

Wall Crossing and M-Theory

by

Mina AGANAGIC, Hiroshi OOGURI, Cumrun VAFA and Masahito YAMAZAKI

Abstract

We study BPS bound states of D0 and D2 branes on a single D6 brane wrapping a Calabi–Yau 3-fold X . When X has no compact 4-cycles, the BPS bound states are organized into a free field Fock space, whose generators correspond to BPS states of spinning M2 branes in M-theory compactified down to 5 dimensions by a Calabi–Yau 3-fold X . The generating function of the D-brane bound states is expressed as a reduction of the square of the topological string partition function, in all chambers of the Kähler moduli space.

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§1. Introduction

The topological string theory gives solutions to a variety of counting problems in string theory and M-theory. From the worldsheet perspective, the A-model topological string partition function \mathcal{Z}_{top} generates the Gromov–Witten invariants, which count holomorphic curves in a Calabi–Yau (CY) 3-fold X . On the other hand, from the target space perspective, \mathcal{Z}_{top} computes the Gopakumar–Vafa (GV) invariants, which count BPS states of spinning black holes in 5 dimensions constructed from M2 branes in M-theory on X [9]. Moreover, the absolute-value-

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M. Aganagic: Center for Theoretical Physics, University of California, Berkeley, CA 94720, USA;
e-mail: mina@math.berkeley.edu

H. Ooguri: California Institute of Technology, Pasadena, CA 91125, USA and
IPMU, University of Tokyo, Chiba 277-8586, Japan;
e-mail: h.ooguri@gmail.com

C. Vafa: Jefferson Physical Laboratory, Harvard University, Cambridge, MA 02138, USA;
e-mail: vafa@physics.harvard.edu

M. Yamazaki: California Institute of Technology, Pasadena, CA 91125, USA and
IPMU, University of Tokyo, Chiba 277-8586, Japan and
Department of Physics, University of Tokyo, Tokyo 113-0033, Japan;
e-mail: masahito@princeton.edu

squared $|\mathcal{Z}_{\text{top}}|^2$ has been related to the partition function of BPS black holes in 4 dimensions, which are bound states of D branes in type II string theory on X [22].

The topological string partition function \mathcal{Z}_{top} also counts the numbers of D0 and D2 brane bound states on a single D6 brane on X , namely the Donaldson-Thomas (DT) invariants defined in [6, 27]. The relation between the GV invariants and DT invariants was suggested and formulated in [13, 16], and its physical explanation was given in [4] using the 4D/5D connection [8]. More recently, a mathematical proof of the GV/DT correspondence was given in [17] when X is a toric CY 3-fold.

However, the number of BPS states has background dependence. As we vary moduli of the background geometry and cross a wall of marginal stability, the number can jump [7, 15]. In this paper we will generalize the results of [4] to include the background dependence of the M-theory computation. We show BPS bound states are organized into a free field Fock space, whose generators correspond to BPS states of spinning M2 branes in M-theory compactified down to 5 dimensions by a Calabi–Yau 3-fold X . This enables us to write the generating function \mathcal{Z}_{BPS} of BPS bound states of D-branes as a reduction of the square of the topological string partition function,

$$(1.1) \quad \mathcal{Z}_{\text{BPS}} = \mathcal{Z}_{\text{top}}^2|_{\text{chamber}},$$

in an appropriate sense described in the following, in all chambers of the Kähler moduli space. Our results apply to the BPS counting for an arbitrary CY (whether toric or non-toric) without compact 4-cycles.

For the conifold, the change of the numbers of BPS states across a wall of marginal stability has been studied by physicists in [14, 2] (see also [5]) and mathematicians [28, 20]. The case of generalized conifold geometries was studied in [19]. The formula (1.1) derived from the perspective of M-theory reproduces these results. Our results also provide a simple derivation of the “semi-primitive” wall crossing formula of Denef and Moore [3], in the present context.

The rest of the paper is organized as follows. In Section 2, we will explain the basic idea to use M-theory to count bound states of a single D6 brane with D0 and D2 branes on a CY 3-fold. In Section 3, we will describe the counting procedure in more detail and derive the generating function for the numbers of BPS bound states using a free field Fock space in any chamber of the background Kähler moduli space. In Section 4, we will compare the Fock space picture with the known results for the resolved conifold and its generalizations. Finally, in Section 5 we point out that our results give a derivation of the Denef–Moore “semi-primitive” wall-crossing formula in the present context.

§2. The basic idea

In this section we will explain the basic idea. We will apply this idea, in the following sections, to find a concrete expression for BPS state degeneracies in various chambers for CY 3-folds with no compact 4-cycles.

We are interested in counting the BPS partition function of one D6 brane bound to arbitrary number of D2 and D0 branes. The idea is the following: In M-theory, the D6 brane lifts to the Taub-NUT space with the unit charge. D2 branes are M2 branes transverse to the S^1 , and D0 branes are gravitons with Kaluza-Klein momenta along the S^1 . The Taub-NUT space is an S^1 fibration over \mathbb{R}^3 , and S^1 shrinks at the position of D6. Thus the problem of finding bound states to the D6 brane becomes simply the problem of finding BPS states in the Taub-NUT geometry. Suppose we have BPS states for flat $\mathbb{R}^{4,1}$ background. Then for each such BPS state we can consider the corresponding possible BPS states in the Taub-NUT geometry. For each single particle BPS state we can consider its normalized wave functions in this geometry. Such states would constitute BPS states which in the type IIA reduction correspond to BPS particles bound to the D6 brane. However, this would only constitute single particle BPS states bound to the D6 brane.

Now consider multiple such particles in the Taub-NUT background. This problem may sound formidable, because now we will have to consider the interaction of such particles with each other and even their potentially forming new bound states. We will now make the following two assumptions:

Assumption 1. We can choose the background moduli of CY as well as the chemical potential so that a maximal set of BPS states have parallel central charge and thus exert no force on one another. Therefore, at far away separation, the bound states correspond to single particle wave functions in the Taub-NUT geometry.

Assumption 2. The only BPS states in 5D are particles. In other words there are no compact 4-cycles in the CY and thus we can ignore BPS *string states* obtained by wrapping 5 branes around 4-cycles.

Assumption 1 can be satisfied as follows: Consider the Euclidean geometry of M-theory in the form of Taub-NUT times S^1 , where we have compactified the Euclidean time on the circle. The BPS central charge for M2 branes wrapping 2-cycles of CY, but with no excitation along the Taub-NUT, is given by

$$Z(M2) = iA(M2) - C(M2),$$

where $A(M2)$ denotes the area of the M2 brane and $C(M2)$ corresponds to the coupling of the M2 brane to the 3-form potential turned on along the CY 2-cycles

as well as the S^1 of the Taub-NUT. However we need to include excitations along the Taub-NUT. As discussed in [4] these are given by the momenta along the Taub-NUT circle. Let us denote the total momentum along the circle by n (as we will review in Section 3, this can arise both due to internal spin as well as the orbital spin in the $SU(2)_L \subset SO(4)_{\text{rotation}}$). Let us denote the radius of the Taub-NUT circle by R . In this case the central charge of the BPS M2 brane becomes¹

$$Z(M2, n) = iA(M2) - C(M2) - n/R.$$

To satisfy Assumption 1, we need to make sure that differently wrapped M2 branes all have the same phase for Z . This in particular means that we need to choose the Kähler classes so that the 2-cycles of CY have all shrunk to zero size, i.e. $A(M2) = 0$ for all the classes. Even though this may sound singular and it could lead to many massless states, by turning on the $C(M2)$ we can avoid generating massless states in the limit. The condition that different states have the same central charge is simply that

$$(2.1) \quad C(M2) + n/R > 0.$$

Note that, in going to type IIA, this condition is simply the statement that the B fields are turned on along 2-cycles of CY and the M2 branes wrapping them will have $B(D2)$ 0 branes induced. Moreover n translates to a D0 brane charge since it is the momentum along the Taub-NUT. Thus, these states correspond to BPS states of the same type, i.e., preserving the same supersymmetry, as long as the net number of 0 branes is positive.

Now we are ready to put together all these mutually BPS states as a gas of particles in the Taub-NUT geometry. By the fact that they are mutually BPS, they will exert no force on one another. Moreover, as long as they are far away, we can simply consider the product of the individual wave functions. One may worry what happens if they come close together. Indeed they can form bound states, but that is already accounted for by including all single particle bound

¹Note that $C(M2)$ is periodic with period $1/R$. To see this, note that we can view it as a holonomy of the gauge field obtained by reducing the 3-form on the 2-cycle of CY, around the Taub-NUT circle. The holonomy of a gauge field on a circle of radius R is periodic, with period $1/R$. In terms of the IIA quantities, we have

$$C(M2) = B(D2)/R,$$

where $B(D2)$ is the NS-NS B-field through the 2-cycle in IIA on the CY wrapped by the corresponding D2 brane (which has periodicity $B \rightarrow B + 1$). Since the central charge of the D0 and D2 brane is in general complex, there is an overall complex number multiplying $1/R$. For simplicity of notation, we will suppress this overall complex number in the following, but this fact has to be kept in mind. Note also that relative to [14] we are keeping the D6 brane charge fixed (for example, to $Z(D6) = 1$), and varying the D0 and D2 brane central charges.

states of M2 brane. Here is where Assumption 2 becomes important: If we in addition had 4-cycles, then wrapped M5 branes along 4-cycles, which also wrap the S^1 of Taub-NUT can now form new bound state with the gas of M2 brane particles on the Taub-NUT. But in the absence of 4-cycles of CY, we can simply take the single particle wave functions (taking their statistics into account) and write the total degeneracy of such BPS states, by taking suitable bosonic/fermionic creation operators, one for each state satisfying $C(M2) + n/R > 0$. Finally, while Assumption 1 is satisfied only for special backgrounds where $A(M2)$ vanishes, the degeneracies are guaranteed to be the same everywhere within a given chamber, and independent of this choice. This is all we need to compute all the degeneracies of BPS states in various chambers as we will show in the following sections.

§3. BPS state counting and wall crossing

We will use this section to spell out, in a little more detail, how to use M-theory to compute the degeneracies of one D6 brane on X bound to D2 branes wrapping 2-cycles in X and D0 branes. The D6-D2-D0 partition function is the Witten index²

$$\mathrm{Tr}[(-1)^F e^{-\epsilon H}]$$

of the theory on

$$X \times \mathbb{R}^3 \times S_t^1,$$

where we have compactified the Euclidian time on a circle of radius ϵ . The type IIA geometry with one D6 brane lifts to M-theory on

$$X \times \text{Taub-NUT} \times S_t^1,$$

where the asymptotic radius of the Taub-NUT circle R is related to IIA string coupling. Since the D6 brane is geometrized, the computation of the BPS bound states of D2 branes and D0 branes with D6 brane lifts to a question of computing the degeneracies of M2 branes with momentum around the Taub-NUT circle.

Suppose we know the degeneracies of M-theory in the

$$X \times \mathbb{R}^4 \times S^1.$$

This corresponds to taking the $R \rightarrow \infty$ limit, where the Taub-NUT just becomes \mathbb{R}^4 . As is clear from the previous section, at *fixed* B the degeneracies are *unchanged* by varying R since no states decay in the process—all the central charges

² Here we are ignoring the fermionic zero modes in the 4 non-compact directions. Otherwise, additional factors need to be inserted to absorb these.

simply get rescaled. Thus, the knowledge of these allows us to compute the degeneracies on $X \times \text{Taub-NUT} \times S^1$ background as well.

The Kaluza–Klein momentum around the Taub-NUT circle gets identified, in terms of the theory in the $R \rightarrow \infty$ limit, with the total spin of the M2 brane. This can be understood by comparing the isometries of the finite and the infinite R theory, as explained in [4]. We can view taking R to infinity as zooming in to the origin of the Taub-NUT. The isometry group is the rotation group $SO(4) = SU(2)_L \times SU(2)_R$ about the origin of \mathbb{R}^4 . The $SU(2)_R$ is identified with the $SO(3)$ that rotates the sphere at infinity of the \mathbb{R}^3 base of the Taub-NUT. Moreover, the rotations around the S^1 of the Taub-NUT end up identified with the

$$U(1) \subset SU(2)_L.$$

Thus, the Kaluza–Klein momentum is identified with the total J_z^L spin of the M2 brane on \mathbb{R}^4 .

Now, let

$$N_\beta^{(m_L, m_R)}$$

be the degeneracy of the *5-dimensional BPS* states of M2 branes of charge β and spin the intrinsic $(2j_L^z, 2j_R^z) = (m_L, m_R)$ (where the spin refers to the spin of the highest state of the multiplet). To get an index, we will be tracing over the $SU(2)_R$ quantum numbers, so we get a net number

$$N_\beta^{m_L} = \sum_{m_R} (-1)^{m_R} N_\beta^{(m_L, m_R)}$$

of 5D BPS states, of the fixed $SU(2)_L$ spin m_L .

Each such 5D BPS particle can in addition have excitations on \mathbb{R}^4 . Namely, for each 5D particle we get a field

$$\Phi(z_1, z_2)$$

on \mathbb{R}^4 with $z_{1,2}$ as the complex coordinates. In the usual way, the modes of this field

$$\Phi(z_1, z_2) = \sum_{\ell_1, \ell_2} \alpha_{\ell_1, \ell_2} z_1^{\ell_1} z_2^{\ell_2}$$

correspond to the ground-state wave functions of the particle with different momenta on \mathbb{R}^4 . (We are suppressing a Gaussian factor that ensures the wave functions are normalizable.) Since $U(1) \in SU(2)_L$ acts on z_1, z_2 with charge 1, the particle corresponding to

$$\alpha_{\ell_1, \ell_2}$$

carries, in addition to the M2 brane charge β and intrinsic momentum m , a total angular momentum³

$$n = \ell_1 + \ell_2 + m.$$

Which of these 1-particle states are mutually BPS? The answer depends on the background, and a priori, we need to consider particles in four dimensions coming from both the M2 branes and the anti-M2 branes in M-theory. Along the slice in the moduli space we have been considering, the central charge of the particle with M2 brane charge β and total spin n is

$$Z(\beta, n) = \beta C + n/R = (\beta B + n)/R.$$

The states with

$$(3.1) \quad Z(\beta, n) > 0$$

all preserve the same supersymmetry and bind to the D6 brane (we could have picked the opposite sign, and then the particles would bind to anti-D6 branes). For example, for

$$B > 0, \quad R > 0$$

alongside M2 branes with $\beta > 0$, and for sufficiently large n also the anti-M2 branes with $\beta < 0$ have positive $Z > 0$ and contribute to the BPS partition function. So in general we need to consider both signs of β . It is important to note that the degeneracies N_β^m of the 5D particles are independent of the background. The choice of background only affects which half of the supersymmetry the states preserve.

Now, we can put these all together and compute the BPS partition function in a given chamber. Simply, in each chamber, the BPS partition function is the character in the Fock space of single particle states preserving the same supersymmetry! In fact, a useful way to go about computing the partition function is in steps:

Step 1. Start with the unrestricted partition function—the character

$$\mathcal{Z}_{\text{Fock}} = \text{Tr}_{\text{Fock}} q^{Q_0} Q^{Q_2}$$

in the full Fock space. The oscillators of charge β and intrinsic spin m and arbitrary 4d momenta contribute a factor

$$(3.2) \quad \prod_{\ell_1 + \ell_2 = n} (1 - q^{\ell_1 + \ell_2 + m} Q^\beta)^{N_\beta^m} = (1 - q^{n+m} Q^\beta)^{n N_\beta^m}.$$

³In the present case, we are restricting to CY manifolds with no compact 4-cycles. When the CY is furthermore toric, as in the cases discussed in Sections 4.1 and 4.2, the genus of the target space curve wrapped by the M2 branes vanishes. This means that the intrinsic spin of all the M2 branes vanishes as well.

In addition, both the M2 branes and the anti-M2 branes contribute, and the total character is

$$\mathcal{Z}_{\text{Fock}} = \prod_{\beta, m} \prod_{n=1}^{\infty} (1 - q^{n+m} Q^{\beta})^{n N_{\beta}^m}.$$

Step 2. The 5d degeneracies N_{β}^m of M-theory on $X \times R^{4,1}$ are computed by the topological string partition function on X [9, 10]. This allows us to write

$$\mathcal{Z}_{\text{Fock}} = \mathcal{Z}_{\text{top}}(q, Q) \mathcal{Z}_{\text{top}}(q, Q^{-1}).$$

In particular, the knowledge of topological string amplitude allows us to compute the BPS degeneracies in any chamber.

The topological string partition function has an expansion

$$\mathcal{Z}_{\text{top}}(q, Q) = M(q)^{\chi(X)/2} \prod_{\beta > 0, m} \prod_{n=1}^{\infty} (1 - q^{m+n} Q^{\beta})^{n N_m^{\beta}}.$$

where q and Q are determined by the string coupling constant g_s and the Kähler moduli t by $q = e^{-g_s}$ and $Q = e^{-t}$. The MacMahon function $M(q)$ is defined by

$$M(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-n}.$$

Above, $\chi(X)$ is the Euler characteristic of X . Note that topological string involves only the M2 states with positive $\beta > 0$. On the other hand, the full Fock space includes also anti-M2 branes. Since M2 branes and anti-M2 branes are CPT conjugates in 5d, this gives another factor of \mathcal{Z}_{top} with $Q \rightarrow Q^{-1}$.

Note that we also have states with $\beta = 0$. These are the pure KK modes, the particles with no M2 brane charge. To count the number of BPS states of this type, we note that, for each R^4 momentum (l_1, l_2) we get a classical particle whose moduli space is the Calabi–Yau X . Quantizing this, we get a particle for each element of the cohomology of X . On a (p, q) form, $SU(2)_R$ acts with the Lefschetz action, and $SU(2)_L$ acts trivially. We find that the m_R eigenvalue of a (p, q) form on X is $m_R = p + q - 3$. Therefore, the pure KK modes contribute with $N_{\beta=0} = -\chi(X)$. This agrees with the power of the MacMahon function $M(q)$ we get from $\mathcal{Z}_{\text{top}}(q, Q) \mathcal{Z}_{\text{top}}(q, Q^{-1})$.

Step 3. We identify the walls of marginal stability as places where the central charge vanishes for one of the oscillators contributing to $\mathcal{Z}_{\text{top}}(q, Q)$ or $\mathcal{Z}_{\text{top}}(q, Q^{-1})$.

Step 4. In any chamber, the BPS partition function is a restriction of $\mathcal{Z}_{\text{Fock}}$ to the subspace of states that satisfy $Z(\beta, n) > 0$ in that chamber:

$$(3.3) \quad \mathcal{Z}_{\text{BPS}}(\text{chamber}) = \mathcal{Z}_{\text{Fock}}|_{\text{chamber}}$$

$$(3.4) \quad = \mathcal{Z}_{\text{top}}(q, Q)\mathcal{Z}_{\text{top}}(q, Q^{-1})|_{\text{chamber}}.$$

There is a simple way to keep track of the chamber dependence. For the book-keeping purposes, it is useful to identify the central charge with the chemical potentials. Then, in a given chamber, the BPS states are those for which

$$q^n Q^\beta < 1$$

where $n = m + k$ is the total spin. As we vary the background, and cross into a chamber where this is no longer satisfied for some (n, β) in $\mathcal{Z}_{\text{top}}(q, Q)$ or in $\mathcal{Z}_{\text{top}}(q, Q^{-1})$, we drop the contribution of the corresponding oscillator.

For example, consider some special cases. When

$$(3.5) \quad R > 0, \quad B \rightarrow \infty,$$

for all Kähler classes, $Z(\beta, n) = (\beta B + n)/R > 0$ implies that

$$\beta > 0.$$

In this case, only M2 branes contribute to the partition function. This is the chamber discussed in [4]. By taking the limit (3.5) in (3.4), we find

$$\mathcal{Z}_{\text{BPS}}(R > 0, B \rightarrow \infty) = \mathcal{Z}_{\text{DT}}(q, Q) = M(q)^{\chi/2} \mathcal{Z}_{\text{top}}(q, Q).$$

The partition function in this chamber computes DT invariants. In [17], it was shown that, for a toric CY, \mathcal{Z}_{BPS} is equal to \mathcal{Z}_{top} up to a factor which depends only on q . Here we derived the relation between \mathcal{Z}_{BPS} and \mathcal{Z}_{top} including the factor of $M(q)^{\chi/2}$.

On the other hand, when $0 < B \ll 1$, the BPS partition function is given by

$$(3.6) \quad \mathcal{Z}_{\text{BPS}}(q, Q) = \mathcal{Z}_{\text{NCDT}}(q, Q) = \mathcal{Z}_{\text{top}}(q, Q)\mathcal{Z}_{\text{top}}(q, Q^{-1}).$$

This gives the non-commutative DT invariants studied in [26, 18, 23]. When X is toric, the partition function is computed using the crystal melting picture [18, 23], generalizing the previous result of [21, 13] for \mathbb{C}^3 . In [24], it was shown that the thermodynamic limit of the partition function of the crystal melting model gives the genus-0 topological string partition function. This result was mysterious since the relation between \mathcal{Z}_{top} and \mathcal{Z}_{BPS} was supposed to hold in the DT chamber discussed in the previous paragraph. We now understand why there is such a relation in the non-commutative DT chamber also as in (3.6).

§4. Examples

In this section we give some examples of geometries without compact 4-cycles. We first study toric cases, namely resolved conifold and generalized conifolds. We also give a simple example of a non-compact, non-toric Calabi–Yau as well. In each of these cases, we will use our methods to lay out the chamber structure, identifying walls where BPS states jump, and the BPS partition function in each chamber. In some of the cases we study, the jumps were studied by other means. We will show that they agree with the M-theory results.

§4.1. Resolved conifold

The topological string partition function for the resolved conifold is given by

$$(4.1) \quad \mathcal{Z}_{\text{top}}(q, Q) = M(q) \prod_{n=1}^{\infty} (1 - q^n Q)^n.$$

This means that the only non-vanishing GV invariants are

$$(4.2) \quad N_{\beta=\pm 1}^0 = 1, \quad N_{\beta=0}^0 = -2,$$

and that no BPS state in 5 dimensions has intrinsic spin [9, 10]. Our formula (3.4) then implies that BPS states are counted by

$$(4.3) \quad \mathcal{Z}_{\text{BPS}}(q, Q) = \mathcal{Z}_{\text{top}}(q, Q) \mathcal{Z}_{\text{top}}(q, Q^{-1})|_{\text{chamber}}$$

$$(4.4) \quad = \prod_{(\beta, n) : Z(\beta, n) > 0} (1 - q^n Q^\beta)^{n N_\beta^0}.$$

The product is over $\beta = 0, \pm 1$ and $n = 1, 2, \dots$ such that $Z(\beta, n) > 0$.

The chamber structure is easy to identify in this case since the Kähler moduli space is one-dimensional. When

$$(4.5) \quad R > 0 \quad \text{and} \quad m - 1 < B < m$$

with some $m \geq 1$, the formula (4.4) gives

$$(4.6) \quad \mathcal{Z}_{\text{BPS}}(q, Q) = M(q)^2 \prod_{n=1}^{\infty} (1 - q^n Q)^n \prod_{n=m}^{\infty} (1 - q^n Q^{-1})^n.$$

In particular, the chamber at $m = \infty$ counts the DT invariants [6],

$$(4.7) \quad \mathcal{Z}_{\text{BPS}}(q, Q) = \mathcal{Z}_{\text{DT}}(q, Q) = M(q)^2 \prod_{n=1}^{\infty} (1 - q^n Q)^n,$$

while the chamber at $m = 1$ counts the non-commutative DT invariants [26],

$$(4.8) \quad \mathcal{Z}_{\text{BPS}}(q, Q) = \mathcal{Z}_{\text{NCDT}}(q, Q) = M(q)^2 \prod_{n=1}^{\infty} (1 - q^n Q)^n \prod_{n=1}^{\infty} (1 - q^n Q^{-1})^n.$$

On the other hand, when

$$(4.9) \quad R < 0 \quad \text{and} \quad -m - 1 < B < -m$$

with $m \geq 1$, we have

$$(4.10) \quad \mathcal{Z}_{\text{BPS}}(q, Q) = \prod_{n=1}^m (1 - q^n Q)^n.$$

In particular, the chamber at $m = \infty$ counts the Pandharipande–Thomas invariants [25],

$$(4.11) \quad \mathcal{Z}_{\text{BPS}}(q, Q) = \mathcal{Z}_{\text{PT}}(q, Q) = \prod_{n=1}^{\infty} (1 - q^n Q)^n.$$

These agree with the results in [26, 14, 2, 20] in all chambers.

§4.2. Toric CY without compact 4-cycles

We can also test our formula (3.4) for a more general toric CY without compact 4-cycles. A toric CY is characterized by a convex polygon on a square lattice, and the absence of compact 4-cycles means that there is no internal lattice point in the polygon. By $SL(2, \mathbb{Z})$ transformations of the lattice, one can move one of the edges of the polygon along the positive x -axis, and one of the vertices to (x, y) with $-y < x \leq 0$. If we require that there is no internal lattice point, there are essentially two possibilities: $(x, y) = (0, 1)$ and $(0, 2)$. In the former case, the polygon is a trapezoid of height 1, and the corresponding CY is the so-called generalized conifold, which has $N - 1$ \mathbb{P}^1 's where N is the area of the trapezoid. We will describe the resolved geometry in more detail below. In the latter case, we have an isosceles right triangle with two legs of length 2, which corresponds to $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$.

For the generalized conifold, the topological string partition function has been computed in [12] using the topological vertex [1]. The counting of BPS states has been carried out in all chambers in [19]. Thus, we will use this case to test our formula (3.4). For $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$, the counting in the non-commutative DT chamber has been done in [29].

Homology 2-cycles of the generalized conifold correspond to the simple roots $\alpha_1, \dots, \alpha_{N-1}$ of the A_{N-1} algebra. To identify them in the toric diagram, we divide

the trapezoid into N triangles of area 1 and label the internal lines dividing the triangles as $i = 1, \dots, N - 1$. Each line i corresponds to the blowing up \mathbb{P}^1 at α_i . We will denote the D2 charge by

$$(4.12) \quad \beta = \sum_i n_i \alpha_i.$$

In general, there are several ways to divide the trapezoid, and they correspond to different crepant resolutions of the singularity. If the two triangles across the line i form a rhombus, we have a resolution by $\mathcal{O}(-1, -1)$. On the other hand, if the two triangles form a triangle of area 2, the resolution is by $\mathcal{O}(-2, 0)$. Both the topological string partition function and the BPS counting depend on the choice of the resolution.

The topological string partition function for this geometry is given by

$$(4.13) \quad \mathcal{Z}_{\text{top}}(q, Q) = M(q)^{N/2} \prod_{n=1}^{\infty} \prod_{i \leq j} (1 - q^n Q_i Q_{i+1} \cdots Q_j)^{n N_{ij}},$$

where

$$(4.14) \quad N_{ij} = (-1)^{1+n_{ij}},$$

$n_{ij} = \#\{k \in I \mid i \leq k \leq j\}$, and I is the set of internal lines of the toric diagram corresponding to the resolution by $\mathcal{O}(-1, -1)$. Thus, the only non-vanishing GV invariants are

$$(4.15) \quad N_{\beta}^{m=0} = (-1)^{1+\sum_{i \in I} n_i}$$

for a root vector $\beta = \sum_i n_i \alpha_i$ of A_{N-1} , and

$$(4.16) \quad N_{\beta=0}^{m=0} = -N.$$

Note that when β is a positive root $\beta_{ij} = \beta_i + \cdots + \beta_j$, (4.15) reduces to (4.14). No BPS states in 5 dimensions carry intrinsic spin.

The central charge $Z(\beta, n)$ is given by

$$(4.17) \quad Z(\beta, n) = R^{-1} \left(n + \sum_i n_i B_i \right),$$

where B_i is the B-field evaluated on α_i . The formula (3.4) predicts that BPS states in the chamber characterized by B_i 's are counted as

$$(4.18) \quad \mathcal{Z}_{\text{BPS}}(q, Q) = M(q)^N \prod_{(\beta, n) : Z(\beta, n) > 0} (1 - q^n Q^{\beta})^{n N_{\beta}^0}.$$

Here the product is over all roots β of A_{N-1} and $n = 1, 2, \dots$ such that $Z(\beta, n) > 0$. This agrees with the result in [19].

§4.3. A non-toric example

Our discussion in Sections 2 and 3 is not limited to toric CYs, and applies to any CY without compact 4-cycles. In order to illustrate this point in a concrete setting, let us describe the geometry shown in Figure 1. This geometry arises by identifying two of the four external legs of the (p, q) -web of the resolved conifold. This is one of the simplest non-toric geometries studied in [11], and it is straightforward to repeat the following analysis for other non-toric geometries discussed in [11].

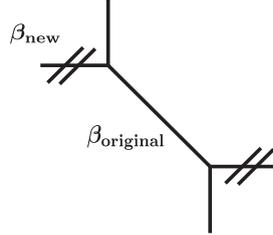


Figure 1. The non-toric CY which arises by identifying two external legs of the (p, q) -web of the resolved conifold.

In addition to the \mathbb{P}^1 of the resolved conifold, the geometry of Figure 1 has another compact \mathbb{P}^1 which arises from identification. Let us denote their homology classes by β_{original} and β_{new} , respectively. As a basis of the homology class, we choose $\beta_1 = \beta_{\text{original}}$ and $\beta_2 = \beta_{\text{original}} + \beta_{\text{new}}$.

The topological string partition function is given by [11]

$$(4.19) \quad \mathcal{Z}_{\text{top}}(q, Q_1, Q_2) = M(q) \left(\prod_{n=1}^{\infty} (1 - Q_1 q^n)^n \right) \prod_{k,n=1}^{\infty} \left(\frac{(1 - q^n Q_1 Q_2^k)(1 - q^n Q_1^{-1} Q_2^k)}{(1 - q^{n-1} Q_2^k)(1 - q^{n+1} Q_2^k)} \right)^n$$

where Q_1 and Q_2 are the variables corresponding to β_1 and β_2 . The GV invariants are therefore given by

$$(4.20) \quad N_{\beta=0}^0 = -2, \quad N_{\pm\beta_1}^0 = 1, \quad N_{\pm\beta_1+k\beta_2}^0 = 1, \quad N_{k\beta_2}^{\pm 1} = -1 \quad (k \in \mathbb{Z} \setminus \{0\}).$$

Notice that genus 1 GV invariants are non-vanishing in this non-toric example.

Again, the general formula gives (notice that $m \neq 0$ in this case)

$$(4.21) \quad \mathcal{Z}_{\text{BPS}}(q, Q) = \mathcal{Z}_{\text{top}}(q, Q) \mathcal{Z}_{\text{top}}(q, Q^{-1})|_{\text{chamber}} = \prod_{(\beta,l,m): Z(\beta,n=m+l) > 0} (1 - q^{l+m} Q^\beta)^{l N_\beta^m}.$$

The formula for the central charge is

$$Z\left(\beta = \sum_{i=1,2} n_i \beta_i, n\right) = R^{-1}\left(\sum_i n_i B_i + n\right),$$

and the equation

$$Z(\beta, n = l + m) = 0,$$

with corresponding GV invariants non-vanishing, determines the position of walls of marginal stability. This is a new result which has not been discussed in the literature to the best of our knowledge.

§5. Relation to the Denef–Moore formula

In this final section, we point out that our M-theory viewpoint discussed in this paper allows us to derive, in the present context of D6, D2 and D0 degeneracies, the “semi-primitive” wall crossing formula of [3]. The latter says the following. Suppose a BPS bound state of charge γ decays into two fragments. Since the D6 brane is non-compact and fills the entire CY, the fragments should have charges $\gamma_1 = (1, 0, \beta', n') \in H_6 \oplus H_4 \oplus H_2 \oplus H_0$ and $\gamma_2 = (0, 0, \beta, n)$ [14]. The position of walls is determined by the condition that the central charges align

$$(5.1) \quad \text{Im}(Z(\gamma_1)\overline{Z(\gamma_2)}) = 0,$$

where $Z(\gamma_i)$ are the central charges for the D brane charges γ_i . The prediction of [3] is that across such a “semi-primitive” wall the partition function jumps by a factor⁴

$$(1 - q^n Q^\beta)^{nN(\beta, n)}.$$

The fact that the same factors enter the topological string partition begged for an explanation. We have provided it by using STS duality to relate both the topological string and the D6-D2-D0 degeneracy counting to M-theory, where the computations unify.⁵ Note however that, while the topological string computes the pieces of the D6-D2-D0 degeneracies, the two partition functions are the same only in one chamber.

⁴We here consider cases where GV invariants are vanishing except for genus 0, such as the toric examples discussed in Sections 4.1 and 4.2.

⁵In this paper, we are considering the subspace of the moduli space where all the central charges are real. The walls are restrictions of the walls discussed by Denef and Moore to this subspace.

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References

- [1] M. Aganagic, A. Klemm, M. Marino and C. Vafa, The topological vertex, *Comm. Math. Phys.* **254** (2005), 425–478. Zbl 1114.81076 MR 2117633
- [2] W. Y. Chuang and D. L. Jafferis, Wall crossing of BPS states on the conifold from Seiberg duality and pyramid partitions, *Comm. Math. Phys.* **292** (2009), 285–301. Zbl 1179.81079 MR 2540079
- [3] F. Denef and G. W. Moore, Split states, entropy enigmas, holes and halos, arXiv:hep-th/0702146.
- [4] R. Dijkgraaf, C. Vafa and E. Verlinde, M-theory and a topological string duality, arXiv:hep-th/0602087.
- [5] T. Dimofte and S. Gukov, Refined, motivic, and quantum, *Lett. Math. Phys.* **91** (2010), 1–27. Zbl 1180.81112 MR 2577296
- [6] S. K. Donaldson and R. P. Thomas, Gauge theory in higher dimensions, in *The geometric universe: science, geometry and the work of Roger Penrose*, Oxford Univ. Press, 1998, 31–47. Zbl 0926.58003 MR 1634503
- [7] D. Gaiotto, G. W. Moore and A. Neitzke, Four-dimensional wall-crossing via three-dimensional field theory, *Comm. Math. Phys.* **299** (2010), 163–224. Zbl pre05789925 MR 2672801
- [8] D. Gaiotto, A. Strominger and X. Yin, New connections between 4D and 5D black holes, *J. High Energy Phys.* **2006**, no. 2, 024, 10 pp. MR 2218039
- [9] R. Gopakumar and C. Vafa, M-theory and topological strings. I, arXiv:hep-th/9809187.
- [10] ———, M-theory and topological strings. II, arXiv:hep-th/9812127.
- [11] T. J. Hollowood, A. Iqbal and C. Vafa, Matrix models, geometric engineering and elliptic genera, *J. High Energy Phys.* **2008**, no. 3, 069, 81 pp. MR 2391051
- [12] A. Iqbal and A. K. Kashani-Poor, The vertex on a strip, *Adv. Theor. Math. Phys.* **10** (2006), 317–343. Zbl 1131.14045 MR 2250273
- [13] A. Iqbal, N. Nekrasov, A. Okounkov and C. Vafa, Quantum foam and topological strings, *J. High Energy Phys.* **2008**, no. 4, 011, 47 pp. MR 2425292

- [14] D. L. Jafferis and G. W. Moore, Wall crossing in local Calabi–Yau manifolds, arXiv:0810.4909 [hep-th].
- [15] M. Kontsevich and Y. Soibelman, Stability structures, motivic Donaldson–Thomas invariants and cluster transformations, arXiv:0811.2435 [math.AG].
- [16] D. Maulik, N. Nekrasov, A. Okounkov and R. Pandharipande, Gromov–Witten theory and Donaldson–Thomas theory. I, *Compos. Math.* **142** (2006), 1263–1285. Zbl 1108.14046 MR 2264664
- [17] D. Maulik, A. Oblomkov, A. Okounkov, R. Pandharipande, Gromov–Witten/Donaldson–Thomas correspondence for toric 3-folds, arXiv:0809.3976 [math.AG].
- [18] S. Mozgovoy and M. Reineke, On the noncommutative Donaldson–Thomas invariants arising from brane tilings, *Adv. Math.* **223** (2010), 1521–1544. Zbl 1191.14008 MR 2592501
- [19] K. Nagao, Derived categories of small toric Calabi–Yau 3-folds and counting invariants, arXiv:0809.2994 [math.AG].
- [20] K. Nagao and H. Nakajima, Counting invariant of perverse coherent sheaves and its wall-crossing, arXiv:0809.2992 [math.AG].
- [21] A. Okounkov, N. Reshetikhin and C. Vafa, Quantum Calabi–Yau and classical crystals, in *The unity of mathematics*, *Progr. Math.* 244, Birkhäuser, 2006, 597–618. Zbl 1129.81080 MR 2181817
- [22] H. Ooguri, A. Strominger and C. Vafa, Black hole attractors and the topological string, *Phys. Rev. D* **70** (2004), 106007, 13 pp. MR 2123156
- [23] H. Ooguri and M. Yamazaki, Crystal melting and toric Calabi–Yau manifolds, *Comm. Math. Phys.* **292** (2009), 179–199. Zbl 1179.81139 MR 2540074
- [24] ———, Emergent Calabi–Yau geometry, *Phys. Rev. Lett.* **102** (2009), 161601. MR 2501279
- [25] R. Pandharipande and R. P. Thomas, Curve counting via stable pairs in the derived category, *Invent. Math.* **178** (2009), 407–447. Zbl 1204.14026 MR 2545686
- [26] B. Szendrői, Non-commutative Donaldson–Thomas theory and the conifold, *Geom. Topol.* **12** (2008), 1171–1202. Zbl 1143.14034 MR 2403807
- [27] R. P. Thomas, A holomorphic Casson invariant for Calabi–Yau 3-folds, and bundles on K3 fibrations, *J. Differential Geom.* **54** (2000), 367–438. Zbl 1034.14015 MR 1818182
- [28] B. Young, Computing a pyramid partition generating function with dimer shuffling, *J. Combin. Theory Ser. A* **116** (2009), 334–350. Zbl 1191.05007 MR 2475021
- [29] B. Young (with an appendix by J. Bryan), Generating functions for colored 3D Young diagrams and the Donaldson–Thomas invariants of orbifolds, *Duke Math. J.* **152** (2010), 115–153. Zbl pre05692596 MR 2643058