Elemente der Mathematik 79 (2024), 25–29
DOI 10.4171/EM/489

Short note Three collinear triangle centres

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1 Introduction

A centre line is a line containing at least three triangle centres. The most prominent one, the Euler line [1,2], carries the centroid G, the orthocentre, the circumcentre and the centre of the Feuerbach circle of a triangle as well as at least four more triangle centres. Hundreds of centre lines are listed in [3].

2 Three collinear triangle centres

We will show that the following three triangle centres also are collinear, i.e. that they lie on a centre line.

(1) *The centroid G.*

(2) *The van Lamoen circle centre L* ([4], Figure 1 left). The three medians divide a triangle in six sub-triangles with a common vertex in the centroid G. The circumcentres of these six triangles lie on a circle, the van Lamoen circle with centre L.

(3) *The A-B-C-line intersection W* ([5], Figure 1 right). In a triangle with vertices A, B, C and corresponding medians s_A, s_B, s_C, let m_A and m_1 be the perpendicular bisectors of the triangle side BC (with midpoint M_A) and of the A-median AM_A (with midpoint M_1) respectively, and let N_1 be the intersection of these two lines. Definition: The A-line is the line parallel to the A-median s_A through the point N_1. In exactly the analogous way, the B- and the C-line are defined. In [5], it was shown that these three lines intersect at a point, the A-B-C-line intersection W.

Remark (On the properties of the A-, B- and C-line (Figure 2)). In a triangle as in (3) above, let G_A be any point different from A on the A-median s_A or its prolongation, and let E, F be the intersections of the lines G_AC, G_AB with the triangle sides AB, AC respectively or their prolongations. In [5], it was shown that, for every such point G_A, the circumcentres of the four triangles ABM_A, AM_AC, AEC, ABF lie on a circle, an A-circle. Since all A-circles pass through the circumcentres P_1, P_2 of the first two triangles, the locus of the A-circle centres is the (punctured) perpendicular bisector of the line seg-

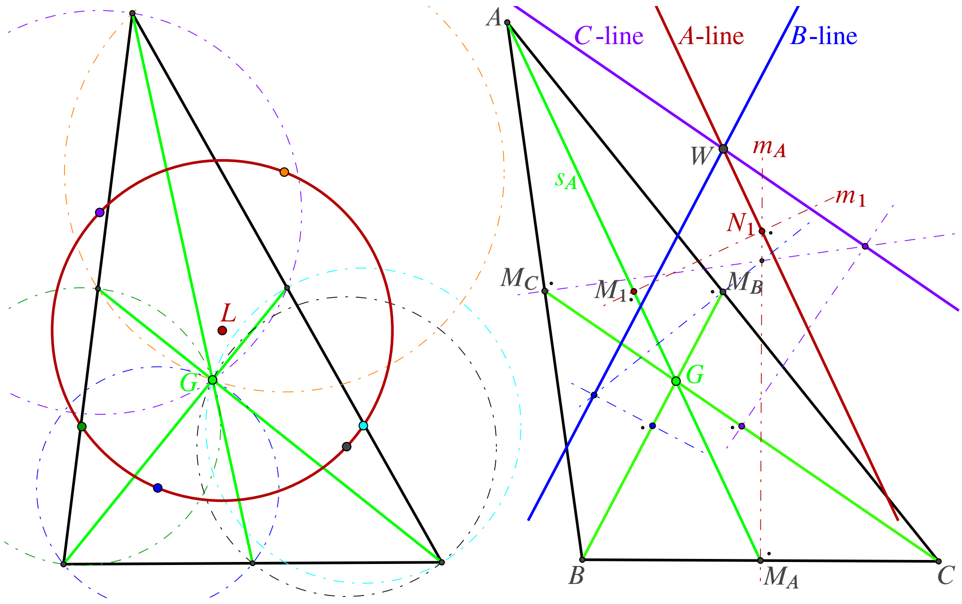

Figure 1

ment P_1P_2. But this is the A-line defined in (3) above since, obviously, it is parallel to s_A and since the points P_1, P_2 lie on the perpendicular bisector m_1 as well as on the perpendicular bisectors m_2, m_3 of the line segment BM_A and M_AC respectively, of which the line m_A is the centre line, such that the midpoint of the line segment delimited by P_1, P_2 is the point N_1.

The A-line is the locus of the A-circle centres when G_A moves on the median s_A.

Of course, the B- and the C-line have the exactly analogous properties.

Theorem. *The centroid G, the van Lamoen circle centre L and the A-B-C-line intersection W in this order lie on a line, and $|WL| : |LG| = 2 : 1$ holds.*

Remarks. (a) The theorem was found using GeoGebra. We give the original proof by elementary geometry. The assertions can be proved by means of the barycentric coordinates, calculated meanwhile by Peter Moses (moparmatic@gmail.com), of the three points G, L and W. They can be found in [3], where these points are named $X(2)$, $X(1153)$, $X(46893)$ and where many other properties of the points are given.

(b) According to [3] (see $X(1153)$), the van Lamoen circle centre $L = X(1153)$ lies on the line through the centroid $G = X(2)$ and the point $X(187)$ (Schoute centre). This means that it is already known that the line through the points G, L, W is a centre line. It is a centre line carrying at least four triangle centres.

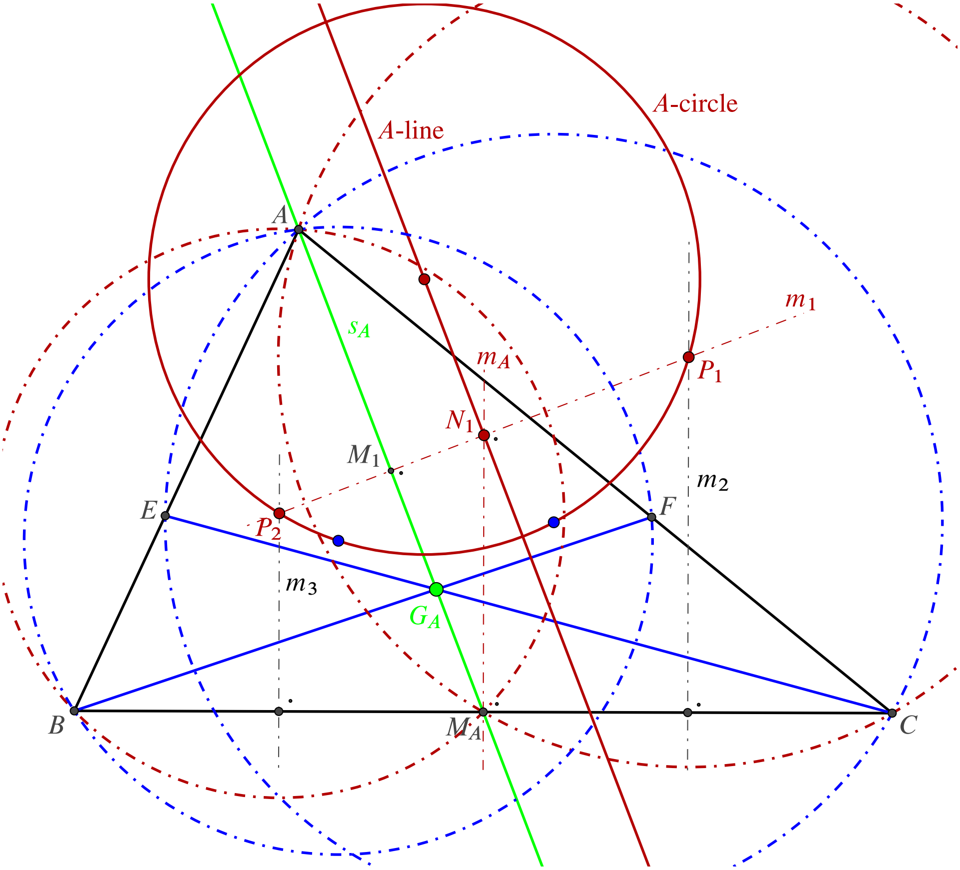

Figure 2

3 Proof

In Figure 3, besides the circumcentres P_1, P_2, we consider further the circumcentres R_1, R_2 of the triangles BM_AG and GM_AC. These are points on the van Lamoen circle. The perpendicular bisector v of the line segment they delimit hence passes through the centre L. The points R_1, R_2 lie on the perpendicular bisector m_4 of the line segment GM_A (with midpoint M_2) which is perpendicular to s_A. So the lines s_A, v and the A-line are parallel. The points R_1, R_2, just as the points P_1, P_2, also lie on the perpendicular bisectors m_2 and m_3 of the line segments BM_A and M_AC respectively. The midpoint N_2 of the line segment they delimit hence lies also on the perpendicular bisector m_A just as the point N_1. So the right triangles $M_AN_2M_2$ and $M_AN_1M_1$ are similar. For the distances $|M_1N_1|$ and $|M_2N_2|$ of the line v and the A-line from the line s_A respectively, we find

$$\frac{|M_1N_1|}{|M_2N_2|} = \frac{|M_1M_A|}{|M_2M_A|} = 3$$

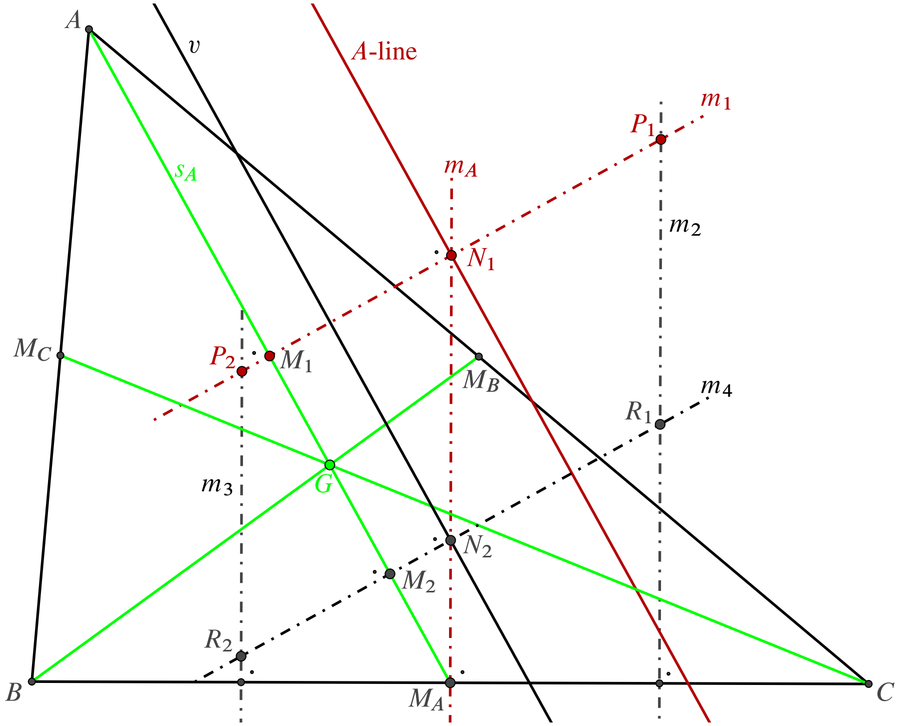

Figure 3

since we have $|M_1 M_A| = \frac{1}{2}|AM_A|$ and $|M_2 M_A| = \frac{1}{6}|AM_A|$. So we have found a line v passing through the point L and which by a central dilation with centre G and factor 3 is mapped to the A-line. In the same way, we can find another line w passing through L and which by the same central dilation is mapped to the B-line. In particular, for the intersections of the corresponding lines, we have the following statement.

> *A central dilation with centre in the centroid G and factor* 3 *maps the van Lamoen circle centre L to the A-B-C-line intersection W.*

This was to be proved.

Acknowledgements. I would like to thank the referee for his helpful remarks, especially for the one which allowed a simplification of the proof. I would like to thank also Professor Norbert Hungerbühler (ETH Zürich) because this text, like some others, would not have been written without his encouragement.

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