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Short note

A geometric property of the Möbius transformation

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Abstract. We show that, for any triple $(z_0, a, a^\star)$ of distinct points in $\mathbb{C}$, there exists a Möbius circle C such that $z_0 \in C$, and a and $a^\star$ are conjugate with respect to C. We use this fact to avoid tedious calculations when constructing a Möbius transformation $f\colon D \to G$, where D and G are Möbius disks, and f satisfies the conditions

$$f(z_0) = w_0, \quad f(a) = b, \quad z_0 \in \partial D, \quad w_0 \in \partial G, \quad a \in D, \quad b \in G.$$

1 Introduction

By a *Möbius disk* in the complex plane $\mathbb{C}$, we mean any open set that is a half-plane, or the interior or exterior of a circle of positive radius. The border of a Möbius disk is called a *Möbius circle* (it is therefore either a line or a circle.) This terminology is motivated by the fact that any Möbius disk corresponds to a spherical cap on the Riemann sphere $\overline{\mathbb{C}}$ under the stereographic projection. For all terms not explicitly defined in this note, see Ahlfors [1].

We recall that a Möbius transformation is a function $f\colon \mathbb{C} \to \mathbb{C}$ of the form $f(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Problem 1. Let $D, G \subseteq \mathbb{C}$ be Möbius disks, and let $a \in D$, $b \in G$, $z_0 \in \partial D$, $w_0 \in \partial G$ be arbitrary points. Find a Möbius transformation $f\colon D \to G$ such that

$$f(z_0) = w_0, \quad f(a) = b. \tag{1}$$

Recall that points $z, z^\star \in \mathbb{C}$ are called *conjugate* with respect to a line L if L is the perpendicular bisector of the segment $[z, z^\star]$; z and $z^\star$ are called *conjugate* with respect to a circle C of radius $R > 0$ centered at c if z and $z^\star$ lie on the same ray proceeding from c and $|z - c||z^\star - c| = R^2$.

We will use the following properties of Möbius transformations throughout this note (see [1, Section 3.3]).

- *Conformal property.* Any Möbius transformation $f(z)$ extended to $\overline{\mathbb{C}}$ via the formulas

$$\text{if } c \neq 0, \text{ then } f\left(-\frac{d}{c}\right) := \infty, \quad f(\infty) := \frac{a}{c}, \quad \text{if } c = 0, \text{ then } f(\infty) := \infty$$

is conformal on $\overline{\mathbb{C}}$. In particular, f is a homeomorphism of $\overline{\mathbb{C}}$.

- *Circular-preserving property.* The image of a Möbius circle under a Möbius transformation is a Möbius circle.
- *Conjugate property*. If C is a Möbius circle in $\mathbb{C}$ and $a, a^\star \in \mathbb{C}$ are conjugate with respect to C, then $f(a)$ and $f(a^\star)$ are conjugate with respect to $f(C)$.
- *Property of three points.* Any Möbius transformation is determined uniquely by its values at any three distinct points of the extended plane. In other words, given any three distinct points $z_i \in \overline{\mathbb{C}}$ and any three distinct points $w_i \in \overline{\mathbb{C}}$, $i = 1, 2, 3$, there exists a unique Möbius transformation f with the property $f(z_i) = w_i$.

We note that if a solution f of Problem 1 exists, then, by the conformal property,

$$f(\partial D) = \partial G. \tag{2}$$

Conversely, suppose f is any Möbius transformation satisfying (1) and (2). Then, by the conjugate property, $f(a^\star) = b^\star$, where $a^\star$ and $b^\star$ are points conjugate to a and b with respect to ∂D and ∂G respectively. Now, however, we know the value of f at three distinct points z_0, a, and $a^\star$, and by the property of three points, $w = f(z)$ is determined uniquely by its values at these points:

$$\frac{w-b}{w-b^\star} \cdot \frac{w_0-b^\star}{w_0-b} = \frac{z-a}{z-a^\star} \cdot \frac{z_0-a^\star}{z_0-a}. \tag{3}$$

In practice, it suffices to find function $w = f(z)$ from (3) and verify (2). As (3) implies (1), it will follow from (2) and the conformal property that $f(D) = G$, and so f is the unique solution of Problem 1. Verifying (2) directly, however, leads to very tedious calculations. In this note, we show in Theorem 2 that this is not necessary since the unique function $w = f(z)$ satisfying (3) also satisfies (2) and is therefore a unique solution of Problem 1. In order to prove our main result (Theorem 2), we first prove a property of conjugate points (Theorem 1). We conclude the note with a numerical example.

2 A property of conjugate points

Proposition 1. *Let C be a circle of radius $R > 0$ centered at $c \in \mathbb{C}$, and let $a, a^\star \in \mathbb{C} \setminus C$ be points conjugate with respect to C. Then the line L which is the orthogonal bisector of the segment $[a, a^\star]$ does not intersect C.*

Proof. Without loss of generality, suppose that $|a - c| < R$. We introduce the Cartesian coordinate system xy such that L is the y-axis, and c and the segment $[a, a^\star]$ are on the x-axis (Figure 1). Then $c > a > 0 > a^\star$, $a^\star = -a$, and

$$c^2 - a^2 = (c-a)(c+a) = |a-c||a^\star - c| = R^2, \tag{4}$$

from which it follows that $c > R$. Hence $L \cap C = \varnothing$. ■

We now prove a property of conjugate points which is used in our solution of Problem 1.

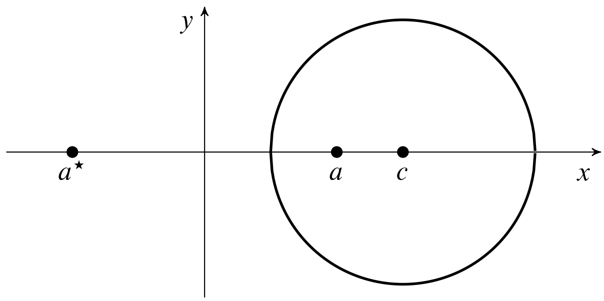

Figure 1

Theorem 1. *Let z_0, a, and $a^\star$ be distinct arbitrary complex numbers. Then there exists a unique Möbius circle in the complex plane containing z_0, with respect to which a and $a^\star$ are conjugate.*

Proof. Let L be the bisecting perpendicular of the segment $[a, a^\star]$. If $z_0 \in L$, then L is the only line satisfying the conditions of the theorem. It follows from Proposition 1 that there does not exist a circle satisfying the conditions of the theorem.

Suppose now that $z_0 \notin L$. Then neither L nor any other line satisfies the conditions of the theorem. Suppose further that z_0 and a belong to the same half-plane determined by L, and introduce a coordinate system as in the proof of Proposition 1 (Figure 1). If there exists a circle C of radius $R > 0$ satisfying the conditions of the theorem, then its center c is a real number. Thus $a, a^\star, c \in \mathbb{R}$ and

$$a > 0 > a^\star, \quad a^\star = -a, \quad \operatorname{Re} z_0 > 0. \tag{5}$$

Using (4), we find the center c and radius R of C from equations

$$R^2 = c^2 - a^2 = |z_0 - c|^2.$$

After elementary algebra, we find

$$c = \frac{|z_0|^2 + a^2}{2 \operatorname{Re} z_0}, \quad R = \sqrt{c^2 - a^2}. \tag{6}$$

Note that (5) and (6) together imply that $0 < R < c$, and so the circle C of radius R centered at the point $(c, 0)$ is a unique circle satisfying the conditions of the theorem. ∎

3 Solution of Problem 1

Below, we use the same notation as in the statement of Problem 1. Our main result is the following theorem.

Theorem 2. *Let $a^\star$ (respectively $b^\star$) be the point conjugate to a (respectively to b) with respect to ∂D (respectively ∂G), and let f be the Möbius transformation determined by (3). Then f is the unique solution of Problem 1.*

Proof. As noted above, f is a homeomorphism of $\overline{\mathbb{C}}$, and so it suffices to show (2).

By the circular-preserving property, $f(\partial D)$ is a Möbius circle and $w_0 = f(z_0) = \partial G \cap f(\partial D)$. By the conjugate property, $b = f(a)$ and $b^\star = f(a^\star)$ are conjugate with respect to $f(\partial D)$. The points b and $b^\star$, however, are also conjugate with respect to ∂G. By Theorem 1, there exists a unique Möbius circle containing w_0 and having the property that b and $b^\star$ are conjugate with respect to it. It follows that $f(\partial D) = \partial G$. ∎

4 Numerical example

We illustrate our main result with a numerical example.

Example 1. Find a Möbius transformation f mapping the exterior D of the circle

$$\partial D := \{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R} \text{ with } (x-3)^2 + (y-4)^2 = 25\}$$

onto the half-plane

$$G := \{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R} \text{ with } 5x + 2y + 10 > 0\}$$

such that f maps the point $a = \frac{1}{2} - i$ of D to the point $b = 0$ of G and the point $z_0 = 0$ of ∂D to the point $w_0 = -2$ of ∂G.

We find the points $a^\star = 1$ and $b^\star = -\frac{100}{29} - \frac{40}{29}i$ conjugate to a and b with respect to ∂D and ∂G respectively. Using (3), we find that

$$f(z) = \frac{(200 + 80i)z + (-180 + 160i)}{(-119 + 22i)z + (90 - 80i)}.$$

By Theorem 2, f is the unique solution satisfying the given conditions. Interestingly, the image of the point $\frac{52}{101} + \frac{86}{101}i \in \partial D$ under f is $\infty \in \partial G$.

We note that, parametrizing ∂D as $z = 3 + 4i + 5e^{i\theta}$, $\theta \in [0, 2\pi]$, it can be verified using a symbolic algebra system (we used SageMath) that indeed $f(\partial D) = \partial G$. As mentioned above, doing this by hand is arduously tedious.

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References

[1] L. Ahlfors, *Complex analysis*. 3rd edn., McGraw–Hill, New York, 1979

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