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Self-Adaptive Methods for PDE

Organised by
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Introduction by the Organisers

This conference was on theoretical and practical aspects of error control and self-adaptivity in the numerical solution of partial differential equations. The organizers were Endre Süli (Oxford), Rolf Rannacher (Heidelberg) and Rüdiger Verfürth (Bochum), and the 44 participants came from 11 different countries.

Many processes in sciences and engineering are formulated in terms of partial differential equations. With the incorporation of more and more complete physics these models become increasingly complex and their accurate numerical simulation requires the use of efficient self-adaptive methods. Such "self-adptivity" is usually based on *a posteriori* estimates for the discretization and iteration errors in terms of local quantities (residuals) obtained from the computed solution. During the last few years, these approaches have seen a rapid development from simple model situations towards real-life applications. The conference's focus was on basic questions of the rigorous mathematical understanding of these methods and their applications. The frame of the conference was set by six invited one-hour survey lectures on the topics

- Adaptive hp finite element methods.
- Adaptivity in numerical optimization.
- Model adaptivity.
- Convergence of adaptive finite element methods.
- Multiscale adaptivity and wavelets.
- Space-time adaptivity in nonstationary problems.

Additionally 18 shorter talks have been given on more specialized aspects not treated in the survey lectures and on other subjects. Important topics addressed by these talks were "adaptivity for nonlinear, nondifferentiable PDEs", "adaptivity for coupled problems", "anisotropic and moving mesh adaptation", "local and

global recovery techniques", "parallelization of adaptive methods", and "mesh generation". The talks showed that much progress has been made in extending the concept of a posteriori error estimation and mesh adaptation from simple elliptic problems to real-life situations including coupled models, multiple scales and nonlinearity. Thanks to the restricted number of talks, there was left plenty of time for discussion which was extensively used by the participants. The discussions concentrated on question related to the convergence and asymptotic complexity of adaptive finite element methods and the various strategies of using "duality-based" a posteriori error estimates. The recent extension of these concepts to optimization problems and to model calibration has opened a whole new world of applications and will significantly affect the future development of PDE numerics.

Workshop on Self-Adaptive Methods for PDE

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Abstracts

A survey of *hp*-adaptive finite element algorithms for elliptic problems

Mark Ainsworth

In this survey lecture, we gave an overview of the main adaptive algorithms that are presently used in adaptive *hp*-finite element algorithms. Here, the goal is to adaptively select the local mesh-size h and the local polynomial order p of the elements in such a way that the method converges exponentially fast in the energy norm, even for problems where the solution has singularities. Three basic strategies were identified: (i) methods based, directly or indirectly, on identification of the local regularity of the solution [3, 1, 2, 9, 10, 7, 8], (ii) methods based on local mesh optimisation [6, 5], and (iii) methods based on 'tagging' geometric features where the solution is known a priori to be non-smooth [3, 4]. Numerical examples illustrating the relative performance of the various strategies were presented.

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Parallel adaptive methods

Randolph E. Bank
(joint work with Michael J. Holst, Shaoying Lu, and Jeff Ovall)

In [2, 3], a general approach to parallel adaptive meshing for systems of elliptic partial differential equations was introduced. This approach was designed to keep communications costs low, and to allow sequential adaptive software to be employed without extensive recoding. The original paradigm has three main components:

> **Step 1: Load Balancing.** We solve a small problem on a coarse mesh, and use a posteriori error estimates to partition the mesh. Each subregion has approximately the same error, although subregions may vary considerably in terms of numbers of elements or gridpoints.
> **Step 2: Adaptive Meshing.** Each processor is provided the complete coarse mesh and instructed to sequentially solve the *entire* problem, with the stipulation that its adaptive refinement should be limited largely to its own partition. The target number of elements and grid points for each problem is the same. At the end of this step, the mesh could be regularized such that the global mesh described in Step 3 will be conforming.
> **Step 3: Global Solve.** A final mesh is computed using the union of the refined partitions provided by each processor. A final solution computed using a domain decomposition or parallel multigrid technique.

With this approach, the load balancing problem is reduced to the numerical solution of a small elliptic problem on a single processor, using a sequential adaptive solver such as PLTMG without requiring any modifications to the sequential solver. The bulk of the calculation in the adaptive meshing step also takes place independently on each processor and can also be performed with a sequential solver with no modifications necessary for communication. The only parts of the calculation requiring communication are (1) the initial fan-out of the mesh distribution to the processors at the beginning of adaptive meshing step, once the decomposition is determined by the error estimator in load balancing; (2) the mesh regularization, requiring communication to produce a global conforming mesh in preparation for the final global solve in Step 3; and (3) the final solution phase, that might require local communication (e.g., boundary exchanges). In some circumstances, it might be useful to avoid the initial fan-out communication step by allowing all processors (which are otherwise idle) to simultaneously compute the coarse solution and load balance in Step 1. Note that a good initial guess for the final global solve is provided by the adaptive meshing step by taking the solution from each subregion restricted to its partition.

A more complete discussion of the overall paradigm as well as some numerical illustrations can be found in [2, 3]. A description of a domain decomposition solver used in Step 3 of the paradigm is given in [4]. In Mitchell [5], a parallel adaptive procedure similar to Step 2 of our procedure is described. Our goal here is to

present a variant of the above approach in which the load balancing occurs on a much finer mesh. This procedure is described in detail in [1].

In Step 1 of the paradigm, we assume that $N_c \gg p$ where N_c is the size of the coarse mesh and p is the number of processors. This is necessary to allow the load balance to do an adequate job of partitioning the domain into regions with approximately equal error. We also assume that the that N_c is sufficiently large and the mesh sufficiently well adapted for the a posteriori error estimates to accurately reflect the true behavior of the error. For the second step of the paradigm, we assume that $N_p \gg N_c$ where N_p is the target size for the adaptive mesh produced in Step 2 of the paradigm. Taking $N_p \gg N_c$ is important to marginalize the cost of redundant computations. For example, if $N_p = 2N_c$, then one could expect that about half of the computation on each processor would be redundant, which is a significant fraction of the total cost. By solving the problem on the entire domain, using a coarse mesh in all but one subregion, we are in effect substituting computation for communication. This trade-off is most effective in situations where N_p is much larger than N_c (e.g., $N_p > 10N_c$) so that the redundant computation represents a small fraction of the total cost. If any of these assumptions is weakened or violated, there might be a corresponding decline the effectiveness of the paradigm. In this case, we consider the possibility of modifying Steps 1 and 2 of the paradigm as follows.

> **Step 1': Load Balancing.** On a single processor we adaptively create a *fine* mesh of size N_p, and use a posteriori error estimates to partition the mesh such that each subregion has approximately equal error, similar to Step 1 of the original paradigm.
>
> **Step 2': Adaptive Meshing.** Each processor is provided the complete adaptive mesh and instructed to sequentially solve the *entire* problem. However, in this case each processor should adaptively *coarsen* regions corresponding to other processors, and adaptively refine its own subregion. The size of the problem on each processor should remain at N_p, but this adaptive rezoning strategy will concentrate the degrees of freedom in the processor's subregion. At the end of this step, the mesh could be regularized such that the global mesh described in Step 3 will be conforming.
>
> **Step 3': Global Solve.** This step is the same as Step 3; the global mesh consists of the refined partitions provided by each processor. A final solution is computed using a domain decomposition or parallel multigrid technique.

With this variant, the initial mesh can be of any size. Indeed, our choice of N_p is mainly for convenience and to simplify notation. Of course, allowing the mesh in Step 1' to be finer increases the cost of both the solution and the load balance, but it allows for flexibility in overcoming potential deficiencies of a very course mesh in the original paradigm. As before, all processors could simultaneously carry out Step 1' in order to avoid the initial fan-out communication step.

Although both procedures have simple communication patterns with small communication costs, it is important to emphasize one caveat. The goal is to create

a final global adaptive mesh in which the error is roughly equilibrated among the elements, and the effort needed to create these meshes is roughly the same on each processor. The idea of creating subregions of approximately equal error for the load balancing steps really amounts to the fragile assumption that this corresponds to approximately equal work for each processor, and that the final adaptive mesh will have a reasonable equilibration of the error among the elements.

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Adaptive finite elements for optimization

Roland Becker

We give an overview over adaptive methods for optimization with partial differential equations. The main tool is an a posteriori error estimator which takes into account the specific needs originating from different types of optimization problems. The estimator is based on the standard techniques in adaptive finite elements: error representation and Galerkin orthogonality. The classical application of the estimator is local mesh refinement, but it can also be used for the adaptation of other discretization parameters and serve as a stopping criterion in an iterative solution algorithm.

The first part of the talk describes three typical optimization problems with different interpretation. The three examples use the stationary Navier-Stokes equations as the state equation.

In the first example the action of the control is used for reduction of the drag coefficient of an immersed body; here the main objective is to compute the value of the drag coefficient at the optimum – which corresponds to the value of the cost functional in the context of optimization.

In the second example we wish to identify the viscosity constant in the Navier-Stokes equations from measurements. Now, the control itself is at the heart of the matter.

The third example deals with a situation where the control parameters are used for model calibration, and the goal of the computation is not directly related to the control (but to another objective functional).

The presented examples lead us to consider the following general case: we are interest in computing an interest function $I(q, u)$ which depends on both control q and state variable u. The interest functional is independent of the optimization problem which determines q and u. The second part of the talk develops this idea. By specialization we obtain estimators for:

- error in the cost functional [3, 1]
- error in a functional of the controls [5]
- error in an independent functional of the state [4]
- norm of the controls [2]

Beside the first estimator, the others require the solution of an additional problem involving the adjoint of the linearized state operator. The right-hand side of this problem depends on the special context.

In the last part of the talk, we discuss the issue of sensitivity analysis. The hope is to provide practical information for the interpretation of the results and design of experiments. The standard procedure in optimal experimental design is to use a statistical model: for example, the measurements are interpreted as true value plus a white noise. Then one tries to maximize the confident region with respect to changes in the experiment.

Here, we propose another concept based on deterministic sensitivity analysis. The first idea is to use again the interest functional I as the guide We define relative condition numbers τ_k which multiply the relative errors in measurements $\frac{\Delta C_k}{C_k}$ to produce relative changes in the value of the interest functional:

$$\frac{\Delta I}{I} = \sum_{k=1}^{n_Z} \tau_k \, \frac{\Delta C_k}{C_k}.$$

These numbers provide an answer to questions like: which parameters have been most important, which measurements have been most important in computing the solution? This can also be seen as a first step towards optimal experiment design.

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Coupling adaptive modeling and mesh refinement

Malte Braack

We propose a twofold adaptive method based on a posteriori control of discretization error and modeling error with respect to functional output $j(u)$, see [1]. Denoting by u the continuous solution of a partial differential equation in variational formulation and by u_h the discrete solution of a discrete equation. The two formulations differ not only by the variational spaces but also with respect to different models entering the partial differential equation. The discrete variational formulation is considered to involve a simpler model. The a posteriori error representation derived in [3] is of the following form:

$$j(u) - j(u_h) \approx \eta_h + \eta_m + R\,,$$

where the terms η_h and η_m are the error estimators of the discretization error and the modeling error, respectively. The part η_h consists of residuals with respect to the simpler model and involves approximations of the interpolation error of the primal solution u and the interpolation error of an associated dual solution z. The modeling error estimator η_m involves the residual with respect to the more accurate model. The remainder term R is (at least formally) of higher-order.

In an adaptive process, both contributions of errors are equilibrated. While the discretization error is reduced by local mesh refinement, the modeling error is reduced by changing to a more exact model locally. As a consequence, the model changes from cell to cell in the computational domain.

The metholodogy is applied to two kinds of modeling errors. In the first one, we apply the methodology to combustion problems, were complicated diffusion models (multicomponent diffusion) are known but rarely used in practice, see [2] and [4], due to the high numerically cost. Therefore, we use also a simpler diffusion model (Fick's law) and measure the introduced error. In the adaptive process, we switch dynamically to the more accurate model (equation) and refine the mesh simultaneously.

In the second type of problems, we consider finite element stabilization as an introduced error, for instance streamline diffusion. We ask if the accuracy of the discretization can be enhanced by adapting the diffusion parameters locally.

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A posteriori error analysis and adaptive mesh refinement for nonlinear reaction diffusion problems

Graham F. Carey
(joint work with Brian Carnes)

Two classes of nonlinear reaction-diffusion problems are considered. In the first class the nonlinearity enters through the reaction term and is scaled by a reaction parameter. The prototype example is the catalytic reaction problem in chemical engineering (e.g. see Aris [1], Carey and Finlayson [2]). Here the parameter is the Thiele modulus. The first problem class exhibits multiple solutions and has a turning point. In the present work we develop a new approach for a posteriori error estimation in finite element approximation in which error indicators are developed for both the spatial concentration field and the scalar parameter within an augmented arc-length continuation algorithm. Computable local element error indicators are derived together with corresponding global error indicators using residual and dual formulations. Numerical results illustrate the effectiveness of the indicators and approach for a nonlinear reaction problem having a parameter at both regular points and turning points. Details are provided in Carnes and Carey [3].

In the second problem class the nonlinearity enters through the diffusion tensor. Here applications include binary and ternary diffusion corresponding to the nonlinear scalar and system cases respectively. Suitable structure conditions on the associated constitutive parameters and regularity assumptions apply. Local residuals for the primal and dual problem are constructed and used to determine associated bubble function error indicators as the solution of corresponding local boundary value problems on each element (see Verfürth [4]). Results of adaptive refinement calculations and corresponding meshes are compared for different error indicator choices including residual bubbles in the primal solution and the dual problem contributions.

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5 further remarks on convergence of AFEM

Carsten Carstensen

The beautiful talk of Nochetto on the convergence of adaptive finite element methods (AFEM) provoked a few open questions which are addressed in this presentation at the end of the very interactive meeting. Assuming that the reader is familiar with the methodology of AFEM the 5 subtle issues are:

1. Removing Inner Node Refinements? This is most likely impossible. However, there is an alternative to bisec5 (i.e. 5 newest-vertex bisections of one triangle). There is a well-known counter example of error reduction and it is explained that this is the only one—generically speaking, every full-star refinement will lead to the error reduction property.

2. AFEM Beyond Fixing Poor Convergence? There is raising empirical (by computer experiments) evidence that AFEM does not just improve the convergence rates. In fact, in linear elasticity the incompressibility locking of P1 FEM seems to be overcome by P1 AFEM. Other examples by Bänsch, Nochetto et al. concern mixed finite element methods and nonconvex minimization problems.

3. Little Theory: η_H VS Error Reduction? The error reduction property is advertised as something we need to prove while in the major part of the literature it is called saturation assumption and then is some hypothesis. The presentation introduces some framework in a strongly convex minimization problem and the equivalence of (1) error reduction (2) residual coverage and (3) reliability of hierarchical error estimator.

4. Convergence 4 Convex Minimization? The presentation provides a generalization of the convergence result of AFEM for the p-Laplacian. The larger class of energy densities in the nonlinear minimization problem of p-th order growth is characterized by a convexity estimate in terms of stress differences. Applications include the Nonlinear Laplacian, Optimal Design Problem, Scalar 2-Well Problem, Vectorial 2-Well Problem, Hencky elastoplasticity with hardening. Even if the minimization problem has multiple minimizers on either the discrete or continuous level, the stress field is unique and converges strongly in some Lebesgue space.

5. AMFEM? Nochetto presented some results of adaptive mixed FEM (AMFEM) in a context of $H(div)$ spaces which required modifications and raised the question of the convergence of AMFEM in general. The presentation concludes with the announcement that this can be proved and will be published in joint work with Ronald Hoppe soon.

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An adaptive method for the Hamilton-Jacobi equations

Bernardo Cockburn
(joint work with Bayram Yenikaya)

We introduce and numerically study an adaptive method for approximating the *viscosity* solution of the following model steady state Hamilton-Jacobi equation

$$u + H(u_x) = f \qquad \text{in } (0,1), \tag{1}$$

with *periodic* boundary conditions. For any given positive parameter τ, the method obtains an approximation u_h satisfying the quality constraint

$$\| u - u_h \|_{L^\infty(G_h)} \leq \tau, \tag{2}$$

where G_h is the mesh on which the approximate solution is computed. Moreover, the method achieves this with *optimal* complexity.

To the knowledge of the authors, there is no other adaptive method for Hamilton-Jacobi equations. One reason might be that a posteriori error estimates in the L^∞-norm were only recently introduced. Indeed, the adaptive method we propose uses an approximate version of the rigorous a posteriori error estimate obtained in [1], namely,

$$\| u - u_h \|_{L^\infty(0,1)} \leq \Phi(u_h),$$

where Φ is a suitably defined non-linear functional; see a similar result in [2] for the time-dependent case. A posteriori error estimates in L^1-like norms for time-dependent Hamilton-Jacobi equation with strictly convex Hamiltonians were obtained in [3, Corollary 2.2]. However, we are not aware of any adaptive method based on them.

On the other hand, there has been considerable amount of work on a posteriori error estimation and adaptive methods for scalar hyperbolic conservation laws, which are equations closely related to the Hamilton-Jacobi equations; for example, the derivative of the viscosity solution of equation (1) is the *entropy* solution of the equation

$$v + (H(v))_x = f_x.$$

A posteriori error estimates for the scalar hyperbolic conservation laws have been obtained in almost all papers concerned with error estimation for these equations; see a review in [4]. However, only in [5] one can find an exhaustive computational study of the corresponding effectivity index which is the ratio of the upper bound of the error given by the a posteriori error estimate to the actual value of the error. It was carried out for the Engquist-Osher scheme on uniform meshes for the equation

$$v_t + (H(v))_x = 0.$$

The effectivity index was shown to remain very close to the ideal value of one in two cases: When the entropy solution is smooth, and when it has a discontinuity and the Hamiltonian H is linear. Unfortunately, no results were given for the difficult case in which H is a non-linear function and the entropy solution has a discontinuity. In [6] this difficult case is treated successfully by using an adaptive method for which the effectivity index turns out to be of order one. No study of the effectivity index is presented for the adaptive algorithm proposed in [7]. In [8], an adaptive method for a schock-capturing discontinuous Galerkin method is considered and an study of two effectivity indexes carried out for functionals of the solution of hyperbolic conservation laws. Here, we also carry out a thorough study of the effectivity index of our method extending in this way the work carried out in [1] for the case of uniform meshes.

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Multiscale adaptivity by wavelet techniques

Wolfgang Dahmen

This talk reviews some recent developments of adaptive numerical strategies based on multiscale decompositions into wavelet bases. The main emphasis is not on any detailed specific applications. Instead some concepts are highlighted that help intertwining the analysis and resolution process concerning, in particular, the interaction of different lenght scales under nonlinear operators.

The nature of an adaptive strategy depends, of course, on the particular goal to be pursued. Roughly, one may distinguish two classes. (i) Calculating a local functional of the (globally defined) solution, (ii) recovering the whole solution within some target tolerance with respect to a given norm. The focus in this talk is on (ii).

In this regard one may distinguish two major directions :
(I) *Adaptive wavelet methods for stationary variational problems that are in a well-posed in a suitable Hilbert space setting.* This covers linear and nonlinear elliptic boundary value problems, boundary integral equations, transmission problems, mixed formulations and indefinite problems, optimal control problems and can be extended in a canonical way to problems of parabolic type [4, 5, 6]. The Stokes system will serve as a model case for the latter class of problems [10].
(II) *Adaptive wavelet methods for non-stationary hyperbolic conservation laws* such as the Euler equations of gas dynamics and corresponding viscous versions such as the compressible Navier Stokes equations.
In (I) the following issues are addressed. While computational experience appears to confirm the high potential of adaptive methods, a rigorous analysis is only now emerging. This analysis centers on deriving error and complexity estimates that relate the computational work and the adaptively generated number of degrees of freedom to the achieved accuracy. The main ingredients of such an analysis will be indicated. It will stress the importance of closely intertwining the analysis-discretization-solution process. This culminates in a new algorithmic paradigm based on adaptive realizations of iterative schemes that are conceptually formulated for the full infinite dimensional problem transformed into wavelet coordinates [5, 6]. Since, due to the specific features of wavelet bases such as the norm equivalences between certain function and sequence norms, the computational process thereby inherits the stability properties of the continuous problem. In turn, compatibility constraints on the choice of trial spaces such as the Ladyshenskaya-Babuska-Brezzi condition in connection with saddle point problems become void

[10]. A central result is to show that these new schemes exhibit an asymptotically optimal complexity in the following sense. If the smallest number of degrees of freedom in the given discretization framework needed to achieve accuracy ϵ grows at most like $N(\epsilon) = \epsilon^{-1/s}$ (for a certain range of s depending on the wavelet basis) then the adaptive scheme produces an approximate solution with accuracy ϵ at a computational expense that also stays proportional to $\epsilon^{-1/s}$, uniformly in ϵ. The analytical tools developed in this context, in turn, suggest new algorithmic ingredients. Some numerical examples e.g. for the Stokes system illustrate these results [2, 10]. These tools come from nonlinear approximation and computational harmonic analysis [12].

The basic approach to (II) is quite different. As originally initiated by A. Harten, Wavelet concepts are used here more in the spirit of post processing conventional discretization schemes such as finite volume discretizations [13, 13, 1, 11]. Nevertheless, the compression of a flow field represented in wavelet coordinates can be shown to lead to a fully adaptive solver based on a realiable prediction of significant flow components when progressing in time [9, 15]. As already in connection with nonlinear problems in (I) a key issue is the interaction of components from different scales under nonlinear mappings. Wavelet concepts offer ways of estimating such interactions in a quantitative manner e.g. for nonlinearities of power growth [7]. This part of the talk wis kept short and mainly restricted to some numerical experiments for aerodynamical applications [3].

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Reliable a posteriori error control for non-conforming finite element approximation of Stokes flow

Willy Dörfler
(joint work with Mark Ainsworth)

We derive computable a posteriori error estimates for the lowest order non-conforming Crouzeix–Raviart element in case of approximation of incompressible Stokes flow. The estimator provides an explicit upper bound that is free of any unknown constants in the leading order. In addition, it is shown that the estimator provides an equivalent lower bound on the error up to a generic constant.

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be an open polygonal or polyhedral domain. For a given *forcing term* $\boldsymbol{f} : \Omega \to \mathbb{R}^d$ we seek a *velocity field* $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ and a *pressure* $p : \Omega \to \mathbb{R}$ satisfying the *Stokes equations,*

$$\begin{aligned} -\Delta \boldsymbol{u} + \boldsymbol{\nabla} p &= \boldsymbol{f} \quad \text{in } \Omega, \\ \boldsymbol{\nabla}\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \partial\Omega \end{aligned}$$

subject to the side-constraint $\int_\Omega p = 0$.

This problem is discretised with the finite element method, using the *non-conforming Crouzeix–Raviart element.* As a solution we get the discrete velocity $\boldsymbol{u}_h$ and the discrete pressure p_h.

We prove computable a upper posteriori bounds for the error in the *energy norm* of the problem. Let $\overline{\boldsymbol{f}}_K$ denote a constant approximation of $\boldsymbol{f}$ on the element K. To deal with the error, it is split into a *conforming* and a *non-conforming* part. The estimator η_{C} for the conforming part of the error is defined by

$$\eta_{\mathrm{C}} := \frac{1}{d^2(d+1)} \Big(\sum_{K\in\mathcal{K}} |K| \left|\overline{\boldsymbol{f}}_K\right|^2 \sum_{\gamma\,:\,\gamma\subset\partial K} \left|\boldsymbol{x}_\gamma - \boldsymbol{x}_K\right|^2 \Big)^{1/2},$$

where $\boldsymbol{x}_K$ and $\boldsymbol{x}_\gamma$ denote the centroid of the element K and an edge γ respectively. The estimator η_{NC} for the non-conforming part is defined by

$$\eta_{\mathrm{NC}} := \|\boldsymbol{\nabla}_h(\boldsymbol{u}^* - \boldsymbol{u}_h)\| + \frac{1}{c_0}\|\boldsymbol{\nabla}\cdot\boldsymbol{u}^*\|,$$

where $\boldsymbol{u}^*$ is any function in $H_0^1(\Omega)^2$, $\boldsymbol{\nabla}_h$ is the piecewise defined gradient, and c_0 is the lower bound in the *inf-sup condition*. Fortunately, bounds for c_0 are available and can be obtained from literature. The estimator for the total error is obtained by summing the estimators

$$\eta := \eta_{\mathrm{C}} + \eta_{\mathrm{NC}}.$$

The influence of data is measured by the *data oscillation* on $\mathcal{K}$ defined by

$$\operatorname*{osc}_{\mathcal{K}}(\boldsymbol{f}) := \Big(\sum_{K\in\mathcal{K}} |K| \, \|\boldsymbol{f} - \overline{\boldsymbol{f}}_K\|_K^2 \Big).$$

This term is typically of higher order.

Theorem 1. (Upper a posteriori bound) *For any choice $\boldsymbol{u}^* \in H_0^1(\Omega)^2$ one has the estimate*

$$\|\boldsymbol{\nabla}_h(\boldsymbol{u} - \boldsymbol{u}_h)\| \le \eta + C \operatorname*{osc}_{\mathcal{K}}(\boldsymbol{f}),$$

where C is a positive constant that only depends on the shape regularity of the mesh.

Thus, the estimator η provides an explicit upper bound in the leading order.

To show a lower bound, one has to be more specific about the choice of $\boldsymbol{u}^*$. Here, some explicit mapping $\boldsymbol{u}^* := \boldsymbol{Q}_h\boldsymbol{u}_h$ into a higher finite element space are given.

Theorem 2. (Lower a posteriori bound) *Suppose that each element $K \in \mathcal{K}$ has at most one edge on the boundary. If the estimator η_{NC} is defined by taking $\boldsymbol{u}^* = \boldsymbol{Q}_h\boldsymbol{u}_h$, where $\boldsymbol{Q}_h$ is any of the operators defined, then*

$$c\,\eta \le \|\boldsymbol{\nabla}_h\boldsymbol{e}\| + \operatorname*{osc}_{\mathcal{K}}(\boldsymbol{f}),$$

where c is a positive constant that depends only on the shape regularity of the mesh.

The result can be extended to the case of *non-homogeneous boundary conditions* as well. Computations of an example with regular and one with singular solution in a two-dimensional domain shows quasi-optimal estimates with an overestimation between 3 and 4.

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Some observations about a posteriori analysis and adaptivity for space-time finite element methods

Donald Estep

Currently, the state of implementation of *a posteriori* error estimates and adaptive error control for evolutionary partial differential equations is relatively unfinished. There are fundamental issues underlying adaptivity for space-time finite elements that present hurdles for future progress. In this talk, we discuss three of these.

We begin by noting that there are fundamental differences between successful approaches to adaptivity for hyperbolic and parabolic problems. This is due in part to the difference between finite and infinite speed of propagation of information and reversibility/irreversibility with respect to time. The issues we discuss are important for both types of problems, though the details are different in the context of each type. In this talk, we focus on parabolic problems.

We consider continuous and discontinuous Galerkin discretizations of potentially singular systems of reaction-diffusion equations. The systems couple one or more parabolic problems with diffusion functions bounded away from zero and some ordinary differential equations in time. Such systems are widely used as models in biology, chemistry, ecology, and population modeling. The discretizations we consider use space-time "slabs" with approximation functions that are weakly or strongly discontinuous piecewise polynomials in time, where on each time interval, the coefficients are in the space of continuous, piecewise linear functions with respect to a triangulation of the space domain. The *a priori* convergence properties of these methods is well established.

We describe an *a posteriori* analysis of the error using residuals and the generalized Green's function. After defining a quantity of interest as a linear function involving the solution at the final time, we pose the adjoint problem corresponding to a linearized version of the forward problem. Then, we state the *a posteriori* error representation formula obtained by a variational argument.

The standard approach to adaptive error control is to pose the problem of discretization selection as an optimal control problem. In order to carry out a calculus of variations argument, the *a posteriori* error representation is replaced by an *a posteriori* bound on the representation written as a sum of nonnegative element contributions that eliminates cancellation of contributions between elements. We derive an appropriate bound after assuming the most favorable regularity for the generalized Green's function that can be expected to hold and the corresponding residuals. This bound has the form of a sum over elements of stability factors, i.e., weights given by seminorms of the generalized Green's function, and norms of residuals.

We then discuss bounds on the stability factors and residuals. We state a theorem that says that generally we can make residuals small on any given time step by discretization enrichment. We point out that in general, boundedness of

the stability factors requires boundedness (at least) of the true solution and the numerical solution.

With this introduction, we present three fundamental issues that are important for any future progress in adaptivity.

- Robust error control appears to require a global approach that uses information about the stability of the solution being approximated. How should this information be obtained and how should it be incorporated into the adaptive error control?
- Adaptive mesh selection is usually posed as an optimal control problem involving a constraint given in terms of an *a posteriori* error **bound** that eliminates cancellation of contributions between elements. Such bounds are frequently orders of magnitude too large in evolution problems, which seriously weakens any claims about optimality. Are there theories for optimal mesh selection that utilize estimates that allow cancellation of errors?
- On many problems, the adaptive criteria must be extended beyond accuracy to include preservation of special stability properties of the solution. This is important not only for computing accurate solutions, but also for computing accurate error estimates. In these cases, what are good criteria and how should a discretization be changed in response to such criteria?

We discuss these issues in the context of several examples, including the chaotic Lorenz ordinary differential equation, the bistable reaction-diffusion equation, and S-I-R model of rabies, and a predator-prey system.

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Self-adaptive hp approach in the DGFEM for avoiding Gibbs phenomenon

Miloslav Feistauer

(joint work with Vít Dolejší)

This paper is concerned with the discontinuous Galerkin finite element method (DGFEM) and its applications in Computational Fluid Dynamics (CFD). Our goal is to develop a sufficiently accurate and robust method for the numerical solution of nonlinear conservation laws, nonlinear convection-diffusion problems

and compressible flow, i.e. compressible Euler or Navier-Stokes equations. During the last decade, mainly two competing techniques have been used in CFD: the finite element method (FEM) and the finite volume method (FVM). A compromise between these techniques is the dicontinuous Galerkin finite element method, using ideas and advantages of the FEM as well as the FVM. For a survey of DGFE techniques we refer to [2]. In CFD the DGFEM was used first in [1].

In the DGFEM the sought solution is approximated by piecewise polynomial functions over meshes consisting of various types of elements. Usually we use tetrahedra or hexahedra, but by [7], one can use elements of the form of general star-shaped polyhedra. The requirement of the continuity on interfaces between neighbouring elements is completely relaxed and replaced by interior and boundary penalty terms combined with suitable stabilization terms. The convective terms are approximated with the aid of a numerical flux (an important ingredient in the FVM). As a result we obtain in a natural way higher order schemes, whose accuracy depends on the degree of the used polynomial approximations.

Unfortunately, numerical solutions obtained by the DGFEM (similarly as by other higher order methods) suffer from the Gibbs phenomenon, manifested by spurious nonphysical overshoots and undershoots in the vicinity of discontinuities or steep gradients (shock waves, contact discontinuities, boundary layers). In order to cure this "disease", various methods have been applied as artificial viscosity, shock capturing streamline diffusion (see, e.g. [8], Section 4.3) or order limiting developed originally in the FVM (see, e.g. [2]). The goal of all these methods is to decrease the order of accuracy of the scheme near discontinuities and steep gradients. However they have some disadvantages, as the necessity to tune various parameters or the danger that the accuracy is diminished also out of regions, where it is necessary, particularly when unstructured and/or anisotropic meshes are used.

Here we are concerned with a new startegy applicable in the framework of the DGFEM on all types of meshes and allowing the limiting of the order of accuracy of the DGFEM only in a small neighbourhood of discontinuities and steep gradients. It is based on a suitable jump indicator marking elements on which the limiting of accuracy is applied. On the marked elements, in the convective terms the approximate solution is replaced by its modification obtained with the aid of an averaging procedure. The detailed derivation and justification of this method is described in [6].

The combination of the limiting procedure with an adaptive mesh refinement can be treated as a special case of the *hp*-version of the DGFEM. In our computations we apply the anisotropic mesh adaptation (AMA) technique described in [3]. A series of numerical experiments carried out for known test problems as well as some technically relevant problems show the reliability, accuracy and efficiency of this method. For details, see [5], [6], [4].

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Towards fourth order accuracy for integral outputs from flows with shocks

Mike Giles

This talk begins with a discussion of possible limits to the effective order of accuracy which may be achieved for integral outputs such as lift from approximations of the Euler equations when there is a shock. A heuristic argument, based on the discrete shock width and the total number of grid nodes, suggests that the best that may be achievable using straight-sided finite elements with anisotropic adaptation is fourth order accuracy in 2D, and third order accuracy in 3D.

The talk then presents an extension of the technique of adjoint error correction [1, 2, 3] to improve the accuracy of integral outputs when there are shocks. The key is to introduce a mesh-dependent regularisation, and correct for the error in the functional due to the regularisation, $J(u_\epsilon) - J(u)$, as well as for the error in approximating the regularised problem, $J(u_h) - J(u_\epsilon)$. Numerical results are presented for a quasi-1D model problem with $\epsilon = h^2$, using adaptive grid redistribution and grid refinement. The accuracy of the lift approximation from the primal calculation is second order, but the adjoint error correction raises the accuracy to fourth order. This remains the case even when there is relatively poor resolution of the regularised shock, a result which is probably connected to recent work on the convergence of linearised functionals and adjoint approximations of Burgers equation with an underlying discontinuity [4, 5].

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Adaptive hp finite element method for flow problems

Vincent Heuveline

For many applications, the hp Finite Element Method offers an extremely effective discretization since it generally allows to gain an *exponential convergence* with regard to the number of unknowns (see e.g. [4]). However for non smooth solution, this convergence behavior relies on an adequate choice of the mesh size h and of the polynomial order p which is not known a priori in many applications.

We propose a goal oriented adaptive scheme for the simultaneous control of h and p in the context of flow problems. The proposed scheme is based on a duality-based a posteriori error analysis which is developed for the conforming hp Galerkin finite element approximation of second-order elliptic problems. Duality arguments combined with Galerkin orthogonality yield representations of the error in arbitrary quantities of interest. From these error estimates, criteria are derived for the simultaneous adaptation of the mesh size h and the polynomial degree p [2].

Numerical experiments considering the incompressible Navier-Stokes equations in two and three dimensional channel around an obstacle are presented. The computation of the drag, lift and torque acting on the obstacle can be computed with an error less than one per cent with less than 20.000 unknowns. This is at least an order of magnitude less than standard lower order finite element approach. These results clearly show the effectivity of the proposed approach. Many open questions still remain expecially in the context of instationary flows.

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Moving adaptive finite elements

Jens Lang

The purpose of this talk is to present a combination of an r–adaptive and an h–adaptive finite element method, [1]. r–adaptivity, i.e., moving grid points through the computational domain without destroying the mesh connectivity, is accomplished by a moving mesh method. This method is based on a moving mesh PDE where the gradient or an a posteriori error estimate of the numerical solution is used to indicate the regions requiring higher mesh density. Although moving methods have a good potential to solve non–trivial problems including free boundaries or time–dependent domains, a fixed number of grid points may become a major disadvantage.

Here, h–adaptivity can be useful to insert new grid points in regions where large solution variations have to be resolved and to delete grid points where they are no longer needed. Thus, the main idea is to run the r–method until an h–method is required to keep the estimated discretization error in space below a certain tolerance.

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Adaptive submodeling for elasticity

Mats G. Larson

Submodeling is a procedure for local enhancement of the resolution of a coarse global finite element solution by solving a local problem on a subdomain containing an area of particular interest. We focus on linear elasticity and computation of local stress levels determined by the local geometry of the domain. We derive a posteriori error estimates for the submodeling procedure using duality techniques. Based on these estimates we propose an adaptive procedure for automatic choice of the resolution and size of the submodel. The procedure is illustrated for problems of industrial interest.

In many industrial engineering applications the initial grid is large due to the complexity of the geometry of the problem. Such large initial grids make it difficult to apply standard automatic adaptive procedures based on a posteriori error estimates since in only a few refinement steps may result in a very large mesh. Nevertheless local mesh refinement may often be necessary to compute accurate

local values of the stress field. In such situations submodeling is an attractive alternative.

Submodeling is based on solving a local problem on a subdomain of the global problem containing an area of particular interest, for instance an area with high stress levels in the coarse grid solution, with improved resolution. Boundary conditions are obtained from the coarse grid solution. The size of the submodel problem may be much smaller than the global coarse grid and may thus be solved in a short amount of time, sometimes close to real time. In complex engineering applications such a technique opens up the possibility of interactive simulation and optimization of local design changes. Another important application is simulation of the effect of small features, for instance a hole, which are not present in the coarse global mesh. Removing small scale features from the CAD data in order to simplify meshing is common in industrial computations.

In this work we consider a submodeling procedure for linear elasticity based on displacement boundary conditions. We let Ω be the global domain and $\mathcal{K}_H$ be a triangulation of Ω. Further we assume that the subdomain is a union of elements K in $\mathcal{K}_H$ and we let $\mathcal{K}^h$ be a finer triangulation of ω, for instance obtained by refinement of $\mathcal{K}_H$. The global finite element solution to the linear elasticity equations is called $\boldsymbol{u}_H$ and the submodel solution $\boldsymbol{u}^h$.

We develop several a posteriori error estimates for this submodeling procedure. Typically, we are interested in localized values of stresses in engineering applications and thus we seek a posteriori error estimates of the error in such specific quantities. We derive the a posteriori error estimates using duality techniques where a certain generalized Green's function is introduced to represent the error in a linear functional to the residual of the computed solution. We refer to the recent book [1] by Bangert and Rannacher for an introduction to duality based a posteriori error estimates.

The most basic estimate of the error in a linear functional $m(\boldsymbol{e}) = m(\boldsymbol{u}) - m(\boldsymbol{u}^h)$ of the submodel solution takes the form

$$\begin{aligned} m(\boldsymbol{e})_\omega = &\sum_{K\in\mathcal{K}^h} \mathcal{R}_K(\boldsymbol{u}^h)\mathcal{W}_K(\boldsymbol{\phi}) \\ &+ \sum_{E\in\partial\omega} \mathcal{R}_E(\boldsymbol{u}^h, \boldsymbol{u}_H)\mathcal{W}_E(\boldsymbol{\phi}). \\ &+ \sum_{K\in\mathcal{K}_H} \mathcal{R}_K(\boldsymbol{u}_H)\cdot\mathcal{W}_K(\boldsymbol{\phi}). \end{aligned}$$

Here $\mathcal{R}_K(\cdot)$ and the weight $\mathcal{W}_K(\cdot)$ are computable estimates of the residual $\nabla\cdot\sigma(\boldsymbol{u}_H)$ and the local interpolation error $\phi - \pi\phi$ in the solution ϕ to the dual problem. Further $\mathcal{R}_E(\boldsymbol{u}^h, \boldsymbol{u}_H)$ is a residual measuring the jump in the normal stress at the submodel boundary and $\mathcal{W}_E$ is the L^2 norm of the dual solution at face E on the submodel boundary. We note that the first term is primarily dependendent on the resolution in the submodel mesh, the second depends on the submodel size, and finally the third depends on the resolution in the coarse global mesh. Assuming that the coarse global mesh is given an adaptive algorithm for

automatic tuning of the submodel resolution and size can be designed based on the three contributions to the error. We focus on applications of industrial interest where the local geometry of the domain in the problem determines local stress levels. We illustrate and evaluate the submodeling procedure on test problems of industrial interest.

The idea of submodeling is old and frequently employed in industrial computations. Submodeling was used as a tool to construct parallel algorithms in the work of Xu and Zhou [7]. A priori and local residual based a posteriori error estimates are also presented for the Poisson equation. A related approach was proposed by Bank and Holst [2] with applications to parallel adaptive meshing and [3] where a duality based domain decomposition method is proposed and analysed. Further submodeling is also related to local error estimates for the finite element method, see Schatz and Wahlbin [5]. Recently submodeling has been exploited by Oden and Venmanganti in [4] and [6] for simulation of elastic bodies with microstructure. Here the material model in the submodel is different from the global coarse problem.

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Adaptive finite element methods for surface diffusion

Pedro Morin

(joint work with Eberhard Bänsch and Ricardo H. Nochetto)

The overall goal of this project is to devise efficient numerical tools for simulating morphological changes in stressed epitaxial films and thereby study their complicated nonlinear dynamics. To model the misfit between the crystalline

structure of the substrate and epitaxial film, the film may be thought of as subjected to mechanical stresses. This causes a plastic deformation of the film, due to the diffusion of atoms along the interface (surface diffusion). This morphological instability may eventually lead to crack formation and fracture, an issue of paramount importance in Materials Science; see for instance [1, 4, 5] and the list of references in [3].

The dynamics of the free surface $\Gamma(t) \subseteq \mathbb{R}^d$ is governed by the (4th order highly nonlinear) geometric law

$$V = -\Delta_\Gamma(\kappa - \varepsilon), \tag{0.3}$$

where $d = 2, 3$, V and κ are the (scalar) normal velocity and mean curvature of Γ, respectively, $\Delta_\Gamma = \mathrm{div}_\Gamma \nabla_\Gamma$ is the Laplace-Beltrami operator and ε is the elastic energy density of the bulk $\Omega(t)$ enclosed by $\Gamma(t)$.

We first presented a novel variational formulation for surface diffusion of parametric surfaces, where ε is a given forcing function. This formulation is based on a semi-implicit time discretization, which requires no explicit parametrization of the surface and yields a linear system of elliptic PDE to approximate at each time step [2]. Already this simplified problem presents several difficulties and interesting features: the former related to mesh distortion, and the latter including topological changes of the bulk surrounded by the free surface $\Gamma(t)$. An algorithm which incorporates time and space adaptivity was discussed, and the advantage when dealing with very disparate time scales as well as different geometric structures was discussed.

Secondly we presented the coupling of this purely geometric evolution with the elasticity problem in the bulk. Issues such as mesh generation, mesh smoothing and mesh transformation arise in dealing with the fully coupled problem, and were discussed in the talk.

The elasticity problem to be solved after each timestep is stationary and for this specific problem we concluded that generating a mesh after modifying the surface in every timestep is more convenient than transforming one mesh into another for the new domain.

Animations of the computational results can be found in

`http://www.math.umd.edu/~rhn/SurfDiff/Movies`

For the case of graphs, a a variational formulation and optimal a priori error estimates for a time-continuous finite element discretization was obtained [3]. We also performed several simulations in 1d and 2d with and without forcing which explore the smoothing effect of surface diffusion as well as the onset of singularities in finite time, such as infinite slopes and cracks.

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Convergence of adaptive finite element methods

Ricardo Nochetto

This was a *survey lecture* on the basic ingredients for convergence of adaptive finite element methods (AFEM) for elliptic PDE of the form

$$-\operatorname{div}(A\boldsymbol{\nabla} u) = f,$$

with piecewise constant positive definite matrix A. The first concept is that of interior node, which allows for error reduction, and the second one is data oscillation. Counterexamples show that the interior node property as well as control of data oscillation are necessary for error reduction. Marking of elements for refinement consists of two steps: the first one, due to W. Dörfler [4], selects a number of elements such that their element indicators amount to a fixed proportion of the total error estimator. The second step adds elements so as to satisfy a similar condition for data oscillation. Under these conditions, the energy error is shown to decrease linearly as the adaptive counter tends to infinity. These results are joint with P. Morin and K. Siebert [6, 7, 8].

The lecture also addressed convergence of AFEM for elliptic PDE with variable coefficients, including convection,

$$-\operatorname{div}(A\boldsymbol{\nabla} u) + \mathbf{b}\cdot\boldsymbol{\nabla} u + cu = f.$$

This requires dealing with energy error and oscillation together since they no longer decouple. This result hinges on a quasi-orthogonality property in the induced energy norm, and an oscillation reduction inequality which relates oscillation and discrete solutions in consecutive meshes; the latter is in the spirit of [2]. This is joint work with K. Mekchay.

Finally the lecture discussed convergence of AFEM in $H(\operatorname{div};\Omega)$, namely for the elliptic PDE

$$A\mathbf{u} + \nabla \operatorname{div}\mathbf{u} = \mathbf{f},$$

with Raviart-Thomas elements. Results similar to those above were presented, along with applications to mixed methods via an Augmented Lagrangian formulation and an Uzawa iteration in infinite dimensions; the latter is in the spirit of [1, 3]. This is joint work with J.M. Cascón and K. Siebert.

Several live simulations were presented. They were all implemented within the finite element toolbox ALBERT of A. Schmidt and K. Siebert [9].

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A posteriori error estimates and adaptive finite elements for meshes with high aspect ratio

Marco Picasso

An anisotropic, a posteriori error estimator is proposed for the Laplace problem solved with continuous, piecewise linear finite elements. Using the anisotropic interpolation estimates derived in [1, 2] (see also [3] for similar results), the error in the energy norm is bounded above by an error estimator, the constant being independent of the mesh size and aspect ratio.

The error estimator contains a term measuring the alignement of the error gradient with the directions of maximum and minimum stretching. This error gradient is approached using Zienkiewicz-Zhu post-processing [4]. A lower bound is proved [5] provided the error is equidistributed in the directions of maximum and minimum stretching. This suggests an adaptive algorithm aiming at equidistributing the error in the directions of maximum and minimum stretching. A numerical study of the effectivity index confirms the theoretical predictions.

The method is extended to various elliptic and parabolic problems [6, 7, 8, 9].

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A review of modeling error estimation and model adaptivity

Serge Prudhomme

Concepts and methods for modeling error estimation and model adaptivity of computer predictions of physical events are reviewed. A motivation for model adaptivity is founded on the observation that one often used too complicated (cost-prohibitive) a model for what needs to be predicted. When performing computer simulations of a physical event, one can generally select a family of mathematical models that are liable to deliver reliable predictions. It is assumed that a base, or reference, model can be picked, one of such details that it allows for very precise predictions, but maybe is too complicated to be efficiently solved on current computers. The main goal of model adaptivity is to devise algorithms which automatically select among the surrogate models the simplest model that meets prescribed tolerances on the modeling error. For example, model adaptivity can be useful to locate the regions, preferably small, that require the use of the reference model rather than a surrogate model. We note that the notion of best or simplest models strongly depends on prescribed tolerances and on the choice of the quantity of interests that need to be computed; hence the development of goal-oriented approaches, in which the adaptive process is driven by the error in quantities of interest to the analyst.

To fix ideas, we suppose that a physical event of interest can be accurately predicted by the reference abstract model problem: find $u \in V$ such that

$$B(u; v) = F(v), \quad \forall v \in V,$$

where $B(\cdot;\cdot)$ denotes a semilinear form, $F(\cdot)$ a linear functional, and V a space of admissible functions. A surrogate model problem, embedded in the form $B_i(\cdot;\cdot)$, may consist in finding $u_i \in V$ such that

$$B_i(u_i; v) = F(v), \quad \forall v \in V.$$

The solution u_i of the surrogate problem is further discretized by means of numerical methods. In the case of the finite element method, the above problem reduces

to finding $u_h \in V^h$ such that

$$B_i(u_h; v) = F(v), \quad \forall v \in V^h,$$

where V^h defines a conforming finite element subspace of V. The goal is then to assess the accuracy of u_h in approximating the best prediction u. In other words, one wants to estimate the error $\mathcal{E} = Q(u) - Q(u_h)$, where $Q(u)$ represents the quantity of interest to be predicted. For adaptation purposes, the main issue is to discern the modeling error $\mathcal{E}_m = Q(u) - Q(u_i)$, due to replacing the base model by a surrogate model, and the discretization error $\mathcal{E}_h = Q(u_i) - Q(u_h)$, due to employing a discretization method for the solution of the surrogate problem, i.e.

$$\mathcal{E} = \mathcal{E}_m + \mathcal{E}_h.$$

Goal-oriented estimation of the approximation error, i.e. $\mathcal{E}_h = Q(u_i) - Q(u_h)$ has been the subject of extensive research works, see for example [1, 2, 4] and references therein. The approach was extended to the modeling error $\mathcal{E}_m = Q(u) - Q(u_i)$ in [4], where u_i was defined either as the solution of the surrogate model or simply as an arbitrary function in V. Since $u^h \in V^h \subset V$, the results can thus be used to estimate the total error $\mathcal{E} = Q(u) - Q(u_h)$ (note that the solution of the surrogate problem is not available in practical applications) and the modeling error can then be derived as:

$$\mathcal{E}_m = \mathcal{E} - \mathcal{E}_h.$$

A similar result is obtained in [3] where the reference model is given as $B(u; v) = a(u, v) + d(u, v)$ and the surrogate model as $B_i(u; v) = a(u, v)$. In other words, the term $d(u, v)$ stands for the part of the model which is difficult to compute and can be viewed as a perturbation between the surrogate and the reference models. The quantities $\mathcal{E}$ and $\mathcal{E}_h$ are usually approximated as products of residuals by solutions of dual problems, as shown in [1, 2, 4, 5], and the major difficulty in order to derive computable estimates of $\mathcal{E}$ and $\mathcal{E}_h$ becomes to obtain accurate approximations of the dual solutions. Variations in goal-oriented methods are actually related to the means by which the dual solutions are approximated.

Several illustrative application examples for modeling error estimation and adaptivity are presented. In particular, we mention the goal-oriented adaptive local solution (GOALS) algorithm proposed by Oden and Vemaganti [7, 10, 9] for the analysis of heterogeneous materials and perforated materials. We also review approaches for the estimation of the modeling error for dimensionally reduced models of elliptic boundary value problems posed on thin domains (see e.g. [8]) and for constitutive law models of nonlinear viscoelasticity problems [6].

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Lepp / terminal-edge algorithms: an integrated approach for mesh generation

Maria-Cecilia Rivara

Unstructured meshes are widely used as basic tools for solving application problems by means of either the finite element or the finite volume methods. In the finite-element context, longest-edge refinement algorithms[1, 2, 3] have been successfully used in practice for performing adaptive finite element computations in two and three dimensions. In 2-dimensions the algorithms were also successfully parallelized in order to deal with big and complex applications.

In recent years, improved Lepp / terminal-edge algorithms for the refinement of non-Delaunay triangulations[4, 7, 8], as well as a new family of algorithms for improving Delaunay triangulations have been developed: Algorithms for triangulation improvement, for automatic quality triangulation, for obtuse triangulation improvement [5, 6](as needed in finite volume methods), and for approximate quality triangulation [9]. These algorithms take advantage of the properties of two related concepts: the Lepp (longest-edge propagation path) of any triangle / tetrahedron t and its associated terminal-edges. Either for improving or refining a mesh, the algorithms use a terminal-edge point selection criteria as follows. For any target element to be improved or refined, the midpoint of an associated terminal-edge is selected for point insertion. Each terminal-edge is a special edge in the mesh which is the common longest-edge of every element (triangle or tetrahedron) that shares this terminal-edge in the mesh. In the case of the refinement algorithm, this is done by longest-edge bisection of all the elements that share the terminal-edge, which is a very local operation. In the case of the improvement algorithms, a constrained Delaunay triangulation is assumed, and terminal-edge midpoint insertion is performed by using a constrained Delaunay algorithm. The process is repeatedly performed until the target element is destroyed in the mesh.

In 2-dimensions the refinement algorithms essentially guarantee that the quality of the input triangulation is maintained, while the improvement algorithms assure that a 30 degrees triangulation can be obtained. In this talk, the algorithms, their mathematical properties, some implementation issues, as well as some of their possible extensions are discussed.

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A posteriori error analysis for the finite element approximation of fluid-structure interaction vibration problems

Rodolfo Rodríguez
(joint work with Ricardo G. Durán, Claudio Padra)

We deal with a posteriori error estimates for the finite element approximation of a second order eigenvalue problem, which arises from the computation of the vibration modes of a coupled acoustic fluid – elastic structure interaction problem ([6, 7, 10]).

First, we consider a simple model problem: the approximation by piecewise linear finite elements of the eigenvalue problem for the Laplace operator. We give an elementary proof of the equivalence between the error and a residual type error estimator, up to higher order terms. In this case, we also prove that the volumetric part of the residual is dominated by a constant times the edge or face residuals, again up to higher order terms ([9]).

Then we consider the approximation by lowest-order Raviart-Thomas elements of the mixed formulation of this problem. This is equivalent at continuous and

discrete level to the displacement formulation of the vibration problem for an acoustic fluid. We present a residual type error estimator which has been shown to be equivalent to the energy norm of the error, up to higher order terms ([4, 8]).

Finally, we consider the approximation of the pure displacement formulation of the structural acoustics vibration problem. We define an error estimator by combining variations of the two previous ones ([5]) with some additional terms to take care of the variational crimes arising in the fluid-structure interface ([2]). We prove that the estimator is equivalent to the energy norm of the error up to higher order terms, for general meshes satisfying the usual regularity assumption ([3]). The analysis shows that the estimator yields global upper and local lower bounds for the error with constants independent of the physical parameters. So, it can be used to design an efficient adaptive scheme. We report numerical experiments which exhibits the efficiency of the method, even when grids non-matching on the fluid-solid interface are used ([1]).

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A posteriori error analysis for elliptic PDEs on domains with complicated structures

Stefan Sauter

(joint work with Carsten Carstensen)

The discretisation of boundary value problems on complicated domains cannot resolve all geometric details such as small holes or pores. The model problem

consists of a triangulated polygonal domain with holes of a size of the mesh-width at most and mixed boundary conditions for the Poisson equation. Reliable and efficient a posteriori error estimates are presented for a fully numerical discretisation with conforming piecewise affine finite elements. Emphasis is on technical difficulties with the numerical approximation of the domain and their influence on the constants in the reliability and efficiency estimates.

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Convergence rates for an adaptive dual weighted residual finite element algorithm

Erik von Schwerin

(joint work with K.-S. Moon and A. Szepessy)

Basic convergence rates are established for an adaptive algorithm based on the dual weighted residual error representation,

$$\text{error} = \sum_{\text{elements}} \text{error density} \times \text{mesh size}^{2+d},$$

applied to piecewise linear tensor finite element approximation of functionals of multi scale solutions to second order elliptic partial differential equations in bounded domains of $\mathbb{R}^d$. In contrast to the usual aim to derive an a posteriori error estimate, this work derives, as the mesh size tends to zero, a uniformly convergent error expansion for the error density, with computable leading order term. It is shown that the optimal adaptive isotropic mesh uses a number of elements proportional to the $d/2$ power of the $L^{\frac{d}{d+2}}$ quasi-norm of the error density; the same error for approximation with a uniform mesh requires a number of elements proportional to the $d/2$ power of the larger L^1 norm of the same error density. A point is that this measure recognizes different convergence rates for multi scale problems, although the convergence order may be the same. The main result is a proof that the adaptive algorithm based on successive subdivisions of elements reduces the maximal error indicator with a factor or stops with the error asymptotically bounded by the tolerance using the optimal number of elements, up to a problem independent factor. An important step is to prove uniform convergence of the expansion for the error density, which is based on localized averages of second order difference quotients of the primal and dual finite element solutions. The averages are used since the difference quotients itself do not converge pointwise for adapted meshes. The proof uses weak convergence techniques and a symmetrizer for the second order difference quotients. Numerical experiments for an elasticity problem with a crack and different variants of the averages show that the algorithm is useful also in practice.

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Polynomial preserving recovery

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Recovery type error estimators are widely used in engineering applications and their practical effectiveness has been recognized by more and more researchers. See, e.g., [1, Chapter 4], [2]-[6], [8, 9], and [13]-[16]. The most popular recovery technique, the Zienkiewicz-Zhu *Superconvergent Patch Recovery* (SPR) [18], has been used in many commercial codes for the purpose of smoothing and adaptive re-meshing. These codes include ANSYS, MCS/NASTRAN-Marc, Pro/MECHANICA (a product of Parametric Technology), I-DEAS (a product of SDRC, part of EDS), and NASA's COMET-AR (COmputational MEchanics Testbed With Adaptive Refinement). In a computer based investigation [2] by Babuška et al., it was found that among all error estimators tested (including the equilibrated residual error estimator [1] and many others), the Zienkiewicz-Zhu error estimator [17] based on SPR is the most robust.

The idea of SPR is to fit higher-order polynomials, in a discrete least-squares sense, with computed gradients on element patches. Recently, we proposed an alternative recovery method [14, 15, 16]. The idea is to fit higher-order polynomials with computed solution values (instead of gradient values) at some local sampling points. The recovered gradient at a nodal point is obtained by evaluating the gradient of the resultant polynomial at the same nodal point. One significant feature of this recovery is *Polynomial Preserving*. For this reason, we call it PPR.

In an earlier work, Wiberg-Li [12] used function value fitting to improve convergence in the L_2-norm. In some much earlier works, e.g. [7], function value interpolation (instead of least-squares fitting) was used to improve accuracy.

Numerical tests have shown PPR comparing favorably with SPR. This method maintains the simplicity, efficiency, and superconvergence property of SPR, and performs better than SPR under certain meshes (such as the Chevron pattern) as well as on boundaries. In fact, PPR is superconvergent for linear element under the uniform Chevron mesh and ultra-convergent (superconvergence of order two) at element edge centers for quadratic element under the uniform regular mesh [16].

The major advantage of PPR is its polynomial preserving property under practical meshes, a property not shared by SPR. To be more precise, let z be an interior

node, ω_z be the associated element patch, and G_h be the recovery operator by PPR, then

$$|(\nabla u - G_h u)(z)| \leq Ch^{k+1}|u|_{W_\infty^{k+2}(\omega_z)};$$

Furthermore, if z is a grid symmetry point and $k = 2r$,

$$|(\nabla u - G_h u)(z)| \leq Ch^{k+2}|u|_{W_\infty^{k+3}(\omega_z)}.$$

Based on the polynomial preserving property, we have established, under translation invariant meshes:

a) PPR is superconvergent for a large class of triangular elements [16].

b) PPR is ultra-convergent for even-order finite elements under uniform triangulation of the regular pattern [14].

As for non-uniform grids, we need to introduce a mesh condition [13, 15]. Triangulation $T_h = T_{1,h} \cup T_{2,h}$ is said to satisfy *Condition* (α, σ), if it fulfils the maximum angle condition and

1) Two adjacent triangles inside $T_{1,h}$ form an $O(h^{1+\alpha})$ $(\alpha > 0)$ parallelogram;

2) $|\Omega_{2,h}| = O(h^\sigma)$, $\sigma > 0$; $\quad \bar{\Omega}_{2,h} = \bigcup_{\tau \in T_{2,h}} \bar{\tau}$.

For the quadrilateral mesh, instead of 1), we have

1') Sides of two adjacent quadrilaterals inside $T_{1,h}$ defer by $O(h^{1+\alpha})$ $(\alpha > 0)$ from parallel; and every quadrilateral $K \in T_{1,h}$ is convex, and the distance between the mid-points of two diagonals $d_K = O(h_K^{1+\alpha})$.

Note that $\alpha = \infty$ is associated with uniform mesh and $\alpha = 0$ represents completely unstructured mesh. The condition (α, σ) allows completely unstructured mesh on a small measured region while permitting some distortions on the rest of the domain. A mesh generated by the Delaunay triangulation usually satisfies *Condition* (α, σ).

Under the aforementioned mesh condition, we have proven that the recovered gradient by PPR is superconvergent with order $O(h^{1+\rho})$, $\rho = \min(\alpha, \sigma/2, 1/2)$, for both linear and bilinear elements. The result is valid for both the energy norm and the maximum norm in an interior region [14, 15]. As a consequence, the recovered gradient results in an asymptotically exact error estimator. The error bound is in the form of

$$\eta_h + O(h^{1+\rho}) \leq \|\nabla(u - u_h)\| \leq \eta_h + O(h^{1+\rho}),$$

rather than

$$\frac{1}{C}\eta_h + \text{higher order term} \leq \|\nabla(u - u_h)\| \leq C\eta_h + \text{higher order term}$$

for most error bounds in the literature, where C is an unknown constant, which may be very large and hence makes the error bound not very meaningful in practice.

Further numerical tests show that PPR is robust under some highly anisotropic meshes. Theoretical analysis along this line is also promising.

As a recovery technique, PPR is problem independent, method independent, and completely local, like SPR. It can be used for finite element method, finite difference method, as well as finite volume method. Furthermore, the underlying

differential equations are not used in the recovery procedure.

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