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Bridges between three remarkable inequalities

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1 Introduction

Let ABC and $A'B'C'$ be two triangles with areas Δ and Δ', respectively. Let their sides be a, b, c and a', b', c', respectively, and let R be the circumradius of triangle ABC. The aim of this paper is to establish links between the following three remarkable inequalities for triangles.

- *Bottema–Kooi–Schoenberg* (BKS) inequality [4, 15]: for any $x, y, z \in \mathbb{R}$, it holds

$$(x+y+z)^2 R^2 \geqslant yza^2 + zxb^2 + xyc^2. \tag{BKS}$$

- *Oppenheim* (O) inequality: for any $x, y, z \in \mathbb{R}$ with $x+y>0$, $y+z>0$, $z+x>0$ and $xy+yz+zx>0$, it holds

$$xa^2 + yb^2 + zc^2 \geqslant 4\Delta\sqrt{xy+yz+zx}. \tag{O}$$

- *Neuberg–Pedoe* (NP) inequality: for two triangles ABC and $A'B'C'$, it holds

$$a^2(-a'^2+b'^2+c'^2) + b^2(a'^2-b'^2+c'^2) + c^2(a'^2+b'^2-c'^2) \geqslant 16\Delta\Delta'. \tag{NP}$$

In der Dreiecksgeometrie sind die Ungleichungen von Bottema-Kooi-Schoenberg und Oppenheim und die Ungleichung von Neuberg-Pedoe für zwei Dreiecke wohlbekannt. Weniger bekannt ist, dass sie alle zueinander äquivalent sind. In der vorliegenden Arbeit werden die Brücken zwischen den drei Ungleichungen gebaut. Anschliessend beleuchten die Autoren den Zusammenhang mit der Ungleichung von Kooi sowie mit weiteren Ungleichungen.

Remark 1. I. J. Schoenberg (1903–1990), the inventor of splines, proved in [15] for n points $A_1, \dots, A_n$ that lie on a sphere of radius R in $\mathbb{R}^n$ and $t_1, \dots, t_n \in \mathbb{R}$ the general inequality

$$2R^2\left(\sum_{i=1}^{n} t_i\right)^2 \geqslant \sum_{i,j=1}^{n} t_i t_j |A_i A_j|^2.$$

We will emphasize the algebraic character of the inequalities. The next section is needed for what we set out to do.

2 The polar moment of inertia inequality

We use the following theorem.

Theorem 1 (see [1, Corollary 2.4] and [13] for the general case). *Let $t_1, \dots, t_n \in \mathbb{R}$ and $z_1, \dots, z_n \in \mathbb{C}$, $n \geqslant 2$. Then we have*

$$\left|\sum_{i=1}^{n} t_i z_i\right|^2 = \left(\sum_{i=1}^{n} t_i\right)\sum_{i=1}^{n} t_i |z_i|^2 - \sum_{1\leqslant i<j\leqslant n} t_i t_j |z_i - z_j|^2.$$

Remark 2. The case $n = 2$ with $t_1 = t_2 = 1$ is the parallelogram law of Pappus [14],

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2.$$

An obvious corollary is the following theorem.

Theorem 2. *Let $t_1, \dots, t_n \in \mathbb{R}$ and $z_1, \dots, z_n \in \mathbb{C}$, $n \geqslant 2$. Then*

$$\left(\sum_{i=1}^{n} t_i\right)\sum_{i=1}^{n} t_i |z_i|^2 \geqslant \sum_{1\leqslant i<j\leqslant n} t_i t_j |z_i - z_j|^2,$$

with equality if, and only if, $\sum_{i=1}^{n} t_i z_i = 0$.

A consequence of this result is the polar moment of inertia inequality of Klamkin.

Theorem 3 (see Klamkin [3]). *Let $A_1, \dots, A_n$ be points in the plane and let $t_1, \dots, t_n \in \mathbb{R}$. Then, for any point M in the plane, we have*

$$\left(\sum_{i=1}^{n} t_i\right)\sum_{i=1}^{n} t_i |MA_i|^2 \geqslant \sum_{1\leqslant i<j\leqslant n} t_i t_j |A_i A_j|^2, \tag{1}$$

with equality if, and only if, M coincides with the centroid of the set of weighted points $\{A_i, t_i\}$, $i = 1, \dots, n$, that is, $\sum_{i=1}^{n} t_i z_i = 0$.

Proof. Let $m, a_1, \dots, a_n$ be complex numbers corresponding to the points $M, A_1, \dots, A_n$. We put $z_i = a_i - m$ and obtain Klamkin's inequality from Theorem 2. ∎

3 The main results

We start by proving the Bottema–Kooi–Schoenberg inequality (BKS).

Theorem 4 (Bottema–Kooi–Schoenberg inequality (BKS)). *For any triangle ABC and any $x, y, z \in \mathbb{R}$, it holds*

$$R^2(x+y+z)^2 \geqslant yza^2 + zxb^2 + xyc^2.$$

Proof. We put $M = O$ in Klamkin's inequality (1), where O is the circumcenter of the triangle, $A_1 = A$, $A_2 = B$, $A_3 = C$ and $t_1 = x$, $t_2 = y$, $t_3 = z$. Hence inequality (BKS) follows. ∎

Theorem 5. *For any triangle ABC and any $x, y, z, x', y', z' \in \mathbb{R}$, inequality* (BKS),

$$R^2(x+y+z)^2 \geqslant yza^2 + zxb^2 + xyc^2,$$

is equivalent to the inequality

$$(x'a^2 + y'b^2 + z'c^2)^2 \geqslant 16\Delta^2(x'y' + y'z' + z'x'). \tag{$\bar{\mathrm{O}}$}$$

Proof. ($\bar{\mathrm{O}}$) $\Rightarrow$ (BKS): In inequality ($\bar{\mathrm{O}}$), we put $x' := x/a^2$, $y' := y/b^2$, $z' := z/c^2$, and we obtain

$$a^2b^2c^2(x+y+z)^2 \geqslant 16\Delta^2(yza^2 + zxb^2 + xyc^2).$$

By $abc = 4\Delta R$, we get inequality (BKS).

(BKS) $\Rightarrow$ ($\bar{\mathrm{O}}$): Now (BKS) turns to ($\bar{\mathrm{O}}$) by setting $x := x'a^2$, $y := y'b^2$, $z := z'c^2$ in (BKS). We obtain

$$R^2(x'a^2 + y'b^2 + z'c^2)^2 \geqslant a^2b^2c^2(x'y' + y'z' + z'x').$$

Again with $abc = 4\Delta R$, the conclusion follows. ∎

As a consequence of inequality (BKS), we obtain the following theorem.

Theorem 6. *Let $x, y, z \in \mathbb{R}$ with $x + y > 0$ and $xy + yz + zx \geqslant 0$. Then*

$$xa^2 + yb^2 + zc^2 \geqslant 0.$$

Proof. From $x + y > 0$ and $xy + yz + zx \geqslant 0$, it follows that $z \geqslant -\frac{xy}{x+y}$. Hence

$$xa^2 + yb^2 + zc^2 \geqslant xa^2 + yb^2 - \frac{xyc^2}{x+y} = \frac{x(x+y)a^2 + y(x+y)b^2 - xyc^2}{x+y} \geqslant 0.$$

The last inequality $x(x+y)a^2 + y(x+y)b^2 - xyc^2 \geqslant 0$ is obtained when we put $x := y$, $y := x$, $z := -x-y$ in inequality (BKS). ∎

A straightforward consequence of the last result is the following theorem.

Theorem 7. *Bottema–Kooi–Schoenberg inequality* (BKS) *and Oppenheim inequality* (O) *are equivalent for $x + y > 0$, $y + z > 0$, $z + x > 0$ and $xy + yz + zx > 0$.*

Proof. (BKS) $\Rightarrow$ (O): By Theorem 5, from (BKS), inequality ($\bar{\mathrm{O}}$) follows. If $x+y>0$, $y+z>0$, $z+x>0$ and $xy+yz+zx>0$, then it is clear that the Oppenheim inequality (O) follows from ($\bar{\mathrm{O}}$).

(O) $\Rightarrow$ (BKS): This is obvious. ■

It remains to show the equivalence of Oppenheim inequality (O) and Neuberg–Pedoe inequality (NP).

Theorem 8. *Let $x, y, z \in \mathbb{R}$ with $xy + yz + zx \geqslant 0$ and let $x + y > 0$. Then*

$$xa^2 + yb^2 + zc^2 \geqslant 4\Delta\sqrt{xy + yz + zx}.$$

Proof. The proof follows from Theorem 4, Theorem 5 and Theorem 6. ■

Lemma 1. *Let $x, y, z \in \mathbb{R}$ be such that $x+y>0$, $y+z>0$, $z+x>0$. Then $\sqrt{x+y}$, $\sqrt{y+z}$, $\sqrt{z+x}$ are sides of a triangle if, and only if, $xy+yz+zx>0$.*

Proof. Let $a = \sqrt{y+z}$, $b = \sqrt{z+x}$, $c = \sqrt{x+y}$ be the sides of a triangle. Then, by the equivalent form of Heron's formula,

$$\begin{aligned}16\Delta^2 &= 2\sum a^2b^2 - \sum a^4\\ &= 2\sum (y+z)(z+x) - \sum (y+z)^2\\ &= 4(xy+yz+zx).\end{aligned}$$

Hence $xy + yz + zx > 0$.

Now let $xy+yz+zx>0$. We put $u = \sqrt{x+y}$, $v = \sqrt{y+z}$, $w = \sqrt{z+x}$. Then

$$\begin{aligned}4(xy+yz+zx) &= \sum (y+z)(z+x) - \sum (y+z)^2\\ &= 2\sum u^2v^2 - \sum u^4\\ &= (u+v+w)(-u+v+w)(u-v+w)(u+v-w) > 0.\end{aligned}$$

Let $U = -u+v+w$, $V = u-v+w$, $W = u+v-w$. The product

$$(-u+v+w)(u-v+w)(u+v-w)$$

is positive when either all three factors U, V, W are positive, in which case u, v, w are sides of a triangle, or two factors are negative and one is positive. The last case is impossible because if, for example, $U<0$, $V<0$, then $2w = U+V<0$, and that is a contradiction. ■

Remark 3. From the proof above, we see that a triangle with sides $\sqrt{x+y}$, $\sqrt{y+z}$, $\sqrt{z+x}$ has area $\Delta = \sqrt{xy+yz+zx}/2$.

Theorem 9. *Let $x, y, z \in \mathbb{R}$ be such that $xy+yz+zx>0$. Then*

$$xa^2 + yb^2 + zc^2 \geqslant 4\Delta\sqrt{xy + yz + zx}$$

for any triangle ABC if, and only if, $x+y>0$, $y+z>0$, $z+x>0$.

Proof. If $x + y > 0$, $y + z > 0$, $z + x > 0$, then the inequality is the Oppenheim inequality (O). Let the inequality hold for all triangles ABC and let $b = c = 1$, $a = 2\varepsilon$, $\varepsilon \in (0, 1)$. Then the inequality becomes

$$x \cdot 4\varepsilon^2 + y + z \geqslant 4\varepsilon\sqrt{1-\varepsilon^2}\sqrt{xy + yz + zx}.$$

Letting $\varepsilon \to 0$, we see that $y + z \geqslant 0$. Similarly, $z + x \geqslant 0$, $x + y \geqslant 0$. The case of equality is excluded since if, for example, $x + y = 0$, then $xy + yz + zx = -x^2 \leqslant 0$, a contradiction. Hence $x + y > 0$, $y + z > 0$, $z + x > 0$. ∎

Theorem 10. *Oppenheim inequality* (O) *and Neuberg–Pedoe inequality* (NP) *are equivalent.*

Proof. (O) ⇒ (NP): In the Oppenheim inequality (O), we put

$$x = -a'^2 + b'^2 + c'^2, \quad y = a'^2 - b'^2 + c'^2, \quad z = a'^2 + b'^2 - c'^2$$

and use the equivalent form of Heron's formula.

(NP) ⇒ (O): We use Lemma 1 and put $a' = \sqrt{y+z}$, $b' = \sqrt{z+x}$, $c' = \sqrt{x+y}$ for the sides of the triangle. Then, from (NP) and Remark 3,

$$2(xa^2 + yb^2 + zc^2) \geqslant 16\Delta\sqrt{xy + yz + zx}/2$$

follows, which is the Oppenheim inequality (O). ∎

4 Relation to Kooi's and other inequalities

The interested readers may wonder if the familiar Kooi inequality (see [9, 11])

$$s^2 \leqslant \frac{R(4R+r)^2}{2(2R-r)}, \tag{2}$$

with s the semiperimeter, r the inradius of the triangle, and inequality (BKS) are related. Indeed, they are, and we can derive Kooi's inequality from (BKS). We put $x = a(s-a)$, $y = b(s-b)$, $z = c(s-c)$ in (BKS) and use the identities $\sum a(s-a) = 2r(4R+r)$ and $\sum a(s-b)(s-c) = 2rs(2R-r)$ which follow from the well-known relations

$$a^2 + b^2 + c^2 = 2(s^2 - 4Rr - r^2) \quad \text{and} \quad ab + bc + ca = s^2 + 4Rr + r^2.$$

Kooi's inequality (2) has many applications in triangle geometry. It implies the important Gerretsen inequality (see [6])

$$s^2 \leqslant 4R^2 + 4Rr + 3r^2, \tag{3}$$

which, for example, can be used for a short proof of the inequality $OH \geqslant OI$, where O, H and I are the circumcenter, orthocenter and the incenter of the triangle. That inequality is crucial in the proof of a conjectured inequality for the altitudes of the excentral triangle; see [16].

Remark 4. The referee brought to our attention the famous article by Euler [2], where the master computed the distances between the four classical triangle centers. The Gerretsen inequality is then a simple corollary of

$$HI^2 = 4R^2 + 4Rr + 3r^2 - s^2.$$

In addition, the referee gave the equality

$$OH^2 - OI^2 - 2HI^2 = 2Rr - 4r^2 \geqslant 0,$$

which follows from the equation $OH^2 = 9R^2 + 8Rr + 2r^2 - 2s^2$ and Euler's formula $OI^2 = R^2 - 2Rr$. As a consequence, we have $OH \geqslant OI$.

Gerretsen's inequality (3) implies the celebrated Finsler–Hadwiger inequality (see [12, 17])

$$a^2 + b^2 + c^2 \geqslant 4\Delta\sqrt{3} + (a-b)^2 + (b-c)^2 + (c-a)^2. \tag{4}$$

Kooi's inequality (2) is actually equivalent to the sharper version of the Finsler–Hadwiger inequality by Euler's $R \geqslant 2r$ (see [11]),

$$a^2 + b^2 + c^2 \geqslant 4\Delta\sqrt{3 + \frac{R-2r}{R}} + (a-b)^2 + (b-c)^2 + (c-a)^2.$$

Even stronger inequalities than Kooi's and Gerretsen's can be derived from the fundamental triangle inequality (see [10])

$$\begin{aligned} &2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R(R-2r)} \\ &\qquad \leqslant s^2 \leqslant 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)}. \end{aligned}$$

For the end, we note the curiosity that the seemingly weaker Weitzenböck inequality (see [18])

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}\Delta,$$

which follows from the Oppenheim inequality (O) for $x = y = z = 1$, is actually equivalent to the Finsler–Hadwiger inequality (4) [7, 8]. For the excentral and circummidarc triangles used in the derivations, see [5].

To conclude, the Bottema–Kooi–Schoenberg inequality and its two avatars – the Oppenheim and Neuberg–Pedoe inequalities – put their stamp on everything.

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