Elemente der Mathematik 79 (2024), 1–7
DOI 10.4171/EM/518

Integers expressible as sums of primes and composites

John M. Campbell

John M. Campbell completed his Ph.D. in pure mathematics at York University and graduated first class with distinction with a Specialized Honours B.Sc. degree in mathematics from York University. He has been awarded the prestigious Killam Postdoctoral Fellowship and the prestigious Carswell Scholarship.

1 Introduction

In 2020, Zhang [8] proved the following remarkable result.

Theorem 1 (Zhang, 2020 [8]). *Each even integer greater than* 30 *is the sum of a composite number* c *and a prime* p *such that* $p \nmid c$.

Naturally, this raises the question as to whether or not this same property holds for odd integers greater than some fixed value. However, Zhang's techniques from [8] can only be directly applied to odd integers so as to obtain the following inequivalent result: every odd integer greater than 23 can be represented in the form $p + q + c$, where p and q are prime and c is a composite number such that p, q, and c are pairwise coprime. Furthermore, it appears that the only research, apart from our current note, currently citing Zhang's work in [8] is due to Alzer and Kwong [1], who did not introduce any results on or related to the expression of integers in the form $p + c$ in the manner we have previously indicated. We succeed, as below, in proving that Zhang's result also holds for odd integers.

Die additive Zahlentheorie ist innerhalb der Zahlentheorie ein wichtiges und umfangreiches Gebiet. Die Zerlegung oder Partition natürlicher Zahlen nach bestimmten Regeln oder Bedingungen ist innerhalb und ausserhalb der Zahlentheorie allgegenwärtig und bildet ein Hauptthema der additiven Zahlentheorie. Im Jahr 2020 bewies Zhang durch Verfeinerung der Rosser-Schoenfeld-Ungleichung das folgende bemerkenswerte Resultat: Jede gerade ganze Zahl grösser als 30 kann als Summe einer zusammengesetzten Zahl c und einer Primzahl p, die c nicht teilt, ausgedrückt werden. Der Autor der vorliegenden Arbeit verbessert die Ergebnisse von Zhang, indem er nachweist, dass jede ganze Zahl grösser als 30, unabhängig von ihrer Parität, in der Form $p + c$ für eine zusammengesetzte Zahl c und eine Primzahl p ausgedrückt werden kann, so dass $p \nmid c$ gilt.

There are many reasons as to the mathematical interest in the expression of integers as $p+c$ in the manner previously specified. In this regard, we may appeal to the importance of the discipline of *additive number theory* as a main subject within number theory. Additive number theory broadly refers to the study as to how integers may be expressed with sums of integers from a given set or sets, with reference to Nathanson's classic texts on additive number theory [4, 5]. Letting $\mathbb{P}$ denote the set of prime numbers and letting $\mathbb{P}^C$ denote the complement of this set within $\mathbb{N}_{\geq 2}$, the evaluation of

$$\{p+c : p \in \mathbb{P},\, c \in \mathbb{P}^C,\, p \nmid c\} \tag{1}$$

is a natural problem in additive number theory, in view of the foregoing considerations concerning Zhang's article [8]. Problems in additive number theory often concern the evaluation of sets of integers of the form

$$A+B = \{a+b : a \in A,\, b \in B\}, \tag{2}$$

and this is indicative of how the problem of evaluating (1) is natural as a variant, relative to (2), in view of the additional condition that $p \nmid c$.

The interest in the evaluation of (1) may be considered in relation to one of the most famous unsolved problems in mathematics, namely, the Goldbach conjecture. In view of the notation in (1) and (2), the Goldbach conjecture may be reformulated so as to state that

$$\{p_1+p_2 : p_1, p_2 \in \mathbb{P}\} \tag{3}$$

equals $2\mathbb{N}_{\geq 2}$. The integer set in (1) seems like a natural variant of (3), and since Zhang [8] has shown that $2\mathbb{N}_{\geq 16}$ is contained in (1), this leads us to consider something of a variant of the Goldbach conjecture for odd integers, as in the problem of expressing odd integers in the form indicated in (1).

Let $\omega(n)$ denote the number of distinct prime factors of n. Let $\pi(n)$ denote the prime-counting function giving the number of primes less than or equal to n. Zhang, in 2020 [8], proved that

$$2\pi(n) > \pi(2n) + \omega(2n)$$

for $n \geq 59$, improving upon Rosser and Schoenfeld's inequality whereby $2\pi(n) > \pi(2n)$ for $n > 2$ (see [7]), recalling Landau's 1909 conjecture [3] that the inequality

$$\pi(2n) \leq 2\pi(n)$$

holds for all $n \geq 2$. As in Zhang's work, we are to apply Dusart's inequalities whereby

$$\frac{n}{\log n - 1} < \pi(n) < \frac{n}{\log n - 1.1}$$

for $n \geq 60184$ (see [2]), together with the inequality proved by Robin [6] whereby

$$\omega(n) \leq 1.38403 \frac{\log n}{\log \log n} \tag{4}$$

for $n \geq 3$.

2 Main result

To begin with, we claim that the inequality

$$\frac{2n}{\log(n)-1} - \frac{2n+1}{\log(2n+1)-1.1} \geq \frac{1.38403\log(2n+1)}{\log(\log(2n+1))} \tag{5}$$

holds for $n \geq 60184$. Writing $c = 1.1$, by rewriting the left-hand side of (5) as

$$\frac{2cn - 2n + 2n\log(n) - 2n\log(2n+1) + \log(n) - 1}{(\log(n)-1)(c - \log(2n+1))},$$

we see that this is greater than

$$\frac{-2cn + 2n - 2n\log(n) + 2n\log(2n) - \log(n)}{\log(n)\log(2n+1)}, \tag{6}$$

and that (6) is equivalent to

$$\frac{-2cn + 2n + 2n\log(2) - \log(n)}{\log(n)\log(2n+1)},$$

which is greater than

$$\frac{1.18629n}{\log(n)\log(2n+1)} - \frac{1}{\log(2n+1)}, \tag{7}$$

and (7) is greater than

$$\frac{1.18629n}{\log(n^2)\log(n^2)} - 1 > \frac{0.29n}{\log^2 n} - 1,$$

We claim that

$$\frac{0.29n}{\log^2(n)} - 1 > \frac{0.25n}{\log^2(n)}$$

within the specified range. This is equivalent to

$$0.04n > \log^2(n) \tag{8}$$

holding in the specified range. Rewriting $n = e^{\sqrt{y}}$, the inequality in (8) is equivalent to $0.04e^{\sqrt{y}} > y$. Rewriting $y = z^2$, by the Maclaurin series for the exponential function, we have $0.04e^z > 0.04\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6}\right)$, and we may find that $0.04\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6}\right) > z^2$ holds in the appropriate range by numerically computing the roots of the left-hand side of $0.04\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6}\right) - z^2 > 0$. The right-hand side of (5) is less than

$$\frac{2\log(2n+1)}{\log(\log(2n+1))} < \frac{2\log(n^2)}{\log(\log(n))} < 4\log(n),$$

and we can show that

$$\frac{n}{4\log^2(n)} > 4\log(n)$$

in much the same way that we had proved (8). So, since the left-hand side of (5) is greater than $\frac{n}{4\log^2(n)}$ and since $4\log(n)$ is greater than the right-hand side of (5), we find that (5) holds.

The inequality in (5) allows us to prove that the inequality

$$2\pi(n) - \pi(2n+1) > \omega(2n+1) \tag{9}$$

holds for $n \geq 60184$, in the following way. According to Dusart's inequalities, we have that

$$2\pi(n) - \pi(2n+1) > \frac{2n}{\log(n)-1} - \frac{2n+1}{\log(2n+1)-1.1}$$

for $n \geq 60184$. So, by (5), we have that

$$2\pi(n) - \pi(2n+1) > \frac{1.38403\log(2n+1)}{\log(\log(2n+1))}$$

for $n \geq 60184$. So, Robin's inequality in (4) for odd arguments then gives us the desired inequality in (9). So we may proceed to check that the inequality $2\pi(n) - \pi(2n+1) > \omega(2n+1)$ holds for all integers $n \geq 59$, by checking the finite number of cases for $59 \leq n < 60184$.

Now, mimicking notation from [8], let $k_n = k$ denote the number of primes that are coprime to $2n+1$ and less than or equal to n. Then

$$k = \begin{cases} \pi(n) - \omega(2n+1) & \text{if } 2n+1 \text{ is composite, and} \\ \pi(n) & \text{if } 2n+1 \text{ is prime} \end{cases}$$

for $n \geq 1$ (cf. [8]). To show this, one may argue that either $2n+1$ is a prime, in which case $k = \pi(n)$, by definition, and if $2n+1$ is not a prime, then its least factor is greater than or equal to 3, and hence all of its prime factors are at most n.

Our proof of the following result is largely based on extending Zhang's proof of Theorem 1 [8].

Theorem 2. *Each integer greater than or equal to* 119 *is the sum of a composite number* c *and a prime* p *such that* $p \nmid c$ *(cf.* [8]*).*

Proof. We have shown that

$$\pi(n) - \pi(2n+1) > \omega(2n+1) - \pi(n)$$

holds for all integers $n \geq 59$. So we have that

$$\pi(2n+1) - \pi(n) < \pi(n) - \omega(2n+1) \tag{10}$$

for all integers $n \geq 59$. So both (10) and the inequality

$$\pi(2n+1) - \pi(n) < \pi(n)$$

hold for all integers $n \geq 59$. So, letting $k_n = k$ be as above, we have that

$$\pi(2n+1) - \pi(n) < k \tag{11}$$

for all integers $n \geq 59$. Recall that $k = k_n$ denotes the number of primes that are coprime to $2n + 1$ and less than or equal to n. Adopting notation from Zhang [8], we let these k primes be denoted with $q_1, q_2, \ldots, q_k$. We see that the family

$$\{2n + 1 - q_i : i = 1, 2, \ldots, k\}$$

consists of k distinct numbers. We claim that the implication

$$k > \pi(2n + 1) - \pi(n) \implies \exists i \in \{1, 2, \ldots, k\},\ 2n + 1 - q_i \text{ is composite} \tag{12}$$

holds true. This is equivalent to the statement that

$$k \leq \pi(2n + 1) - \pi(n) \vee \exists i \in \{1, 2, \ldots, k\},\ 2n + 1 - q_i \text{ is composite.}$$

By way of contradiction, suppose that

$$\pi(2n + 1) - \pi(n) < k \wedge \forall i \in \{1, 2, \ldots, k\},\ 2n + 1 - q_i \text{ is prime.}$$

By definition, we have that $q_i \leq n$ for all i. So $2n + 1 - q_i \geq n + 1$. However, this would imply that there would exist k distinct primes in $[n + 1, 2n + 1)$, contradicting the assumption that $\pi(2n + 1) - \pi(n) < k$. So we have shown that the implication displayed in (12) holds true. However, we have shown that (11) holds true for integers $n \geq 59$. So, from the conditional statement shown in (12), we can conclude that, for $n \geq 59$, there is a prime q_i such that $2n + 1 - q_i$ is composite and such that q_i is coprime to $2n + 1$. So, for every odd integer m greater than or equal to 119, the integer m may be written as $p + c$ for a prime p and a composite c, and furthermore, we have that p does not divide c since q_i cannot divide $2n + 1 - q_i$.

So we have shown that the statement given in the theorem under consideration holds for odd integers greater than or equal to 119. So the full statement of this theorem then follows from Zhang's result [8] reproduced as Theorem 1. ∎

As a consequence, we have that each integer greater than or equal to 31 is the sum of a composite number c and a prime p such that $p \nmid c$ (cf. [8]) since we may simply check the finite cases for the integers among $\{118, 117, \ldots, 31\}$, and one may verify that the integer partitions listed as follows satisfy the required conditions:

$$\begin{array}{llll}
118 = 57 + 61, & 117 = 58 + 59, & 116 = 57 + 59, & 115 = 56 + 59, \\
114 = 55 + 59, & 113 = 54 + 59, & 112 = 51 + 61, & 111 = 53 + 58, \\
110 = 53 + 57, & 109 = 53 + 56, & 108 = 53 + 55, & 107 = 53 + 54, \\
106 = 45 + 61, & 105 = 52 + 53, & 104 = 51 + 53, & 103 = 50 + 53, \\
102 = 49 + 53, & 101 = 48 + 53, & 100 = 43 + 57, & 99 = 47 + 52, \\
98 = 47 + 51, & 97 = 47 + 50, & 96 = 47 + 49, & 95 = 47 + 48, \\
94 = 43 + 51, & 93 = 46 + 47, & 92 = 45 + 47, & 91 = 44 + 47, \\
90 = 41 + 49, & 89 = 43 + 46, & 88 = 43 + 45, & 87 = 43 + 44, \\
86 = 41 + 45, & 85 = 42 + 43, & 84 = 29 + 55, & 83 = 41 + 42,
\end{array}$$

$$\begin{array}{llll}
82 = 39 + 43, & 81 = 40 + 41, & 80 = 39 + 41, & 79 = 38 + 41, \\
78 = 35 + 43, & 77 = 37 + 40, & 76 = 37 + 39, & 75 = 37 + 38, \\
74 = 33 + 41, & 73 = 36 + 37, & 72 = 35 + 37, & 71 = 34 + 37, \\
70 = 33 + 37, & 69 = 32 + 37, & 68 = 29 + 39, & 67 = 31 + 36, \\
66 = 31 + 35, & 65 = 31 + 34, & 64 = 31 + 33, & 63 = 31 + 32, \\
62 = 29 + 33, & 61 = 30 + 31, & 60 = 11 + 49, & 59 = 29 + 30, \\
58 = 27 + 31, & 57 = 28 + 29, & 56 = 27 + 29, & 55 = 26 + 29, \\
54 = 25 + 29, & 53 = 24 + 29, & 52 = 21 + 31, & 51 = 23 + 28, \\
50 = 23 + 27, & 49 = 23 + 26, & 48 = 23 + 25, & 47 = 23 + 24, \\
46 = 19 + 27, & 45 = 22 + 23, & 44 = 21 + 23, & 43 = 20 + 23, \\
42 = 17 + 25, & 41 = 19 + 22, & 40 = 19 + 21, & 39 = 19 + 20, \\
38 = 17 + 21, & 37 = 18 + 19, & 36 = 11 + 25, & 35 = 17 + 18, \\
34 = 15 + 19, & 33 = 16 + 17, & 32 = 15 + 17, & 31 = 14 + 17.
\end{array}$$

We may verify that the integer 30 cannot be written in the manner indicated in the above theorem. So 30 is the greatest integer that cannot be written as $p + c$ for a composite c and a prime p such that $p \nmid c$.

3 Conclusion

Since additive number theory broadly refers to the discipline within number theory concerning the partitioning/decomposition of integers using summands of a specified form, this motivates the exploration of variants and generalizations of the problem of expressing natural numbers in the form $p + c$ for $p \in \mathbb{P}$ and $c \in \mathbb{P}^C$ with $p \nmid c$. For $S \subseteq \mathbb{N}$, the set S is said to be a *basis of order h* if each element of $\mathbb{N}$ may be expressed as the sum of h members in S, and such additive bases form a main object of study in additive number theory. This motivates the exploration of higher-order analogues of decompositions of the form $p + c$ that we have specified, i.e., with the use of additional terms. So we are led to consider the exploration as to how the techniques due to Zhang [8] may be extended so as to express integers in the form $p_1 + p_2 + \cdots + p_i + c$ for fixed i and for primes p_j and a composite c such that the primes of the form p_j and c satisfy certain divisibility conditions, as in [8, Corollary 2]. Alternatively, how can we extend the techniques from [8] so as to obtain results on expressing natural numbers in the form $p + c_1 + c_2$ for a prime p and composites c_1 and c_2 satisfying certain divisibility conditions? More generally, how can we express integers in the form

$$p_1 + p_2 + \cdots + p_i + c_1 + c_2 + \cdots + c_j$$

for fixed i and j and for primes p_ℓ and composites c_ℓ satisfying certain divisibility conditions?

Acknowledgements. The author wants to thank Joel E. Cohen for helpful comments concerning this article. The author is also thankful for the referee feedback that has been provided.

References

[1] H. Alzer and M. K. Kwong, On Landau's inequality for the prime counting function, *J. Integer Seq.* **25** (2022), article no. 22.7.3.

[2] P. Dusart, Estimates of some functions over primes without R.H., preprint (2010), arXiv:1002.0442v1.

[3] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen. Vol. 1*, Teubner, Leipzig, 1909.

[4] M. B. Nathanson, *Additive Number Theory: The Classical Bases*, Springer, New York, 1996.

[5] M. B. Nathanson, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, Springer, New York, 1996.

[6] G. Robin, Sur la différence $Li(\theta(x)) - \pi(x)$, *Ann. Fac. Sci. Toulouse Math.* **6** (1984), 257–268.

[7] J. B. Rosser and L. Schoenfeld, Abstracts of scientific communications, *Intern. Congr. Math. Moscow* **3** (1966).

[8] S. Zhang, An improved inequality of Rosser and Schoenfeld and its application, *Integers* **20** (2020), article no. A103.

John M. Campbell
Toronto Metropolitan University
Toronto, Canada
jmaxwellcampbell@gmail.com