t for $s \neq t$ as follows:

$$\begin{cases} R_s = (r_1, \dots, r_{n-1}), \\ R_t = (r'_1, \dots, r'_{n-1}); \end{cases}$$

then

$$\begin{cases} k_1 r_1 + \cdots + k_{n-1} r_{n-1} = -k_n r_n, \\ k_1 r'_1 + \cdots + k_{n-1} r'_{n-1} = -k_n r'_n. \end{cases}$$

Now if $R_s = R_t$, then $r_n = r'_n$, and we get $(r_1, \dots, r_{n-1}, r_n) = (r'_1, \dots, r'_{n-1}, r'_n)$, a contradiction (since each two rows in M are different)! Therefore, overlooking the n-th column (and in fact every j-th column C_j with $k_j \neq 0$) makes all the new rows distinct; in other words, any C_j with $k_j \neq 0$ is a *solution*. This completes the proof. ■

Corollary. *Let $M = M_{n \times n}$ be a $(0, 1)$-matrix with all rows distinct; then there exists one permutation of the columns of M such that if N is the matrix M under this permutation, then for any $k \in \{1, \dots, n-1\}$, all rows of the submatrix $N_{(k+1) \times k}$ are distinct, too. Here $N = N_{n \times n} = (b_{ij})_{i,j=1}^{n}$, and $N_{(k+1) \times k} = (b_{ij})$ with $(i, j) \in \{1, \dots, k+1\} \times \{1, \dots, k\}$.*

Proof. This is an obvious and direct conclusion of Bondy's theorem. ■

We conclude this article with the following two examples, two questions, and one small application.

Examples, Questions, Application

Bondy's theorem ensures the existence of at least one column whose deletion makes all the new rows distinct (i.e. at least one *solution* exists).

Example 1. One case of the minimal number ($= 1$) of *solutions* (up to equivalency) holds, for example, for the following matrix:

$$M_{n \times n} = (a_{ij})_{i,j=1}^{n} \quad \text{with } a_{ij} = \begin{cases} 0 & \text{for all } i = j \in \{2, \dots, n\}, \\ 1 & \text{otherwise,} \end{cases}$$

such that the first column is the only *solution*.

Example 2. One case of the maximal number ($= n$) of *solutions* (up to equivalency) holds, for instance, for the identity matrix I_n such that any one of the n columns is one *solution*.

Question 1. Are there other examples of $(0, 1)$-matrices $M_{n\times n}$ (up to equivalency) for the above minimal and maximal cases?

Question 2. Could one find some examples or all of the $(0, 1)$-matrices $M_{n\times n}$ such that, for any integer k with $1 < k < n$, there exist exactly k *solutions* (up to equivalency)?

Application. This is only one small application of Theorem 1 to determinants: let M be any square $(0, 1)$-matrix with the mentioned property in Theorem 1; then $\det(M) = 0$.

Acknowledgment. Many thanks to the referee for the useful suggestions.

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