Mathematisches Forschungsinstitut Oberwolfach

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Mini-Workshop: Geometry and Duality in String Theory

Organised by

Xenia de la Ossa (Oxford)
Sylvie Paycha (Clermont)
Tsou Sheung Tsun (Oxford)

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Introduction by the Organisers

This mini-workshop brought together 13 women geometers and physicists interested in recent developments in String Theory.

The topic of this meeting, Geometry and Duality in String Theory, was chosen because of its present central role in theoretical physics, and because of the richness of the geometric tools involved in the various notions of duality arising in string theory. The scientific aim was to strengthen the bridge between mathematicians and physicists applying and developing these tools to analyse and exploit these notions of duality. The workshop's interdisciplinary nature also aimed to encourage further interaction between women mathematicians and physicists. Every participant gave a talk in a subject relevant to the main topics of the workshop. The areas covered by the talks were: AdS/CFT correspondence (C. Nuñez, A. Grassi), mirror symmetry (X. de la Ossa, A. Grassi, K. Wendland), open-closed string dualities (A. Grassi), electric-magnetic duality (Tsou S. T.), K-theory (S. Shafer-Nameki), Singularities of Calabi-Yau manifolds and the McKay correspondence (T. Friedmann, A. Degeratu, Y. Ito), Conformal Field Theories (K. Wendland, S. Shafer-Nameki, C. Nuñez), symmetries of M-Theory (A. Taormina), current cosmological results and phenomenological constraints on String Theory from cosmological data (S. Paban), arithmetic of Calabi-Yau manifolds (S. Kadir), and discrete curves and the Toda-Lattice (N. Kutz). It was very satisfying to see that the dynamics of the workshop was so positive and interesting, and all talks were very well delivered. We had plenty of time for informal discussions, and collaborations were highly encouraged. Indeed, several promising collaborations did start then.

The idea of such a mini-workshop, with only women speakers, arose from a scientific gathering of women mathematicians and physicists present at the European Women in Mathematics meeting held in Malta in August 2001. Clearly, the reasons as to why there is such a low representation of women in physics and mathematics are very complex and this is not the place to address them. The discussion of these issues, while interesting and relevant to justify this scientific meeting, were not the purpose of the workshop. However, we felt that this small scale scientific workshop with women speakers contributed to the exchange of information on latest developments in the area of this workshop, thus helping with the advancement of participants' careers, in a first class professional environment. We also believe this high standard scientific workshop has contributed to increase the visibility of women working in this field, and we hope would further encourage women to enter and stay in this rapidly developing area of research. We hope this increased visibility would also encourage women starting their careers or who might have had interruptions to their careers. We were therefore very proud that this meeting gathering women mathematicians and physicts from different parts of the world interested in various aspects of duality in string theory took place, and we are grateful to the MFO in Oberwolfach for their support. This workshop was an excellent opportunity for us to establish and further develop scientific and personal contacts.

Sadly, one of the organizers of this workshop, Sylvie Paycha, was not able to participate due to an accident her son had just before the beginning of the workshop. It is regretable because it was Sylvie who first had the idea of this workshop and it was she who invested a lot of energy to bring this to fruition.

Finally, we would like to thank Natalia de la Ossa, an expert on gender issues, for her advise in writing the proposal for this mini-workshop, and for her comments on this introduction.

Workshop on Mini-Workshop: Geometry and Duality in String Theory

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Abstracts

Introduction to Mirror Symmetry

Xenia de la Ossa

This lecture focuses on an example of the type of new mathematical ideas that have emerged as a consequence of String Theories. I present here an introduction to the concept of *mirror symmetry* between pairs of Calabi–Yau manifolds[1, 2, 3, 4]. Mirror Symmetry is but one example of a *duality symmetry*, a deep relation between *different* types of String Theories. I also give an introduction to the geometry of the moduli space [5, 6] of certain string theories relevant to the discussion of mirror symmetry.

In this lecture we say that the pair $(\mathcal{X}, \mathcal{Y})$ of Calabi–Yau manifolds is a *mirror pair* if the effective Quantum Field Theory of a String Theory compactified on $\mathcal{X}$, $\Gamma_{\mathcal{X}}$, is the same as the effective Quantum Field Theory of a String Theory compactified on $\mathcal{Y}$, $\Gamma_{\mathcal{Y}}$. The mirror symmetry conjecture ascertains that for every Calabi–Yau manifold $\mathcal{X}$, there exists another Calabi–Yau manifold $\mathcal{Y}$, such that $\Gamma_{\mathcal{X}} = \Gamma_{\mathcal{Y}}$. The most elementary consequence of this conjecture is that the spectrum of both theories coincide. However, when mirror symmetry is stated in terms of an isomorphism between the parameter spaces of the different theories involved, isomorphism which is apparent only after all the quantum corrections to the theory have been included, one finds new and surprising identities between the physical quantities (correlation functions) of the different theories. Mathematically, these identities between correlation functions correspond to generating functions for Gromov-Witten invariants. I illustrate this by explaining how the generating function for the number of rational curves (genus-zero Gromov-Witten invariants) for a quintic three-fold was originally found [7]. For a proof of the identities in [7], see [8, 9]. A puzzle that remains to be solved is the integrality of the isomorphism map between the parameter spaces [10]. A beautiful generalization of the original ideas of mirror symmetry was proposed by Kontsevich [11] in his Homological Mirror Conjecture, in which it was proposed an isomorphism between Fukaya's A_∞–category of Lagrangian Submanifolds of $\mathcal{X}$ and the bounded derived category of coherent sheaves on $\mathcal{Y}$.

One interesting question has to do with the relation between the Calabi–Yau manifolds in a mirror pair. That is, given a Calabi–Yau manifold, how do we construct its mirror? Batyrev [12] has constructed a large family of Calabi–Yau manifolds which are hypersurfaces of toric varieties with the feature that it is mirror symmetric, in other words, given a Calabi–Yau manifold which is defined as a hypersurface of a toric variety, he found a procedure to construct the mirror manifold. By considering also the degrees of freedom corresponding to D-branes which appear in String Theory, Strominger, Yau and Zaslow [13], proposed a geometric description of the mirror of a Calabi–Yau manifold. They conjecture that if a Calabi–Yau manifold has a mirror, then it must be a fibration with general fibers being three-tori which are Special Lagrangian submanifolds. Furthermore, they conjecture that the mirror manifold has general fibers which are the "T-dual" of the three-tori. Mirror symmetry remains a conjecture however progress has been made in understanding its deep mathematical structure [14].

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Introduction to the McKay Correspondence and its Manifestations in Physics

Tamar Friedmann

The purpose of this talk was to present the McKay correspondence, which relates the topology of certain blown-up spaces to the representation theory of certain finite groups, and to discuss some of its manifestations in string theory.

We started by presenting the relevant finite groups, which are the discrete subgroups Γ of SU(2); they have an ADE classification into cyclic (A_n), binary dihedral (D_n), binary tetrahedral (E_6), binary octahedral (E_7), and binary icosahedral (E_8) groups. Following this we showed how to construct the corresponding ADE singularities $\mathbf{C}^2/\Gamma$ and how to blow them up, leading to an exceptional divisor whose dual graph turns out to be the Dynkin diagram of the Lie algebra of the corresponding A, D, or E type.

We discussed the topology of the exceptional divisor in the blown-up space, which has an important place in string theory via the method known as geometric engineering. This method enables us to answer the following question: what kind of physics do we get if our compactification manifold has an ADE singularity? The answer is that the physics is a gauge theory with gauge group given by the corresponding A, D, or E Lie group. We showed how the gauge fields of this theory arise in the process of compactification, namely via Kaluza-Klein reduction of 3-form fields on the 2-cycles of the exceptional divisor, and via wrapping D2-branes on these 2-cycles.

Next, we presented what is known as the McKay graph or McKay quiver, which is defined in terms of data coming from the representation theory of Γ. We explained how the observation that the McKay graph is identical to the corresponding (extended) ADE Dynkin diagram leads to the McKay correspondence.

Finally, we explained how the McKay graph appears in physics through what are known as quiver theories: these theories, in which the group Γ acts in a specified way on a given configuration of D-branes, are direct physical manifestations of Kronheimer's hyper-Kahler quotient construction of ALE spaces, which uses the McKay correspondence in an essential way.

A Stringy View on K-Theory

Sakura Shafer Nameki

Recently K-theory has made its appearance at various places in string theory and conformal field theory. In this talk I shall explain two such instances:

1) It has been conjectured that charges of D-branes in string theory are classified by K-theory. This proposal is motivated by tachyon condensation and the thereby resulting D-brane descent relations of Sen. For various backgrounds the conjecture has been tested and been confirmed. I shall examplify this with curved backgrounds, such as D-branes on group manifolds and coset models. In this case, due to the non-trivial B-field in the background, the relevant K-theories are twisted. To illustrate this, I shall discuss the $SU(2)$ WZW model in detail, and present both the charge computation in the CFT as well as give a derivation of the twisted K-theory ${}^{\tau}K(SU(2))$ using the Rosenberg spectral sequence.

2) Twisted equivariant K-theory has a profound connection to conformal field theory, via the theorem by Freed, Hopkins and Teleman (FHT). For a simple, simply-connected, connected Lie group G, the statement of FHT is, that the twisted G-equivariant K-theory of G is isomorphic as an algebra to the Verlinde algebra of the WZW model associated to G. The product on the K-theory side is given by the Pontriyagin product, and the level of the WZW is related to the twist class $\tau \in H^3_G(G, \mathbb{Z})$. Of key importance is that the equivariance is with respect to the conjugation action of G on itself. I shall present an extension of this theorem to $\mathcal{N} = 2$ coset conformal field theories for G and a maximal rank subgroup H, stating that the chiral ring is isomorphic to ${}^{\tau}K_{H/Z}(G)$, where Z is the common

centre of G and H. This will be illustrated with the super-parafermion theories $SU(2)/U(1)$.

AdS/CFT correspondence: the three dimensional case

Carmen Nuñez

This lecture is a short introduction to some ideas supporting the conjectured duality between gauge theories and gravity, currently known as the AdS/CFT correspondence.

The fundamental interactions of nature are explained by the Standard Model of the electroweak and strong forces and by Einstein's theory of General Relativity. In the framework of field theory there is no connection between the gauge theories of the Standard Model and Einstein's description of gravity, but any field theory involving gravity suffers from the problem of non-renormalizability. In the framework of string theory instead, where quantum gravity makes sense, we see not only that they naturally occur together, but also that the presence of both gravitational and gauge interactions is perhaps unavoidable in a consistent string theory.

The original motivation for string theory was the physics of the strong interactions. The large number of mesons and hadrons that were experimentally discovered in the 1960's could be interpreted as different oscillation modes of a string. But it was later discovered that they are actually described by QCD, a gauge theory based on SU(3). QCD is asymptotically free: the effective coupling constant decreases as the energy increases. At low energy the theory becomes strongly coupled and it is not easy to perform calculations. It was suggested by 't Hooft that the theory might simplify when the number of colors N is large [1]. The diagrammatic expansion suggests that the large N limit of QCD is a free string theory if the string coupling constant is 1/N. This feature is very general and applies to different kinds of gauge theories. In this way the large N limit connects gauge theories with string theories: the gauge theory description is useful for the high energy behavior of gauge theories and the string theory description is useful for low energy issues such as confinement. This is an example of duality.

The indications for this duality do not specify which string theory is dual to a particular gauge theory. For four dimensional Yang-Mills theory one would naively expect to get a bosonic string theory in four dimensions. But we know that this is inconsistent. The Polyakov action has a Weyl anomaly: under a conformal change in the metric, the action changes by a Liouville contribution which behaves like an extra dimension, so if we are interested in four dimensional gauge theories we have to look for strings at least in five dimensions and we have to specify the space where the string moves. The AdS/CFT correspondence realizes this idea but with five additional dimensions, leading to a ten dimensional string theory.

The insight on the correspondence emerged from the study of D-branes. Here we present a short introduction to D-branes and consider Type IIB superstring theory in ten flat dimensions with N parallel D3-branes. String theory on this background contains two kinds of perturbative excitations, closed strings and open strings. The closed strings are excitations of empty space and the open strings end on the D-branes and describe excitations of the D-branes. If we consider the system at low energies, energies lower than the string scale, then only massless modes can be excited and an effective lagrangian describes their interactions. The closed string massless states give a gravity supermultiplet in 10 dimensions, and their low energy effective lagrangian is that of type IIB supergravity. Close to the D-branes, in the near horizon region, the spacetime metric is $AdS_5 \times S^5$. The open string massless states give an $\mathcal{N} = 4$ vector supermultiplet in 3+1 dimensions and their low energy dynamics is described by an $\mathcal{N} = 4$ $U(N)$ super Yang-Mills theory. This is the original duality discovered by Maldacena [3].

We discuss some of the calculations that support this conjecture, which turns out to fit well into the holographic description of AdS gravity [4]. We briefly describe both sides of the correspondence: CFT and AdS spaces, and present the three dimensional example where the conjecture can be worked out beyond the supergravity approximation, in the full string theory.

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Crepant Resolutions of Calabi-Yau Orbifolds

Anda Degeratu

A Calabi-Yau manifold is a complex Kähler manifold with trivial canonical bundle. In the attempt to construct such manifolds it is useful to take into consideration singular Calabi-Yaus. One of the simplest singularities which can arise is an orbifold singularity. An orbifold is the quotient of a smooth Calabi-Yau manifold by a discrete group action which generically has fixed points. Locally such an orbifold is modeled on $\mathbb{C}^n/G$, where G is a finite subgroup of $SL(n, \mathbb{C})$.

From a geometrical perspective we can try to resolve the orbifold singularity. A resolution (X, π) of $\mathbb{C}^n/G$ is a nonsingular complex manifold X of dimension n with a proper biholomorphic map $\pi : X \to \mathbb{C}^n/G$ that induces a biholomorphism between dense open sets. We call X a *crepant resolution*[1] if the canonical bundles are isomorphic, $K_X \cong \pi^*(K_{\mathbb{C}^n/G})$. Since Calabi-Yau manifolds have trivial canonical bundle, to obtain a Calabi-Yau structure on X one must choose a crepant resolutions of singularities.

It turns out that the amount of information we know about a crepant resolution of singularities of $\mathbb{C}^n/G$ depends dramatically on the dimension n of the orbifold:

[1]Etymology: For a resolution of singularities we can define a notion of *discrepancy* [R1]. A crepant resolution is a resolution without discrepancy.

$n = 2$: A crepant resolution always exists and is unique. Its topology is entirely described in terms of the finite group G (via the McKay Correspondence).

$n = 3$: A crepant resolution always exists but it is not unique; they are related by flops. However all the crepant resolutions have the same Euler and Betti numbers: the *stringy* Betti and Hodge numbers of the orbifold [DHVW].

$n \geq 4$: In this case very little is known; crepant resolutions exist in rather special cases. Many singularities are terminal, which implies that they admit no crepant resolution.

We would like to completely understand the topology of crepant resolutions in the case $n = 3$. We are concerned with the study of the ring structure in cohomology. This is related to the generalization of the McKay Correspondence.

The case $n = 2$. The quotient singularities $\mathbb{C}^2/G$, for G a finite subgroup of $SL(2, \mathbb{C})$, were first classified by Klein in 1884 and are called *Kleinian singularities* (they are also known as *Du Val singularities* or *rational double points*). There are five families of finite subgroups of $SL(2, \mathbb{C})$: the cyclic subgroups $\mathcal{C}_k$, the binary dihedral groups $\mathcal{D}_k$ of order $4k$, the binary tetrahedral group $\mathcal{T}$ of order 24, the binary octahedral group $\mathcal{O}$ of order 48, and the binary icosahedral group $\mathcal{I}$ of order 120. A crepant resolution exists for each family and is unique. Moreover the finite group completely describes the topology of the resolution. This is encoded in the McKay Correspondence [McK1], which establishes a bijection between the set of irreducible representations of G and the set of vertices of an extended Dynkin diagram of type ADE (the Dynkin diagrams corresponding to the simple Lie algebras of the following five types: A_{k-1}, D_{k+2}, E_6, E_7 and E_8). Using McKay's correspondence it is easy to describe the crepant resolution $\pi : X \to \mathbb{C}^2/G$. The exceptional divisor $\pi^{-1}(0)$ is the dual of the Dynkin diagram: the vertices of the Dynkin diagram correspond naturally to rational curves C_i with self-intersection -2. Two curves intersect transversally at one point if and only if the corresponding vertices are joined by an edge in the Dynkin diagram, otherwise they do not intersect. The curves above form a basis for $H_2(X, \mathbb{Z})$. The intersection form with respect to this basis is the negative of the Cartan matrix.

The first geometrical interpretation of the McKay Correspondence was given by Gonzalez-Sprinberg and Verdier [GV]. To each of the irreducible representations R_i they associated a locally free coherent sheaf $\mathcal{R}_i$. The set of all these coherent sheaves form a basis for $K(X)$, the K-theory of X. Moreover, the first Chern classes $c_1(\mathcal{R}_i)$ form a basis in $H^2(X, \mathbb{Q})$ and the product of two such classes in $H^*(X, \mathbb{Q})$ is given by the formula

$$\left[\int_X c_1(\mathcal{R}_i)c_1(\mathcal{R}_j)\right]_{i,j=1,\ldots,r} = -\,C^{-1}, \tag{1}$$

where C^{-1} is the inverse of the Cartan matrix. The proof given by Gonzalez-Sprinberg and Verdier uses a case by case analysis and techniques from algebraic geometry. Kronheimer and Nakajima gave a proof of the formula using techniques from gauge theory [KroN].

To summarize, in the case of surface singularities, $\mathbb{C}^2/G$, the representation theory of the finite group G completely determines the topology the crepant resolution. The Dynkin diagram and the Cartan matrix (and hence the simple Lie algebra $\mathfrak{g}$ associated to it) encode everything we want to know about the topology of the crepant resolution.
The case $n = 3$. The finite subgroups of $SL(3, \mathbb{C})$ were classified by Blichfeldt in 1917 [Bl]: there are ten families of such finite subgroups. In the early 1990's a case by case analysis was used to construct a crepant resolution of $\mathbb{C}^3/G$ with the given stringy Euler and Betti numbers (see [Ro] and the references therein). As a consequence of these constructions, we know that all the crepant resolutions of $\mathbb{C}^3/G$ have the Euler and Betti numbers given by the stringy Euler and Betti numbers of the orbifold (since these numbers are unchanged under flops). In 1995 Nakamura made the conjecture that $\mathrm{Hilb}^G(\mathbb{C}^3)$ is a crepant resolution of $\mathbb{C}^3/G$. In general, for G a finite subgroup of $SL(n, \mathbb{C})$, the algebraic variety $\mathrm{Hilb}^G(\mathbb{C}^n)$ parametrizes the 0-dimensional G-invariant subschemes of $\mathbb{C}^n$ whose space of global sections is isomorphic to the regular representation of G. Nakamura made the conjecture based on his computations for the case $n = 2$ [INak]; then he proved it in dimension $n = 3$ for the case of abelian groups [Nak]. In 1999 Bridgeland, King and Reid gave a general proof of the conjecture in the case $n = 3$, relying heavily on derived category techniques [BKR]. In 2002 Craw and Ishii proved that (at least in the case G abelian) all the crepant resolutions arrive as moduli spaces [CI].

In the case of surface singularities, an important feature of the McKay Correspondence is that it gives the ring structure in cohomology in terms of the finite group. For the case $n \geq 3$, nothing is known about the multiplicative structures in cohomology or K-theory.

Let $G \subset SL(3, \mathbb{C})$ be a finite subgroup acting with an isolated singularity on $\mathbb{C}^3/G$. Let X be a crepant resolution of $\mathbb{C}^3/G$. On this resolution we associate a vector bundle $\mathcal{R}_i$ to each irreducible representation of G – this is the extension of the Gonzalez-Sprinberg-Verdier sheaves. These bundles form a basis of the K-theory of X, and via the Chern character isomorphism, $\{\mathrm{ch}(\mathcal{R}_0), \mathrm{ch}(\mathcal{R}_1), \ldots, \mathrm{ch}(\mathcal{R}_{\mathrm{r}})\}$ basis of $H^*(X; \mathbb{Q})$.

The idea is to use the Atiyah-Patodi-Singer (APS) index theorem for studying multiplicative properties of the (Chern classes of the) bundles $\mathcal{R}_i$. In [De2] we show a that a generalization of Kronheimer and Nakajima's formula (1) holds in the compactly supported cohomology of X:

$$\left[\int_X (\mathrm{ch}(\mathcal{R}_{\mathrm{i}}) - \mathrm{rk}(\mathcal{R}_{\mathrm{i}}))\left(\mathrm{ch}(\mathrm{R}_{\mathrm{j}}^*) - \mathrm{rk}(\mathcal{R}_{\mathrm{i}})\right)\right]_{i,j=1,\ldots,r} = C^{-1}. \tag{2}$$

Here C is a matrix associated to the finite group G and its embedding into $SL(3, \mathbb{C})$, generalizing the Cartan matrix of the case $n = 2$.

However, the relations (2) are common for all the crepant resolutions of $\mathbb{C}^3/G$. They do not give any insight about what changes when two crepantresolutions differ by a flop. Using the analytical approach developed in [De2] we prove that given X with the tautological line bundles $\mathcal{R}_0, \ldots, \mathcal{R}_{r-1}$, and given X' obtained from

X via the flop of a $(-1,-1)$ curve C with its tautological bundles $\mathcal{R}'_0, \ldots, \mathcal{R}'_{r-1}$ we have, [De3]:

$$\int_X c_1(\mathcal{R}'_j)^3 = \int_X c_1(\mathcal{R}_j)^3 - \deg \mathcal{R}_j|_C. \tag{3}$$

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Three dimensional McKay correspondence and Mirror Symmetry

Yukari Ito

In my talk, I explained the McKay correspondence for 3-dimensional Quotient singularities. Original McKay correspondence was observed in dimension two by John McKay in 1979 and developed as a geometrical correspondence between a set of irreducible representations of the acting group G and the Grothendieck group of the minimal resolution of quotient singularities by Gonzalez-Sprinberg and Verdier.

I discussed a result by Nakajima and myself which generalizes the McKay correspondence to 3-dimensional case. Then I showed an intersection formula in terms of the multiplicities of the irreducible representations in the tensor product between an irreducible representation and a 3-dimensional regular representation. We do not yet understand the how to interpret this formula mathematically. However, I mentioned that one can find the same formula in some papers on local mirror symmetry, a similarity which I commented in my talk.

I hope that this similarity can be explained both in physics and Mathematics in the near future.

Open-closed String dualities in geometry

Antonella Grassi

In my lecture I discussed the geometric aspects of open/ closed string dualities (also known as Large N-dualities). I presented some of the main results obtained in this area in the past 6 years or so, focusing on particular examples to illustrate some of the many ideas involved. I emphasized, in particular, applications to algebraic and symplectic geometry and connections with the topics of other talks presented at the workshop.

The open/closed dualities involve transformations among Calabi-Yau threefolds, holomorphic curves with and without boundaries, Chern Simons theory on (real) 3 manifolds and knot invariants. Vafa, Gopakumar and Ooguri noted, via a string theory analysis, that topological and knots invariants of S^3 determine, and are determined, by "local" Gromov-Witten invariants of a certain Calabi–Yau manifold X. The key point is that S^3 is a vanishing cycle in a Calabi–Yau Y which is deformed to a singular Calabi–Yau Y_0; X is a Calabi–Yau resolution of Y_0. The topological and knot invariants of S^3 are determined via $\mathrm{U}(N)$ Chern–Simons theory on S^3 (as introduced by Witten); the local Gromov-Witten invariants of X are in a neighborhood of the birational contraction $X \to Y_0$. The transformation between Y and X is a "Calabi–Yau transition".

It is natural to ask if there exists an appropriate generalization for global, rather than local, invariants. It turns out that the (closed) Gromov-Witten invariants of X agree, with a suitable identification of the parameters, with "modified open Gromov-Witten invariants" of Y. There is no general theory to define "open Gromov-Witten" invariants, which should compute enumerative invariants of maps from open Riemann surfaces to Y, where the image of the boundary is constrained to be on Lagrangian submanifolds (the vanishing cycles, in our case). However, under some assumptions it is possible to compute the open invariants. The modified invariants arise by combining the open enumerative invariants with knots and links invariants determined in the vanishing cycles by the boundaries of open holomorphic curves, via Chern–Simons theory. I also discussed briefly the mirror dual of this set up.

Electric–Magnetic Duality: abelian and nonabelian

TSOU Sheung Tsun

It is well known that Maxwell's theory of electromagnetism is dual symmetric between electricity and magnetism, where duality is defined by the Hodge star operation on the 2-form field strength $F_{\mu\nu}$

$$F_{\mu\nu} = -\tfrac{1}{2}\,\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}.$$

Schematically, we can relate:

$$\begin{array}{ccc}
\boxed{\begin{array}{c} A_\mu \text{ exists as} \\ \text{potential for } F_{\mu\nu} \\ (F = \mathrm{d}\,A) \end{array}} & \overset{\text{Poincaré}}{\Longleftrightarrow} & \boxed{\begin{array}{c} \text{Defining constraint} \\ \partial_\mu F^{\mu\nu} = 0 \\ (\mathrm{d}\,F = 0) \end{array}} \\
\Big\Updownarrow & & \Big\Updownarrow \text{Gauss} \\
\boxed{\begin{array}{c} \text{Principal } A_\mu \\ \text{bundle trivial} \end{array}} & \overset{\text{Dirac}}{\Longleftrightarrow} & \boxed{\begin{array}{c} \text{No magnetic} \\ \text{monopole } \tilde{e} \end{array}} \\
& & \\
\text{GEOMETRY} & & \text{PHYSICS}
\end{array}$$

It is also known (through the work of Gu and Yang, for example) that this duality does not generalize to the nonabelian Yang–Mills case. We reformulate the Yang–Mills action in loop space and use a constrained variation to prove that Yang–Mills theory is dual-symmetric, where now the duality operation is in terms of loop space variables and generalizes the Hodge star operation (but reduces to it in the abelian case). Schematically, we can draw a parallel with the abelian case:

$$\begin{array}{ccc}
\boxed{\begin{array}{c} A_\mu \text{ exists} \\ \rightsquigarrow \text{ phase factor} \\ \rightsquigarrow \text{ loop space curvature} \end{array}} & \overset{\text{ext.Poincaré}}{\Longleftrightarrow} & \boxed{\begin{array}{c} \text{Defining constraint} \\ G_{\mu\nu} = 0 \\ \text{(loop space flat)} \end{array}} \\
\Updownarrow & & \Updownarrow \\
\boxed{\begin{array}{c} \text{Principal } A_\mu \\ \text{bundle trivial} \end{array}} & \overset{\text{definition}}{\Longleftrightarrow} & \boxed{\begin{array}{c} \text{No magnetic} \\ \text{monopole } \tilde{g} \end{array}} \\
\text{GEOMETRY} & & \text{PHYSICS}
\end{array}$$

A hint that this dual symmetry persists in a quantum theory is indicated by proving the 't Hooft commutation relation for the corresponding quantum operators. Moreover, the resulting dual symmetry suggests a way to solve the generation puzzle in particle physics: applied to confined colour $SU(3)$, we get a broken $\widetilde{SU(3)}$ which can be identified with the three generations of fermions, as observed in nature. The resulting model we call the Dualized Standard Model (for a review, see Acta Physica Polonica 33B (2002) 4041–4100, updated in European Physical Journal C30 (2003), 51–54.).

The Entropy of the Microwave Background and the Acceleration of the Universe

Sonia Paban

While String Theory has to make progress in the absence of new high energy experimental results, the field of Cosmology is enjoying an abundance of data, in particular, from the increasingly precise measurements on the microwave background. One of the most surprising new findings has been that the expansion of the universe is accelerating. Other results include the very approximate flatness of the universe, and strong bounds on its topology. This talk presented an introduction to current cosmological results as well as a brief introduction to inflation. The development of String Theory has been seriously affected by this new data. Though the current precision is not enough to single out the cause of the acceleration it is possible that its origin is a cosmological constant. Understanding the magnitude of such a contribution as well as its compatibility with the framework of String Theory has spurred an intellectual activity among string theorists. The most relevant points of this debate were only briefly presented during the talk due to a lack of time.

The Symmetry of M-Theories

Anne Taormina

A maximally oxydised theory associated with a simple group $\mathcal{G}$ is a theory of gravity coupled to forms and dilatons defined in the highest possible space-time dimension D which, upon dimensional reduction to three, is expressible in terms of a coset space $\mathcal{G}/\mathcal{H}$ where $\mathcal{H}$ is the maximally compact subgroup of $\mathcal{G}$. The maximally oxydised actions corresponding to all simple groups $\mathcal{G}$ have been classified [1] and they comprise in particular pure gravity in D dimensions, the bosonic part of the low energy effective action of M-theory and the low energy effective action of the bosonic string. It has been conjectured that these actions possess the much larger triply-extended Kac-Moody symmetry $\mathcal{G}^{+++}$. $\mathcal{G}^{+++}$ algebras are defined from the Dynkin diagrams obtained by adding three nodes to those of $\mathcal{G}$ [2]. One first adds the affine node, then a second node connected to it by a single line to define the doubly-extended $\mathcal{G}^{++}$ algebras, then similarly a third node connected to the second one to define the triply-extended algebras $\mathcal{G}^{+++}$. Such $\mathcal{G}^{+++}$ symmetries were first conjectured in [3] for pure gravity in D dimensions (A_{D-3}^{+++}), the low energy effective action of M-theory ($E_8^{+++} \equiv E_{11}$) and of the 26-dimensional bosonic string ($D_{24}^{+++} \equiv K_{27}$). The generalisation to all $\mathcal{G}^{+++}$ was proposed in [4].

The study of specific classical solutions of maximally oxydised theories provides some ways of testing the existence of such $\mathcal{G}^{+++}$ symmetries. On the one hand, explicit representations of their *Weyl group* for Kasner-type solutions have been obtained for all simple $\mathcal{G}^{+++}$ [4], and their relation to cosmological billiards [5] has been brought to light. On the other hand, the maximally oxydised theories also admit zero binding energy configurations of intersecting closed extremal branes. In such configurations, some branes may open on host closed branes. Properties of extremal branes reveal symmetries of the underlying theory, which are compatible with the presence of a $\mathcal{G}^{+++}$ Kac-Moody algebra [6].

The results reported here point towards the existence of triply-extended symmetries not only for M-theory and for the bosonic string but for all oxydised theories. The M-theory quest is generally viewed as the privileged way to reach a unified theory of gravity and matter. From the point of view developed here, the bosonic part of the M-theory effective action is just one amongst many sharing the same universal type of symmetry. In addition, the embedding of superstrings into the bosonic string [7, 8] whose effective action is also an oxydised theory, suggests that the degrees of freedom hidden in different $\mathcal{G}^{+++}$ may be related to each other.

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On degenerating sequences of CFTs and their geometric interpretation

Katrin Wendland
(joint work with Daniel Roggenkamp)

Phenomena like mirror symmetry [1] are closely linked to degeneration phenomena, both in geometry and conformal field theory (CFT): Consider a CFT obtained from a non-linear sigma model construction on a Calabi-Yau variety X. If for X one performs a large complex structure or a large radius limit, as needed in the analysis of mirror symmetry, instanton contributions are suppressed in the correlation functions of the CFT. That is, the CFT yields exact data from geometry in such a limit. Degeneration phenomena can therefore be expected to provide the key to the decoding of geometry from CFT.

While geometric degeneration phenomena have been studied in detail by mathematicians, see e.g. [2, 5, 7, 4], their counter parts in CFT are less well understood. In [9] we give an intrinsic notion of limiting processes in CFTs. According to our definition the limit of a sequence of CFTs is not necessarily a full-fledged CFT: Though we construct a limiting pre-Hilbert space which carries limiting OPE-coefficients and a representation of two commuting copies of limiting Virasoro algebras, e.g. the torus partition functions need not converge. Indeed, as motivated above, we have to allow for degenerate limits of CFTs in order to build the bridge to geometry. We characterize such degenerate limits by the existence of an infinite dimensional subspace $\mathbb{H}$ of the limiting pre-Hilbert space which is generated by states with vanishing left- and right dimensions with respect to the limiting Virasoro actions. This is in accord with the notion of classical limits introduced by Moore and Seiberg in [8]. The limiting OPE-constants allow us to recover the structure of an algebra $\mathcal{A}$ which acts on $\mathbb{H}$ and as we prove is commutative. Moreover, an appropriate rescaling of the energy operator obtained from the limiting Virasoro algebra provides us with a self-adjoint operator H on $\mathbb{H}$ which according to unpublished results by Kontsevich [6] has the properties of a generalized Laplacian without constant term. We adapt ideas by Fröhlich and Gawędzki [3] to implement Connes' approach to non-commutative geometry and interpret a closure of $\mathcal{A}$ as space of continuous functions $C^0(M)$ on a manifold M. H is then interpreted as generalized Laplacian with respect to a Riemannian metric g on M together with a dilaton $\Phi \in C^\infty(M)$ such that H acts on $L^2(M, \mathrm{dvol}_{\widetilde{g}})$ with $\widetilde{g} = \left(e^{2\Phi}\right)^{2/\dim_{\mathbb{R}}(M)} g$. The limiting data of our CFTs allow us to explicitly determine g and Φ, i.e. to determine a geometric interpretation (M, g, Φ) of the limit.

Our notion of limiting processes of CFTs surpasses (but is compatible with) known results from deformation theory: We apply our techniques to the discrete family of (non-supersymmetric) diagonal unitary Virasoro minimal models, prove that they give a convergent sequence of CFTs according to our definition, and explicitly determine the geometric interpretation of its large level limit. A different large level limit of these Virasoro minimal models was proposed in [10].

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The Arithmetic of Calabi-Yau Manifolds

Shabnam Kadir

We are interested in studying arithmetic properties of Mirror symmetry. Mirror symmetry is a conjecture in string theory according to which certain 'mirror pairs' of Calabi-Yau manifolds give rise to isomorphic physical theories. (A Calabi-Yau manifold is a complex variety of dimension d with trivial canonical bundle and vanishing Hodge numbers $h^{i,0}$ for $0 < i < d$, e.g. a one dimensional Calabi-Yau variety is an elliptic curve, a 2-dimensional Calabi-Yau is a K3 surface, and in dimensions three and above there are many thousands of Calabi-Yau manifolds).

Physicists concerned with mirror symmetry usually deal with Calabi-Yau manifolds defined over $\mathbb{C}$, here however, in order to study the arithmetic, we shall reduce these algebraic Calabi-Yau varieties over discrete finite fields,$\mathbb{F}_q$, $q = p^r$, where p is prime and $r \in \mathbb{N}$; these are the extensions of degree r of the finite field, $\mathbb{F}_p$. The data of the number of rational points of the reduced variety, $N_{r,p}(X) = \#(X/\mathbb{F}_{p^r})$, can be encoded in a generating function known as Artin's Congruent Zeta Function, which takes the form:

$$Z(X/\mathbb{F}_{p^r}, t) \equiv \exp\left(\sum_{r\in\mathbb{N}} N_{r,p}(X)\frac{t^r}{r}\right). \tag{4}$$

The motivation for choosing the above type of generating function is due to the fact that this expression leads to rational functions in the formal variable, t. This result is part of the famous Weil conjectures (no longer conjectures for at least the last 30 years [D1, Del]). The Weil conjectures show that the Artin's Zeta function for a smooth variety is a rational function determined by the cohomology of the variety, in particular, that the degree of the numerator is the sum of the odd Betti numbers of the variety and the denominator, the sum of the even Betti numbers. As mirror symmetry interchanges the odd and even Betti numbers of the Calabi-Yau variety, it is a natural question to investigate the zeta functions of pairs of mirror symmetric families. Hence, there is speculation as to whether a "quantum modification" to the Congruent Zeta function can be defined such that the zeta function of mirror pairs of Calabi-Yau varieties are inverses of each other. Notice that the above conjecture cannot hold using the "classical definition" (4) above because this would mean that for a pair of manifolds, (X, Y), we would have to have $N_{r,p}(X) = -N_{r,p}(Y)$, which is not possible.

In order to study these questions, we shall be considering families with up to two parameters, and use methods very similar to [CdOV1, CdOV2]. In particular, we study a one parameter family of K3 surfaces and a two parameter family of Calabi-Yau threefolds, octic hypersurfaces in weighted projective space $\mathbb{P}_4{}^{(1,1,2,2,2)}[8]$, the mirror symmetry of which was studied in detail in [CdOFKM]. We shall be concerned with Calabi-Yau manifolds which are hypersurfaces in toric varieties, as this provides a powerful calculational tool, and it enables one to use the Batyrev formulation of mirror symmetry [Bat].

The method of computation involves use of Gauss sums, that is the sum of an additive character and a multiplicative character. For the purpose of counting rational points the Dwork character is a very suitable choice for the additive character along with the Teichmüller character as multiplicative character. The Dwork character was first used by Dwork [D1] to prove the rationality of the congruent zeta function for varieties (this is one of the Weil conjectures). The number of rational points can be written in terms of these Gauss sums, thus enabling computer aided computation of the zeta function.

The mirror of the Calabi-Yau manifolds can be found using the Batyrev mirror construction [Bat]. This is a construction in toric geometry in which a dual pair of reflexive polytopes can be related via toric geometry to mirror symmetric pairs of Calabi-Yau manifolds. Using this construction and using Gauss sums it is possible to also find the zeta function of the mirror Calabi-Yau manifold. In [CdOV2], the mirror zeta function was found to have some factors in common with the original zeta function, namely a contribution to the number of points associated to the unique interior lattice point of the polyhedra. We shall find the same phenomenon for the octic where there is a sextic, $R_{(0,0,0,0,0)}$, that appears in both the original family and the mirror. For the octic, on the mirror side, there was also a contribution related to a zero-dimensional Calabi-Yau manifold(studied in [CdOV1]) which was sensitive to a particular type of (non-conifold) singularity. This contrasts with the the fact that zeta function of the original family had a contribution related to a particular monomial (not on the polyhedron) that is sensitive to the presence of conifold points.

In [CdOV1] it was shown that the number of rational points of the quintic Calabi-Yau manifolds over $\mathbb{F}_p$ can be given in terms of the periods; our calculations verify this relation for the octic. The periods satisfy a system of differential equations known as the Picard-Fuchs equations, with respect to the parameters. The Picard-Fuchs equations for the quintic (and also the octic) simplify considerable because of the automorphisms of the manifold, $\mathcal{A}$. The elements of the polynomial ring can be classified according to their transformation properties under the automorphisms, that is into representations of $\mathcal{A}$, and owing to the correspondence with the periods, their periods can be classified accordingly.

We shall be particularly interested in the behaviour of zeta functions of singular members of such families, as for singular varieties there is no guiding principle similar to the Weil conjectures for the smooth case. In the computations involving the two parameter family of Octic Calabi-Yau manifolds, it is observed that the zeta function degenerates in a consistent way at singularities. The nature of the degeneration depended on the type of singularity. In this case there were two types of singularity: a one dimensional locus (in the base space) of conifold points and another one dimensional locus of points where the Calabi-Yau manifold was birational to a one dimensional family, which had previously been studied by Rodriguez-Villegas [R-V].

Number theorists have been interested in the cohomological L-series of Calabi-Yau varieties over $\mathbb{Q}$ or number fields [Yui]. One important question is the modularity of Calabi-Yau varieties, i.e. are their cohomological L-series completely determined by certain modular cusp forms? and Number Theory is the proof of the Taniyama-Shimura-Weil conjecture of the so-called modularity of elliptic curves defined over $\mathbb{Q}$ by A. Wiles et al. Wiles' idea is to exploit 2-dimensional Galois representations arising from elliptic curves and modular forms of weight 2 on some congruence sup-groups of $PSL(2,\mathbb{Z})$, and establish their equivalence. Number theorists are trying to use his methods to explore the arithmetic of Calabi-Yau threefolds. In particular, rigid Calabi-Yau threefolds defined over the field of rational numbers are equipped with 2-dimensional Galois representations, which are conjecturally equivalent to modular forms of one variable of weight 4 on some congruence subgroup of $PSL(2,\mathbb{Z})$ [Yui]. For not necessarily rigid Calabi-Yau threefolds over the rationals, the Langlands Program predicts that there should be some automorphic forms attached to them. Modularity was observed for the octic family of Calabi-Yau threefolds at the special values in the moduli space where there was birationality with the one parameter family, and simultaneously, a conifold singularity. Not only were the results in [R-V] verified, but new examples of modularity were observed, cusp forms for which were found using the tables of Stein [St].

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Discrete Curves and the Toda lattice

Nadja Kutz

The one-dimensional Toda lattice hierarchy can be interpreted as being generated by certain flows on discrete curves in $\mathbb{C}^2$ which are maps:

$$\begin{aligned} \gamma : \mathbb{Z} &\rightarrow \mathbb{C}^2 \\ k &\mapsto \gamma_k = \begin{pmatrix} x_k \\ y_k \end{pmatrix} \end{aligned} \tag{5}$$

The determinants of two neighbouring curve points:

$$g_k = \det(\gamma_k, \gamma_{k+1}) = x_k y_{k+1} - y_k x_{k+1} \tag{6}$$

$$u_k = \det(\gamma_{k-1}, \gamma_{k+1}) \tag{7}$$

are in this interpretation more or less the Flaschka-Manakov variables of the Toda lattice hierarchy. In the talk it was shown, how to construct flows on discrete Curves in $\mathbb{C}^2$, which yield the Toda flows on the Flaschka-Manakov variables. In looking for a geometrical interpretation it was displayed that curves, which show certain invariance under the flows or reductions therof have a special geometric shape. In particular curves in $\mathbb{C}^2$ whose determinants are constant under the Toda flows are socalled discrete quadrics, i.e. curves, whose points are lying on a quadric. It is a wellknown fact that the Toda lattice hierarchy reduces to the Volterra hierarchy for every second flow in the hierarchy. This is obtained by requiring $u_k = \lambda g_k g_{k-1}$. In this reduction the crossratio of 4 neighbouring points evolves according to the equations of the Volterra hierarchy. Yet there exist another family of flows on discrete curves which give the flows of the Volterra hierarchy. These are obtained in the reduction: $g_k = 1$. Here again the crossratio of 4 neighbouring points evolves according to the equations of the Volterra hierarchy. In that sense there exist two families of flows on discrete curves which are dual two each other. The above mentioned quadrics are the only curves whose shape is preserved by the two dual flows.

Another hint for the geometrical nature of the Toda flows was given by the fact that curves in $\mathbb{C}$ (obtained by interpreting x and y as homogenous coordinates for $\mathbb{C}P^1$ and the assumption that the curve doesn't hit infinity), with the constraint $g_k = 1$ whose determinants are invariant under the discrete KdV flow (= "second" flow of the Volterra hierarchy) are generalized discrete elastic curves.

In addition a Poisson structure for closed discrete curves was given, which yield all three known local Poisson brackets of the Toda lattice. The simplicity of the structure gives another indication that the above interpretation of the Toda hierarchy is rather natural.

The talk was based on the two articles: "Discrete curves in $\mathbb{C}P^1$ and the Todalattice," Tim Hoffmann, Nadja Kutz, to appear in Stud. Appl. Math.

(math.DG/0208190)
"Tri-hamiltonian Toda lattice and a canonical bracket for closed discrete curves," Nadja Kutz, Lett. Math. Phys. 64, Issue 3, 229-234 (2003) (math.DG/0304083)

Finally at the end of the talk it was shown how special time-discretized flows on discrete curves can be used to construct a doubly discrete Liouville equation, which plays an important role in String theory. We followed here the construction as given in:

"Strongly coupled quantum discrete Liouville Theory" L.D. Faddeev, R.M. Kashaev, A. Yu. Volkov; Commun.Math.Phys. 219 (2001) 199-219.

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Participants

Dr. Anda Degeratu
anda@math.duke.edu
andad@msri.org
Dept. of Mathematics
Duke University
P.O.Box 90320
Durham, NC 27708-0320
USA

Dr. Tamar Friedmann
tamarf@lns.mit.edu
Center for Theoretical Physics
Massachusetts Institute of Techn.
Bldg. 6-308A
77 Massachusetts Avenue
Cambridge MA 02139-4307
USA

Prof. Dr. Antonella Grassi
grassi@math.upenn.edu
Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104-6395
USA

Prof. Dr. Yukari Ito
y-ito@math.nagoya-u.ac.jp
Graduate School of Mathematics
Nagoya University
Chikusa-ku
Nagoya 464-8602
Japan

Prof. Dr. Shabnam Nargis Kadir
kadir@maths.ox.ac.uk
Mathematical Institute
Oxford University
24 - 29, St. Giles
GB-Oxford OX1 3LB

Dr. Nadja Kutz
nadja@math.tu-berlin.de
Fakultät II-Institut f. Mathematik
Technische Universität Berlin
Sekr. MA 3-2
Straße des 17. Juni 136
10623 Berlin

Prof. Dr. Carmen A. Nunez
carmen@iafe.uba.ar
Instituto de Astronomia y Fisica
del Espacio
Ciudad Universitaria
C.C.67 - Suc. 28
1428 Buenos Aires
Argentina

Prof. Dr. Xenia de la Ossa
delaossa@maths.ox.ac.uk
Mathematical Institute
Oxford University
24 - 29, St. Giles
GB-Oxford OX1 3LB

Prof. Dr. Sonia Paban
paban@zippy.ph.utexas.edu
Department of Physics
University of Texas
1 University Station
Austin TX 78712-0269
USA

Prof. Dr. Sakura Schafer-Nameki
S.Schafer-Nameki@damtp.cam.ac.uk
II. Institut für Theoretische
Physik
Universität Hamburg
Luruper Chaussee 149
22761 Hamburg

Dr. Anne Taormina
anne.taormina@durham.ac.uk
Dept. of Mathematical Sciences
The University of Durham
Science Laboratories
South Road
GB-Durham, DH1 3LE

Prof. Dr. Sheung Tsun Tsou
tsou@maths.ox.ac.uk
Mathematical Institute
Oxford University
24 - 29, St. Giles
GB-Oxford OX1 3LB

Dr. Katrin Wendland
wendland@maths.warwick.ac.uk
Mathematics Institute
University of Warwick
GB-Coventry CV4 7AL