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On M. Sato's Classification of Some Reductive Prehomogeneous Vector Spaces

Dedicated to Professor Mikio Sato with our deepest admiration

by

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Abstract

Under some condition, M. Sato classified reductive prehomogeneous vector spaces of the form $(G_0 \times G, \Lambda_1 \otimes \rho, V(n) \otimes V)$. In this paper, under another condition, we classify the prehomogeneous vector spaces of the same form. We consider everything over the complex number field $\mathbb{C}$.

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Introduction

For the definition and basic properties of prehomogeneous vector spaces (abbrev. PV), see [K2]. Although the classification of irreducible PV's has been completed in [SK], to classify all the non-irreducible reductive PV's still looks almost impossible.

In the 1960s, Professor Mikio Sato considered the reductive PV's of the form $(G_0 \times G, \Lambda_1 \otimes \rho, V(n) \otimes V)$ with a connected semisimple subgroup G_0 of $SL(n)$. Although this form looks special, we show later that any reductive triplet with a scalar multiplication is PV-equivalent to a triplet of this form. Here $\rho : G \to GL(V)$ is a d-dimensional representation of a connected reductive algebraic group G. Then we have $\rho = \rho_1 + \cdots + \rho_m$ and $V = V_1 + \cdots + V_m$ where $\rho_\mu : G \to GL(V_\mu)$ is an irreducible representation $(1 \leq \mu \leq m)$. For each μ, we have $V_\mu = V_{\mu 1} \otimes \cdots \otimes V_{\mu k_\mu}$ where some simple component of G acts on $V_{\mu\nu}$ irreducibly. Put $d_\mu = \dim V_\mu$

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and $d_{\mu\nu} = \dim V_{\mu\nu}$. Then we have $d = d_1 + \cdots + d_m$ and $d_\mu = d_{\mu 1} \cdots d_{\mu k_\mu}$. Here if $d_\mu = 1$, we put $k_\mu = 0$. If $d_\mu \geq 2$, we have $k_\mu \geq 1$ and we may assume $d_{\mu\nu} \geq 2$ $(1 \leq \nu \leq k_\mu)$. Now put $\delta = \max\{d_{\mu\nu}\}$. We may assume that $\delta = d_{11}$ by renumbering if necessary. Then $k_1 = 0$ implies that $\delta = 1$.

Professor Mikio Sato proved that if $(G_0 \times G, \Lambda_1 \otimes \rho, V(n) \otimes V)$ is a PV for $\delta \leq n \leq d - \delta$, then k_1 must be one of $0, 1, 2$, and classified such PV's when $k_1 = 2$ as follows. Here and throughout, to simplify notation, we write

$$G^{\times(m)} \quad \text{instead of} \quad \overbrace{G \times \cdots \times G}^{m},$$

and similarly for other binary operations in place of $\times$.

Theorem 0.1 (M. Sato). *Assume that $(G_0 \times G, \Lambda_1 \otimes \rho, V(n) \otimes V)$ is a PV with $\delta \leq n \leq d - \delta$ and $k_1 = 2$. Then it is one of the following regular PV's.*

(i) $(SL(n) \times ((GL(2) \times SL(2)) \times GL(2)^{\times(m-1)}), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1^{\boxplus(m-1)}), V(n) \otimes (V(2) \otimes V(2) + V(2)^{+(m-1)}))$ *with* $m \geq 1$ *and* $n = 2$ *or* $n = 2m$ $(= d - 2)$.

(ii) $(SL(n) \times ((GL(3) \times SL(2)) \times GL(3)^{\times(m-1)}), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1^{\boxplus(m-1)}), V(n) \otimes (V(3) \otimes V(2) + V(3)^{+(m-1)}))$ *with* $m \geq 1$ *and* $n = 3$ *or* $n = 3m$ $(= d - 3)$.

(iii) $(SL(3) \times ((GL(2) \times SL(2)) \times GL(2)), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1), V(3) \otimes (V(2) \otimes V(2) + V(2)))$.

(iv) $(SL(n) \times ((GL(3) \times SL(2)) \times GL(3)), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1), V(n) \otimes (V(3) \otimes V(2) + V(3)))$ $(n = 4, 5)$.

(v) $(SL(n) \times ((GL(3) \times SL(2)) \times GL(k) \times GL(3)^{\times(m-2)}), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1 \boxplus \Lambda_1^{\boxplus(m-2)}), V(n) \otimes (V(3) \otimes V(2) + V(k) + V(3)^{+(m-2)}))$ *with* $m \geq 2$*;* $n = 3$ *or* $n = k + 3m - 3$ $(= d - 3)$*;* $k = 1$ *or* 2.

(vi) $(SL(n) \times ((GL(2) \times SL(2)) \times GL(1) \times GL(2)^{\times(m-2)}), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1 \boxplus \Lambda_1^{\boxplus(m-2)}), V(n) \otimes (V(2) \otimes V(2) + V(1) + V(2)^{+(m-2)}))$ *with* $m \geq 2$*; and* $n = 2$ *or* $n = 2m - 1$ $(= d - 2)$.

Proof. See p. 239 in [K1]. □

It is easy to see that when $k_1 = 0$, only the triplet $(SL(m-1) \times (GL(1)^{\times(m)}), \Lambda_1 \otimes (\Lambda_1^{\boxplus(m)}), V(m-1) \otimes (V(1)^{+(m)}))$ with $m \geq 3$ is a PV.

Hence we shall consider the remaining case $k_1 = 1$ which implies that $(G, \rho_1, V_1) = ((GL(1) \times) G_s, (\Lambda_1 \otimes) \sigma, V(\delta))$ where G_s is a simple algebraic group.

First, we give a complete classification of these PV's when G_s is an exceptional simple algebraic group. Our first main result is as follows.

Theorem 0.2. *Assume that* $\mathbf{T} := (G_0 \times G, \Lambda_1 \otimes \rho, V(n) \otimes V)$ *with* $k_1 = 1$ *is a PV with* $(G, \rho_1, V_1) = ((GL(1)\times)G_s, (\Lambda_1\otimes)\sigma, V(\delta))$ *where* G_s *is an exceptional simple algebraic group. Then* $\mathbf{T}$ *is one of the following regular PV's.*

(i) $(SL(n) \times ((GL(1) \times G_s) \times GL(\delta)^{\times(m-1)}), \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) \boxplus \Lambda_1^{\boxplus(m-1)}), V(n) \otimes (V(\delta) + V(\delta)^{+(m-1)}))$ *with* $m \geq 2$*;* $n = \delta$ *or* $n = (m-1)\delta$ *where* σ *is any irreducible representation of* G_s *with* $\deg \sigma = \delta$*.*

(ii) $(SL(n) \times ((GL(1) \times (G_2)) \times GL(t) \times GL(7)^{\times(m-2)}), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_2) \boxplus \Lambda_1 \boxplus \Lambda_1^{\boxplus(m-2)}), V(n) \otimes (V(7) + V(t) + V(7)^{+(m-2)})\ (m \geq 3))$ *with* $t = 1, 2, 5, 6$ *where* $n = 7$ *or* $n = t + 7(m-2)$*.*

(iii) $(SL(n) \times ((GL(1) \times E_6) \times GL(t) \times GL(27)^{\times(m-2)}), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1 \boxplus \Lambda_1^{\boxplus(m-2)}), V(n) \otimes (V(27) + V(t) + V(27)^{+(m-2)}))\ (m \geq 3)$ *with* $t = 1, 2, 25, 26$ *where* $n = 27$ *or* $n = t + 27(m-2)$*.*

(iv) $(SL(n) \times ((GL(1) \times E_7) \times GL(t) \times GL(56)^{\times(m-2)}), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_6) \boxplus \Lambda_1 \boxplus \Lambda_1^{\boxplus(m-2)}), V(n) \otimes (V(56) + V(t) + V(56)^{+(m-2)}))\ (m \geq 3)$ *with* $t = 1, 55$ *where* $n = 56$ *or* $n = t + 56(m-2)$*.*

The proof of Theorem 0.2 will be given in Section 2. Secondly we give a classification of the extreme case $n = \delta$ or $n = d - \delta$ when G_s is a classical simple algebraic group with $G_s \neq SL(\delta)$. This restriction is in a sense natural because the case $G_s = SL(\delta)$ contains all reductive PV's with a scalar multiplication. Actually a triplet $(GL(1) \times H,\ \Lambda_1 \otimes \sigma, V(k))$ with $k \geq 3$ is a PV if and only if $(G_0 \times G, \Lambda_1 \otimes \rho, V(n) \otimes V)$ is a PV where $G_0 = SL(n)$ with $n = \delta = k(k-1) - 1 < d - \delta = k(k-1)$ and $G = (GL(1) \times SL(\delta)) \times (H \times GL(k-1)), \rho = (\Lambda_1 \otimes \Lambda_1) \otimes (1 \otimes 1) + (1 \otimes 1) \otimes (\sigma \otimes \Lambda_1)$.

Our second main result is as follows.

Theorem 0.3. *Assume that* $\mathbf{T} := (G_0 \times G, \Lambda_1 \otimes (\rho_1 + \cdots + \rho_m), V(n) \otimes (V(\delta) + V(d_2) + \cdots + V(d_m))$ *with* $n = \delta$ *or* $n = d - \delta = d_2 + \cdots + d_m$ *is a PV where* $(G, \rho_1, V(\delta)) = (GL(1) \times G_s, \Lambda_1 \otimes \sigma, V(\delta))\ (\neq (GL(\delta), \Lambda_1, V(\delta)))$ *with a classical simple algebraic group* G_s *and each* $V(d_\mu)$ *has an independent scalar multiplication. Then* $\mathbf{T}$ *is one of the following PV's.*

(I) *Regular PV's:*

(i) $(SL(n) \times ((GL(1) \times G_s) \times GL(\delta)^{\times(m-1)}), \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) \boxplus \Lambda_1^{\boxplus(m-1)}), V(n) \otimes (V(\delta) + V(\delta)^{+(m-1)}))$ *with* $m \geq 2$*;* $n = \delta$ *or* $n = (m-1)\delta$ *where* σ *is any irreducible representation of* G_s *with* $\deg \sigma = \delta$*.*

(ii) $(SL(n) \times ((GL(1) \times G_s) \times (GL(1) \times T_s) \times GL(\delta)^{\times(m-2)}), \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) \boxplus (\Lambda_1 \otimes \tau) \boxplus \Lambda_1^{\boxplus(m-2)}), V(n) \otimes (V(\delta) + V(t) + V(\delta)^{+(m-2)}))$ *with* $m \geq 3$*,* $\delta > t \geq 1$*;* $n = \delta$ *or* $n = t + (m-2)\delta$ *where* T_s *is a simple algebraic group such that*

$(GL(1)\times G_s\times T_s, \Lambda_1\otimes\sigma\otimes\tau, V(1)\otimes V(\delta)\otimes V(t))$ *is a non-trivial irreducible regular* 2*-simple PV.*

(iii) $(SL(n)\times((GL(1)\times Sp(t))\times GL(u)\times GL(v)\times GL(2t)^{\times(m-3)}), \Lambda_1\otimes((\Lambda_1\otimes\Lambda_1)\boxplus\Lambda_1\boxplus\Lambda_1\boxplus\Lambda_1^{\boxplus(m-3)}), V(n)\otimes(V(2t)+V(u)+V(v)+V(2t)^{+(m-3)}))$ *with* $t\geq 2$*;* $n=2t$ *or* $n=u+v+2t(m-3)$ *where* $(u,v)=(1,1),(1,k)$ *with* $m\geq 4$*, or* $(u,v)=(1,2t-1),(2t-1,2t-1),(k,2t-1)$ *with* $m\geq 3$*. Here* k *is an odd integer satisfying* $3\leq k\leq 2t-3$*.*

(iv) $(SL(n)\times((GL(1)\times Spin(10))\times GL(u)\times GL(u)\times GL(16)^{\times(m-3)}), \Lambda_1\otimes((\Lambda_1\otimes\text{a half-spin rep.})\boxplus\Lambda_1\boxplus\Lambda_1\boxplus\Lambda_1^{\boxplus(m-3)}), V(n)\otimes(V(16)+V(u)+V(u)+V(16)^{+(m-3)}))$ *with* $n=16$ *or* $n=2u+16(m-3)$ *where* ($u=1$ *and* $m\geq 4$) *or* ($u=15$ *and* $m\geq 3$)*.*

(v) $((SL(2t-1)\times SL(1))\times((GL(1)\times Sp(t))\times GL(2t)^{\times(m-1)}), (\Lambda_1\boxplus\Lambda_1)\otimes((\Lambda_1\otimes\Lambda_1)\boxplus\Lambda_1^{\boxplus(m-1)}), (V(2t-1)+V(1))\otimes(V(2t)+V(2t)^{+(m-1)}))$ *with* $t\geq 2$ *and* $m\geq 2$*.*

(II) *Non-regular PV's:*

(i) $(SL(n)\times((GL(1)\times G_s)\times(GL(1)\times T_s)\times GL(\delta)^{\times(m-2)}), \Lambda_1\otimes((\Lambda_1\otimes\sigma)\boxplus(\Lambda_1\otimes\tau)\boxplus\Lambda_1^{\boxplus(m-2)}), V(n)\otimes(V(\delta)+V(t)+V(\delta)^{+(m-2)}))$ *with* $m\geq 3$*,* $\delta>t\geq 1$*;* $n=\delta$ *or* $n=t+(m-2)\delta$ *where* T_s *is a simple algebraic group such that* $(GL(1)\times G_s\times T_s, \Lambda_1\otimes\sigma\otimes\tau, V(1)\otimes V(\delta)\otimes V(t))$ *is a non-trivial irreducible non-regular* 2*-simple PV.*

(ii) $(SL(n)\times((GL(1)\times Sp(t))\times GL(2r)\times GL(2k+1)), \Lambda_1\otimes((\Lambda_1\otimes\Lambda_1)\boxplus\Lambda_1\boxplus\Lambda_1), V(n)\otimes(V(2t)+V(2r)+V(2k+1)))$ *with* $t\geq 2$*;* $2t-2\geq 2r\geq 2$*;* $2t-1\geq 2k+1\geq 1$*;* $2r+2k+1>2t$*;* $n=2t$ *or* $n=2r+2k+1$*.*

(iii) $(SL(n)\times((GL(1)\times Sp(t))\times GL(2r)\times GL(2k+1)\times GL(2t)^{\times(m-3)}), \Lambda_1\otimes((\Lambda_1\otimes\Lambda_1)\boxplus\Lambda_1\boxplus\Lambda_1\boxplus\Lambda_1^{\boxplus(m-3)}), V(n)\otimes(V(2t)+V(2r)+V(2k+1)+V(2t)^{+(m-3)}))$ *with* $t\geq 2$*;* $m\geq 4$*;* $2t-2\geq 2r\geq 2$*;* $2t-1\geq 2k+1\geq 1$*;* $n=2t$ *or* $n=2r+2k+1+2t(m-3)$*.*

(iv) $(SL(n)\times((GL(1)\times Sp(t))\times GL(u)\times GL(v)\times GL(w)\times GL(2t)^{\times(m-4)}), \Lambda_1\otimes((\Lambda_1\otimes\Lambda_1)\boxplus\Lambda_1\boxplus\Lambda_1\boxplus\Lambda_1\boxplus\Lambda_1^{\boxplus(m-4)}), V(n)\otimes(V(2t)+V(u)+V(v)+V(w)+V(2t)^{+(m-4)}))$ *with* $t\geq 2$*;* $n=2t$ *or* $n=u+v+w+2t(m-4)$ *where* $(u,v,w)=(1,1,1),(1,1,k))$ *with* $m\geq 5$*, or* $(u,v,w)=(1,1,2t-1),(1,2t-1,2t-1),(2t-1,2t-1,2t-1),(1,k,2t-1),(k,2t-1,2t-1))$ *with* $m\geq 4$*. Here* k *is an odd integer satisfying* $3\leq k\leq 2t-3$*.*

(v) $(SL(n)\times((GL(1)\times SL(2t+1))\times GL(u)\times GL(u)\times GL(\delta)^{\times(m-3)}), \Lambda_1\otimes((\Lambda_1\otimes\Lambda_2)\boxplus\Lambda_1\boxplus\Lambda_1\boxplus\Lambda_1^{\boxplus(m-3)}), V(n)\otimes(V(\delta)+V(u)+V(u)+V(\delta)^{+(m-3)}))$ *with* $\delta=t(2t+1)$*,* $t\geq 2$*;* $n=\delta=t(2t+1)$ *or* $n=2u+t(2t+1)(m-3)$ *where* ($u=1$ *and* $m\geq 4$) *or* ($u=\delta-1=t(2t+1)-1$ *and* $m\geq 3$)*.*

The proof of Theorem 0.3 will be given in Section 3.

Remark 0.4. In the case of exceptional simple algebraic groups, as we see in Theorem 0.2, there does not exist a non-extreme PV. However in the classical case, there exist non-extreme PV's. See Proposition 3.10.

Remark 0.5. There are other methods of classification of PV's. See [Kac] and [R].

Notation. We denote by $M(m,n)$ (resp. $M(n)$) the totality of $m \times n$ (resp. $n \times n$) matrices. For the classical algebraic groups, we denote by $GL(n)$ (resp. $SL(n)$, $Sp(n), SO(n), Spin(n)$) the general linear group (resp. the special linear group, the symplectic group, the special orthogonal group, the spin group).

The exceptional simple algebraic group of rank 2 is denoted by (G_2) instead of G_2 to distinguish it from the second group in G_i $(i = 1, \dots, m)$. We denote by E_i (resp. F_4) the exceptional simple algebraic group of rank i $(6 \leq i \leq 8)$ (resp. 4).

Now for the exceptional simple algebraic group $G_s = (G_2)$ (resp. F_4, E_6, E_7, E_8), we denote its least representation degree (resp. its next least representation degree) by δ_0 (resp. δ_1). Then we have $\delta_0 = 7$ (resp. $26, 27, 56, 248$) and $\delta_1 = 14$ (resp. $52, 78, 133, 3875$). Since $\dim G_s = 14$ (resp. $52, 78, 133, 248$), we have $\dim G_s \leq \delta_1$.

We denote by Λ_1 the standard representation of $GL(n)$ on $\mathbb{C}^n$. For a subgroup H of $GL(n)$, the restriction $\Lambda_1|_H$ (= the inclusion $H \hookrightarrow GL(n)$) is also simply denoted by Λ_1. More generally, Λ_k $(k = 1, \dots, r)$ denotes the fundamental irreducible representation of a simple algebraic group of rank r.

Since $\otimes$ and $\oplus$ are sometimes difficult to distinguish, we use the notation $+$ for the direct sum $\oplus$. Let $\rho_i : G_i \to GL_{m_i}$ be a rational representation of an algebraic group G_i $(i = 1, \dots, m)$. Then we denote the representation $\rho = (\rho_1 \otimes 1 \otimes \cdots \otimes 1) + \cdots + (1 \otimes \cdots \otimes 1 \otimes \rho_m)$ of $G_1 \times \cdots \times G_m$ by $\rho_1 \boxplus \cdots \boxplus \rho_m$.

In general, we denote by ρ^* the dual representation of a rational representation ρ. We denote by $V(n)$ an n-dimensional vector space in general. If $V(n)$ and $V(n)^*$ appear at the same time, $V(n)^*$ denotes the dual space of $V(n)$.

§1. Preliminaries

Let $(G_0 \times G, \Lambda_1 \otimes \rho, V(n) \otimes V)$ be as in Introduction where $\rho = \rho_1 + \cdots + \rho_m$ and $V = V_1 + \cdots + V_m$. Let G_μ be the image of $\rho_\mu : G \to GL(V_\mu)$. Recall that $k_\mu = 0$ implies $G_\mu = GL(1)$ and $d_\mu = 1$. Since $G \to G_1 \times \cdots \times G_m$ is injective, we have

$$\dim G \leq \dim G_1 + \cdots + \dim G_m.$$

If $(G_0 \times G, \Lambda_1 \otimes \rho, V(n) \otimes V)$ is a PV, we have $\dim G_0 + \dim G \geq nd$ (see p. 41 in [SK]). Since $n^2 - 1 \geq \dim G_0$, we have $\dim G_1 + \cdots + \dim G_m \geq n(d - n) + 1$.

If $\delta \leq n \leq d-\delta$, we have $n(d-n)-\delta(d-\delta)=(d-\delta-n)(n-\delta)\geq 0$. Hence $\dim G_1+\cdots+\dim G_m \geq \delta(d-\delta)+1$ for $\delta \leq n \leq d-\delta$. We can express this as

$$0 \geq N_1+\cdots+N_m \quad (\delta \leq n \leq d-\delta) \tag{1}$$

where $N_1=-\dim G_1+\delta(d_1-\delta)+1$ and $N_\mu=-\dim G_\mu+\delta d_\mu$ $(2\leq \mu \leq m)$.

In particular, if $k_1=1$ and $G_1=GL(1)\times G_s$ with a simple algebraic group G_s, we have

$$\dim G_s \geq N_2+\cdots+N_m \quad (\delta \leq n \leq d-\delta) \tag{2}$$

In the case $\dim G_0 \leq \frac{1}{2}n(n\pm 1)$, put $N_1^{\pm}=-\dim G_1+\delta(d_1-\delta)+\frac{1}{2}\delta(\delta\mp 1)$. When $\delta \leq n \leq 2d-\delta\mp 1$, we can see similarly that

$$0 \geq N_1^{\pm}+\cdots+N_m \quad (\delta \leq n \leq 2d-\delta\mp 1). \tag{3}$$

Lemma 1.1. *For $2\leq \mu \leq m$, we have $N_\mu \geq 0$.*

Proof. If $k_\mu=0$, we have $N_\mu=-1+\delta\geq 0$. If $k_\mu\geq 1$, we may assume that $d_{\mu 1}\geq d_{\mu\nu}\geq 2$ $(2\leq \mu\leq m)$. Then $\dim G_\mu \leq 1+(d_{\mu 1}^2-1)+\cdots+(d_{\mu k_\mu}^2-1)\leq 1+k_\mu(d_{\mu 1}^2-1)$ and $d_\mu \geq d_{\mu 1}2^{k_\mu-1}$, and hence $N_\mu=-\dim G_\mu+\delta d_\mu \geq (2^{k_\mu-1}-k_\mu)d_{\mu 1}^2+(k_\mu-1)$ ≥ 0. □

Proposition 1.2 (M. Sato). *Assume that $(G_0\times G, \Lambda_1\otimes\rho, V(n)\otimes V)$ is a PV. Then we have the following assertions.*

(i) *If $\delta\leq n\leq d-\delta$, then $k_1\leq 2$.*
(ii) *If $\delta\leq n\leq 2d-\delta-1$ and $\dim G_0\leq \frac{1}{2}n(n+1)$, then $k_1\leq 2$.*
(iii) *If $\delta\leq n\leq 2d-\delta+1$ and $\dim G_0\leq \frac{1}{2}n(n-1)$, then $k_1\leq 2$.*

Proof. Under these conditions on n, by (1), (3) and Lemma 1.1, we have $0\geq N_1^-\geq N_1^+\geq N_1=-\dim G_1+\delta(d_1-\delta)+1\geq -k_1(\delta^2-1)+\delta^2(2^{k_1-1}-1)=(2^{k_1-1}-1-k_1)\delta^2+k_1$. Hence $2^{k_1-1}-1-k_1<0$ and $k_1\leq 2$. □

In the case $\delta\leq n\leq d-\delta$ and $k_1=2$, the classification has been completed by M. Sato (see Theorem 0.1). We shall consider the case $\delta\leq n\leq d-\delta$ and $k_1=1$.

Proposition 1.3. (i) *For any $\sigma: H\to GL(V)$ and any $n\geq \deg\sigma=\dim V$, a triplet $(H\times GL(n), \sigma\otimes\Lambda_1, V\otimes V(n))$ is a PV. Such a PV is called trivial.*

(ii) *For any σ and any $n\geq m=\deg\sigma$, a triplet (G,ρ,V) is a PV if and only if $(G\times GL(n), \rho\otimes 1+\sigma\otimes\Lambda_1, V+V(m)\otimes V(n))$ is a PV.*

(iii) *For any $(G,\rho,V(n))$ and $(H,\sigma,V(m))$, a triplet $(GL(n)\times(G\times H), \Lambda_1\otimes(\rho\boxplus\sigma), V(n)\otimes(V(n)+V(m)))$ is a PV if and only if $(G\times H, \rho^*\otimes\sigma, V(n)^*\otimes V(m))$ is a PV. Moreover if G is reductive, then it is a PV if and only if $(G\times H, \rho\otimes\sigma, V(n)\otimes V(m))$ is a PV.*

Proof. For (i), see p. 43 in [SK]; and (ii) is obvious from (i). Since the $GL(n)$-part of the generic isotropy subgroup at I_n of $(GL(n) \times G, \Lambda_1 \otimes \rho, M(n))$ is $\rho^*(G)$, we have (iii). □

Proposition 1.4. *Let G be a reductive algebraic subgroup of $GL(n)$.*

(i) *If $G \neq GL(n), SL(n)$ and the inclusion $\Lambda_1 : G \hookrightarrow GL(n)$ is an irreducible representation, then* $\dim G \leq 1 + \frac{1}{2}n(n+1)$ $(= \dim(GL(1) \times Sp(n'))$ *with* $n = 2n')$. *Moreover if n is an odd integer, then* $\dim G \leq 1 + \frac{1}{2}n(n-1)$ $(= \dim(GL(1) \times SO(n)))$.

(ii) *If G_0 is a semisimple algebraic proper subgroup of $SL(n)$ and the inclusion $\Lambda_1 : G_0 \hookrightarrow SL(n)$ is an irreducible representation, then* $\dim G_0 \leq \frac{1}{2}n(n+1)$.

(iii) *If $G \neq GL(n)$ and the inclusion $\Lambda_1 : G \hookrightarrow GL(n)$ is not irreducible, then* $\dim G \leq (n-1)^2 + 1$.

Proof. For (i) and (ii), see Lemma 17, p. 52 in [SK]. For (iii), we may assume that $G \subset GL(k) \times GL(n-k)$ $(1 \leq k \leq n-1)$, and hence $\dim G \leq k^2 + (n-k)^2 \leq 1 + (n-1)^2$. □

Proposition 1.5. *For $d_2 + d_3 \geq n > d_2, d_3$ and $n \geq \delta$, the following conditions are equivalent.*

(i) $(GL(n) \times (H \times GL(d_2) \times GL(d_3)), \Lambda_1 \otimes (\sigma \boxplus \Lambda_1 \boxplus \Lambda_1), V(n) \otimes (V(\delta) + V(d_2) + V(d_3)))$ *is a PV.*

(ii) $(H \times (GL(n-d_2) \times GL(n-d_3)), \sigma \otimes (\Lambda_1 \boxplus \Lambda_1), V(\delta) \otimes (V(n-d_2) + V(n-d_3)))$ *is a PV.*

Proof. Since $\delta \leq n$ and $(n-d_2)+(n-d_3) \leq n$, we obtain this result by Theorem 7.8 in [K2]. □

Proposition 1.6. *If $(G \times GL(n), \rho \otimes \Lambda_1, V(\delta) \otimes V(n))$ with $\delta > n \geq 1$ is an irreducible PV, then $(G \times GL(1), \rho \otimes \Lambda_1, V(\delta) \otimes V(1))$ is also a PV. However if the former triplet is not irreducible, this conclusion does not hold in general.*

Proof. See Proposition 3.2 in [KTK]. A counterexample in the non-irreducible case is given by Remark 3.3 in [KTK]. □

Proposition 1.7. *Let G_s be an exceptional simple algebraic group.*

(i) $\mathbf{T}_1 := (SO(n) \times (GL(m_1) \times GL(m_2)), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1), V(n) \otimes (V(m_1) + V(m_2)))$ $(n > m_1 \geq m_2 \geq 1)$ *is a non-PV.*

(ii) $\mathbf{T}_2 := (G_s \times (GL(r) \times GL(s)), \sigma \otimes (\Lambda_1 \boxplus \Lambda_1), V(\delta) \otimes (V(r) + V(s)))$ *is a non-PV for $\delta > r \geq s \geq 1$.*

(iii) *For a semisimple algebraic subgroup* $G_0 \subsetneq SL(\delta)$*, a triplet* $(G_0 \times (GL(1) \times G_s), \Lambda_1 \otimes (\Lambda_1 \otimes \sigma), V(\delta) \otimes V(\delta))$ *is a non-PV.*

Proof. The triplet $\mathbf{T}_1$ is PV-equivalent to $((SO(m_1) \times SO(n-m_1)) \times GL(m_2), (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1, (V(m_1) + V(n-m_1)) \otimes V(m_2))$ (see p. 109 in [SK]). Note that we may assume $n/2 > m_1 \geq m_2 \geq 1$ by castling transformation. Thus $\mathbf{T}_1$ is PV-equivalent to $(SO(n-m_1) \times SO(m_2), \Lambda_1 \otimes \Lambda_1, V(n-m_1) \otimes V(m_2))$, which is a non-PV (see p. 53 in [SK]).

If the triplet $\mathbf{T}_2$ is a PV, then $(G_s \times GL(1), \sigma \otimes \Lambda_1, V(\delta))$ must be a PV by Proposition 1.6. Hence by [SK], we have $G_s \neq F_4, E_8$, and for $G_s = (G_2)$ (resp. E_6, E_7), $\delta = 7$ (resp. $27, 56$). Since $((G_2), \Lambda_2, V(7)) \subset (SO(7), \Lambda_1, V(7))$, we obtain the case for $G_s = (G_2)$ by (i). For E_6, since $(E_6 \times GL(t), \Lambda_1 \otimes \Lambda_1, V(27) \otimes V(t))$ is a non-PV for $3 \leq t \leq 24$, $\mathbf{T}_2$ is a non-PV if one of $r, s, r+s, 54-r-s$ is in between 3 and 24. However if $r, s = 1$ or 26, it is a non-PV by [K3]. For E_7, note that $(E_7 \times GL(t), \Lambda_6 \otimes \Lambda_1, V(56) \otimes V(t))$ is a non-PV for $2 \leq t \leq 54$. Hence if one of $r, s, r+s, 112-r-s$ is in between 2 and 54, then $\mathbf{T}_2$ is a non-PV. However if $r, s = 1$ or 55, it is a non-PV by [K3]. For (iii), if $G_0 \neq GL(\delta)$ and the inclusion $\Lambda_1 : G_0 \hookrightarrow SL(\delta)$ is irreducible, then by Proposition 1.4, we have $\dim G_0 \leq \frac{1}{2}\delta(\delta+1)$. Therefore if $(G_0 \times G_1, V(\delta) \otimes V(\delta))$ is a PV, then $\frac{1}{2}\delta(\delta+1) + 1 + \dim G_s \geq \dim G_0 + \dim G_1 \geq \delta^2$ and hence $\delta_1 \geq \dim G_s \geq -1 + \frac{1}{2}\delta(\delta-1) \geq -1 + \frac{1}{2}\delta_0(\delta_0 - 1) > \delta_1$, a contradiction. If the inclusion $\Lambda_1 : G_0 \hookrightarrow SL(\delta)$ is not irreducible, we may assume that $G_0 \subset GL(r) \times GL(\delta - r)$. Then by (ii), we obtain our result. □

Proposition 1.8. $((GL(m_1) \times GL(m_2)) \times (GL(n_1) \times GL(n_2)), (\Lambda_1 \boxplus \Lambda_1) \otimes (\Lambda_1 \boxplus \Lambda_1), (V(m_1) + V(m_2)) \otimes (V(n_1) + V(n_2)))$ *is a PV if and only if* $m_1 + m_2 \neq n_1 + n_2$.

Proof. See Theorem 9.6 in [K1]. □

Proposition 1.9. $(SL(n) \times (GL(d_1) \times GL(d_2) \times GL(d_3) \times GL(d_4)), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1), V(n) \otimes (V(d_1) + V(d_2) + V(d_3) + V(d_4)))$ *is a PV if and only if* $d_1 + d_2 + d_3 + d_4 \neq 2n$ *or* $n \leq \max\{d_1, d_2, d_3, d_4\}$.

Proof. See Theorem 9.10 in [K1]. □

Proposition 1.10. *Assume* $k_1 = 1$ *and* $(G, \rho_1, V(d_1)) = (GL(1) \times G_s, \Lambda_1 \otimes \sigma, V(\delta))$ *where* G_s *is any simple algebraic group. Assume that* G_0 *is a reductive subgroup of* $GL(n)$ *such that the inclusion* $\Lambda_1 : G_0 \hookrightarrow GL(n)$ *is not irreducible. Then* $\mathbf{T} := (G_0 \times G, \Lambda_1 \otimes (\rho_1 + \cdots + \rho_m), V(n) \otimes (V(\delta) + V(d_2) + \cdots + V(d_m)))$ *with* $n = d_2 + \cdots + d_m$ $(= d - \delta)$ *and* $m \geq 3$ *is a non-PV.*

Proof. By assumption, we have $G_0 \subset GL(k) \times GL(d-\delta-k)$ with $1 \leq k < d-\delta$. If $\mathbf{T}$ is a PV, then $((GL(k) \times GL(d-\delta-k)) \times (GL(d_2) \times GL(d_3+\cdots+d_m)), (\Lambda_1 \boxplus \Lambda_1) \otimes (\Lambda_1 \boxplus \Lambda_1), (V(k)+V(d-\delta-k)) \otimes (V(d_2)+V(d_3+\cdots+d_m)))$ must be a PV, which is a contradiction by Proposition 1.8. □

Proposition 1.11 (Regularity of PV's). (i) *Assume that G is a reductive algebraic group. For $m > n \geq 1$, $(G \times GL(n), \rho\otimes 1+\sigma\otimes\Lambda_1, V+V(m)\otimes V(n))$ is a regular PV if and only if $(G\times GL(m-n), \rho\otimes 1+\sigma^*\otimes\Lambda_1, V+V(m)^*\otimes V(m-n))$ is a regular PV.*

(ii) *For $n=\delta$ or $n=t+k\delta$, $(SL(n) \times ((GL(1) \times G_s) \times T \times GL(\delta)^{\times(k)}), \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) \boxplus \tau \boxplus \Lambda_1^{\boxplus(k)}), V(n) \otimes (V(\delta)+V(t)+V(\delta)^{+(k)}))$ is a regular PV if and only if $(G_s \times T, \sigma\otimes\tau, V(\delta)\otimes V(t))$ is a regular PV where G_s is a simple algebraic group.*

(iii) *Let $\rho: G \to GL(V)$ be a representation and σ its restriction to a subgroup H of G. Assume that (H,σ,V) is a PV. Then (G,ρ,V) is a PV. Moreover if (G,ρ,V) is a regular PV, then (H,σ,V) is also a regular PV.*

Proof. A reductive PV is regular if and only if the generic isotropy subgroup is reductive. Since the generic isotropy subgroup is invariant up to isomorphism under castling transformation, we have (i). For (ii), it is enough to see the case $n=\delta$ since the case $n=t+k\delta$ is its castling transform. The generic isotropy subgroup at I_δ of $(SL(\delta)\times(GL(1)\times G_s), \Lambda_1\otimes(\Lambda_1\otimes\sigma), M(\delta))$ is $\{(\sigma^*(A),1,A) \mid A \in G_s\} \cong G_s$, and for $H \subset GL(\delta)$, the generic isotropy subgroup at I_δ of $(H\times GL(\delta), \Lambda_1\otimes\Lambda_1, M(\delta))$ is $\{(h, {}^t h^{-1}) \mid h \in H\} \cong H$, and we have our result. Note that $(G_s \times T, \sigma^*\otimes\tau, V(\delta)^*\otimes V(t)) \cong (G_s\times T, \sigma\otimes\tau, V(\delta)\otimes V(t))$ since G_s is reductive. By the definition of regularity, (iii) is clear. □

Proposition 1.12. (I) *The following triplets are regular PV's.*

(i) $(G_2 \times GL(t), \Lambda_2\otimes\Lambda_1, V(7)\otimes V(t))$ *with* $t=1,2,5,6$.

(ii) $(E_6 \times GL(t), \Lambda_1\otimes\Lambda_1, V(27)\otimes V(t))$ *with* $t=1,2,25,26$.

(iii) $(E_7 \times GL(t), \Lambda_6\otimes\Lambda_1, V(56)\otimes V(t))$ *with* $t=1,55$.

(iv) $(Sp(t)\times(GL(u)\times GL(v)), \Lambda_1\otimes(\Lambda_1\boxplus\Lambda_1), V(2t)\otimes(V(u)+V(v)))$ *with* $(u,v) = (1,1),(1,2t-1),(2t-1,2t-1),(1,k),(k,2t-1)$ *where k is an odd integer satisfying* $3 \leq k \leq 2t-3$.

(v) $(Spin(10)\times(GL(u)\times GL(u))$, *a half-spin rep.* $\otimes(\Lambda_1\boxplus\Lambda_1), V(16)\otimes(V(u)+V(u)))$ *with* $u=1,15$.

(II) *The following triplets are non-regular PV's.*

(i) $(Sp(t) \times (GL(2r) \times GL(2k+1)), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1), V(2t) \otimes (V(2r) + V(2k+1))$ *with* $t \geq 2,\ 2t-2 \geq 2r \geq 2,\ 2t-1 \geq 2k+1 \geq 1$.

(ii) $(Sp(t) \times (GL(u) \times GL(v) \times GL(w)), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1), V(2t) \otimes (V(u) + V(v) + V(w)))$ *with* $(u, v, w) = (1,1,1), (1,1,2t-1), (1,2t-1,2t-1), (2t-1, 2t-1, 2t-1), (1,1,k), (1,2t-1,k), (2t-1,2t-1,k)$ *where* k *is an odd integer satisfying* $3 \leq k \leq 2t-3$.

(iii) $(SL(2t+1) \times (GL(u) \times GL(u)), \Lambda_2 \otimes (\Lambda_1 \boxplus \Lambda_1), V(t(2t+1)) \otimes (V(u) + V(u)))$ *with* $u = 1$ *or* $u = t(2t+1) - 1$.

Proof. Use Proposition 1.11, [Ka] and [KUY]. □

§2. The exceptional case

In this section, we shall give the proof of Theorem 0.2. We assume that $k_1 = 1$ and $G_1 = (\Lambda_1 \otimes \sigma)(GL(1) \times G_s)$ where G_s is an exceptional simple algebraic group. For simplicity, we write $G_1 = GL(1) \times G_s$. In this case, we have $d_1 = \delta$ and hence $N_1 = -\dim G_1 + 1 = -\dim G_s$. We may assume that $d_{\mu 1} \geq d_{\mu\nu}$ $(1 \leq \nu \leq k_\mu)$ when $k_\mu \geq 1$. For δ_0 and δ_1, see Notation in Introduction.

Lemma 2.1. *Let* G_s *be an exceptional simple algebraic group.*

(i) *If* $k_\mu \geq 3$, *then* $N_\mu \geq 4d_{\mu 1}(\delta - d_{\mu 1}) + d_{\mu 1}^2 + 2$.

(ii) *If* $k_\mu = 2$ *and* $d_{\mu 1} \geq d_{\mu 2} \geq 3$, *then* $N_\mu \geq 1 + 2(\delta - d_{\mu 1})d_{\mu 1} + \delta d_{\mu 1} \geq 3\delta$.

(iii) *If* $k_\mu = 2$ *and* $d_{\mu 2} = 2$, *then* $N_\mu \geq (\delta - d_{\mu 1})d_{\mu 1} + (\delta d_{\mu 1} - 3) \geq 3\delta$.

Proof. As in the proof of Lemma 1.1, we have $\dim G_\mu \leq 1 + k_\mu(d_{\mu 1}^2 - 1)$ and $d_\mu \geq d_{\mu 1} 2^{k_\mu - 1}$ $(2 \leq \mu \leq m)$. Hence $N_\mu = -\dim G_\mu + \delta d_\mu \geq k_\mu - 1 - k_\mu d_{\mu 1}^2 + 2^{k_\mu - 1}\delta d_{\mu 1} = 2^{k_\mu - 1}(\delta - d_{\mu 1})d_{\mu 1} + (2^{k_\mu - 1} - k_\mu)d_{\mu 1}^2 + (k_\mu - 1)$. If $k_\mu \geq 3$, we have $2^{k_\mu - 1} \geq 4$ and $2^{k_\mu - 1} - k_\mu \geq 1$, and we obtain (i). If $k_\mu = 2$, we have $\dim G_\mu \leq 1 + (d_{\mu 1}^2 - 1) + (d_{\mu 2}^2 - 1)$ and $d_\mu = d_{\mu 1}d_{\mu 2}$. Hence $N_\mu \geq 1 - d_{\mu 1}^2 - d_{\mu 2}^2 + \delta d_{\mu 1} d_{\mu 2}$. If $d_{\mu 1} \geq d_{\mu 2} \geq 3$, we have $N_\mu \geq 1 - 2d_{\mu 1}^2 + 3\delta d_{\mu 1}$, i.e., (ii). If $d_{\mu 2} = 2$, we have $N_\mu \geq 2\delta d_{\mu 1} - d_{\mu 1}^2 - 3 \geq \delta d_{\mu 1} - 3$. In particular, $N_\mu \geq 4\delta - 3$ $(d_{\mu 1} \geq 4)$, $N_\mu \geq 6\delta - 12$ $(d_{\mu 1} = 3)$, $N_\mu \geq 4\delta - 7$ $(d_{\mu 1} = 2)$. Since $\delta \geq 7$, we have $N_\mu \geq 3\delta$. □

Proposition 2.2. *If* $G_1 = GL_1 \times G_s$ *with* G_s *an exceptional simple algebraic group, then* $k_\mu = 0$ *or* $k_\mu = 1$ $(2 \leq \mu \leq m)$.

Proof. For the exceptional simple algebraic group $G_s = (G_2)$ (resp. F_4, E_6, E_7, E_8), put $t = 4$ (resp. $8, 9, 12, 16$). By Lemma 2.1, if $k_\mu \geq 3$ and $d_{\mu 1} \geq t$, then we have $\dim G_s = 14$ (resp. $52, 78, 133, 248$) $\geq N_\mu \geq d_{\mu 1}^2 \geq t^2 = 16$ (resp. $64, 81, 144, 256$), a contradiction. If $k_\mu \geq 3$ and $t - 1 \geq d_{\mu 1} \geq 2$, we have $\delta - d_{\mu 1} \geq 4$ (resp.

$19, 19, 45, 233$), and $\dim G_s \geq N_\mu \geq 4d_{\mu 1}(\delta - d_{\mu 1}) \geq 8(\delta - d_{\mu 1}) \geq 32$ (resp. $152, 152, 360, 1864$), a contradiction. If $k_\mu = 2$, we have $\dim G_s \geq N_\mu \geq 3\delta \geq 3\delta_0 = 21$ (resp. $78, 81, 168, 744$), a contradiction. Hence $k_\mu \leq 1$. $\square$

By Proposition 2.2, first we shall investigate the prehomogeneity of

$$(4) \qquad (G_0 \times ((GL(1) \times G_s) \times G_2 \times \cdots \times G_m), \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) + \rho_2 + \cdots + \rho_m), V(n) \otimes (V(\delta) + V(d_2) + \cdots + V(d_m)))$$

with $\delta \geq d_\mu$ $(2 \leq \mu \leq m)$ and $\delta \leq n \leq d_2 + \cdots + d_m$ where G_s is an exceptional simple algebraic group and each G_μ contains the scalar action $GL(1)$. Then we shall find the groups $\rho(G) \subsetneq (GL(1) \times \sigma(G_s)) \times G_2 \times \cdots \times G_m$ such that $(G_0 \times G, \Lambda_1 \otimes \rho, V(n) \otimes V(d))$ is a PV.

Lemma 2.3. *If* (4) *is a PV with* $n = d - \delta$ $(= d_2 + \cdots + d_m)$, *then* $G_0 = SL(n)$.

Proof. Assume that $G_0 \neq SL(n)$ and the inclusion $\Lambda_1 : G_0 \hookrightarrow SL(n)$ is irreducible; then by (ii) of Proposition 1.4, we have $\dim G_0 \leq \frac{1}{2}n(n+1)$. Hence if (4) is a PV, we have $\frac{1}{2}n(n+1) + 1 + \dim G_s + d_2^2 + \cdots + d_m^2 \geq n(\delta + n)$, and hence $\delta_1 \geq \dim G_s \geq \frac{1}{2}n(n-1) - 1 + d_2(\delta - d_2) + \cdots + d_m(\delta - d_m) \geq \frac{1}{2}\delta(\delta - 1) - 1 > \delta_1$, a contradiction. If the inclusion $\Lambda_1 : G_0 \hookrightarrow SL(n)$ is not irreducible and $m \geq 3$, then by Proposition 1.10, it is a non-PV. If $m = 2$, then $\delta \leq n \leq d_2 \leq \delta$ implies that $n = \delta$. Then by (ii) of Proposition 1.7, it is a non-PV. $\square$

Lemma 2.4. *If* (4) *is a PV, then* $1 \leq d_\mu \leq 3$ *or* $\delta - 3 \leq d_\mu \leq \delta$, *and only the following cases are possible for* G_s, δ *and* G_μ $(2 \leq \mu \leq m)$.

- (i) *If* $d_\mu = \delta$, *then* $N_\mu = 0$ *and* $G_\mu = GL(\delta)$ *for any* σ *with* $\deg \sigma = \delta$.
- (ii) *If* $d_\mu = \delta - 1$ *or* $d_\mu = 1$, *then* $N_\mu = \delta - 1$, $G_\mu = GL(d_\mu)$ *and either* $\delta = \delta_0$, *or* $\delta = \delta_1$ *with* $G_s \neq E_8$.
- (iii) *If* $d_\mu = \delta - 2$ *or* $d_\mu = 2$, *then* $N_\mu = 2(\delta_0 - 2)$, $G_\mu = GL(d_\mu)$ *and* $\delta = \delta_0$ *with* $G_s \neq E_8$.
- (iv) *If* $d_\mu = \delta - 3$ *or* $d_\mu = 3$, *then* $N_\mu = 3(\delta_0 - 3)$, $G_\mu = GL(d_\mu)$ *for* $G_s = (G_2)$ *and* E_6 *with* $\delta = \delta_0$; *or* $N_\mu = 77$, $G_\mu = GL(1) \times SO(3)$ *for* $G_s = E_6$.

Proof. Let $G_s = (G_2)$ (resp. F_4, E_6, E_7, E_8). For (i), if $G_\mu \neq GL(\delta)$, then by Proposition 1.4, we have $\dim G_\mu \leq 1 + \frac{1}{2}\delta(\delta + 1)$ and hence by (2), we have $\dim G_s = 14$ (resp. $52, 78, 133, 248$) $\geq N_\mu = -\dim G_\mu + \delta^2 \geq \frac{1}{2}\delta(\delta - 1) - 1 \geq 20$ (resp. $324, 350, 1539, 30627$), a contradiction. Hence $N_\mu = -\dim GL(\delta) + \delta d_\mu = 0$. For (ii), if $d_\mu = \delta - 1$ and $G_\mu \neq GL(\delta - 1)$, then similarly we have $\dim G_\mu \leq 1 + \frac{1}{2}\delta(\delta - 1)$ and hence $\dim G_s \geq N_\mu \geq \frac{1}{2}\delta(\delta - 1) - 1$, a contradiction. It follows that $N_\mu = -\dim GL(\delta-1)+\delta(\delta-1) = \delta-1$. If $d_\mu = 1$, then $k_\mu = 0, G_\mu = GL(1)$ and $N_\mu = \delta - 1$. If $G_s \neq E_8$, then $\dim G_s = \delta_1 \geq N_\mu = \delta - 1$, and we have $\delta = \delta_0, \delta_1$. If $G_s = E_8$, we have $\dim G_s = \delta_0 \geq N_\mu = \delta - 1$ and hence $\delta = \delta_0$.

Now if $G_s = E_8$ with $\delta - 2 \geq d_\mu \geq 2$, we have $248 \geq N_\mu \geq d_\mu(\delta - d_\mu) \geq d_\mu(248 - d_\mu) > 248$, a contradiction. Hence we assume that $G_s \neq E_8$ so that $\dim G_s = \delta_1$. Assume that $\delta - 2 \geq d_\mu \geq 2$. Then $\delta = \delta_0$ since $\dim G_s = \delta_1 \geq N_\mu \geq d_\mu(\delta - d_\mu) > \delta_1$ if and only if $\frac{1}{2}(\delta + \sqrt{\delta^2 - 4\delta_1}) > d_\mu > \frac{1}{2}(\delta - \sqrt{\delta^2 - 4\delta_1})$, and if $\delta \geq \delta_1$, we have $\frac{1}{2}(\delta + \sqrt{\delta^2 - 4\delta_1}) > \delta - 2 \geq d_\mu \geq 2 > \frac{1}{2}(\delta - \sqrt{\delta^2 - 4\delta_1})$. Thus $\delta = \delta_0$ and if $\delta_0 - 4 \geq d_\mu \geq 4$, then $\frac{1}{2}(\delta_0 + \sqrt{\delta_0^2 - 4\delta_1}) \geq \delta_0 - 4 > d_\mu \geq 4 > \frac{1}{2}(\delta_0 - \sqrt{\delta_0^2 - 4\delta_1})$ since $4\delta_0 > \delta_1 + 16$. This implies that $\dim G_s = \delta_1 \geq N_\mu \geq d_\mu(\delta_0 - d_\mu) > \delta_1$, a contradiction. It follows that $\delta_0 \geq d_\mu \geq \delta_0 - 3$ or $d_\mu = 1, 2, 3$. If $G_s = F_4$ (resp. E_7) with $d_\mu = \delta_0 - 3$ or 3, then $\dim G_s = 26$ (resp. 56) $\geq N_\mu \geq 3(\delta_0 - 3) = 69$ (resp. 159) > 26 (resp. 133), a contradiction. Hence if $d_\mu = \delta - 3$ or 3, then $G_s = (G_2)$ or E_6 with $\delta = \delta_0$. Assume that $d_\mu = \delta_0 - 2$ (resp. $\delta_0 - 3$) with $G_\mu \neq GL(d_\mu)$. Then by Proposition 1.4, we have $\dim G_\mu \leq 1 + \frac{1}{2}(\delta_0 - 2)(\delta_0 - 1)$ (resp. $1 + \frac{1}{2}(\delta_0 - 3)(\delta_0 - 2)$) and hence $\dim G_s = \delta_1 \geq N_\mu \geq -1 - \frac{1}{2}(\delta_0 - 2)(\delta_0 - 1) + \delta_0(\delta_0 - 2) = -1 + \frac{1}{2}(\delta_0 - 2)(\delta_0 + 1)$ (resp. $-1 + \frac{1}{2}(\delta_0 - 3)(\delta_0 + 2)) > \delta_1$, a contradiction, and hence $G_\mu = GL(d_\mu)$ for $d_\mu = \delta_0 - 2$ or $\delta_0 - 3$. If $d_\mu = 2$, then G_μ must be $GL(2)$. If $d_\mu = 3$ and $G_\mu \neq GL(3)$, then $G_\mu = GL(1) \times SO(3)$. For $G_s = (G_2)$, we have $\dim(G_2) = 14 \geq N_\mu = -\dim(GL(1) \times SO(3)) + 3\delta_0 = 17$, a contradiction. □

Proposition 2.5. *If* (4) *is a PV, then only the following cases are possible.*

(i) $G_2 = \cdots = G_m = GL(\delta)$ *for any* σ *with* $\deg \sigma = \delta$.

(ii) $G_2 = GL(\delta - 1)$ *or* $GL(1)$*;* $G_3 = \cdots = G_m = GL(\delta)$ *with* $\delta = \delta_0$*, or* $\delta = \delta_1$ *with* $G_s \neq E_8$.

(iii) $G_2, G_3 = GL(\delta_0 - 1)$ *or* $GL(1)$*;* $G_4 = \cdots = G_m = GL(\delta_0)$ *with* $\delta = \delta_0$ *and* $G_s \neq E_8$.

(iv) $G_2 = GL(\delta_0 - 2)$ *or* $GL(2)$*;* $G_3 = \cdots = G_m = GL(\delta_0)$ *with* $\delta = \delta_0$ *and* $G_s \neq E_8$.

(v) $G_2 = GL(\delta_0 - 3)$ *or* $GL(3)$*;* $G_3 = \cdots = G_m = GL(\delta_0)$ *with* $\delta = \delta_0$ *and* $G_s = (G_2)$ *or* E_6.

(vi) $G_2 = GL(1) \times SO(3)$*;* $G_3 = \cdots = G_m = GL(27)$ *with* $G_s = E_6$.

Proof. Use (2) and Lemma 2.4. □

Lemma 2.6. *Let* G_s *be any simple algebraic group, and* G_0 *a semisimple algebraic subgroup of* $SL(n)$*. For* $\delta \leq n \leq (m-1)\delta$*, let* $\mathbf{T} := (G_0 \times ((GL_1 \times G_s) \times GL(\delta)^{\times(m-1)}, \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) \boxplus \Lambda_1^{\boxplus(m-1)}), V(n) \otimes (V(\delta)^{+(m)}))$.

(i) *If* $n = \delta$, *then* $\mathbf{T}$ *is a PV if and only if* $(G_0 \times (GL(1) \times G_s), \Lambda_1 \otimes (\Lambda_1 \otimes \sigma), V(\delta) \otimes V(\delta))$ *is a PV. In particular if* $G_0 = SL(n)$, *then* $\mathbf{T}$ *is always a PV.*

(ii) *If* G_s *is an exceptional simple algebraic group and* $n = \delta$, *then* $\mathbf{T}$ *is a PV if and only if* $G_0 = SL(n)$.

(iii) *If* $n = k\delta$ $(2 \leq k \leq (m-2))$, *then* $\mathbf{T}$ *is a non-PV.*

(iv) *If* $n = (m-1)\delta$ *with* $m \geq 3$, *then* $\mathbf{T}$ *is a PV if and only if* $G_0 = SL(n)$.

(v) *If* $\mathbf{T}$ *is a PV with* $\delta < n < (m-1)\delta$, *then* $(GL(1) \times G_s, \Lambda_1 \otimes \sigma, V(\delta))$ *must be a PV.*

Proof. By (ii) of Proposition 1.3, we obtain (i). We obtain (ii) by (ii) of Proposition 1.7. For (iii), if $\mathbf{T}$ is a PV for $n = k\delta$, it is also a PV when $G_0 = GL(k\delta)$. Then by (iii) of Proposition 1.3, it is PV-equivalent to $((GL(\delta)^{\times(k)}) \times ((GL(1) \times G_s) \times GL(\delta)^{\times(m-1-k)}), (\Lambda_1^{\boxplus(k)}) \otimes ((\Lambda_1 \otimes \sigma) + \Lambda_1^{\boxplus(m-k-1)}), V(k\delta) \otimes V((m-k)\delta))$. In particular $((GL(\delta) \times GL(\delta)) \times (GL(\delta) \times GL(\delta)), (\Lambda_1 \boxplus \Lambda_1) \otimes (\Lambda_1 \boxplus \Lambda_1), (V(\delta) + V(\delta)) \otimes (V(\delta) + V(\delta)))$ is a PV, but it is a non-PV by Proposition 1.8, a contradiction. For (iv), if $G_0 \neq SL(n)$, it is a non-PV by Lemma 2.3. If $G_0 = SL(n)$, then by castling transformation, $\mathbf{T}$ reduces to (i). For (v), we may assume that $k\delta < n < (k+1)\delta$ $(1 \leq k \leq m-2)$ by (iii). Then $(GL(n) \times ((GL(1) \times G_s) \times GL(\delta)^{\times(k)}), \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) \boxplus \Lambda_1^{\boxplus(k)}), V(n) \otimes (V(\delta) + V(\delta)^{+(k)}))$ must be a PV. By castling transformation, we have $n \mapsto n' = (k+1)\delta - n = \delta - (n - k\delta)$. Hence $(G_s \times GL(n'), \sigma \otimes \Lambda_1, V(\delta) \otimes V(n'))$ with $\delta > n' \geq 1$ must be an irreducible PV. Hence by Proposition 1.6, we have (v). □

Proposition 2.7. *For* $m \geq 3$ *and* $\delta \leq n \leq (m-1)\delta$, $\mathbf{T} := (G_0 \times ((GL(1) \times G_s) \times GL(\delta)^{\times(m-1)}), \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) \boxplus \Lambda_1^{\boxplus(m-1)}), V(n) \otimes (V(\delta) + V(\delta)^{+(m-1)}))$ *is a PV if and only if* $G_0 = SL(n)$ *with* $n = \delta$ *or* $n = (m-1)\delta$.

Proof. By (ii) and (iv) of Lemma 2.6, we have the case $n = \delta$ or $n = (m-1)\delta$. For $\delta < n < (m-1)\delta$, by (v) of Lemma 2.6, $\mathbf{T}$ is a non-PV for $G_s = F_4$ and E_8. For $G_s = (G_2)$ (resp. E_6, E_7), if it is a PV, then $\delta = 7$ (resp. $27, 56$), and $n \neq k\delta$ $(2 \leq k \leq (m-2))$ by (iii) and (v) of Lemma 2.6. Hence there is t satisfying $3 \leq t \leq m$ and $(t-2)\delta < n < (t-1)\delta$, i.e., $\delta < (n' =)\ t\delta - n < 2\delta$ and $(GL(n) \times (G_s \times GL(\delta)^{\times(t-1)}), \Lambda_1 \otimes (\sigma \boxplus \Lambda_1^{\boxplus(t-1)}), V(n) \otimes (V(\delta)^{+(t)}))$ is a PV. By castling transformation, we have $n \mapsto n' = t\delta - n$ with $\delta < n' < 2\delta$, and since $t-1 \geq 2$, $(GL(n') \times (G_s \times GL(\delta) \times GL(\delta)), \Lambda_1 \otimes (\sigma \boxplus \Lambda_1 \boxplus \Lambda_1), V(n') \otimes (V(\delta) + V(\delta) + V(\delta)))$ must be a PV which is PV-equivalent to $(G_s \times (GL(n'-\delta) \times GL(n'-\delta)), \sigma \otimes (\Lambda_1 \boxplus \Lambda_1), V(\delta) \otimes (V(n'-\delta) + V(n'-\delta)))$ with $1 \leq n' - \delta < \delta$ by Proposition 1.4. However it is a non-PV by Proposition 1.7, a contradiction. □

Proposition 2.8. *Let $d_2 = 1, \delta - 1$ with $\delta = \delta_0, \delta_1$. For $m \geq 3$ and $\delta \leq n \leq d_2 + (m-2)\delta$, we have the following assertions.*

(i) $\mathbf{T} := (G_0 \times ((GL(1) \times G_s) \times GL(d_2) \times GL(\delta)^{\times(m-2)}), \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) \boxplus \Lambda_1 \boxplus \Lambda_1^{\boxplus(m-2)}), V(n) \otimes (V(\delta) + V(d_2) + V(\delta)^{+(m-2)}))$ *is a PV if and only if* $G_0 = SL(n)$ *with* $n = \delta_0$ *or* $n = d_2 + (m-2)\delta_0$ *for* $G_s = (G_2), E_6, E_7$.

(ii) $\mathbf{T}$ *is a non-PV for* $G_s = F_4$ *and* E_8.

Proof. If $\mathbf{T}$ is a PV with $n = \delta$, then by (ii) of Lemma 2.6, we have $G_0 = SL(n)$. Then by (ii) and (iii) of Proposition 1.3, $\mathbf{T}$ is PV-equivalent to $(G_s \times GL(d_2), \sigma \otimes \Lambda_1, V(\delta) \otimes V(d_2))$ with $d_2 = 1, \delta - 1$, which is a PV if and only if $G_s = (G_2), E_6, E_7$. By castling transformation, we have the result for $n = d_2 + (m-2)\delta$ and $G_0 = SL(n)$. $\mathbf{T}$ is a non-PV for $n = d_2 + (m-2)\delta$ and $G_0 \neq SL(n)$ by Lemma 2.3. Assume that $\delta < n < d_2 + (m-2)\delta$ and $m \geq 4$. If $\delta < n < (m-2)\delta$, then $\mathbf{T}$ is a non-PV by Proposition 2.7. So assume that $(m-2)\delta \leq n < d_2 + (m-2)\delta$. To prove the non-prehomogeneity, it is enough to assume that $G_0 = GL(n)$. Then by castling transformation, we have $n \mapsto (\delta <) n' = d_2 + (m-1)\delta - n = (d_2 + \delta) - (n - (m-2)\delta) \leq d_2 + \delta < (m-2)\delta$, and hence $\mathbf{T}$ is a non-PV. Finally assume that $m = 3$ and $\delta < n < d_2 + \delta$. Then $d_2 \neq 1$ and $d_2 = \delta - 1$. Hence by Proposition 1.5, $\mathbf{T}$ is PV-equivalent to $(G_s \times (GL(n-\delta+1) \times GL(n-\delta)), \sigma \otimes (\Lambda_1 \boxplus \Lambda_1), V(\delta) \otimes (V(n-\delta+1) + V(n-\delta)))$ with $\delta > n - \delta + 1 > n - \delta \geq 1$. By Proposition 1.7, the latter is a non-PV. $\square$

Proposition 2.9. *Let $d_2, d_3 = 1$ or $\delta - 1$. For $m \geq 3$ and $\delta \leq n \leq d_2 + d_3 + (m-3)\delta$, $\mathbf{T} := (G_0 \times ((GL(1) \times G_s) \times GL(d_2) \times GL(d_3) \times GL(\delta)^{\times(m-3)}), \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) \boxplus \Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1^{\boxplus(m-3)}), V(n) \otimes (V(\delta) + V(d_2) + V(d_3) + V(\delta)^{+(m-3)}))$ is a non-PV.*

Proof. To prove non-prehomogeneity, we may assume that $G_0 = GL(n)$. If $n = \delta$, by (ii) and (iii) of Proposition 1.3, $\mathbf{T}$ is PV-equivalent to $(G_s \times (GL(d_2) \times GL(d_3)), \sigma \otimes (\Lambda_1 \boxplus \Lambda_1), V(\delta) \otimes (V(d_2) + V(d_3)))$ with $1 \leq d_2, d_3 < \delta$, which is a non-PV by Proposition 1.7. Hence by castling transformation, $\mathbf{T}$ is a non-PV for $n = d_2 + d_3 + (m-3)\delta$. Therefore we may assume that $\delta < n < d_2 + d_3 + (m-3)\delta$ with $d_2 \leq d_3$. By Proposition 2.8, $\mathbf{T}$ is a non-PV for $\delta < n < d_3 + (m-3)\delta$. Hence we may assume that $d_3 + (m-3)\delta \leq n < d_2 + d_3 + (m-3)\delta$. By castling transformation, we have $n \mapsto n' = d_2 + d_3 + (m-2)\delta - n$ with $\delta < n' \leq d_2 + \delta$. Hence if $n' \leq d_2 + \delta < d_3 + (m-3)\delta$, i.e., $m \geq 5$ or $m = 4$ with $d_2 = 1 < d_3 = \delta - 1$, then $\mathbf{T}$ is a non-PV. Assume that $m = 4$ and $d_2 = d_3$. If $\delta < n' < d_3 + \delta$, $\mathbf{T}$ is a non-PV. If $n = d_3 + \delta$, then $\mathbf{T}$ is a non-PV by Proposition 1.9. If $m = 3$, then $\delta < n < d_2 + d_3$ gives $d_2 = d_3 = \delta - 1$, and by Proposition 1.5, $\mathbf{T}$ is PV-equivalent to $(G_s \times (GL(n-\delta+1) \times GL(n-\delta+1)), \sigma \otimes (\Lambda_1 \boxplus \Lambda_1), V(\delta) \otimes (V(n-\delta+1) + V(n-\delta+1)))$ with $\delta > n - \delta + 1 > 1$. The latter is a non-PV by Proposition 1.7. $\square$

Proposition 2.10. *Let $d_2 = 2, 3, \delta_0 - 3$ or $\delta_0 - 2$ with $\delta = \delta_0$. For $m \geq 3$ and $\delta_0 \leq n \leq d_2 + (m-2)\delta_0$, we have the following assertions.*

(i) $\mathbf{T} := (G_0 \times ((GL(1) \times G_s) \times GL(d_2) \times GL(\delta_0)^{\times(m-2)}), \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) \boxplus \Lambda_1 \boxplus \Lambda_1^{\boxplus(m-2)}), V(n) \otimes (V(\delta_0) + V(d_2) + V(\delta_0)^{+(m-2)}))$ *is a PV if and only if $G_0 = SL(n)$ with $n = \delta_0, d_2 + (m-2)\delta_0$ and $d_2 = 2, \delta_0 - 2$ for $G_s = (G_2)$ and E_6.*

(ii) $\mathbf{T}$ *is a non-PV for $G_s = F_4, E_7$ and E_8.*

Proof. Assume that $n = \delta_0$. Then by (ii) of Proposition 1.3, $\mathbf{T}$ is PV-equivalent to $(G_0 \times ((GL(1) \times G_s) \times GL(d_2)), \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) \boxplus \Lambda_1), V(\delta_0) \otimes (V(\delta_0) + V(d_2)))$. By the proof of (ii) of Lemma 2.6, it is a non-PV when $G_0 \neq SL(\delta_0)$. If $G_0 = SL(\delta_0)$, then by (iii) of Proposition 1.3, $\mathbf{T}$ is PV-equivalent to $(G_s \times GL(d_2), \sigma \otimes \Lambda_1, V(\delta_0) \otimes V(d_2))$ which is a PV if and only if $G_s = (G_2), E_6$ and $d_2 = 2, \delta_0 - 2$. Hence by castling transformation, we have the same result when $G_0 = SL(n)$ with $n = d_2 + (m-2)\delta_0$. If $n = d_2 + (m-2)\delta_0$ and $G_0 \neq SL(n)$, then $\mathbf{T}$ is a non-PV by Lemma 2.3. Now assume that $\delta_0 < n < d_2 + (m-2)\delta_0$ with $m \geq 4$. By Proposition 2.7, $\mathbf{T}$ is a non-PV for $\delta_0 < n < (m-2)\delta_0$. So we may assume that $(m-2)\delta_0 \leq n < d_2 + (m-2)\delta_0$. To prove the non-prehomogeneity, it is enough to assume that $G_0 = GL(n)$. Then by castling transformation, we have $n \mapsto n' = d_2 + (m-1)\delta_0 - n$ with $\delta_0 < n' < d_2 + \delta_0 < (m-2)\delta_0$, and hence $\mathbf{T}$ is a non-PV. Finally assume that $m = 3$ and $\delta_0 < n < d_2 + \delta_0$. Again to prove the non-prehomogeneity, it is enough to assume that $G_0 = GL(n)$. Then by Proposition 1.5, $\mathbf{T}$ is PV-equivalent to $(G_s \times (GL(n-d_2) \times GL(n-\delta_0)), \sigma \otimes (\Lambda_1 \boxplus \Lambda_1), V(\delta_0) \otimes (V(n-d_2) + V(n-\delta_0)))$ with $1 \leq n-d_2, n-\delta_0 < \delta_0$, which is a non-PV by Proposition 1.7. Finally note that since case $GL(d_2) = GL(3)$ is a non-PV, case (vi) in Proposition 2.5 is a non-PV. □

Now we shall give the proof of Theorem 0.2. By Propositions 2.7 to 2.10, the triplets appearing in (i)–(iv) of Theorem 0.2 are PV's and it is enough to find the group G of those $(GL(1) \times G_s) \times G_2 \times \cdots \times G_m$ such that $(SL(n) \times G, \Lambda_1 \otimes \rho, V(n) \otimes V(d))$ is a PV. By castling transformation, we may assume that $n = \delta$ (resp. $\delta = 7, 27, 56$). If we restrict $GL(\delta)$ to $SL(\delta)$ (resp. $GL(1) \times G_s$ to G_s), then the triplet becomes a non-PV since $(SL(\delta) \times SL(\delta), \Lambda_1 \otimes \Lambda_1, V(\delta) \otimes V(\delta))$ is a non-PV. Now if we restrict $(GL(\delta) \times GL(\delta), \Lambda_1 \boxplus \Lambda_1, V(\delta) + V(\delta))$ to $(GL(\delta), \Lambda_1 + \Lambda_1, V(\delta) + V(\delta))$ and if it is still a PV, then $(SL(\delta) \times ((GL(1) \times G_s) \times GL(\delta)), \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) \boxplus (\Lambda_1 + \Lambda_1)), V(\delta) \otimes (V(\delta) + V(\delta) + V(\delta)))$ must be a PV. Then by (iii) of Proposition 1.3, it is PV-equivalent to $(G_s \times GL(\delta), \sigma \otimes (\Lambda_1 + \Lambda_1), V(\delta) \otimes (V(\delta) + V(\delta)))$ which is also PV-equivalent to $(G_s, \sigma \otimes \sigma^*, V(\delta) \otimes V(\delta)^*)$ which is a non-PV for dimensional reasons, a contradiction. Thus any proper restriction of the group gives a non-PV. By Propositions 1.11 and 1.12, these PV's are regular.

§3. The extreme case for classical simple algebraic groups

In this section, we shall give the proof of Theorem 0.3. Let $(G_0 \times G, \Lambda_1 \otimes \rho, V(n) \otimes V)$ with $\delta \leq n \leq d - \delta$ be as in Introduction. The remaining case is $k_1 = 1$ so that $(G, \rho_1, V_1) = (GL(1) \times G_s, \Lambda_1 \otimes \sigma, V(1) \otimes V(\delta))$ where G_s is a simple algebraic group. Note that $V(1) \otimes V \cong V$ in general. In this section, we deal with the extreme case $n = \delta$ or $n = d - \delta$ when G_s is a classical simple algebraic group with $(G, \rho_1, V_1) \neq (GL(\delta), \Lambda_1, V(\delta))$. First we assume that $G_0 = SL(n)$ with $n = \delta$ or $n = d - \delta$ and $\rho(G) = G_1 \times \cdots \times G_m$ (see Preliminaries) where each $V(d_\mu)$ $(1 \leq \mu \leq m)$ has an independent scalar action. Then by castling transformation, it is enough to consider the case $n = \delta$. After the classification of this case, we shall classify $(G_0 \times G, \Lambda_1 \otimes \rho, V(n) \otimes V)$ for $G_0 \subsetneq SL(n)$ and $\rho(G) \subset G_1 \times \cdots \times G_m$; or $G_0 = SL(n)$ and $\rho(G) \subsetneq G_1 \times \cdots \times G_m$ where each $V(d_\mu)$ $(1 \leq \mu \leq m)$ has an independent scalar action.

Lemma 3.1. *Under the above assumption,* $(G_0 \times G, \Lambda_1 \otimes \rho, V(n) \otimes V) = (SL(\delta) \times ((GL_1 \times G_s) \times G_2 \times \cdots \times G_m), \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) \boxplus \rho_2 \boxplus \cdots \boxplus \rho_m), V(\delta) \otimes (V(\delta) + V(d_2) + \cdots + V(d_m)))$ *is PV-equivalent to*

(5) $\quad (G_s \times (G_2 \times \cdots \times G_m), \sigma \otimes (\rho_2 \boxplus \cdots \boxplus \rho_m), V(\delta) \otimes (V(d_2) + \cdots + V(d_m)))$

with $\delta \leq d_2 + \cdots + d_m$ *where* G_s *is a classical simple algebraic group* $\neq SL(\delta)$.

Proof. Use (iii) of Proposition 1.3. □

Therefore we shall investigate the prehomogeneity of the triplet (5).

Proposition 3.2. *If* $m = 2$*, then* (5) *is a PV if and only if* $(G_2, \rho_2, V(d_2)) = (GL(\delta), \Lambda_1, V(\delta))$.

Proof. If $k_2 = 0$, then $3 \leq \delta \leq d_2 = 1$, a contradiction. If $k_2 = 1$, then $(\delta \leq)\ d_2 \leq \delta$ and hence $d_2 = \delta$. Thus $(G_s \times G_2, \sigma \otimes \rho_2, V(\delta) \otimes V(\delta))$ must be an irreducible PV, and by [SK], we have $(G_2, \rho_2, V(\delta)) = (GL(\delta), \Lambda_1, V(\delta))$. In this case, (5) is a trivial regular PV. Finally assume that $k_2 \geq 2$. If (5) is reduced, by [SK], it must be an irreducible trivial PV. Then $G_2 = GL(N) \times H$ and $V(d_2) = V(N) \otimes V(h)$ with $\delta \geq N \geq \delta h\ (> \delta)$, a contradiction. Hence (5) is not reduced. Assume that $SL(M)$ has the highest rank among the simple factors of G_2. Note that $\delta \geq M$. Then $(G_s \times G_2, \sigma \otimes \rho_2, V(\delta) \otimes V(\delta)) = (G_s \times GL(M) \times K, \sigma \otimes \Lambda_1 \otimes \kappa, V(\delta) \otimes V(M) \otimes V(k))$ is reduced since $\delta k - M\ (\geq 2\delta - M) \geq M$, a contradiction, and hence $k_2 \ngeq 2$. □

Note that Proposition 3.2 corresponds to (i) of Lemma 2.6 with $m = 2$ and $G_0 = SL(n)$.

Now consider (5) with $m \geq 3$. If it is a PV, then for any μ_1 and μ_2 with $2 \leq \mu_1 < \mu_2 \leq m$, $(G_s \times (G_{\mu_1} \times G_{\mu_2}), \sigma \otimes (\rho_{\mu_1} \boxplus \rho_{\mu_2}), V(\delta) \otimes (V(d_{\mu_1}) + V(d_{\mu_2})))$ must be a 2-irreducible PV.

Lemma 3.3. *If* $(G_s \times (G_{\mu_1} \times G_{\mu_2}), \sigma \otimes (\rho_{\mu_1} \boxplus \rho_{\mu_2}), V(\delta) \otimes (V(d_{\mu_1}) + V(d_{\mu_2})))$ *is a PV with* $(G_{\mu_i}, \rho_{\mu_i}, V(d_{\mu_i})) \neq (GL(\delta), \Lambda_1, V(\delta))$ $(i = 1, 2)$*, then it is castling equivalent to one of the following PV's.*

(i) $(Sp(t) \times (GL(2k+1) \times GL(2r)), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1), V(2t) \otimes (V(2k+1) + V(2r)))$ *with* $t \geq 2r \geq 2$ *and* $t \geq 2k+1 \geq 1$.

(ii) $(Sp(t) \times (GL(2k+1) \times GL(1)), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1), V(2t) \otimes (V(2k+1) + V(1)))$ *with* $t \geq 2$ *and* $t \geq 2k+1 \geq 1$.

(iii) $(Spin(10) \times (GL(1) \times GL(1))$, *a half-spin rep.* $\otimes (\Lambda_1 \boxplus \Lambda_1), V(16) \otimes (V(1) + V(1)))$.

(iv) $(SL(2t+1) \times (GL(1) \times GL(1)), \Lambda_2 \otimes (\Lambda_1 \boxplus \Lambda_1), V(t(2t+1)) \otimes (V(1) + V(1)))$ *with* $t \geq 2$.

Proof. Use [Ka]. Note that (5) is not the unsolved case of [Ka]. If it is of trivial type, then $(G_{\mu_i}, \rho_{\mu_i}, V(d_{\mu_i})) = (GL(\delta), \Lambda_1, V(\delta))$ for some $i = 1, 2$. □

Proposition 3.4. *If* (5) *with* $m \geq 3$ *is a PV, then it is one of the following PV's.*

(i) $(G_s \times (GL(\delta)^{\times(m-1)}), \sigma \otimes (\Lambda_1^{\boxplus(m-1)}), V(\delta) \otimes (V(\delta)^{+(m-1)}))$.

(ii) $(G_s \times ((GL(1) \times T_s) \times GL(\delta)^{\times(m-2)}), \sigma \otimes ((\Lambda_1 \otimes \tau) \boxplus \Lambda_1^{\boxplus(m-2)}), V(\delta) \otimes (V(t) + V(\delta)^{+(m-2)}))$ *with* $\delta > t \geq 1$ *where* T_s *is a simple algebraic group such that* $(GL(1) \times G_s \times T_s, \Lambda_1 \otimes \sigma \otimes \tau, V(1) \otimes V(\delta) \otimes V(t))$ *is a non-trivial irreducible* 2*-simple PV.*

(iii) $(Sp(t) \times (GL(u) \times GL(v) \times GL(2t)^{\times(m-3)}), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1^{\boxplus(m-3)}), V(2t) \otimes (V(u) + V(v) + V(2t)^{+(m-3)}))$ *with* $t \geq 2$ *where* $(u, v) = (1, 1), (1, k)$ *with* $m \geq 4$*, or* $(u, v) = (1, 2t-1), (2t-1, 2t-1), (k, 2t-1)$ *with* $m \geq 3$*. Here* k *is an odd integer satisfying* $3 \leq k \leq 2t-3$.

(iv) $(Sp(t) \times (GL(u) \times GL(v) \times GL(w) \times GL(2t)^{\times(m-4)}), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1^{\boxplus(m-4)}), V(2t) \otimes (V(u) + V(v) + V(w) + V(2t)^{+(m-4)}))$ *with* $t \geq 2$ *where* $(u, v, w) = (1, 1, 1), (1, 1, k)$ *with* $m \geq 5$*, or* $(u, v, w) = (1, 1, 2t-1), (1, 2t-1, 2t-1), (2t-1, 2t-1, 2t-1), (1, k, 2t-1), (k, 2t-1, 2t-1)$ *with* $m \geq 4$*. Here* k *is an odd integer satisfying* $3 \leq k \leq 2t-3$.

(v) $(Sp(t) \times (GL(2r) \times GL(2k+1)), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1), V(2t) \otimes (V(2r) + V(2k+1)))$ *with* $t \geq 2$, $t-1 \geq r \geq 1$, $2t-1 \geq 2k+1 \geq 1$ *and* $2r + 2k + 1 > 2t$.

(vi) $(Sp(t) \times (GL(2r) \times GL(2k+1) \times GL(2t)^{\times(m-3)}), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1^{\boxplus(m-3)}), V(2t) \otimes (V(2r) + V(2k+1) + V(2t)^{+(m-3)}))$ *with* $t \geq 2$, $m \geq 4$, $t-1 \geq r \geq 1$ *and* $2t-1 \geq 2k+1 \geq 1$.

(vii) $(Spin(10) \times (GL(u) \times GL(u) \times GL(16)^{\times(m-3)})$, *a half-spin rep.* $\otimes (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1^{\boxplus(m-3)}), V(16) \otimes (V(u) + V(u) + V(16)^{+(m-3)}))$ *with* ($u = 1$ *and* $m \geq 4$) *or* ($u = 15$ *and* $m \geq 3$).

(viii) $(SL(2t+1) \times (GL(u) \times GL(u) \times GL(\delta)^{\times(m-3)}), \Lambda_2 \otimes (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1^{\boxplus(m-3)}), V(\delta) \otimes (V(u) + V(u) + V(\delta)^{+(m-3)}))$ *with* $t \geq 2$, ($u = 1$ *and* $m \geq 4$) *or* ($u = \delta - 1 = t(2t+1) - 1$ *and* $m \geq 3$).

Proof. Put $M = \{\mu \mid (G_\mu, \rho_\mu, V_\mu) \neq (GL(\delta), \Lambda_1, V(\delta)),\ 2 \leq \mu \leq m\}$. If $M = \emptyset$, then (5) is a PV by Proposition 1.3 and we obtain (i). If $\sharp M = 1$, we may assume that $M = \{2\}$ and by Proposition 1.3, (5) is PV-equivalent to $(G_s \times G_2, \sigma \otimes \rho_2, V(\delta) \otimes V(d_2))$. If this is a PV, then by Proposition 3.2, we have $\delta > d_2 \geq 1$. Since $(G_s, \sigma, V(\delta)) \neq (SL(\delta), \Lambda_1, V(\delta))$, this is a non-trivial irreducible PV. Then by the list of [SK], we see that this is a 2-simple PV, and we obtain (ii) where we put $G_2 = GL(1) \times T_s$ and $d_2 = t$. When $\sharp M \geq 2$, we may assume that $M = \{2, 3, \ldots, r\}$ with $r \geq 3$. Then (5) is PV-equivalent to

$$(6) \qquad (G_s \times (G_2 \times \cdots \times G_r), \sigma \otimes (\rho_2 \boxplus \cdots \boxplus \rho_r), V(\delta) \otimes (V(d_2) + \cdots + V(d_r)))$$

with $(G_\mu, \rho_\mu, V(d_\mu)) \neq (GL(\delta), \Lambda_1, V(\delta))$ $(2 \leq \mu \leq r)$ and $r \geq 3$. Then by Lemma 3.3, $(G_s, \sigma, V(\delta))$ must be one of $(Sp(t), \Lambda_1, V(2t))$ $(t \geq 2)$, $(Spin(10)$, a half-spin rep., $V(16))$, $(SL(2t+1), \Lambda_2, V(t(2t+1)))$ $(t \geq 2)$. First assume that $(G_s, \sigma, V(\delta)) = (Sp(t), \Lambda_1, V(2t))$ $(t \geq 2)$. Since $(Sp(t), \Lambda_1, V(2t)) = (Sp(t), \Lambda_1^*, V(2t)^*)$, we may assume that each $(Sp(t) \times G_\mu, \Lambda_1 \otimes \rho_\mu, V(2t) \otimes V(d_\mu))$ is reduced by castling transformations. Then by Lemma 3.3, we may assume that $(G_\mu, \rho_\mu, V(d_\mu)) = (GL(d_\mu), \Lambda_1, V(d_\mu))$ with $d_2 \leq \cdots \leq d_r \leq t$. Now assume that all d_μ are odd. Then by Lemma 3.3, we have $(d_2, \ldots, d_r) = (1, \ldots, 1)$ or $(1, \ldots, 1, k)$ where k is odd satisfying $3 \leq k \leq t$. When $(d_2, \ldots, d_r) = (1, \ldots, 1)$, by [K3], we have $r = 3$ or $r = 4$. This gives (iii) and (iv) related with 1 and $2t - 1$. When $(d_2, \ldots, d_r) = (1, \ldots, 1, k)$, by [KKIY], it is a PV (resp. non-PV) for $r = 3$ and 4 (resp. $r \geq 5$) by Theorem 2.24 (resp. Lemma 2.22) in [KKIY]. This gives (iii) and (iv) related with k. If there is an even number among $(d_2, \ldots, d_r)$, it is unique by Lemma 3.3. By Lemma 2.20 in [KKIY] and Lemma 3.27 in [KUY], we have $r = 3$. If $m = 3$ (resp. $m \geq 4$), this gives (v) (resp. (vi)). Next assume that $(G_s, \sigma, V(\delta)) = (Spin(10)$, a half-spin rep.,$V(16))$. By [K3], we have $r = 3$ and we obtain (vii). Finally assume that $(G_s, \sigma, V(\delta)) = (SL(2t+1), \Lambda_2, V(t(2t+1)))$ $(t \geq 2)$. In this case, we have (viii) similarly. $\square$

By Lemma 3.1, Proposition 3.2 and Proposition 3.4, we obtain the case for $G_0 = SL(n)$ and $\rho(G) = G_1 \times \cdots \times G_m$. Now we shall classify $(G_0 \times G, \Lambda_1 \otimes \rho, V(n) \otimes V)$ for $G_0 = SL(n)$ and $\rho(G) \subsetneq G_1 \times \cdots \times G_m$; or $G_0 \subsetneq SL(n)$ and $\rho(G) \subset G_1 \times \cdots \times G_m$ where each $V(d_\mu)$ $(1 \leq \mu \leq m)$ has an independent scalar action.

Lemma 3.5. *Let G_{ss} be a semisimple proper algebraic subgroup of $SL(\delta)$ and $\sigma : G_s \to GL(\delta)$ an irreducible representation of a simple algebraic group G_s. Then $\mathbf{T} := (GL(1) \times G_{ss} \times G_s, \Lambda_1 \otimes \Lambda_1 \otimes \sigma, V(1) \otimes V(\delta) \otimes V(\delta))$ is a PV if and only if $(G_{ss}, \Lambda_1, V(\delta)) = (SL(2t-1) \times SL(1), \Lambda_1 \boxplus \Lambda_1, V(2t-1) + V(1))$ and $(G_s, \sigma, V(\delta)) = (Sp(t), \Lambda_1, V(2t))$ with $t \geq 2$.*

Proof. First assume that the inclusion $\Lambda_1 : G_{ss} \hookrightarrow SL(\delta)$ is irreducible. Then $\mathbf{T}$ is irreducible and reduced since otherwise $G_{ss} = SL(r) \times H, V(\delta) = V(r) \otimes V(h)$ with $\delta > h \geq 2$ and $r > h\delta - r \geq 2\delta - r > r$, a contradiction. Then by [SK], $\mathbf{T}$ is a non-PV. Next assume that the inclusion $\Lambda_1 : G_{ss} \hookrightarrow SL(\delta)$ is non-irreducible. Then by Lemma 3.3 and Proposition 3.4, the only possible case is $(G_{ss}, \Lambda_1, V(\delta)) = (SL(2t-1) \times SL(1), \Lambda_1 \boxplus \Lambda_1, V(2t-1) + V(1))$ and $(G_s, \sigma, V(\delta)) = (Sp(t), \Lambda_1, V(2t))$ with $t \geq 2$. By (20) in p. 97 of [K3], $(Sp(t) \times (GL(1) \times GL(1)), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1), V(2t) + V(2t))$ is a PV with one irreducible relative invariant. It is castling equivalent to $(Sp(t) \times (GL(2t-1) \times GL(1)), \Lambda_1^* \otimes \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1 \otimes \Lambda_1), M(2t, 2t-1) + V(2t))$. Since $\Lambda_1^* = \Lambda_1$ for $Sp(t)$, $(Sp(t) \times (GL(2t-1) \times GL(1)), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1), M(2t))$ is a PV with unique irreducible relative invariant $f(X) = \det X$ $(X \in M(2t))$. Hence $\mathbf{T}$ is a PV (see Proposition 2.12 in [K2]). □

Lemma 3.6. $\mathbf{T} := ((SL(2t-1) \times SL(1)) \times ((GL(1) \times Sp(t)) \times GL(d_2)), (\Lambda_1 \boxplus \Lambda_1) \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1), (V(2t-1) + V(1)) \otimes (V(2t) + V(d_2)))$ *is a non-PV for* $2t > d_2 \geq 1$ *and* $t \geq 2$.

Proof. One can easily check that the $SL(2t-1)$-part of a generic isotropy subgroup of $((SL(2t-1) \times SL(1)) \times (GL(1) \times Sp(t)), (\Lambda_1 \boxplus \Lambda_1) \otimes (\Lambda_1 \otimes \Lambda_1), (V(2t-1) + V(1)) \otimes V(2t))$ at the identity matrix $I_{2t} \in M(2t) = (V(2t-1) + V(1)) \otimes V(2t)$ is isomorphic to $A = \left(\begin{smallmatrix} Sp(t-1) & 0 \\ 0 & 1 \end{smallmatrix}\right)$ $(\subset SL(2t-1))$. Therefore $\mathbf{T}$ is PV-equivalent to $((A \times SL(1)) \times GL(d_2)), (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1, (V(2t-1) + V(1)) \otimes V(d_2)) \cong (Sp(t-1) \times GL(d_2), \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1)$. By castling transformation, we may assume that $1 \leq d_2 \leq t$. If d_2 is even and $t \geq 3$, then $2t - 2 > t \geq d_2$ and by (10), p. 396 in [KKIY], this space has two irreducible relative invariants, so that this is a non-PV due to lack of $GL(1)$. If d_2 is even and $t = 2$, then $d_2 = 2$ and $\dim(Sp(1) \times GL(2)) = 7 < 8 = \deg(\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1)$, hence it is a non-PV. If d_2 is odd $(= 2r+1)$, by p. 102 in [SK], it is PV-equivalent to $\left(\left(\begin{smallmatrix} Sp(r) & * \\ 0 & * \end{smallmatrix}\right), \Lambda_1 + \Lambda_1\right)$, which is PV-equivalent to $\left(\left(\begin{smallmatrix} Sp(r) & 0 \\ 0 & 1 \end{smallmatrix}\right), \Lambda_1\right)$, which is a non-PV. □

Lemma 3.7. *The following triplets are non-PV's.*

(i) $(GL(r) \times GL(r), \Lambda_1 \otimes (\Lambda_1 + \Lambda_1^{(*)}), M(r) + M(r))$ *with* $r \geq 2$.

(ii) $((GL(1) \times Sp(t)) \times GL(2t-1), (\Lambda_1 \otimes \Lambda_1) \otimes (\Lambda_1 + \Lambda_1^{(*)}), M(2t, 2t-1) + M(2t, 2t-1))$ *with* $t \geq 2$.

(iii) $((GL(1) \times Spin(10)) \times GL(15), (\Lambda_1 \otimes \textit{a half-spin rep.}) \otimes (\Lambda_1 + \Lambda_1^{(*)}), M(16, 15) + M(16, 15))$.

(iv) $(GL(2t+1) \times GL(t(2t+1)-1), \Lambda_2 \otimes (\Lambda_1 + \Lambda_1^{(*)}), M(\delta, \delta-1) + M(\delta, \delta-1))$ *with* $\delta = t(2t+1)$ *and* $t \geq 2$.

Here $(G, \rho^{(*)})$ *implies* (G, ρ) *or* (G, ρ^*).

Proof. By [KKTI], we have (i), and by [KKIY], we have (ii)–(iv). □

Lemma 3.8. *Let* $\sigma : G_s \to GL(\delta)$ $(\delta \geq 3)$ *be an irreducible representation of a simple algebraic group* G_s *such that* $\sigma(G_s) \subsetneq SL(\delta)$. *For* $\delta \geq d_2 \geq 1$ *and* $m \geq 3$, *let* G_{ss} *be a semisimple algebraic proper subgroup of* $SL((m-2)\delta + d_2)$ *such that the inclusion* $\Lambda_1 : G_{ss} \hookrightarrow SL((m-2)\delta + d_2)$ *is irreducible. Put* $G = G_{ss} \times ((GL(1) \times G_s) \times GL(d_2) \times GL(\delta)^{\times(m-2)}), \rho = \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) \boxplus \Lambda_1 \boxplus \Lambda_1^{\boxplus(m-2)})$ *and* $V = V((m-2)\delta + d_2) \otimes (V(\delta) + V(d_2) + V(\delta)^{+(m-2)})$. *Then* $\dim G < \dim V$. *In particular* (G, ρ, V) *is a non-PV.*

Proof. By (ii) of Proposition 1.4, we have $\dim G_s \leq \frac{1}{2}\delta(\delta+1)$ and $\dim G_{ss} \leq \frac{1}{2}((m-2)\delta + d_2)((m-2)\delta + d_2 + 1)$. Since $\dim G = \dim G_{ss} + 1 + \dim G_s + d_2^2 + (m-2)\delta^2$ and $\dim V = ((m-2)\delta + d_2)((m-1)\delta + d_2)$, we have $\dim V - \dim G \geq \frac{1}{2}(m-1)(m-3)\delta^2 + \frac{1}{2}(2d_2-1)(m-1)\delta - \frac{1}{2}d_2(d_2+1) - 1 \geq \frac{1}{2}(2d_2-1)(m-1)\delta - \frac{1}{2}d_2(d_2+1) - 1$ $(= A)$. If $d_2 = 1$, then $A = \frac{1}{2}(m-1)\delta - 2 \geq \delta - 2 \geq 1$. If $d_2 \geq 2$, then $A \geq \frac{1}{2}(3d_2^2 - 3d_2 - 2) \geq 2$. Hence $\dim V > \dim G$. □

Lemma 3.9. (i) $\dim(GL(r) \times GL(s)) < \dim GL(r+s)$ *for* $1 \leq r \leq s < r+s \leq \delta$.

(ii) $\dim(GL(r) \times GL(s)) < \dim(GL(r+s-\delta) \times GL(\delta))$ *for* $1 \leq r \leq s < \delta < r+s$.

Proof. For (i), we have $\dim GL(r+s) - \dim(GL(r) \times GL(s)) = 2rs > 0$. For (ii), we have $\dim(GL(r+s-\delta) \times GL(\delta)) - \dim(GL(r) \times GL(s)) = 2(\delta-r)(\delta-s) > 0$. □

Now we shall prove Theorem 0.3. For the case $G_0 = SL(n)$ and $\rho(G) = G_1 \times \cdots \times G_m$, by Lemma 3.1, Proposition 3.2 and Proposition 3.4, we have (I)(i)–(iv) and (II)(i)–(v) of Theorem 0.3. Next we shall consider the case $G_0 \subsetneq SL(n)$ with $n = \delta$. By Lemmas 3.5 and 3.6, only the restriction of (I)(i) of Theorem 0.3 is possible and we obtain (I)(v) of Theorem 0.3. If $n = \delta$ and $\rho(G) \subsetneq G_1 \times \cdots \times G_m$, **T** is a non-PV by Lemma 3.7. Now assume that $n = d - \delta$. Since we may assume that $n \neq \delta$, we assume that $m \geq 3$. If $G_{ss} \subsetneq SL(d-\delta)$ and the inclusion $\Lambda_1 : G_{ss} \hookrightarrow SL(d-\delta)$ is irreducible, then by Lemma 3.9, we can reduce cases (I)(i)–(iv) and (II)(i)–(v) to the case of Lemma 3.8, and hence $\mathbf{T}$ is a non-PV. If the inclusion $\Lambda_1 : G_{ss} \hookrightarrow SL(d-\delta)$ is not irreducible, then by Proposition 1.10, $\mathbf{T}$ is a non-PV. Finally assume that it is a PV when $\rho(G) \subsetneq G_1 \times \cdots \times G_m$. Then it is also a PV when $G_0 = GL(d-\delta)$, and hence by castling transformation, it is a PV for $G_0 = GL(\delta)$. Since we assume that each $V(d_\mu)$ has an independent scalar action, $\mathbf{T}$ is still a PV for $G_0 = SL(\delta)$, a contradiction. The regularity follows from Propositions 1.11 and 1.12. Hence we obtain our result.

According to Remark 0.4, we shall give some examples of non-extreme PV's.

Proposition 3.10. *For $t \geq 2$, $\delta = 2t \geq 2r \geq 2$, $\delta = 2t > 2s+1 \geq 1$ and $2t < n < 2r+2s+1$, the triplet*

$$\mathbf{T} := (SL(n) \times ((GL(1) \times Sp(t)) \times GL(2r) \times GL(2s+1)), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1 \boxplus \Lambda_1))$$

is a PV of the non-extreme case. For example, $(SL(35) \times ((GL(1) \times Sp(16)) \times GL(18) \times GL(19)), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1 \boxplus \Lambda_1), V(35) \otimes (V(32) + V(18) + V(19)))$ is a PV.

Proof. Note that $\mathbf{T}$ is PV-equivalent to $(GL(n) \times (Sp(t) \times GL(2r) \times GL(2s+1)), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1), V(n) \otimes (V(2t) + V(2r) + V(2s+1)))$. Then it is castling equivalent to $(GL(n) \times (Sp(t) \times GL(n-2r) \times GL(n-2s-1)), \Lambda_1 \otimes \Lambda_1 \otimes 1 \otimes 1 + \Lambda_1^* \otimes 1 \otimes (\Lambda_1 \boxplus \Lambda_1), V(n) \otimes V(2t) + V(n)^* \otimes (V(n-2r) + V(n-2s-1)))$. Since $2t < n$ and $(n-2r)+(n-2s-1) < n$, by Theorem 7.8 in [K2], $\mathbf{T}$ is PV-equivalent to $(Sp(t) \times (GL(n-2r) \times GL(n-2s-1)), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1), V(2t) \otimes (V(n-2r) + V(n-2s-1))$. Since $2t > n-2r \geq 1$ and $2t > n-2s-1 \geq 1$, it is a PV (see (149), p. 197 in [KUY]).

□

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