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Geomathematics

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Introduction by the Organisers

The present workshop gave an overview of recent results and current trends in geomathematics. The organizers and participants would like to take the opportunity to thank again the "Mathematisches Forschungsinstitut Oberwolfach" for having provided an inspiring environment for the meeting and the scientific work. The pleasant atmosphere contributed to the overall success of the workshop.

Workshop on Geomathematics

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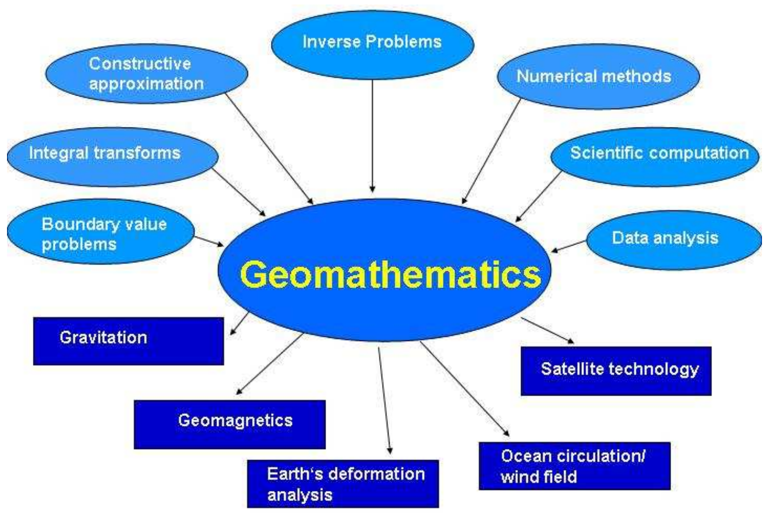
Abstracts

Some Remarks on the Geomathematics Workshop

Willi Freeden

During the last decades technological progress has changed completely the observational methods in all fields of geosciences and -engineering with a trend to achieve immediate results, thus reducing time and costs. Modern high speed computers and satellite based techniques are entering more and more disciplines like geomagnetics, geodesy, geology, meteorology, navigation and many others. The increasing observational accuracy demands adequate mathematical tools; mathematics concerned with geoscientific problems, i.e., geomathematics, is becoming more and more indispensable. Geomathematics offers appropriate means of assimilating, assessing, and reducing the comprehensible form the readily increasing flow of data from geomagnetic, geochemical, geodetic, geological, and satellite sources and providing an objective basis for scientific interpretation, classification, testing of concepts and solution of problems. Undoubtedly, the stage is set for geomathematics to play a major role in all Earth's sciences.

The purpose of the meeting was to encourage and enhance the dialogue and the collaboration between actual research fields on geomathematics (i.e., gravitation, geomagnetics, Earth's deformation analysis, ocean circulation/wind field, and satellite technology) and relevant mathematical methods and tools in geomathematics (i.e., special functions of mathematical (geo)physics, differential equations, boundary value problems, integral transforms, constructive approximation, inverse problems, numerical methods, scientific computation, data analysis).



The meeting was well attended with over 45 participants from all continents.

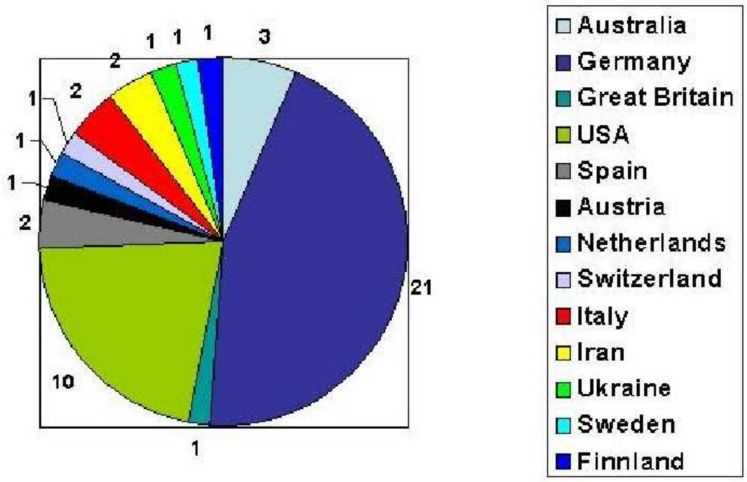


FIGURE 1. Participants of the workshop: Geomathematics

In the actual research fields five lectures gave an overview of recent developments and current trends. The other talks concentrated on specific mathematical techniques in geosciences. All talks demonstrated the diversity as well as the inter-relationships of the areas of research.

In detail, the Geomathematics workshop was concerned with the following research projects:

- Partial Differential Equations.
  - (i) Potential Theory
    geoid and geopotential determination from oblique-derivative boundary value problems, gravimetry (determination of density and discontinuities in the Earth's interior from gravity data), inverse problems from satellite applications (determination of the gravitational field from measurements of the CHAMP (2000), GRACE (2001) and GOCE (2005) satellite missions), time-dependent gravitational field determination (from GOCE data), pseudo-differential equations in 'Satellite-to-Satellite Tracking' and 'Satellite Gravity Gradiometry'
  - (ii) Theory of Elasticity
    Cauchy-Navier-equations of the elastic field (boundary value problems of elasticity, loading problems at reservoirs, causality to seismic phenomena)
  - (iii) Electromagnetism
    geomagnetic field determination (determination of the magnetic induction, modelling of electric current densities in the iono- and magnetosphere from satellite data, regularization), refraction (i.e. determination of atmospheric refraction via CCD-camera data, turbulence, fractal structure)
  - (iv) Navier-Stokes equations on the sphere (wind field modelling)
- Constructive Approximation

(scalar, vectorial and tensorial) radial basis functions, uncertainty principles, space-frequency behaviour, multivariate approximation (splines, wavelets and their application to partial differential equations), data analysis, vectorial spline deformation analysis of the Earth's crust

- Numerical Methods

  numerical integration on the sphere and geoscientifically relevant surfaces, domain decomposition methods, fast multipole methods (FMM), fast wavelet transform (FWT), tree algorithms (pyramid schemes), spline interpolation and smoothing, best approximation, wavelet denoising (multiscale signal-to-noise response),
- Scientific Computing/Data Analysis

  multiscale modelling of the Earth's gravitational field (from CHAMP, GRACE and GOCE data), multiscale modelling of the geomagnetic field and electric current distributions (from MAGSAT and CHAMP data), multiscale modelling of density variations in the Earth's interior from gravity data (using OSA91a, EGM96a), multiscale modelling of the wind field (from data of the Deutscher Wetterdienst)

In what follows the abstracts of the talks are included in the order of the presentation by the speakers.

Interplay between Moment Problems, Inverse Problems and Sampling Theory

M. Zuhair Nashed

We consider the class of operator equations

$$Ax = y \tag{1}$$

where A is a linear operator on a Hilbert space X into a function space of real-valued continuous functions on a set T with the property that $|(Ax)(t)| \leq M_t \|x\|$, $x \in X$, $x \in T$, where M_t is a constant that does not depend on x. Then there exists a family $\{a_t : t \in T\}$ of elements in X such that

$$(Ax)(t) =< a_t, x >, \tag{2}$$

so the operator equation (1) becomes

$$< a_t, x >= y(t). \tag{3}$$

Within this framework we give examples of moment problems, inverse problems and sampling expansions $f(t) = \sum_n f(t_n) S_n(t)$ that can studied in a unified approach. Reproducing kernel Hilbert spaces play a key role. An extended version of Backus–Gilbert method for moment problem is formulated and used to solve the inverse problem (1) in form

$$< a_{t_i}, x >= y(t_i) + \varepsilon_i, \quad i = 1, 2, \ldots, n \tag{4}$$

and establish convergence. A sampling theory approach is also considered for the solution of integral equations of the first kind when the range space and solution space are reproducing kernel Hilbert spaces that admit sampling expansions of the form stated above.

On adaptive inverse estimation of linear functionals from random noisy data – case study: GOCE data processing.

Sergei V. Pereverzyev

The talk is prompted by the question of how to choose the parameter in regularization of geopotential determination from the data, which will be collected during future satellite mission GOCE. It is common belief that the choice of the regularization parameter will be a severe topic for this mission and variety of choice strategies will definitely be necessary in GOCE data processing, because of the lack of information about noise characteristic. In the talk we discussed two scenarios. One of them is that no information about noise is available. In such a situation one can use only the so-called heuristic methods for the choice of the regularization parameter. We propose several heuristically motivated rules which have not been applied so far to satellite gravity gradiometry problem (SGG).
Another scenario is that only the level of random noise is known, but no information about the noise covariance structure is available. We discuss an estimation procedure that adapts to unknown smoothing properties of covariance operator. To the best of our knowledge the first result in this direction has been obtained in 2003 in cooperation with Alex Goldenshluger (see Bernoulli v. 9(5), 2003, pp. 783-807). This research addressed the problem of estimating the value of a linear functional from indirect random noisy observations with finite degree of ill-posedness, and an estimation procedure was proposed which adapts to unknown smootheness of the solution and of the noise covariance operator. It has been shown that accuracy of this adaptive estimator is worse only by a logarithmic factor than one could achieve in the case of known characteristics. On the other hand, it is known that in general SGG-problem has infinite degree of ill-posedness. Nevertheless, we argue that the parameters of forthcoming GOCE-mission allow to treat SGG-problem as moderately ill-posed problem with degree of ill-posedness $a = 5.5$. It means that above mentioned adaptation procedure could be successfully applied.

Multi-scale Approaches for the Determination of the Earth's Interior – from Gravitational and Seismic Data

Volker Michel

Gravity data and seismic data are the most important sources of information for the recovery of the structures in the Earth's interior. As it is already well-known, the mass density function cannot be uniquely determined from pure gravitational data. The non-reconstructible part, which is infinite-dimensional, is called the anharmonic density.

The unique calculation of the harmonic density from gravitational data is nevertheless ill-posed since the solution is instable, i.e., it does not depend continuously on the given data. In this talk a multi-scale regularization technique for the solution of this so-called gravimetry problem is presented. Particular systems of wavelets are constructed that allow a multiresolution analysis of the harmonic density based on gravity models.
Moreover, a new regularization technique for this problem is demonstrated. This new approach uses certain harmonic spline spaces on the three-dimensional ball. An outstanding advantage of this technique is that different types of gravitational data can be merged. So, one can use the gravitational potential, its first radial derivative (derived from SST) and its second radial derivative (derived from SGG). Furthermore, those data can be located at many different heights above the surface (airborne and spaceborne data) as well as on the surface of the Earth (terrestrial data).
Typical seismological data for the investigation of the Earth's composition are travel-times of earthquake waves. From those data, that are related to the positions of source (hypocenter) and receiver (seismograph), models of the velocity of the propagating waves are determined. Primarily, there exist two classes of methods for solving this inverse problem. First, a spherical harmonics expansion is used for the slowness S. Based on the integral equation

$$T = \int_L S(x)\,\mathrm{d}x,$$

where L is the path of a ray associated to a seismic wave (on the surface of the Earth or inside the planet) and T is the corresponding travel-time, the expansion coefficients of S are determined. The use of this approach has become rare because the global character of a polynomial does not fit the locally varying structure of the Earth's crust.
Second, blocks with, e.g., constant or linear slowness in each block are used as a model. Based on the same integral equation the parameters of the blocks are determined. Of course, the corresponding linear equation system can show numerical instabilities if the block sizes are not chosen appropriately.
In this talk it is demonstrated that spherical wavelets, which are based on spherical harmonics but are strongly space-localizing, can be an interesting alternative for future research on this topic. For the case of surface waves an appropriate new method is proposed.

Gravitation

Erwin Groten

With new dedicated LEO (= Low Earth Orbiting)-type satellite projects as well as numerous altimetric satellite missions the precise determination of gravity variations and the detailed gravity field of the Earth gained increasing interest.

Whenever supplemented by new air borne gravity field observations and local surveys we are now able to use substantially improved gravity data to obtain deeper knowledge on mass distribution and mass transport within the Earth, as far as solid, liquid and fluid density distribution is concerned. Moreover, much better information recently became available on Newtonian Gravitational Constant as well as its product with the mass of the Earth, i.e. the terrestrial gravitational constant which act as scale constants. Care is necessary in dealing with related reference frames, so gravity has to be clearly separated from gravitation. Whereas geodesy is based on Newtonian gravitation, space geodesy and astrometric as well as astrophysical considerations are usually based on Einsteinian relativistic concepts. Deviations from classical theories and recent developments in terms of Yukawa's corrections, E. Majorana's shielding concept etc. as well as gravitational waves and related detectors and experimental concepts are discussed. A clear distinction is necessary in dealing with temporal changes of gravity (such as Earth's rotation changes) and changes of gravitation generated by mass transport (sea level and hydrological effects) where the latter can now be detected with high precision from GRACE and similar observations. However, also steric sea level variations and non-steric ocean circulation effects need to be clearly distinguished and separated. CHAMP substantially contributes to separate the model space of harmonic functions and potential theory from observation space (i.e. reality) by delivering reliable information on atmospheric density distribution and related temporal variations, based on atmospheric limb studies. GOCE will basically improve the knowledge on the higher harmonics of the Earth's gravitational field.
Numerical aspects and related phenomena and accuracies are discussed, and future prospects are outlined; see also [1], [2], [3].
Mathematical forms of representations are described in terms of wavelet, spherical and spheroidal analysis. Problems associated with the introduction of approximations, such as Somigliana's field of a level ellipsoid are outlined. It is demonstrated that with satellite altimetry, airborne gravity measurements and satellites of LEO-type the importance of ill-posed problem and related regularizations in geodesy has tremendously increased. Moreover, integral equations have widely replaced relatively simple integral transformations.
Thus, besides the need to solve very large linear (and, to some extent, non-linear) equation systems the necessity to apply sophisticated mathematical techniques has strongly increased in geosciences.
Besides the terrestrial aspects, also the precise determination of orbits in space, mainly in relativistic celestial frames, became of prime relevance. Also gravity field determination within the solar system (Moon, Planets, Comets) is a topic within gravitation theory which gains increasing interest. Thus gravitation as the prime force in space geodesy deserves a revival.

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Weighted Coorbit Spaces and Banach Frames on Homogeneous Spaces

Stephan Dahlke
(joint work with Gabriele Steidl and Gerd Teschke)

General Setting. One of the classical tasks in applied analysis is the efficient representation/analysis of a given signal. Usually, the first step is the decomposition of the signal into suitable building blocks. Current interest especially centers around Riesz bases of wavelet type. However, in recent studies, it has turned out that the use of Riesz bases may have some serious drawbacks, e.g., their lack of flexibility. Therefore, one natural way out suggests itself: why not using a slightly weaker concept and allowing some redundancies, i.e., why not working with frames? In general, given a Hilbert space $\mathcal{H}$, a collection of elements $\{e_i\}_{i\in\mathbb{Z}}$ is called a *frame* if there exist constants $0 < A_1 \leq A_2 < \infty$ such that

$$A_1\|f\|^2_{\mathcal{H}} \leq \sum_{i\in\mathbb{Z}} |\langle f, e_i\rangle_{\mathcal{H}}|^2 \leq A_2\|f\|^2_{\mathcal{H}}.$$

Our aim is to construct (Banach) frames for specific smoothness spaces on domains and manifolds, the so–called coorbit spaces.

Group Theoretical Background. Let $\mathcal{G}$ be a locally compact, topological Hausdorff group which possesses a unitary, irreducible and strongly continuous representation U in a Hilbert space $\mathcal{H}$. Consider the homogeneous space $X = \mathcal{G}/\mathcal{P}$, where $\mathcal{P}$ is a closed subgroup of $\mathcal{G}$, and fix a Borel section $\sigma : X \to \mathcal{G}$. Assume that U is *strictly square integrable* mod $(\mathcal{P}, \sigma)$, i.e., there exists $\psi \in \mathcal{H}$ such that

$$\int_X \langle f, U(\sigma(h)^{-1})\psi\rangle_{\mathcal{H}}\, U(\sigma(h)^{-1})\psi \, d\mu(h) = f,$$

where μ denotes some $\mathcal{G}$–invariant measure on X. Then

$$V_\psi : \mathcal{H} \to L_2(X), \qquad V_\psi f(h) := \langle f, U(\sigma(h)^{-1})\psi\rangle_{\mathcal{H}} \tag{5}$$

is an isometry from $\mathcal{H}$ onto the reproducing kernel Hilbert space

$$\mathcal{M}_2 := \{F : \langle F \in L_2(X),\ R(h,\cdot)\rangle = F(h)\} \qquad R(h,l) := V_\psi(U(\sigma(h)^{-1})\psi)(l).$$

Weighted Coorbit Spaces. Fix a positive, continuous weight function w on $\mathcal{G}$ satifying $w(g \circ \tilde{g}) \leq w(g)\, w(\tilde{g})$, $g, \tilde{g} \in \mathcal{G}$, and consider the weighted L_p–spaces

$$L_{p,w}(X) := \{f \text{ measurable on } X : \|f\|_{L_{p,w}} := \left(\int_X |f(h)|^p w(\sigma(h))^p d\mu(h)\right)^{1/p} < \infty\}.$$

Let us impose the fundamental condition

$$(6)\qquad \int_X |R(h,l)| \frac{w(\sigma(h))}{w(\sigma(l))}\, d\mu(h) \le C_\psi .$$

We define the space

$$H_{1,w} := \{f \in \mathcal{H} :\ V_\psi f \in L_{1,w}(X)\}, \quad \|f\|_{H_{1,w}} := \|V_\psi f\|_{L_{1,w}},$$

which is densely embedded in $\mathcal{H}$ and therefore induces a Gelfand triple $H_{1,w} \hookrightarrow \mathcal{H} \hookrightarrow H'_{1,w}$. By using (6), the operator V_ψ in (5) can be extended to an operator on $H'_{1,w}$ by

$$V_\psi f(h) := \langle f, U(\sigma(h)^{-1})\psi\rangle_{H'_{1,w}\times H_{1,w}} .$$

Therefore, similar to [3, 4], we can define smoothness spaces, the so–called *weighted coorbit spaces* by

$$M_{p,w} := \{f \in H'_{1,w} :\ V_\psi f \in L_{p,w}(X)\}, \qquad \|f\|_{M_{p,w}} := \|V_\psi f\|_{L_{p,w}} .$$

Banach Frames for Weighted Coorbit Spaces. Given some compact neighborhood $\mathcal{U}$ of the identity in $\mathcal{G}$, a family $\mathcal{X} = (x_i)_{i\in\mathcal{I}}$ in $\mathcal{G}$ is called *$\mathcal{U}$–dense* if $\bigcup_{i\in\mathcal{I}} \mathcal{U}x_i = \mathcal{G}$. Let us consider the subset

$$\mathcal{I}_\sigma := \{i \in \mathcal{I} :\ \sigma(X) \cap \mathcal{U}x_i \ne \emptyset\} .$$

We define the *$\mathcal{U}$–oscillation* with respect to the analyzing wavelet ψ as

$$\mathrm{osc}_{\mathcal{U}}(l,h) \quad := \quad \sup_{u\in\mathcal{U}} |\langle \psi, U(\sigma(l)\sigma(h)^{-1})\psi - U(u^{-1}\sigma(l)\sigma(h)^{-1})\psi\rangle_{\mathcal{H}}| .$$

In this setting, we can formulate our main theorems. The first one is a decomposition theorem which says that discretizing the representation $U(\sigma(\cdot)^{-1})$ by means of a $\mathcal{U}$–dense set indeed produces an atomic decomposition of $M_{p,w}$.

Theorem 1. *Let a compact neighborhood $\mathcal{U}$ of the identity in $\mathcal{G}$ be chosen such that*

$$(7)\qquad \int_X \mathrm{osc}_{\mathcal{U}}(l,h) \frac{w(\sigma(h))}{w(\sigma(l))} d\mu(l) \le \gamma \quad \textit{and} \quad \int_X \mathrm{osc}_{\mathcal{U}}(l,h) \frac{w(\sigma(h))}{w(\sigma(l))} d\mu(h) \le \gamma ,$$

where $\gamma < 1$. Let $\mathcal{X} = (x_i)_{i\in\mathcal{I}}$ be a $\mathcal{U}$–dense family. Furthermore, suppose that for some compact neighborhood $\mathcal{Q} \subseteq \mathcal{U}$ of the identity

$$(8)\qquad \mu\{h \in X : \sigma(h) \in \mathcal{Q}\sigma(h_i)\} \ge C_{\mathcal{Q}} > 0$$

holds for all $i \in \mathcal{I}_\sigma$ and that

$$(9)\qquad \int_X \sup_{q\in\mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle_{\mathcal{H}}| \frac{w(\sigma(h))}{w(q^{-1}\sigma(l))}\, d\mu(l) \le \tilde{C}_{\mathcal{Q}}$$

with a constant $\tilde{C}_{\mathcal{Q}} < \infty$ independent of $h \in X$. Then $M_{p,w}$, $1 \le p \le \infty$, has the following atomic decomposition: if $f \in M_{p,w}$, $1 \le p \le \infty$, then f can be represented as

$$f = \sum_{i\in\mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi,$$

where the sequence of coefficients $(c_i)_{i\in\mathcal{I}_\sigma} = (c_i(f))_{i\in\mathcal{I}_\sigma} \in \ell_{p,w}$ *depends linearly on* f *and satisfies*

$$||(c_i)_{i\in\mathcal{I}_\sigma}||_{\ell_{p,w}} \le A||f||_{M_{p,w}}.$$

If $(c_i)_{i\in\mathcal{I}_\sigma} \in \ell_{p,w}$*, then* $f = \sum_{i\in\mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi$ *is contained in* $M_{p,w}$ *and*

$$||f||_{M_{p,w}} \le B||(c_i)_{i\in\mathcal{I}_\sigma}||_{\ell_{p,w}}.$$

Given such an atomic decomposition, the problem arises under which conditions a function f is completely determined by the moments or coefficients $\langle f, U(\sigma(h_i)^{-1})\psi\rangle_{H'_{1,w}\times H_{1,w}}$ and how f can be reconstructed from these coefficients. This question is answered by the following theorem which shows that our generalized coherent states indeed give rise to Banach frames.

Theorem 2. *Impose the same assumptions as in Theorem 1 with*

(10)
$$\int_X \mathrm{osc}_{\mathcal{U}}(h,l)\frac{w(\sigma(h))}{w(\sigma(l))}d\mu(l) \le \frac{\tilde\gamma}{C_\psi} \quad \text{and} \quad \int_X \mathrm{osc}_{\mathcal{U}}(h,l)\frac{w(\sigma(h))}{w(\sigma(l))}d\mu(h) \le \frac{\tilde\gamma}{C_\psi},$$

where $\tilde\gamma < 1$*, instead of* (7) *and with*

$$\int_X \sup_{q\in\mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle_{\mathcal{H}}|\frac{w(q^{-1}\sigma(l))}{w(\sigma(h))}\,d\mu(l) \le \tilde C_{\mathcal{Q}} \tag{11}$$

where $\tilde C_{\mathcal{Q}} < \infty$ *is a constant independent of* $h \in X$*, instead of* (9)*. Let* R *fulfill the additional property*

$$\int_X |R(h,l)|\frac{w(\sigma(h))}{w(\sigma(l))}\,d\mu(l) \le C_\psi.$$

Then the set

$$\{\psi_i := U(\sigma(h_i)^{-1})\psi : i \in \mathcal{I}_\sigma\}$$

is a Banach frame for $M_{p,w}$*. This means that*

i) $f \in M_{p,w}$ *if and only if* $(\langle f,\psi_i\rangle_{H'_{1,w}\times H_{1,w}})_{i\in\mathcal{I}_\sigma} \in \ell_{p,w}$*;*

ii) *there exist two constants* $0 < A' \le B' < \infty$ *such that*

$$A'\,\|f\|_{M_{p,w}} \le \|(\langle f,\psi_i\rangle_{H'_{1,w}\times H_{1,w}})_{i\in\mathcal{I}_\sigma}\|_{\ell_{p,w}} \le B'\,\|f\|_{M_{p,w}};$$

iii) *there exists a bounded, linear reconstruction operator* $\mathcal{S}$ *from* $\ell_{p,w}$ *to* $M_{p,w}$ *such that* $\mathcal{S}\left((\langle f,\psi_i\rangle_{H'_{1,w}\times H_{1,w}})_{i\in\mathcal{I}_\sigma}\right) = f$.

A detailed description can be found in [1, 2].

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Polynomial Interpolation, Approximation, Cubature and Point Designs on the Sphere

Robert S. Womersley

This talk looks at point distributions of the unit sphere S^2 which are good for polynomial interpolation and cubature, and also have good geometric properties.

Choices include approximation method (interpolation, hyperinterpolation, least squares), basis functions (spherical harmonics, reproducing kernel functions, non-polynomial locally supported functions), point sets (product rules, Lebedev, minimum energy, extremal) and cubature weights (from polynomial exactness, equal).

A good approximation/integration scheme must be implementable (not involve unknown integrals over the sphere), have good numerical properties (well-conditioned linear systems), be efficient (fast to evaluate, low number of points, basis functions with (close to) local support) and have good theoretical properties (high degree of polynomial exactness, low operator norm, good worst case error, positive cubature weights, geometrical regularity). Extremal systems of points [12] and extremal spherical designs [2] satisfy many of these criteria.

Let $\mathbb{P}_n(S^2)$ denote the space of all spherical polynomials of degree at most n, and let $d_n = \dim \mathbb{P}_n(S^2) = (n+1)^2$. Let ϕ_i, $i = 1, \ldots, d_n$ be a basis for $\mathbb{P}_n(S^2)$. A system of points $x_j \in S^2$, $j = 1, \ldots, d_n$ is a fundamental system if and only if the basis matrix $\Phi = \phi_i(x_j), i, j = 1, \ldots, d_n$ is nonsingular. Let $\phi : S^2 \to \mathbb{R}^{d_n}$ have components $\phi_i(x), i = 1, \ldots, d_n$. The norm of the interpolation operator as a map from $C(S^2)$ to $C(S^2)$, is the Lebesgue constant

$$\|\Lambda_n\| = \max_{x \in S^2} \|\Phi^{-1}\phi(x)\|_1.$$

One important criterion is how $\|\Lambda_n\|$ grows with n.

For projections on S^2, the minimal operator norm, $O(n^{\frac{1}{2}})$, is achieved by orthogonal projection, which is not implementable. Hyperinterpolation replaces the inner product in orthogonal projection by a discrete inner product using m points with positive weights which is exact for all polynomials of degree up to $2n$. Sloan and Womersley [11] showed that hyperinterpolation achieves the optimal $O(n^{\frac{1}{2}})$ order. This was extended to higher dimensional spheres and a regularity condition removed by Reimer[7]. Hyperinterpolation produces a polynomial approximation, but is not interpolatory unless $m = d_n$, which is not possible for $n \geq 3$. Numerical experiments [13] suggest a growth of $O(n)$ for modest values of n, but achieving this is still an open question.

Extremal fundamental systems are chosen to maximize the determinant of a basis matrix Φ, and are independent of the choice of basis. They have the nice property that the Lebesgue constants are bounded by $d_n = (n+1)^2$. Numerically [12] the growth looks more like $O(n+1)$ for n up to 100. Extremal systems also have nice geometrical properties. For any system of m points $x_1, \dots, x_m$ on S^2, the packing radius (half the minimum angle between points) is

$$q_m = \frac{1}{2} \min_{i \neq j} \cos^{-1}(x_i \cdot x_j).$$

For extremal systems $q_{d_n} \geq \pi/4n$, although numerical evidence [13] suggests q_{d_n} grows more like $\pi/2n$. The covering radius (mesh norm) is

$$h_m = \max_{x \in S^2} \min_{j=1,\dots,m} \cos^{-1}(x \cdot x_j).$$

A measure of the geometric regularity of a point set is the mesh ratio $\rho_m = h_m/q_m \geq 1$. For calculated extremal points $\rho_{d_n} \leq 1.9$ for all $n \leq 100$ $(10,201$ points) and $n = 127, 128, 191$.

For a fundamental system the cubature weights w are the unique solution of the linear system

$$\Phi w = b,$$

where $b_j = \int_{s^2} \phi_j(x) d\omega(x)$, $j = 1, \dots, d_n$. Ideally we would like Gauss rules, with d_n points, but exact for all polynomials of degree up to $2n$. This is not possible for $n \geq 3$ due to the non-existence of tight spherical n-designs. Lebedev rules [6] with octahedral symmetry have very high degrees of precision for low numbers of points. However the octahedral symmetry concentrates points in certain areas, as do product Gauss rules. Extremal systems of d_n points numerically have positive weights, with $w_i/w_{\text{avg}} \geq 0.5$ for $n \leq 100$, where $w_{\text{avg}} = |S^2|/d_n$. However there is currently no proof that the weights are positive for all n.

Another measure of the quality of cubature rules is the behaviour of the worst case error. Recent work by Hesse and Sloan [4], reported at this conference, give an error estimate of $O(d_n^{\frac{s}{2}})$ for the worst case error in H^s. Their results include positive weight cubature rules based on extremal systems. Moreover they show that this estimate is optimal, giving integrands which achieve this upper bound.

The condition number of the basis matrix is critically dependent on the choice of the interpolation points. Systems of d_n points may theoretically be fundamental systems, but have such large condition numbers that in practice they are useless. The calculated extremal systems have spherical harmonic basis matrices with condition number less than 25 for degree $n \leq 100$. Thus there are no numerical difficulties in solving the linear systems for the interpolation or cubature weights.

A spherical n-design [3] is a set of m points on S^2 such that equal weight cubature $w_j = |S^2|/m = w_{\text{avg}}$ for $j = 1, \dots, m$ is exact for all polynomials $p \in \mathbb{P}_n(S^2)$. Classically the interested has been in finding the minimum number of points m to be exact for polynomials of degree $\leq n$, with lower bounds $n^2/4 + O(n)$ on m. A tight spherical n-design is one which achieves these lower bounds on the number of points, but these do not exist for $n \geq 3$ [1]. The smallest number of points for which existence of spherical n-designs is known is $m = O(n^3)$ [5]. Instead of trying to minimize the number of points, we look for spherical n-designs with $m = d_n = (n+1)^2$ (the optimal order, but not the optimal constant).

As the cubature weights for extremal systems are close to equal and have very well-conditioned basis matrices, they provide excellent starting points for finding spherical n-designs. This has been done [2] for $n \leq 50$. Moreover the equal cubature weight condition is a system of $d_n - 1$ nonlinear equations, in $2d_n - 3$ variables (using a spherical parametrization with some rotational invariance removed). This leaves some degrees of freedom, which can be used to maximize the determinant of the basis matrix. Thus an extremal spherical n-design is a set of d_n points on S^2 which maximizes the determinant of a basis matrix subject to the constraint that the equal weight cubature rule at these points is exact for all polynomials $p \in \mathbb{P}_n(S^2)$. Calculated extremal spherical designs are close to the extremal systems, and also numerically have good geometrical properties.

Many other quantities, such as the Reisz s-energy [10] for $s > 0$,

$$E_s(x_1, \ldots, x_m) = \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} \frac{1}{|x_i - x_j|^s}$$

for $s > 0$ are used to characterize well-distributed points on the sphere. For $s = 1$ the asymptotic energy [10] is $m^2 - cm^{3/2}$ where $c \approx -1.106$. Both the extremal systems and extremal spherical designs numerically have $c \approx -1.1$, giving another indication of their good geometric distribution.

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Numerical Integration on the Sphere

Kerstin Hesse
(joint work with Ian H. Sloan)

The most common and widely used cubature rules on the sphere S^2 are *product rules* with positive weights. In this talk I am particularly concerned with positive weight product rules Q_n which integrate all spherical polynomials up to degree n exactly, that is, $Q_n p = Ip$ for all $p \in \mathbb{P}_n(S^2)$, where $If := \int_{S^2} f(\mathbf{x})\, d\omega(\mathbf{x})$. Such product rules can for example be generated as follows: (a) an equal weight rule with $n+1$ equally spaced points is used to discretize the integral with respect to the azimuthal coordinate $\phi \in [0, 2\pi)$; such a rule integrates all trigonometric polynomials of degree $\leq n$ exactly, (b) a positive weight cubature rule with $O(n)$ points and algebraic polynomial degree of exactness n is used to discretize the integral with respect to the coordinate $t = \cos\theta$, $\theta \in [0, \pi]$. Clearly such product rules Q_n have positive weights, use $O(n^2)$ points, and integrate all spherical polynomials of degree $\leq n$ exactly. However they have one huge disadvantage: the geometrical distribution of the points is rather 'uneven', as points cluster at the poles. Also we would like to obtain information about the convergence behaviour of such product rules (in comparison to other types of cubature rules). That is, how does the worst-case cubature error of a sequence of product rules $(Q_n)_{n\in\mathbb{N}}$ behave depending on the degree of exactness n?

In this talk I compare interpolatory cubature based on extremal fundamental systems with such product rules, and present recent results from joint work with Ian H. Sloan. These results establish an upper bound for the worst-case cubature error for a class of cubature rules, and show that this estimate is optimal. This class contains product rules and, assuming positivity of the weights, interpolatory cubature based on extremal fundamental systems.

Let $d_n := \dim(\mathbb{P}_n(S^2)) = (n+1)^2$ be the dimension of the space $\mathbb{P}_n(S^2)$ of all spherical polynomials of degree $\leq n$. A fundamental system $\{\mathbf{x}_j\}_{j=1,\ldots,d_n} \subset S^2$ is a point set for which the interpolation problem to find $\Lambda_n f \in \mathbb{P}_n(S^2)$ such that $\Lambda_n f(\mathbf{x}_j) = f(\mathbf{x}_j)$ for all $j = 1, \ldots, d_n$ is uniquely solvable for all continuous functions f. In other words, it is a point set for which the determinant of the interpolation matrix $[\Phi_k(\mathbf{x}_j)]_{j=1,\ldots,d_n}^{k=1,\ldots,d_n}$ is non-zero, where $\Phi_1, \ldots, \Phi_{d_n}$ is any basis for $\mathbb{P}_n(S^2)$. A fundamental system is called an *extremal fundamental system* if it maximizes the determinant of the interpolation matrix (with respect to any basis of $\mathbb{P}_n(S^2)$). An *interpolatory cubature rule* based on an (extremal) fundamental system is defined by

$$Q_n f := \int_{S^2} \Lambda_n f(\mathbf{x})\, d\omega(\mathbf{x}), \tag{12}$$

that is, the interpolating polynomial $\Lambda_n f$ of f with respect to the (extremal) fundamental system is integrated exactly. The interpolating polynomial can be written in Lagrange representation as $\Lambda_n f = \sum_{j=1}^{d_n} f(\mathbf{x}_j)\, L_j$, where $L_j \in \mathbb{P}_n(S^2)$ is the jth Lagrange polynomial, given by $L_j(\mathbf{x}_k) = \delta_{jk}$, $k = 1, \ldots, d_n$. Substituting this Lagrange representation of $\Lambda_n f$ in (12) yields

$$Q_n f = \sum_{j=1}^{d_n} w_j\, f(\mathbf{x}_j), \qquad \text{where} \qquad w_j := \int_{S^2} L_j(\mathbf{x})\, d\omega(\mathbf{x}).$$

Obviously, such a cubature rule has polynomial degree of exactness n and uses $O(n^2)$ points. There is also strong numerical evidence that interpolatory cubature rules based on extremal fundamental systems have positive weights. However, a proof has yet not been found. Extremal fundamental systems have a *very nice geometric point distribution*. This is theoretically verified by the fact that the minimal angle between any two distinct points has a lower bound of the order $O(n^{-1})$ and that the mesh norm has an upper bound of the same order $O(n^{-1})$. The mesh norm result means intuitively that the point set has no large holes. As for the positive weight product rules, an important question is the rate of convergence of the worst-case cubature error depending on the degree of exactness n. Also it would be desirable to know the optimal rate of convergence for sequences of cubature rules on S^2, as the degree of exactness n tends to infinity, and to identify sequences of cubature rules with that rate of convergence.

The two main results are two theorems. The first establishes an upper bound for the worst-case cubature error. The second shows that this estimate is optimal by constructing a function which achieves this bound. In the following two theorems the space $H^s = H^s(S^2)$, with norm $\|\cdot\|_{H^s}$, is roughly the space of those functions on S^2 whose generalized derivatives up to order s are square-integrable. For $s > 1$, the space H^s is a subset of the space of continuous functions on S^2, and it is also a reproducing kernel Hilbert space.

Theorem 1. *For $s > 1$ there exists a constant $\tilde{c}_s > 0$ such that for any sequence $(Q_n)_{n\in\mathbb{N}}$ of positive weight cubature rules, which satisfies (i) $Q_n p = Ip$ for all $p \in \mathbb{P}_n(S^2)$ and (ii) $m(n) = O(n^2)$, where $m = m(n)$ is the point number of Q_n, the worst-case cubature error in H^s satisfies*

$$\sup_{f\in H^s,\, \|f\|_{H^s}\le 1} |Q_n f - If| \le \tilde{c}_s\, n^{-s} = \hat{c}_s\, (m(n))^{-s/2} \qquad \text{for all } n \in \mathbb{N}. \tag{13}$$

The assumptions in Theorem 1 are fulfilled by a sequence of positive weight product rules. Assuming positivity of the weights, they are also satisfied by a sequence of interpolatory cubature rules based on extremal fundamental systems.

Theorem 2. *For $s > 1$ there exists a constant $c_s > 0$ such that for every m-point cubature rule $Q_m := \sum_{j=1}^{m} w_j\, f(\mathbf{x}_j)$ on S^2 the worst-case cubature error in H^s satisfies*

$$\sup_{f\in H^s,\, \|f\|_{H^s}\le 1} |Q_m f - If| \ge c_s\, m^{-s/2}. \tag{14}$$

Theorem 2 shows the limitations of m-point cubature in H^s: asymptotically we can never achieve a better rate of convergence than $O(m^{-s/2})$. As the order of the upper bound (13) in Theorem 1 and the lower bound (14) in Theorem 2 coincide, both estimates are *optimal.* Any sequence of cubature rules satisfying the assumptions in Theorem 1 has an optimal rate of convergence.

In the talk I give a brief sketch of the proof of Theorem 2. The proof is constructive. The idea is to construct a 'bad' function f_m (which is chosen individually for each m-point cubature rule Q_m) such that the cubature error for $f_m/\|f_m\|_{H^s}$, satisfies $\|f_m\|_{H^s}^{-1} |Q_m f_m - If_m| \geq c_s\, m^{-s/2}$, where c_s does not depend on the particular cubature rule Q_m and on the point number m. The construction of f_m involves a packing of the sphere with $2m$ spherical caps of an appropriate size. As these caps touch at most at the boundary there will be at least m caps that do not contain any cubature points in their respective interiors. The function f_m is constructed such that its support is the union of m such caps that contain no cubature points in the interior. A crucial part of the proof is the estimation of the norm $\|f_m\|_{H_s}$ in terms of orders of m.

That interpolatory cubature rules based on extremal fundamental systems have a much nicer point distribution than product rules and that they, assuming positivity of the weights, have an optimal rate of convergence makes them good candidates for numerical integration. An example to illustrate this is given at the end of the talk.

Theorem 1 has firstly been verified for $s = 3/2$, and this result is reported in [1]. The extension of part of the proof in the case $s = 3/2$ to the case of general $s > 1$ is not straightforward and needs a new argument. A report of Theorem 1 for general $s > 1$ and also a paper presenting Theorem 2 are in preparation and will soon be submitted. A nice survey of interpolatory cubature based on extremal fundamental systems can be found in [2].

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On Radon's convergence proof of Neumann's method for double layer potentials

Wolfgang L. Wendland

In 1837, C.F. Gauss proposed for the construction of the solution u to the Dirichlet problem of the Laplacian with given boundary values φ the use of a double layer potential

$$u(x) = -\frac{1}{4\pi} \int_\Gamma \mu(y) d\Omega_x(y) \quad \text{for} \quad x \in \Omega\,.$$

With the jump relation, this leads to C. Neumann's boundary integral equation

$$(*) \qquad \mu = L\mu + \varphi$$

where

$$(L\mu)(x) = -\frac{1}{4\pi}\int_{\Gamma}(\mu(y)-\mu(x))d\Omega_x(y) \quad \text{for} \quad x \in \Gamma$$

is defined by J. Radon in 1919 [18, 19] as a Stieltjes integral with the signed Radon measure $\Omega_x(E)$ for measurable sets $E \subseteq \Gamma$ (the solid angel). For piecewise smooth Γ including corners and edges, a review is given on Radon's treatment of the boundary integral equation and the extensions by V. Maz'ya, J. Kral, D. Medkova and O. Jansen if the equation is considered on the Banach space of continuous functions μ on Γ. For the corresponding two-dimensional problem, J. Radon in his famous papers 1919 defined closed boundary curves of bounded rotation and showed that for such curves without sharp cusps, the essential norm of L generated by the supremum norm is less than 1, he also showed the relation between eigenvalues of L and exterior and interior Dirichlet integrals of the eigensolution potentials, and that the spectral radius of L is less than 1. Hence, Neumann's classical successive approximation can be applied to the boundary integral equation($*$). In three dimensions, however, the corresponding results are by no means complete yet. Here J. Kral and D. Medkova have introduced the family of weighted supremum norms

$$\|\mu\|_{C^0_w(\Gamma)} := \sup_{x\in\Gamma} |w(x)\mu(x)|$$

with a weight function $w(x)$ satisfying $0 < c_- \leq w(x) \leq 1$ in order to generalize the results by V. Maz'ya [11], J. Kral [9] and the author [23]for $\Gamma \in \mathbb{R}^3$. As it turns out, the essential spectral radius $r_{\text{ess}}(L) < 1$ for piecewise smooth Γ can be shown in the following cases:

Γ is convex [15, 16] with $w \equiv 1$;
Γ is $C^{1+\alpha}$-smooth [17] with $w \equiv 1$;
Γ has edges but no corners [4] with $w \equiv 1$;
Γ has corners and edges such that $r_{\text{ess}}(L) < 1$ holds [2, 3, 8, 9, 23] with $w \equiv 1$;
Γ has isolated conical points [6] and $w \equiv 1$;
Γ is a rectangular surface[1, 10] and w sectorially constant can be constructed;
Γ is polyhedral [20, 21], existence of w but no construction;
Γ is polyhedral, O. Hansen [7] constructs sectorially constant w under additional conditions;
D. Medkova showed in [12, 13] the invariance of $r_{\text{ess}}(L)$ under locally conformal $\mathbb{R}^3$ diffeomorphisms.

In all these cases the Fredholm alternative is valid for the boundary integral equation ($*$) and for piecewise constant trial functions on a triangulation of Γ which is compatible with the weight function w. Moreover, stability and convergence of the classical collocation (or panel) method can be proved [10].

If boundary element Galerkin methods are used for ($*$) in the $L^2(\Gamma)$ setting, then only for convex polyhedrons and for polyhedrons satisfying specific edge conditions, the spectral radius generated by the L^2 norm is known to be less than 1 [5, 14], whereas for general polyhendrons the corresponding result is yet not known.

If, however, the boundary integral equation is treated with an appropriate Galerkin-Petrov method and the equation is considered on the trace space $H^{\frac{1}{2}}(\Gamma)$, then an appropriate norm of L on $H^{\frac{1}{2}}(\Gamma)$ is less than 1 and Neumann's classical successive approximation converges for the corresponding Petrov-Galerkin equations of $(*)$ and, moreover, the method is stable and convergent [22]. These properties are of great value for practical computations and some corresponding results from industrial applications are presented in the lecture.

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Numerical aspects of diffraction coefficient computations

Ivan G. Graham
(joint work with B.D. Bonner and V.P. Smyshlyaev)

The computation of diffraction coefficients for the scattering of high-frequency waves by conical scatterers can be reduced to the solution of a family of homogeneous boundary value problems for the Laplace-Beltrami-Helmholtz equation on a portion of the unit sphere bounded by a simple closed contour (in fact the intersection of the sphere with the conical scatterer). Distance on the contour is geodesic distance on the sphere. The diffraction coefficient may be determined by then integrating the resulting solutions with respect to the wave number (cf. [1]).

In this talk we discuss the numerical computation of the diffraction coefficients using the boundary integral method, with the classical double layer potential approach. The evaluation of the kernel of the integral equation involves computing the (derivative of the) Legendre function with complex index and for this we employ a method which combines solving Legendre's differential equation (when this equation is not singular), together with suitable asymptotic expansions near singular points.

We give an analysis of the scalar integral equation arising in acoustic scattering, which shows its relation to the corresponding integral equation for the planar Helmholtz equation. This allows us to prove, using the results of [2], optimal convergence for piecewise polynomial collocation methods of arbitrary order even when the scatterer has non-smooth cross-section. We also derive efficient quadrature techniques for assembling the boundary element matrices. In practice we employ an $h-p$ approximation scheme, which converges with exponential order.

The scattering of electromagnetic waves is also discussed; the resulting system of integral equations can be analysed by similar techniques to those used for the acoustic case.

We illustrate the talk with computations on both smooth and non-smooth scatterers for both the acoustic and electromagnetic cases.

At the end of the talk we also indicated briefly how asymptotic information could be incorporated into the ansatz functions of standard numerical schemes in order to produce methods which work well for both low and high frequency applications.

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Spectral approximations of boundary integral equations on slender spheroids

Mahadevan Ganesh

We consider some important computational issues associated with solving boundary value problems on three dimensional slender domains, using boundary integral equations. Our model potential theory problem is to compute harmonic functions defined in the interior (or exterior) of a prolate spheroidal domain Ω satisfying Dirichlet or Neumann boundary conditions on the surface Γ, given by

$$\Gamma := \left\{ (x,y,z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 \right\}.$$

The domain Ω is slender in the following sense: the aspect ratio $r := a/b << 1$.

This paper is motivated by the work of Rodin and Steinbach (SIAM J. Sci. Comp, 2003) on the development of boundary element preconditioners defined on slender *two dimensional* domains. The condition number of boundary element matrices depend linearly on the aspect ratio a/b, where in a general slender domain, a is the radius of the smallest circumscribed ball and b is the radius of the largest inscribed ball. Preconditioners were developed by Rodin and Steinbach, based on the idea that geometric proximity of two slender domains translates into spectral proximity. Accordingly, inverse boundary element matrices corresponding to *elliptical* domains (with similar aspect ratio) were proposed in their work as suitable preconditioners. Our approach (in a future work) to develop preconditioners defined on three dimensional slender bodies (such as submarines and fibers), leading to the model problem on the spheroid Γ.

The major part of work in the two dimensional paper is to develop spectral properties of boundary integral operators on a slender ellipse. In this work, we study spectral approximations of boundary integral equations on the slender spheroid Γ.

It is well known that solutions to potential problems can be obtained by solving boundary integral equations Using the fundamental solution $\Phi(\mathbf{x},\mathbf{y}) := \frac{1}{4\pi}|\mathbf{x}-\mathbf{y}|$ of the Laplace operator, the standard boundary integral operators on the spheroid are the single-, double-, adjoint double-layer and hypersingular potential operators, defined respectively as

$$\begin{aligned}
(\mathcal{S}\psi^-)(\mathbf{x}) &:= \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\psi^-(\mathbf{y})ds(\mathbf{y}),\\
(\mathcal{K}\psi^+)(\mathbf{x}) &:= 2\int_\Gamma \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial \boldsymbol{n}(\mathbf{y})}\psi^+(\mathbf{y})ds(\mathbf{y}),\\
(\mathcal{K}'\psi^-)(\mathbf{x}) &:= 2\int_\Gamma \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial \boldsymbol{n}(\mathbf{x})}\psi^-(\mathbf{y})ds(\mathbf{y}),\\
(\mathcal{D}\psi^+)(\mathbf{x}) &:= -2\frac{\partial}{\partial \boldsymbol{n}(\mathbf{x})}\int_\Gamma \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial \boldsymbol{n}(\mathbf{y})}\psi^+(\mathbf{y})ds(\mathbf{y}),
\end{aligned}$$

where $\boldsymbol{n}(\mathbf{y})$ denotes the unit outward normal to Γ at the point $\mathbf{y} \in \Gamma$, $ds(\mathbf{y})$ is the surface measure on Γ, $\mathbf{x} \in \Gamma$ and the density function $\psi^\pm \in H^{\pm 1/2}(\Gamma)$.

The interior (exterior) potential problem with Dirichlet or Neumann boundary condition on Γ can be reformulated as a second– or first–kind boundary integral equation. The second kind equation is defined using the operator $I-\mathcal{K}$ or $I-\mathcal{K}'$, and the first kind formulation is based on $\mathcal{S}$ or $\mathcal{D}$. We refer to $I-\mathcal{K}$ and $I-\mathcal{K}'$ as second kind operators, and $\mathcal{S}$ and $\mathcal{D}$ as first kind operators.

It is well known that on the *unit sphere* U, we have

$$\mathcal{S}^U = -\mathcal{K}^U, \qquad \mathcal{S}^U Y_n^m(\hat{\mathbf{x}}) = \frac{1}{2n+1} Y_n^m(\hat{\mathbf{x}}), \qquad \hat{\mathbf{x}} \in U,$$

where $\mathcal{K}^U$ and $\mathcal{S}^U$ respectively denote the single– and double–layer operator on the unit sphere and Y_n^m are the orthonormal spherical harmonics. Hence from the above equations, the eigenvalues of $\mathcal{K}^U$ lie in $[-1,0)$. This classical result for the double layer operator on the unit sphere was extended to the electrostatic operator $\mathcal{K}'$ on Γ by Ahner and Arenstrof in 1986.

If we replace $\mathcal{S}^U$ and $\mathcal{K}^U$ by $\mathcal{S}$ and $\mathcal{K}$, the identities in the above equations do not hold on Γ. In fact, the spherical harmonics are **not** eigenfunctions of the single layer and hypersingular operators on Γ. However, they are solutions of associated generalised eigenvalue problems. We discuss this topic, after demonstrating some major computational difficulties of spherical coordinates based superalgebraically convergent spectral integral methods to solve potential problems on slender spheroids. We further develop approximation theory results using spheroidally appropriate eigenfunctions, generalising some fundamental approximation results for the sphere case.

It is an open problem to propose and prove stability and convergence of a fully discrete spectral method for the first kind equations on geometries other than the sphere. We solve this problem for spheroids. by developing and analysing fully discrete boundary integral methods. Our numerical results demonstrate advantages of using spheroidal coordinates in boundary integral methods. Such advantages have already been used recently in *infinite element methods* to solve exterior problems.

Fast Fourier transform at nonequispaced knots and applications

Daniel Potts

We use the recently developed fast Fourier transform at nonequispaced knots (NFFT) in a variety of applications. The NFFT realized the the fast computation of the sums

$$f(w_j) = \sum_{k \in I_N^d} f_k \, \mathrm{e}^{-2\pi \mathrm{i} k w_j} \qquad (j = -M/2, \ldots, M/2-1)$$

and

$$h(j) = \sum_{k=-M/2}^{M/2-1} f_k \, \mathrm{e}^{-2\pi \mathrm{i} j w_k} \qquad (j \in I_N^d),$$

where $w_j \in [-1/2, 1/2)^d$. (Software: http://www.math.uni-luebeck.de/potts/nfft)

- **Fast summation (joint work with Gabriele Steidl)**

  The fast computation of special structured discrete sums

  $$f(y_j) := \sum_{k=1}^{N} \alpha_k K(\|y_j - x_k\|) \quad (j = 1, \ldots, M)$$

  or from the linear algebra point of view of products of vectors with special structured dense matrices is a frequently appearing task. We develop a new algorithm for the fast computation of discrete sums based on NFFTs. Our algorithm, in particular our regularisation procedure, is simply structured and can easily be adapted to different kernels K, e.g.

  $$\frac{1}{x},\ \frac{1}{x^2},\ x^2 \log x,\ \log x,\ \mathrm{e}^{-\sigma x^2},\ (x^2 + c^2)^{\pm 1/2}.$$

  Our method utilises the widely known FFT and can consequently incorporate advanced FFT implementations. In summary it requires $O(N \log N + (N+M))$ or $O(N+M)$ arithmetic operations. We prove error estimates to obtain clues about the choice of the involved parameters.

- **Fast spherical Fourier algorithms (joint work with Stefan Kunis)**

  Spherical Fourier series play an important role in many applications. A numerically stable fast transform analogous to the Fast Fourier Transform is of great interest. For a standard grid of $O(N^2)$ points on the sphere, a direct calculation has computational complexity of $O(N^4)$, but a simple separation of variables reduces the complexity to $O(N^3)$. Here we improve well-known fast algorithms for the discrete spherical Fourier transform with a computational complexity of $O(N^2 \log^2 N)$. Furthermore we present, for the first time, a fast algorithm for scattered data on the sphere. For arbitrary $O(N^2)$ points on the sphere, a direct calculation has a computational complexity of $O(N^4)$, but we present an approximate algorithm based on bivariate NFFTs with a computational complexity of $O(N^2 \log^2 N)$.

- **Spherical Filter (joint work with Martin Böhme)**
  We develop a new fast algorithm for uniform-resolution filtering of functions defined on the sphere. We use a fast summation algorithm based on NFFTs. The resulting algorithm performs a triangular truncation of the spectral coefficients while avoiding the need for fast spherical Fourier transforms. The method requires $O(N^2 \log N)$ operations for $O(N^2)$ grid points.
  Furthermore, we apply these techniques to obtain a fast wavelet decomposition algorithm on the sphere. We present the results of numerical experiments to illustrate the performance of the algorithms.

Swarm: a Constellation of Satellites to Investigate the Earth Magnetic Field

Roger Haagmans

Swarm is the fifth Earth Explorer mission. The objective of the **Swarm** mission is to provide the best ever survey of the geomagnetic field and its temporal evolution, in order to gain new insights into the Earth system by improving our understanding of the Earth's interior and climate. The mission is scheduled for launch in 2009. After release from a single launcher, a side-by-side flying lower pair of satellites at an initial altitude of 450 km and a single higher satellite at 530 km will form the **Swarm** constellation. High-precision and high-resolution measurements of the strength, direction and variation of the magnetic field, complemented by precise navigation, accelerometer and electric field measurements, will provide the necessary observations that are required to separate and model various sources of the geomagnetic field. This results in a unique "view" inside the Earth from space to study the composition and processes in the interior. It also allows analysing the Sun's influence within the Earth system. In addition practical applications in many different areas, such as space weather, radiation hazards, navigation and resource exploration, benefit from the **Swarm** concept.
Magnetic fields play an important role in many of the physical processes throughout the Universe. The Earth in particular has a large and complicated magnetic field, the major part of which is produced by a self-sustaining dynamo, operating in the fluid outer-core. However, measurements taken at or near the surface of the Earth are the superposition of magnetic field originating from the outer core as well as the fields caused by magnetised rocks in the Earth's crust, electric currents flowing in the ionosphere, magnetosphere and oceans, and by currents induced in the Earth by time-varying external fields.
Magnetic field changes in internal as well as external origin occur on a variety of time scales, and separating them relies on their different temporal variations. For example, over the last 150 years it has been observed that the axial dipole

component of the Earth's magnetic field has decayed by nearly 10%. This fast decay rate is characteristic of magnetic reversals which occur on average about once ever half million years. Geographically, this recent dipole decay is largely due to changes in the field beneath the South Atlantic Ocean, connected to the growth of the South Atlantic anomaly. Within the Earths interior the core field and, in particular, its temporal changes, known as "secular variation", are among the few means available for probing the properties of the outer core. This secular variation directly reflects the fluid flow in the outmost core and provides a unique experimental constraint on "geodynamo theory". However, the only part of the core field that varies on time scales longer than around one year is observable at the Earth's surface. Studies of the electromagnetic core-mantle coupling require a better knowledge of the electrical conductivity of the lowermost mantle – this can be obtained from the analysis of "jerks", which are sudden changes in the secular variation that last for 1 or 2 years. An improved determination of the core's contribution to the Earth's angular momentum budget will allow for a better estimation of changes in atmospheric and ocean circulation pattern.

It is clear that the nature of the Earths magnetic field is complicated. It is also therefore clear that there is the need for a comprehensive separation and understanding of the external and internal processes that contribute to the Earth's magnetic fields – the **Swarm** mission aims to address such needs as well as allowing for new and exciting studies of the lithospheric field.

The magnetic field is also of importance for the Earth's external environment. While it is known that the air density in the thermosphere is related to geomagnetic activity, recent results from the German CHAMP mission have indicated that air density is locally affected by geomagnetic activity in a specific way that is still to be explored and understood. Furthermore, the magnetic field acts as a shield against high-energy particles from the Sun and outer Space. Continuous space-borne monitoring of the magnetic field at low Earth orbit, and the derivation of field models play an important role in predicting radiation hazards within the space environment.

The scientific and technical background of the mission the expected performance can be found on the page of the **Swarm** mission [1]. The links on the right hand side of the full page lead to pdf-files of the mission report, the technical annex and the presentation material.

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[1] **Swarm** homepage: `http://www.esa.int/esaLP/swarm.html`

Locally Supported Wavelets on the Sphere

Michael Schreiner

A new class of locally supported radial basis functions on the (unit) sphere is introduced by forming an infinite number of convolutions of "isotropic finite elements". The resulting up functions show useful properties: They are locally supported and are infinitely often differentiable. The main properties of these kernels are studied in detail. In particular, the development of a multiresolution analysis within the reference space of square–integrable functions over the sphere is given.

Starting point of our considerations are the functions

$$B_h^\lambda(t) = \begin{cases} 0 & \text{for} \quad -1 \le t \le h \\ \frac{(t-h)^\lambda}{(1-h)^\lambda} & \text{for} \quad h < t \le 1 \end{cases}$$

which we consider for $t \in [-1,1]$, $h \in (-1,1)$ and $\lambda > -1$. Note that in contrast to earlier investigations of these kernels (see e.g. [4]) we let the parameter λ be real, and allow the functions to be unbounded (for $-1 < \lambda < 0$), but with finite integral. Letting $\eta \in \Omega$ be fixed, we get a radial basis function

$$\Omega \ni \xi \mapsto B_h^\lambda(\eta \cdot \xi)$$

which in accordance with our construction has the local support

$$\mathrm{supp} B_h^\lambda(\eta \cdot\ ) = \{\xi \in \Omega | h \le \xi \cdot \eta \le 1\}.$$

Here, $\Omega = \{x \in \mathbb{R}^3 \mid |x| = 1\}$ denotes the unit sphere embedded in $\mathbb{R}^3$. The Legendre series according to the Legendre Polynomials P_n is denoted by

$$B_h^\lambda \sim \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (B_h^\lambda)^\wedge(n) P_n,$$

where $(B_h^\lambda)^\wedge(n) = 2\pi \int_h^1 B_h^\lambda(t) P_n(t) dt$. We scale the kernel B_h^λ so that the Legendre transform of order zero is 1. We define $L_h^\lambda(t) = \frac{1}{(B_h^\lambda)^\wedge(0)} B_h^\lambda(t)$, $t \in [-1,1]$.

Iterated kernels have some appealing properties. To be more concrete, they are still locally supported, their Legendre transform is non–negative, and they show a certain degree of smoothness. They are defined as follows: Let $h \in (0,1)$ and $\lambda > -1$. Then it is known (see e.g. [1]) that the iterated kernel

$$(L_h^\lambda)^{(2)} = L_h^\lambda * L_h^\lambda = \int_\Omega L_h^\lambda(\ \cdot \xi) L_h^\lambda(\xi \cdot\ ) d\omega(\xi)$$

has the support

$$\mathrm{supp}(L_h^\lambda)^{(2)}(\eta \cdot\ ) = \left\{\xi \in \Omega | 2h^2 - 1 \le \xi \cdot \eta \le 1\right\}. \tag{15}$$

Since the support of the aforementioned radial basis functions will become an important issue when we consider infinite convolutions, the statement (15) should be explained in more detail: The support of $L_h^\lambda(t)$ is $[h,1]$, so that the function $\vartheta \mapsto L_h^\lambda(\cos\vartheta)$, $\vartheta \in [0,\pi]$, is supported in $[0, \arccos h]$. The support of the iterated kernel $\vartheta \mapsto (L_h^\lambda)^{(2)}(\cos\vartheta)$ is then twice as large, i.e. $[0, 2\arccos h]$, which is obvious when the kernel is considered as a radial basis function over the sphere Ω. Thus, the support of $t \mapsto (L_h^\lambda)^{(2)}(t)$ is $[\cos(2\arccos h), 1] = [2h^2 - 1, 1]$. We can verify that

(i) If $\lambda > -1$ then $(L_h^\lambda)^{(2)}(\eta \cdot\ ) \in L^2(\Omega)$.
(ii) If $\lambda > -1/2$ then $(L_h^\lambda)^{(2)}(\eta \cdot\ ) \in C(\Omega)$.

(iii) If $\lambda > k/2 - 1/2$ then $(L_h^\lambda)^{(2)}(\eta\cdot\ ) \in C^{(k)}(\Omega)$, $k \in \mathbb{N}$.

Now, we deal with a spherical counterpart of the so–called up function which is, for one dimensional problems, described e.g. in [3]. The main idea is to build an infinite convolution of locally supported functions, where the support of each of the building blocks is chosen carefully to ensure that the resulting convolution is additionally locally supported. Even more, the infinite convolution turns out to be infinitely often differentiable. The reason is that the symbol of the up function decays for increasing n faster than any rational function (in n).

Suppose that $h \in (-1, 1)$, and $\lambda > -1$. We let $\varphi_0 = \arccos h$ and introduce

$$\varphi_i = 2^{-i}\varphi_0, \ , \ h_i = \cos\frac{\varphi_i}{2}, \quad i = 1, 2, \dots. \tag{16}$$

Then Up_h^λ defined by

$$\mathrm{Up}_h^\lambda = (L_{h_1}^{(\lambda)})^{(2)} * (L_{h_2}^{(\lambda)})^{(2)} * \dots = \mathop{\Large *}_{i=1}^{\infty} (L_{h_i}^{(\lambda)})^{(2)} \tag{17}$$

is called *up function* (more precisely: (h, λ)–*up function*).

The basic properties of the up functions are:

(i) Up_h^λ is locally supported with $\mathrm{supp}\mathrm{Up}_h^\lambda = [h, 1]$.

(ii) For every $\eta \in \Omega$: $\mathrm{Up}_h^\lambda(\eta\cdot\ )$ is of class $C^{(\infty)}(\Omega)$.

(iii) $\mathrm{Up}_h^\lambda : [-1, 1] \to \mathbb{R}$ admits the uniformly convergent orthogonal expansion in terms of Legendre polynomials

$$\mathrm{Up}_h^\lambda = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (\mathrm{Up}_h^{(\lambda)})^\wedge(n) P_n \tag{18}$$

where $(\mathrm{Up}_h^{(\lambda)})^\wedge(0) = 1$ and

$$0 \le (\mathrm{Up}_h^{(\lambda)})^\wedge(n) = \prod_{i=1}^{\infty} \left((L_{h_i}^{(\lambda)})^\wedge(n) \right)^2 \le 1, \ n = 0, 1, 2, \dots. \tag{19}$$

(iv) For $n = 1, 2, \dots$

$$\lim_{h\to 1} (\mathrm{Up}_h^{(\lambda)})^\wedge(n) = 1. \tag{20}$$

(v) For all $t \in [-1, 1]$

$$0 \le \mathrm{Up}_h^\lambda(t) \le \mathrm{Up}_h^\lambda(1) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (\mathrm{Up}_h^{(\lambda)})^\wedge(n). \tag{21}$$

(vi) For any $k \in \mathbb{N}$,

$$(\mathrm{Up}_h^{(\lambda)})^\wedge(n) = O(n^{-k}), \quad n \to \infty. \tag{22}$$

We assume from now on, that $h \in (-1, 1)$ and $\lambda > -1$ are fixed. For this h, the numbers h_i, $i = 1, 2, \dots$ are defined as in (16). The scaling function $\Phi_{h,\lambda}^j : [-1, 1] \to \mathbb{R}$ is introduced by

$$\Phi_{h,\lambda}^j = (\mathrm{Up}_h^\lambda)^{j,\dots,\infty} = \mathop{*}_{i=j}^{\infty} L_{h_j,\lambda}^{(2)}, \quad j = 1, 2, \dots. \tag{23}$$

By construction, $\mathrm{supp}\Phi^j_{h,\lambda} = [h_{j-1}, 1]$, and we have the refinement equation

$$\Phi^{j+1}_{h,\lambda} * (L^{(\lambda)}_{h_j})^{(2)} = \Phi^j_{h,\lambda}, \quad j \geq 1. \tag{24}$$

Then it follows for $h \in (-1, 1)$, $\lambda > -1$ that the scale spaces

$$V_j = \{\Phi^j_{h,\lambda} * F | F \in L^2(\Omega)\}.$$

define a multiresolution of $L^2(\Omega)$ in the following sense:

(i) $V_j \subset L^2(\Omega)$ is a linear subspace with $V_j \subset C^{(\infty}(\Omega)$
(ii) $V_1 \subset V_2 \subset V_3 \subset \ldots$
(iii) $\bigcap_{j=1}^{\infty} V_j = V_1$
(iv) $\overline{\bigcup_{j=1}^{\infty} V_j} = L^2(\Omega)$

Based on this multiresolution, locally supported wavelets and the corresponding detail spaces can be found. Decomposition and reconstruction schemes involving the up functions can be developed, see [2] or [5].

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On the representation of smooth functions on the sphere using finitely many bits

Hrushikesh N. Mhaskar

In this paper, $q \geq 1$ is a fixed integer, $\mathbb{S}^q$ denotes the unit sphere of the Euclidean space $\mathbb{R}^{q+1}$, μ^*_q is the volume element measure on $\mathbb{S}^q$. We are interested in a parsimoneous representation of smooth functions on $\mathbb{S}^q$ using finitely may *bits (binary digits).* The minimal number of bits to represent a class of smooth functions within an accuracy of ϵ is given by the metric entropy H_ϵ defined in [1, Chapter 15]. Our representation (Theorem 5) utilizes the characterization of local Besov spaces using certain polynomial operators (Theorem 3). The metric entropy of these classes is described in Theorem 4. The full manuscript is available in [2].

The class of restrictions to $\mathbb{S}^q$ of all homogeneous harmonic polynomials of $q+1$ variables of degree ℓ will be denoted by $\mathbf{H}^q_\ell$, and for any $x \geq 0$, the class of all spherical polynomials of degree $\ell \le x$ will be denoted by Π_x^q. The dimension of $\mathbf{H}_\ell^q$ is denoted by d_ℓ^q, and $\{Y_{\ell,k}\}$ is an orthonormal basis for $\mathbf{H}_\ell^q$. For a signed measure ν, its variation measure is denoted by $|\nu|$. The symbols $c, c_1, \cdots$ denote positive constants depending only on the fixed parameters of the problem; their values may be different at different occurrences. A possibly signed measure ν on $\mathbb{S}^q$ will be called an *M–Z quadrature measure of order* N if $\|P\|_{|\nu|;\mathbb{S}^q,p} \le c\|P\|_{\mu_q^*;\mathbb{S}^q,p}$ and $\int_{\mathbb{S}^q} P(\mathbf{x})d\nu(\mathbf{x}) = \int_{\mathbb{S}^q} P(\mathbf{x})d\mu_q^*(\mathbf{x})$ for each $P \in \Pi_N^q$ and $1 \le p \le \infty$. The existence of M–Z quadrature measures based on discrete sets of scattered points is established in [3]. For a sequence of (signed) measures μ_n, we write $\mu_n \preceq_p \mu_q^*$ if every μ_q^* measurable function f is also μ_n measurable, and $\|f\|_{|\nu|;\mathbb{S}^q,p} \le c\|f\|_{\mu_q^*;\mathbb{S}^q,p}$.

For any $x \ge 0$, $1 \le p \le \infty$ and $f \in L^p(\mu_q^*;\mathbb{S}^q)$, we write $E_{\mathbb{S}^q,x,p}(f) := \min_{P\in\Pi_x^q} \|f - P\|_{\mu_q^*;\mathbb{S}^q,p}$. We will define the Besov spaces in terms of the sequence $\{E_{\mathbb{S}^q,2^n,p}(f)\}$. For $0 < \rho \le \infty$, $\gamma > 0$, we define

$$\mathsf{b}_{\rho,\gamma} := \{\{a_n\}_{n=0}^\infty \ : \ \|\{a_n\}\|_{\rho,\gamma} := \|\{2^{n\gamma}a_n\}\|_{\ell^\rho} < \infty\}.$$

If $1 \le p \le \infty$, the Besov space $B_{\mathbb{S}^q,p,\rho,\gamma}$ consists of all functions $f \in L^p(\mathbb{S}^q)$ for which $\{E_{\mathbb{S}^q,2^n,p}\} \in \mathsf{b}_{\rho,\gamma}$. A spherical cap centered at a point $\mathbf{x}_0 \in \mathbb{S}^q$, and radius $\alpha \in [0,\pi]$ is defined by

$$\text{(25)} \quad \mathbb{S}_\alpha^q(\mathbf{x}_0) := \{\mathbf{x} \in \mathbb{S}^q \ : \ \mathbf{x}\cdot\mathbf{x}_0 \ge \cos\alpha\} = \{\mathbf{x} \in \mathbb{S}^q \ : \ \|\mathbf{x} - \mathbf{x}_0\| \le 2\sin(\alpha/2)\},$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^{q+1}$. For a cap C, the space $C_0^\infty(C)$ consists of infinitely differentiable functions ϕ on $\mathbb{S}^q$ such that $\phi(\mathbf{x}) = 0$ if $\mathbf{x} \notin C$. If $\mathbf{x}_0 \in \mathbb{S}^q$, the local Besov space $B_{\mathbb{S}^q,p,\rho,\gamma}(\mathbf{x}_0)$ consists of functions $f \in L^p(\mathbb{S}^q)$ for which there exists a cap C centered at $\mathbf{x}_0$ such that for every $\phi \in C_0^\infty(C)$, $f\phi \in B_{\mathbb{S}^q,p,\rho,\gamma}$.

Let $h : [0,\infty) \to \mathbb{R}$, $h(x) = 0$ if $x > c$. We define the kernels

$$\text{(26)} \quad \Phi_n(h, \mathbf{x}\cdot\xi) := \sum_{\ell=0}^\infty h(\ell/2^n) \sum_{k=1}^{d_\ell^q} Y_{\ell,k}(\mathbf{x})\overline{Y_{\ell,k}(\xi)}, \qquad \mathbf{x}, \xi \in \mathbb{S}^q, \ n = 0, 1, \cdots.$$

Let $\{\mu_n\}_{n=0}^\infty$ be a sequence of (possibly signed) finite, Borel measures on $\mathbb{S}^q$, $\mu_{-1} := 0$. The *polynomial frame operator* is defined for $n = 0, 1, \cdots$, $\mathbf{x} \in \mathbb{S}^q$, $f \in L^1(|\mu_n|;\mathbb{S}^q) \cap L^1(|\mu_{n-1}|;\mathbb{S}^q)$ by

$$\text{(27)} \quad \tau_n(\mu_n, h, f, \mathbf{x}) := \int_{\mathbb{S}^q} \Phi_n(h, \mathbf{x}\cdot\xi)f(\xi)d\mu_n(\xi) - \int_{\mathbb{S}^q} \Phi_{n-1}(h, \mathbf{x}\cdot\xi)f(\xi)d\mu_{n-1}(\xi).$$

Let $Q \ge 1$ be an integer. We will write $h \in \mathcal{A}_Q^*$ if each of the following conditions is satisfied: (i) $h : [0,\infty) \to \mathbb{R}$, (ii) for some integer $K \ge Q + q$, h is a K times iterated integral of a function of bounded variation, (iii) $h(x) = 1$ for $x \in [0, 1/2]$, and (iv) $h(x) = 0$ if $x > 1$.

Theorem 3. *Let* $1 \le p \le \infty$, $f \in L^p(\mu_q^*;\mathbb{S}^q)$ *(*$C(\mathbb{S}^q)$ *if* $p = \infty$*),* $\gamma > 0$, $0 < \rho \le \infty$, $Q > \max(1,\gamma)$, $h \in \mathcal{A}_Q^*$, *and* $\mathbf{x}_0 \in \mathbb{S}^q$. *For* $n \ge 0$, *let* μ_n, ν_n *be M–Z quadrature measures of order* $6(2^n)$, *and in addition,* $\mu_n \preceq_p \mu_q^*$. *Then the following are equivalent.*

(a) $f \in B_{\mathbb{S}^q,p,\rho,\gamma}(\mathbf{x}_0)$.

(b) *There exists* $\alpha > 0$ *such that for every* $\phi \in C_0^\infty(\mathbb{S}^q_\alpha(\mathbf{x}_0))$, $\{\|\tau_n(\mu_n, h, f\phi)\|_{\mu_q^*;\mathbb{S}^q,p}\} \in \mathsf{b}_{\rho,\gamma}$.
(c) *There exists* $\alpha > 0$ *such that for every* $\phi \in C_0^\infty(\mathbb{S}^q_\alpha(\mathbf{x}_0))$, $\{\|\tau_n(\mu_n, h, f\phi)\|_{|\nu_n|;\mathbb{S}^q,p}\} \in \mathsf{b}_{\rho,\gamma}$.
(d) *There exists* $\alpha > 0$ *such that* $\{\|\tau_n(\mu_n, h, f)\|_{\mu_q^*;\mathbb{S}^q_\alpha(\mathbf{x}_0),p}\} \in \mathsf{b}_{\rho,\gamma}$.
(e) *There exists* $\alpha > 0$ *such that* $\{\|\tau_n(\mu_n, h, f)\|_{|\nu_n|;\mathbb{S}^q_\alpha(\mathbf{x}_0),p}\} \in \mathsf{b}_{\rho,\gamma}$.

The set $\overline{B}_{\mathbb{S}^q,p,\rho,\gamma}$ consists of f such that $\|f\|_{\mu_q^*;\mathbb{S}^q,p} + \|\{E_{\mathbb{S}^q,2^n,p}\}\|_{\rho,\gamma} \le 1$.

Theorem 4. *Let* $0 < \epsilon \le 1$, $1 \le p \le \infty$, $0 < \gamma < \infty$, $0 < \rho \le \infty$. *Then the metric entropy* $H_\epsilon(\overline{B}_{\mathbb{S}^q,p,\rho,\gamma}, L^p(\mu_q^*;\mathbb{S}^q))$ *of* $\overline{B}_{\mathbb{S}^q,p,\rho,\gamma}$ *in* $L^p(\mu_q^*;\mathbb{S}^q)$ *satisfies*

$$c_1(\log(1/\epsilon))^{-(3q)/(2\gamma\rho)}(1/\epsilon)^{q/\gamma} \le H_\epsilon(\overline{B}_{\mathbb{S}^q,p,\rho,\gamma}, L^p(\mu_q^*;\mathbb{S}^q)) \le c_2(1/\epsilon)^{q/\gamma}. \tag{28}$$

Theorem 5. *Let* $p, \gamma, \rho, Q, h, \mu_n$ *be as in Theorem 3. Suppose that for each integer* $n \ge 0$, $\mathcal{C}_n$ *is a finite set of points on* $\mathbb{S}^q$ *such that there exists an M–Z quadrature measure* ν_n *of order* $6(2^n)$, *supported on a subset of* $\mathcal{C}_n$ *as in* [3]. *Let* C *be a spherical cap. If* $n \ge 0$, *and* $f \in L^p(\mu_n;\mathbb{S}^q)$, *we define*

$$I_n(\mu_n, h, f, \xi) := \lfloor 2^{nQ}\sigma_n(\mu_n, h, f, \xi)\rfloor, \qquad \xi \in \mathbb{S}^q, \tag{29}$$

and

$$\begin{aligned} \sigma_n^\circ(C, h, f, \mathbf{x}) &:= \sigma_n^\circ(\mu_n, \nu_n; C, h, f, \mathbf{x}) \\ &:= \ 2^{-nQ}\int_C I_n(\mu_n, h, f, \xi)\Phi_{n+1}(h, \mathbf{x}\cdot\xi)d\nu_n(\xi), \quad \mathbf{x} \in \mathbb{S}^q. \end{aligned} \tag{30}$$

Let $\|f\|_{\mu_q^*;\mathbb{S}^q,p} + \left\|\{\|\tau_n(\mu_n, h, f)\|_{\mu_q^*;C,p}\}\right\|_{\rho,\gamma} \le 1$. *Then for a cap* C', *concentric with* C *and having radius strictly less than that of* C,

$$\{\|f - \sigma_n^\circ(C, h, f)\|_{\mu_q^*;C',p}\} \in \mathsf{b}_{\rho,\gamma}, \tag{31}$$

and in particular, $\|f - \sigma_n^\circ(C, h, f)\|_{\mu_q^*;C',p} \le c(C, C')2^{-n\gamma}$.

If n *is chosen so that* $c(C, C')2^{-n\gamma} \le \epsilon$, *then the number of bits needed to represent all the integers* $\{I_n(\mu_n, h, f, \xi),\ \xi \in C \cap supp(\nu_n)\}$ *does not exceed* $c_1(\log(1/\epsilon))^c(1/\epsilon)^{q/\gamma}\mu_q^*(C)$.

If $n \ge 0$ *and* $g \in L^p(\mu_n;\mathbb{S}^q)$, *then*

$$\|f - \sigma_n^\circ(C, h, g)\|_{\mu_q^*;C',p} \le c(C, C')\{2^{-n\gamma} + \|f - g\|_{\mu_n;\mathbb{S}^q,p}\}. \tag{32}$$

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A New Class of Localized Frames on Spheres

Francis J. Narcowich
(joint work with Pencho Pertushev & Joseph D. Ward)

In this talk we wish to present a new class of tight frames on the sphere. These frames, like the wavelets for the sphere introduced by Freeden and others [2], are based on expansions in ultraspherical harmonics. There are two novel features here. The first is that they have excellent localization properties, and the second is a surprising connection between them and the masks for the Daubechies wavelets. We will discuss what these new frames are and how they can be implemented using a quadrature formula for the sphere that has been recently developed [3]. The present work has been done jointly with Professor Pencho Petrushev, University of South Carolina (Columbia, SC, USA) and Professor Joseph D. Ward, Texas A&M University (College Station, TX, USA). Details and more results may be found in the preprint [5].

Frames were introduced in the 1950s to represent functions via over-complete sets. More recently, they feature prominently in wavelet analysis, especially discretizations of continuous wavelet transforms [1]. Frames on spheres have also been developed; see [4] for references and more discussion.

Let us review the basic facts about them when the target functions belong to a Hilbert space $\mathcal{H}$ with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. In that case, a set $\{\psi_j\}_{j\in\mathcal{J}}$ is a *frame* if there are constants $c, C > 0$ such that for all $f \in \mathcal{H}$

$$c\|f\|^2 \le \sum_{j\in\mathcal{J}} |\langle f, \psi_j\rangle|^2 \le C\|f\|^2.$$

The smallest C and largest c are called upper and lower *frame bounds.* If $C = c$, we say the frame is *tight.* If $C = c = 1$ *and* $\|\psi_j\| = 1$ for all j, then the frame is actually an orthonormal set.

Our Hilbert space will be $L^2(\mathbb{S}^n)$, with the measure being the standard one. Constructing a frame starts with a function $a(t)$ in $C^k(\mathbb{R})$ that is supported on $[\frac{1}{2}, 2]$. We use $a(t)$ to define kernels on $\mathbb{S}^n$. In the case of $\mathbb{S}^2$, these kernels have the form,

$$A_j(\xi\cdot\eta) = \frac{1}{4\pi}\left\{\begin{array}{l} a(1)P_0(\xi\cdot\eta) + 3a(3/2)P_1(\xi\cdot\eta) \\ \sum_{\ell=0}^{\infty}(2\ell+1)a\left(\frac{2\ell+1}{2^{j+1}}\right)P_\ell(\xi\cdot\eta), \end{array}\right.$$

where the P_ℓ's are the usual Legendre polynomials and $\xi\cdot\eta$ is the standard "dot" product. We can replace it by $\xi\cdot\eta = \cos(\theta)$, where θ is the geodesic distance between ξ and η. On $\mathbb{S}^n$, they are somewhat more complicated, with ultraspherical polynomials replacing the Legendre polynomials. In any case, these kernels are themselves spherical polynomials of degree less than 2^{j+1}.

The kernels have two useful features. The first is localization for j large.

Theorem [5]. *Let θ be in $[0,\pi]$, $n \geq 2$, and $k > \max\{1, n-2\}$. Then there is a constant $C_{n,k,a}$ such that*

$$|A_j(\cos\theta)| \leq \frac{2^{nj} C_{n,k,a}}{1 + 2^{jk}(\theta/\pi)^k}.$$

We remark that the large j behavior of $|A_j|$ near $\theta \approx 0$ is $\mathcal{O}(2^{nj})$. For θ bounded away from 0, it is $\mathcal{O}(2^{(n-k)j})$.

The second of the two features mentioned above requires a special choice for $a(t)$. Let $m_0(\xi)$ be a mask for a wavelet with at least $k+1$ vanishing moments. Recall that this means that $m(\xi)$ is a 2π periodic function for which $|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 \equiv 1$, $m_0(0) = 1$, and $m_0^{(j)}(\pi) = 0$ for $j = 0, \ldots, k$. We then define the C^k function

$$a(t) := \begin{cases} m_0(\pi \log_2(t)) & \frac{1}{2} \leq t \leq 2 \\ 0 & t < \frac{1}{2} \text{ or } t > 2. \end{cases}$$

With this choice of a, the operators A_j associated with the kernels A_j satisfy

$$\sum_{j=0}^{\infty} \mathsf{A}_j \mathsf{A}_j^* = I$$

in the strong operator topology. We will call the A_j *frame operators* and the A_j we will call *frame kernels*.

We define the *frame transform* via

$$f \mapsto w_j = \mathsf{A}_j^* f$$

This is our *decomposition formula*. The operator identity in the previous equation then gives us the *reconstruction formula* when applied to the w_j's:

$$f = \sum_{j=0}^{\infty} \mathsf{A}_j w_j.$$

The frames themselves will be obtained by discretizing these formulas. Let $X := \{x_1, \ldots, x_N\}$ be a discrete set of distinct points on S^n; we will call these the *centers*. There are several important quantities associated with this set: the *mesh norm*, $h_X = \sup_{y \in S^n} \inf_{x_j \in X} d(x_j, y)$, where $d(\cdot,\cdot)$ is the geodesic distance between points on the the sphere; *separation radius*, $q_X = \frac{1}{2} \min_{j \neq k} d(x_j, x_k)$; and the *mesh ratio*, $\rho_X := h_X/q_X \geq 1$. The set of centers X is called ρ-uniform if $\rho_X \leq \rho$. For $\rho \geq n+1$, there exists a ρ–uniform X with h_X arbitrarily small. Let $\mathcal{X}$ to be the Voronoi partition of $\mathbb{S}^n$ for X. The region containing x_j will be called R_{x_j}.

The following quadrature formula is essential to our construction.

Theorem [3]. *There exists a constant $c^\diamond > 0$ (depending only on n) such that for any $L \geq 1$ and a ρ-uniform set X in $\mathbb{S}^n$ with $h_X \leq c^\diamond/L$, there exist positive* *coefficients* $\{c_\eta\}_{\eta\in X}$ *such that the quadrature formula*

$$\int_{\mathbb{S}^n} f(\xi)\,d\mu(\xi) \doteq \sum_{\eta\in X} c_\eta f(\eta)$$

is exact for all spherical polynomials of degree $\le L$*. In addition,* $c_\eta \approx L^{-n}$ *with constants of equivalence depending only on* n*.*

Fix $\rho \ge n+1$. Pick a sequence of ρ-uniform sets X_j so that $h_{X_j} \le c^\diamond\, 2^{-j-2}$. Then the quadrature formula above is exact for all spherical harmonics of degree $\ell \le 2^{j+2}$. Also, $c_\eta \approx 2^{-jn}$ and $\#X \approx 2^{nj}$.

The frame transform has the form $w_j(\xi) = \mathsf{A}_j^* f(\xi) = \langle f(\zeta), A_j(\zeta\cdot\xi)\rangle$. The point is that $w_j(\xi)$ is a spherical polynomial of degree less than 2^{j+1}, because $A_j(\zeta\cdot\xi)$ is a spherical polynomial with degree less than 2^{j+1}. In the reconstruction formula this then contributes the term

$$\mathsf{A}_j w_j(\omega) = \int_{\mathbb{S}^n} A_j(\omega\cdot\xi) w_j(\xi) d\mu(\xi).$$

The product $A_j(\omega\cdot\xi)w_j(\xi)$ is a spherical polynomial of degree less than $2^{j+1} + 2^{j+1} = 2^{j+2}$. It can thus be integrated *exactly* with the quadrature formula, so that

$$\mathsf{A}_j w_j(\omega) = \sum_{\eta\in X_j} c_\eta A_j(\eta\cdot\omega) w_j(\omega) = \sum_{\eta\in X_j} \langle f, \psi_{j,\eta}\rangle \psi_{j,\eta},$$

where $\psi_{j,\eta}(\xi) := \sqrt{c_\eta} A_j(\xi\cdot\eta)$, $\eta \in X_j$, is the analysis frame function at level j. Using this, our earlier reconstruction formula, and doing a little more work, we have the following:

Theorem [5] *Let* $f \in L^2(\mathbb{S}^n)$*, then* $f = \sum_{j=0}^\infty \sum_{\eta\in X_j} \langle f, \psi_{j,\eta}\rangle \psi_{j,\eta}$*. Moreover, the frame* $\{\psi_{j,\eta}\}_{j\in\mathbb{Z}_+,\eta\in X_j}$ *is tight,*

$$\|f\|^2 = \sum_{j=0}^\infty \sum_{\eta\in X_j} |\langle f, \psi_{j,\eta}\rangle|^2.$$

Finally, the frame functions have vanishing moments that increase with j*.*

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Interpolation and Least Squares Approximation on the Sphere via Locally Supported Functions

Joseph D. Ward

The goal of my talk at the Oberwolfach Geomathematics conference (May 24-28, 2004) was to discuss potential new tools applicable to the problem of reconstructing functions from scattered data sites on the n-sphere. Common methods employed for reconstruction include interpolation and least squares approximation. Such methods are generally considered "good" if the methods are stable (i.e. condition numbers of the matrices involved are well-behaved), the matrices are sparse (or banded) and if the method approximates smooth functions well (i.e. good error estimates for functions belonging to "smooth" spaces).

Many of these properties hold true when reconstructing functions from scattered data on a compact subset in R^n by means of translates of a given radial basis function. For the current status of these results, one should consult the recent book of M. Buhmann or the upcoming book of H. Wendland. More recently, people have realized that one can "transport" many of the approximation results on R^n onto the n-sphere by simply restricting the RBF defined on R^{n+1} to S^n , i.e., define the zonal function ϕ by means of $\phi(x \cdot y) := \Phi(||x - y||_2)|_{x,y \in S^n}$.

This approach works for many of the well-known RBFs including Thin-Plate splines, Hardy multi-quadrics and the compactly supported Wendland functions. The fact that this approach works so well is for the following reason. Much of the theory which supports the numerics behind RBFs depends on the fact that RBfs are positive definite functions (or at least conditionally positive definite). The restriction to the n-sphere of such functions are then also positive definite (or CPD).

We next illustrate how one would interpolate scattered data on the sphere, the data derived from some underlying function. Given a collection X of scattered sites on the sphere, one constructs the interpolant $I_X f(x) = \sum_{x_k \in X} c_k \phi(x \cdot x_k)$. The interpolation matrices will be invertible with the norm of the inverse depending primarily on the minimal separation of the data sites. Error estimates concerning how well the interpolant fits the underlying data are given by the following:

Theorem 1. *Let X be any point set on S^n with mesh norm h_X, and let ϕ be an SBF. If for some $\tau > n/2$ we have $\hat{\phi}(\ell) \leq c(1 + \lambda_\ell)^{-\tau}$ as $\ell \to \infty$, then for all $f \in N_\phi$ there is a constant C that is independent of X and f for which*

$$\|f - I_X f\|_\infty \leq C h_X^{\tau - n/2} \|f\|_\phi.$$

The norm $\|\cdot\|_\phi$ is associated with the RBF ϕ and gives rise to a reproducing kernel Hilbert space. However, in the case ϕ is either a compactly supported Wendland function or a Thin-Plate spline, the estimates in Theorem 1 still hold if $\|\cdot\|_\phi$ is replaced by the more traditional Sobolev norm $\|\cdot\|_{W_2^s}$.

We next (briefly) discuss the least squares theory applicable to SBF approximation on S^n. Rather than giving the most general result we present a sample result. Full details are available in the papers in the references. Notice that there are two discrete sets in the theorem below. The coarser set Y is where interpolation occurs. The finer set B determines the discrete least squares norm. Also the error is given in terms of interpolation. But this estimate, of course, also gives upper bounds on discrete least squares approximation.

Theorem 2. *Suppose $\tau = k + s$, where k is a positive integer and $0 < s \leq 1$. Let $Y \subset S^n$ be a discrete set with given mesh norm $h = h_{Y,S^n}$. Let $B = \{b_1, \ldots, b_M\}$ be a discrete set on the unit sphere with $h_B \leq h$. If $g \in H^\tau(S^n)$ and*

$$I_Y g(x) = \sum_{\chi_\varepsilon \in Y} c_\varepsilon \phi(x \cdot x_\varepsilon)$$

is constructed from a positive definite kernel Φ satisfying

$$c_1(1+\lambda_\ell)^{-\tau} \leq \hat{\phi}(\ell) \leq c_2(1+\lambda_\ell)^{-\tau}$$

then there is a constant C independent of g and h such that

$$\|g - I_Y g\|_{\ell_2(B)} \leq C h^\tau_{Y,S^n} \|g\|_{H^\tau(S^n)}.$$

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Polynomial Interpolation on the Sphere

Noemí Laín Fernández

The problem of reconstructing a continuous signal on the sphere from discrete data arises in many areas including geophysics and meteorology where the sphere is taken as a model of the surface of the Earth.

A classical way of addressing data fitting problems on the sphere is by *polynomial interpolation*: given a set of nodes $\{\xi_i\}_{i=1,\ldots,N}$ on the two-dimensional sphere $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$ and certain real-valued data $\{y_i\}_{i=1,\ldots,N}$, the goal is to construct a polynomial in the space V_n of spherical polynomials of degree at most n which interpolates the known data. The appropriate analog of polynomials on the sphere are the so-called *spherical harmonics* $\{Y_k^j\}_{j=-k,\ldots,k,\ k\in\mathbb{N}_0}$ (see e.g. Freeden, Gervens and Schreiner [1], Müller [3] or Reimer [5]). Specifically, making use of this explicit basis of spherical harmonics and denoting with $N := (n+1)^2$ the dimension of the space V_n, the interpolation problem on the sphere reads:

find a spherical polynomial $P := \sum_{k=0}^{n} \sum_{j=-k}^{k} \alpha_k^j Y_k^j$ in V_n, such that for given real-valued data $\{y_i\}_{i=1,\dots,N}$, the interpolation conditions

$$P(\xi_i) = \sum_{k=0}^{n} \sum_{j=-k}^{k} \alpha_k^j Y_k^j(\xi_i) = y_i, \quad i = 1, \dots, N,$$

are satisfied. Unfortunately, not for any given set of pairwise distinct points $\{\xi_i\}_{i=1,\dots,N}$, the polynomial interpolation problem has a unique solution and hence it is of interest to identify those point sets for which the above system of equations is nonsingular. While it is clear that such point systems – which we will call *fundamental systems* – exist, they have not yet been extensively treated in the literature. However, von Golitschek and Light [2], Sündermann [6] and Xu [7, 8] present and analyze two special distribution strategies for points on parallel circles on the sphere. While the first construction features little symmetry with respect to the equator because any two different latitudes have to carry a different number of points, the second distribution strategy only works when the underlying polynomial degree n is even. To overcome this restriction, we focus here on the interpolation problem in V_{2k+1}, and present a fundamental system construction, in which the $(n+1)^2$ points are located on $n+1$ parallel latitudes, each of them containing $n+1$ equidistantly distributed points. In the following, $\Psi : [0, \pi] \times [0, 2\pi) \longrightarrow \mathbb{S}^2$ denotes the parameterization of $\mathbb{S}^2$ in spherical coordinates.

Theorem.1 Let $n \in \mathbb{N}$ be odd and let $0 < \rho_1 < \rho_2 < \cdots < \rho_{(n+1)/2} < \pi/2$ and $\rho_{n+2-j} := \pi - \rho_j$ $(j = 1, \dots, (n+1)/2)$ denote a system of symmetric latitudinal angles. Then the set of points $S(\alpha) := \{\xi_{j,k} := \Psi(\rho_j, \theta_k^j) \ : \ j, k = 1, \dots, n+1\}$, where

$$\theta_k^j = \begin{cases} \frac{2\pi k}{n+1}, & \text{if } j \text{ is odd}, \\ \frac{(2(k-1)+\alpha)\pi}{n+1}, & \text{if } j \text{ is even}, \end{cases}$$

and $\alpha \in (0, 2)$, constitutes a fundamental system for V_n.

The symmetric distribution of the resulting points not only generates a clear and regular geometry of the grid of nodes, but also simplifies theoretical and technical matters, as the involved Gram matrices attain a circulant structure. Making then use of the theory of circulant matrices in combination with classical matrix factorization techniques by means of Fourier matrices, we obtain a more manageable expression for the dense interpolation matrices corresponding to our fundamental systems. On the other hand, from the numerical point of view, a point distribution on a structured grid allows the construction of spherical multiscale methods, leading us to the introduction of spherical *polynomial wavelets*.

For fixed $s \in \mathbb{N}$, we define the wavelet space W_n^s $(n \in \mathbb{N})$ as the orthogonal complement of V_n in V_{n+s}, i.e., W_n^s is spanned by the spherical harmonics of degree at most $n + s$ that are orthogonal to V_n. In this talk, we present explicit fundamental systems for W_n^2 and W_n^n.

Theorem 2. Let $\rho \in (0, \pi)$ be such that

$$P_k^m(\pm \cos \rho) \neq 0, \qquad m = 0, \ldots, k,\ k = n+1, n+2. \tag{33}$$

Then

$$S := \left\{\eta_{1,k} := \Psi(\rho, \theta_k^1)\right\}_{k=1,\ldots,2n+3} \cup \left\{\eta_{2,k} := \Psi(\pi - \rho, \theta_k^2)\right\}_{k=1,\ldots,2n+5},$$

with $\theta_k^1 = 2\pi k/(2n+3)$ $(k=1,\ldots,2n+3)$ and $\theta_k^2 = 2\pi k/(2n+5)$ $(k = 1, \ldots, 2n+5)$, constitutes a fundamental system for W_n^2.

While this construction principle reminds us of the strategy followed in [2, 6], the construction of fundamental systems for W_n^n is along the same lines as the one proposed in Theorem 1. Note that in this case we cannot choose the heights of the latitudinal circles arbitrarily anymore. The following theorem is joint work with Jürgen Prestin.

Theorem. 3 Let n be an even integer and let

$$-1 < \cos \rho_n < \cos \rho_{n-1} < \cdots < \cos \rho_1 < 1$$

denote the zeros of the Legendre polynomial P_n. Then the set of points

$$M_n(\alpha) := \{\eta_{j,k} := \Psi(\rho_j, \theta_k^j)\ :\ j = 1, \ldots, n,\ k = 1, \ldots, 3n+2\},$$

with

$$\theta_k^j = \begin{cases} \frac{2\pi k}{3n+2}, & \text{if } j \text{ is odd}, \\ \frac{(2(k-1)+\alpha)\pi}{3n+2}, & \text{if } j \text{ is even}, \end{cases}$$

and $\alpha \in (0, 2)$, constitutes a fundamental system for W_n^n.

In the last part of this talk, we study the localization behavior of the polynomials

$$\varphi_c^n(\circ) := \sum_{k=0}^{n} \frac{2k+1}{4\pi} c_k\, P_k(\xi \cdot \circ), \text{ and } \psi_d^{n,s}(\circ) := \sum_{k=n+1}^{n+s} \frac{2k+1}{4\pi} d_k\, P_k(\xi \cdot \circ),$$

where $\xi \in \mathbb{S}^2$ is a fixed point and $\{c_k\}_{k=0,\ldots,n} \subset \mathbb{R}^{n+1}$ and $\{d_k\}_{k=n+1,n+s} \subset \mathbb{R}^s$ are sets of nonzero real-valued coefficients.

However, since there exist different ways of measuring the *localization* of a function, we cannot generally speak of a unique *optimally localized* polynomial, but have to consider the optimal functions with respect to the localization criteria that we have in mind for our applications.

In the present talk, we basically concentrate on two localization criteria: on the one hand, for $c_k = 1$ $(k = 0, \ldots, n)$ and $d_k = 1$ $(k = n+1, \ldots, n+s)$ the polynomials φ_1^n and $\psi_1^{n,s}$ are the reproducing kernels of V_n and W_n^s, respectively, and have minimal $L^2(\mathbb{S}^2)$-norm among all polynomials in V_n or in W_n^s that attain the same value when they are evaluated at the prescribed point ξ.

A second way of measuring the localization of a function is by means of the uncertainty principle on the two-sphere which was introduced by Narcowich and Ward in [4]. In particular, if we choose the coefficient c_k as the evaluation of the normalized Legendre polynomial of degree k at the greatest zero of the Legendre polynomial P_{n+1}, i.e., $c_k := p_k(x^{n+1}_{\max})$ $(k = 0, \ldots, n)$, we come up with the optimally space-localized polynomial in V_n, where the space-localization is measured by the space-variance factor in the uncertainty product according to [4]. In a similar way, choosing d_k as the evaluation of the normalized *associated* Legendre polynomial $p_{k-(n+1)}(\circ; n+1)$ at the greatest zero of the associated Legendre polynomial $P_s(\circ; n+1)$, i.e. $d_k = p_{k-(n+1)}(y^s_{\max}; n+1)$ $(k = n+1, \ldots, n+s)$, we obtain the polynomial in W^s_n with minimal variance in space domain.

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A positive quadrature rule on the sphere

Jürgen Prestin

(joint work with Daniela Roşca)

In the talk we described an interpolatory quadrature rule for the sphere based on a fundamental set of points introduced by N. Laín Fernández. Let

$$\begin{aligned} \Psi : [0,\pi] \times [0,2\pi) &\to \mathbb{S}^2 \\ (\rho,\theta) &\mapsto (\sin\rho\cos\theta, \sin\rho\sin\theta, \cos\rho) \end{aligned}$$

be the parametrization of $\xi \in \mathbb{S}^2$ in coordinates (ρ,θ). Furthermore, let P_k, $k = 0, 1, \ldots$, denote the Legendre polynomials of degree k, normalized by $P_k(1) = 1$ and let V_n be the space of spherical polynomials of degree less than or equal to n. The dimension of V_n is $\dim V_n = (n+1)^2 = N$ and the reproducing kernel of V_n is

$$K_n(\xi,\eta) = \sum_{k=0}^{n} \frac{2k+1}{4\pi} P_k(\xi\cdot\eta) = k_n(\xi\cdot\eta), \quad \xi,\eta \in \mathbb{S}^2.$$

For given n we consider a set of points $\{\xi_i\}_{i=1,\ldots,N} \subset \mathbb{S}^2$ and the polynomial functions $\varphi_i^n : \mathbb{S}^2 \to \mathbb{C},\ i = 1,\ldots,N$ defined by

$$\varphi_i^n(\circ) = K_n(\xi_i, \circ) = \sum_{k=0}^{n} \frac{2k+1}{4\pi} P_k(\xi_i \cdot \circ),\ i = 1,\ldots,N.$$

A set of points $\{\xi_i\}_{i=1,\ldots,N}$ for which these functions $\{\varphi_i^n\}_{i=1,\ldots,N}$ constitute a basis for V_n is called a *fundamental system* for V_n.

In 2002 N. Laín Fernández proved the following result.

Proposition. *Let $n \in \mathbb{N}$ be an odd number and let $0 < \rho_1 < \rho_2 < \ldots < \rho_{\frac{n+1}{2}} < \pi/2,\ \rho_{n+2-j} = \pi - \rho_j,\quad j = 1,\ldots,(n+1)/2,$ denote a system of symmetric latitudes. Then the set of points $S(\alpha) = \left\{\xi_{j,k} = \Psi(\rho_j, \theta_k^j) : \quad j,k = 1,\ldots,n+1\right\}$, where*

$$\theta_k^j = \begin{cases} \frac{2k\pi}{n+1}, & \text{if } j \text{ is odd}, \\ \frac{2(k-1)+\alpha}{n+1}\pi, & \text{if } j \text{ is even}, \end{cases}$$

with $\alpha \in (0,2)$, constitutes a fundamental system for V_n.

In the following we will study the quadrature formula, for odd n, with the nodes in $S(\alpha)$. The Gram matrix associated to the scaling functions $\{\varphi_i^n\}_{i=1,\ldots,N}$ has the entries

$$\mathbf{\Phi}_n(r,s) = \langle \varphi_r^n, \varphi_s^n \rangle = K_n(\xi_r, \xi_s)$$

and it is positive definite when $\{\xi_i\}_{i=1,\ldots,N}$ is a fundamental system for V_n. Given the fundamental system $\{\varphi_i^n\}_{i=1,\ldots,N}$ of V_n, we can construct unique spherical polynomials $L_j^n : \mathbb{S}^2 \to \mathbb{C}$ in V_n satisfying the condition $L_j^n(\xi_i) = \delta_{ij}$. The set $\{L_j^n\}_{j=1,\ldots,N}$ constitutes a basis of V_n. Furthermore, any $f \in V_n$ can be written as

$$f = \sum_{i=1}^{N} f(\xi_i) L_i^n$$

If $\mathbf{L}_n$ is the Gram matrix of the Lagrangians, defined by $\mathbf{L}_n = (\langle L_i^n, L_j^n \rangle)_{i,j=1,\ldots,N} \in \mathbb{C}^{N\times N}$, then it holds

$$\mathbf{\Phi}_n \mathbf{L}_n = \mathbf{I}_N.$$

Here $\mathbf{I}_N$ denotes the $N \times N$ dimensional identity matrix. This means that the Lagrangians $\{L_j^n\}_{j=1,\ldots,N}$ are the dual functions of the scaling functions $\{\varphi_i^n\}_{i=1,\ldots,N}$.

Let $f \in V_n$ and let $\{L_i^n\}_{i=1,\ldots,N}$ be the Lagrangians associated to a fundamental system $\{\xi_i\}_{i=1,\ldots,N}$. By integration we get

$$\int_{\mathbb{S}^2} f(\xi)\, d\omega(\xi) = \sum_{i=1}^{N} f(\xi_i) \int_{\mathbb{S}^2} L_i^n(\xi)\, d\omega(\xi).$$

Therefore, the weights can be defined as

$$w_i^n = \int_{\mathbb{S}^2} L_i^n(\xi) d\omega(\xi) = \langle L_i^n, 1 \rangle, \quad i = 1,\ldots,N,$$

yielding the following quadrature formula

$$\int_{\mathbb{S}^2} f(\xi)\, d\omega(\xi) = \sum_{i=1}^{N} w_i^n f(\xi_i) + R_n(f).$$

On the other hand, taking $f \equiv 1 \in V_n$, we obtain $\sum_{i=1}^{N} L_i^n \equiv 1$ and therefore

$$w_i^n = \langle L_i^n, 1\rangle = \langle L_i^n, \sum_{k=1}^{N} L_k^n\rangle = \sum_{k=1}^{N} \langle L_i^n, L_k^n\rangle.$$

This means that the weight w_i^n can be calculated as the sum of the entries of the i-th row of the matrix $\mathbf{L}_n$, which is the inverse of the Gram matrix $\mathbf{\Phi}_n$. Consequently, the following equality holds

$$\mathbf{\Phi}_n \left(w_1^n, w_2^n, \ldots, w_N^n\right)^T = (1, 1, \ldots, 1)^T.$$

Theorem 1. *Let $n \in \mathbb{N}$ be an odd number and let P_n be the Legendre polynomial of degree n. For $\gamma < 1$ we consider the polynomial*

$$Q_{n+1}(x) = P_{n+1}(x) - \gamma P_n(x)$$

and its positive roots $q_1 > q_2 > \ldots > q_{\frac{n+1}{2}}$. If in the set $S(\alpha)$ the latitudes ρ_i are taken such that

$$\cos\rho_i = q_i,$$

then the weights w_i^n of the quadrature formula

$$\int_{\mathbb{S}^2} f(\xi)\, d\omega(\xi) = \sum_{i=1}^{N} w_i^n f(\eta_i) + R_n(f), \;\; \text{with} \;\; \{\eta_i\}_{i=1,\ldots,N} = S(\alpha),$$

are positive.

Let us mention that the weights w_i^n are constant for points on the same latitude but for different latitudes they behave as weights in the classical Gauss-Legendre quadrature rule. So, they are far from being constant.

Finally, we are interested in the case of spherical designs. A spherical design is a set of points of $\mathbb{S}^2$ which generates a quadrature formula with equal weights. It is an open question whether there exist spherical designs with $(n+1)^2$ points and which are exact for polynomials of degree n.

Here we try to find conditions on the latitudes ρ_i, which assure that the set $S(\alpha)$ is a spherical design. So we suppose

$$w_1^n = w_2^n = \ldots = w_N^n = w_n.$$

Therefore, $\cos\rho_i = r_i$, where the numbers r_i should satisfy the following conditions

$$\begin{array}{ccccccccc} r_1^2 & + & r_2^2 & + & \ldots & + & r_q^2 & = & q/3, \\ r_1^4 & + & r_2^4 & + & \ldots & + & r_q^4 & = & q/5, \\ & & & & & & & \ldots & \\ r_1^{2q-2} & + & r_2^{2q-2} & + & \ldots & + & r_q^{2q-2} & = & q/n. \end{array}$$

This is a system with $q-1$ equations and q unknowns and the solutions can be described in the following result.

Theorem 2. *Let $n \in \mathbb{N}$ be an odd number and consider the set $S(\alpha)$, defined in the Proposition, with arbitrary $\alpha \in (0,2)$ and with the latitudes $\{\rho_i,\ i = 1,\dots,q\}$, $q = (n+1)/2$, taken such that $\cos\rho_i = \sqrt{\gamma_i}$.*

Then $S(\alpha)$ is not a spherical design for $n \geq 11$.

For $n < 11$ we have a spherical design iff we take the γ_i with $0 < \gamma_i < 1$ as zeros of the following polynomials:

1. *For $n = 3$:*
$$T_2(x) = x^2 - \frac{2x}{3} + \frac{1}{2}\left(\frac{4}{9} - \beta\right) \quad \textit{with} \quad \beta \in \left(\frac{2}{9}, \frac{4}{9}\right)$$
2. *For $n = 5$:*
$$T_3(x) = x^3 - x^2 + \frac{x}{5} + \frac{1}{3}\left(\frac{2}{5} - \beta\right) \quad \textit{with} \quad \beta \in (0.4, 0.433996\dots)$$
3. *For $n = 7$:*
$$T_4(x) = x^4 - \frac{4x^3}{3} + \frac{22x^2}{45} - \frac{148x}{2835} + \frac{1}{4}\left(\frac{18728}{42525} - \beta\right)$$
with $\beta \in (0.4336145\dots, 0.4403997\dots)$
4. *For $n = 9$:*
$$T_5(x) = x^5 - \frac{5x^4}{3} + \frac{8x^3}{9} - \frac{100x^2}{567} + \frac{17x}{1701} + \frac{1}{5}\left(\frac{2300}{5103} - \beta\right)$$
with $\beta \in (0.4507152\dots, 0.4515677\dots)$.

Denoising for Imaging

Otmar Scherzer

Denoising is an important preprocessing step in many applications such as pattern recognition, feature extraction, and segmentation.

In this talk we give an overview on diffusion filtering techniques and variational principles for denoising and deblurring image data. Especially we emphasize on some recent trends in nonlinear, non differentiable, as well as nonconvex variational data smoothing principles.

For more background on Diffusion filtering methods we refer to Weickert [16]. For the relation to variational principles see [14, 13].

Nonconvex data smoothing principles are derived from statistical considerations, sampling, and multiplicative error noise models (cf. [6, 3]).

Recently Y. Meyer [8] introduced the G-norm as a similarity measure for oscillating patterns. Vese & Osher [15] and Aujol et al [1] implemented this ideas for noise removal applications. This novel regularization techniques will be reviewed as well and a relation to a statistical data smoothing method, the taut-string algorithm (cf. Mammen & Geer [7] and Davies & Kovac [2]). This requires the concept of tube methods, which has been developed by Hinterberger et al. [5]. Y. Meyer also gave a characterization of minimizers of the Rudin-Osher-Fatemi functional [10, 12] functional in terms of the G-norm. This result has been generalized to various other statistical nonparametric regression models. This is joint work with S. Osher (UCLA) and A. Obereder [11, 9].

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New Components in Geomagnetic Field Modelling – Wavelet-Mie-Representation and Wavelet-Variances

Thorsten Maier

Dealing with satellite measurements of the geomagnetic field b encounters the difficulty that the field is sampled within a magnetic source region, i.e. in an environment with non-vanishing electric current densities j. Consequently, assuming the quasi-static approximation of Maxwell's equations, data of low-orbiting satellites do usually not meet the prerequisites for the classical Gauss representation of the magnetic field as the gradient of a scalar harmonic potential. The resolution of the magnetic field and the electric currents by means of the so-called Mie representation for solenoidal vector fields is an adequate replacement of the Gauss approach [1, 2, 8, 11]. A vector field f on an open subset $\mathcal{U} \subset \mathbb{R}^3$ is called solenoidal if and only if the integral $\int_S f(x) \cdot \nu(x) d\omega(x)$ vanishes for every closed surface S lying entirely in U (ν denotes the outward normal of S). Every such solenoidal vector field admits a representation in terms of two (uniquely defined) scalar functions P_f, Q_f, with vanishing zero order moment, such that:

$$f = \nabla \wedge LP_f + LQ_f. \tag{34}$$

with the operator L given by $L_x = x \wedge \nabla_x$. Equation (34) is known as the Mie representation of f; $\nabla \wedge LP_f$ and LQ_f are called the poloidal and toroidal part of f, respectively. As far as geomagnetism is concerned, the magnetic field as well as the electric currents are both solenoidal such that the Mie representation can be used. This is advantageous since this representation can equally be applied in regions of vanishing as well as non-vanishing electric current densities. It turns out (e.g. [1]) that the poloidal fields are due to toroidal current densities below and above the satellite's track, whereas the toroidal fields are created by the radial currents which are crossing the satellite's orbit.

There remains the question of how to numerically obtain the Mie representation of a given set of data. A common approach (e.g. [1, 2, 5, 10] is based on expansions of the poloidal and toroidal scalars in terms of spherical harmonics. On the one hand, this approach is advantageous since it admits the possibility to incorporate radial dependencies of magnetic fields and electric currents in a natural way. On the other hand, the global support of the spherical harmonics limits the practicability of this technique since it cannot cope with electric currents (and corresponding magnetic effects) that vary rapidly with latitude or longitude, or that are confined to certain regions. In fact, Backus [1] states that it might be advantageous to find a field parametrization in terms of functions that take efficient account of the specific concentration of the current densities in space. Instead of using spherical harmonics we present so-called Wavelet-Mie-Representations (see [9]) in terms of *space localizing* vectorial scaling functions $\varphi_J^{(i)}$ and wavelets $\psi_J^{(i)}$

of scale J, which are kernels of the form

$$k_J^{(i)}(\xi,\eta) = \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1}(k_J^{(i)})^\wedge(n)Y_{n,k}(\xi)y_{n,k}^{(i)}(\eta), \quad \xi,\eta\in\Omega.$$

Ω represents the unit sphere, $\{Y_{n,k}\}$ denotes a system of scalar spherical harmonics, while $\{y_{n,k}^{(i)}\}$, $i=1,2,3$ denotes a system of vector spherical harmonics which are – in accordance to the Helmholtz-theorem for spherical vector fields (cf. [6]) – radial, surface curl free and surface divergence free, respectively. The properties of the real sequence $(k_J^{(i)})^\wedge(n)$ define whether the kernel is a scaling function or a wavelet (e.g. [3, 4]). Using such ansatz functions, the toroidal field, for example, admits a natural representation in terms of type 3 kernels, i.e. given the magnetic field b at some (roughly constant) altitude, we can approximate the toroidal mode b_{tor} via a series expansion in terms of certain convolutions '$\star$' and '$*$' with vectorial wavelets:

$$b_{tor}(r\xi) \simeq \left(\sum_{J=0}^{J_{max}} \psi_J^{(3)} \star \left(\psi_J^{(3)} * b\right)(r)\right)(\xi), \tag{35}$$

with suitably chosen maximum scale J_{max}. Figure 2 (left) shows the toroidal field as calculated via a wavelet-Mie-representation from a given set of vectorial MAGSAT data (see [9] and the references therein). The typical ribbon-like structures in the polar as well as the equatorial regions are clearly visible. From a physical point of view it is also interesting to know the spatial energy distribution of the toroidal field, since this hints at the geometry of the respective field sources, i.e. the electric radial current densities. It is reasonable to formulate energy measures that suit the field representation which, in our case, means that instead of the well-known degree-variances we should use wavelet-based measures. In [7, 9] we have introduced the scale and position variance of a vector field as

$$Var_{J;\eta}^{(i)}(f) = \int_\Omega\int_\Omega \left(\psi_J^{(i)}(\xi,\eta)\otimes\psi_J^{(i)}(\zeta,\eta)\right)\cdot\left(f(\xi)\otimes f(\zeta)\right)d\omega(\xi)d\omega(\zeta). \tag{36}$$

Since $||f||^2_{l^2(\Omega)} = \sum_J\sum_i\int_\Omega Var_{J,\eta}^{(i)}(f)\,d\omega(\eta)$ the scale and position variances can be interpreted as spatially localized measures for the energy contained in the signal. Figure 2 (right) shows $Var_{J;\eta}^{(i)}(b_{tor})$ integrated along bands of constant latitude. As one can see, the energy is mainly concentrated in the polar regions (scales 4 to 6), while at scales 4 and 5 there is some small concentration of energy in the equatorial region. These results strongly indicate that there are large radial electric currents confined to the vicinity of the poles, while there are some weaker radial currents confined to the equatorial region.

Applying the wavelet-Mie-representation to the electric currents as well, we end up with a expansion of these radial currents using the wavelet-coefficients of the toroidal magnetic field:

$$j_{rad}(r\xi) \simeq \frac{1}{r}\left(\sum_{J=0}^{J_{max}} \tilde{\psi}_J^{(1)} * \left(\psi_J^{(3)} * b\right)(r)\right)(\xi),$$

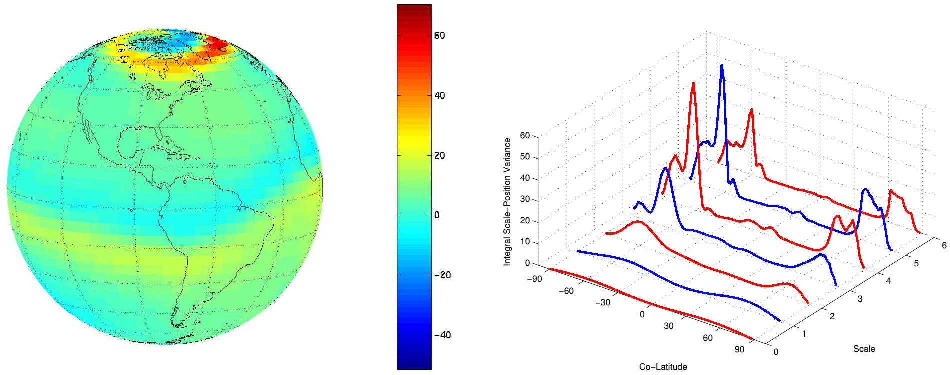

FIGURE 2. Left: East-west component of toroidal field from MAGSAT data, calculated via Wavelet-Mie-Representation [nT]. Right: Integrated scale and position variances of the toroidal field from MAGSAT data.

where $\tilde{\psi}_J^{(1)}$ is a special wavelet related to the Beltrami differential equation. Figure 3 shows the radial current densities calculated from the toroidal field in Figure 2 (left). The largest radial current densities ($|J_{rad}| \lesssim 150$ nA/m^2) are present in the polar regions. In agreement with the results in [10] the main current flow in the polar cap is directed into the ionosphere ($J_{rad} > 0$) during evening. At the poleward boundary of the polar oval the currents flow out of the ionosphere while the main current direction is into the ionosphere at the equatorward boundary. At the magnetic dip equator one realizes comparatively weak upward currents ($|J_{rad}| \lesssim 25$ nA/m^2) accompanied by even weaker downward currents at low latitudes. These current distributions are the radial components of the so-called meridional current system of the equatorial electrojet.

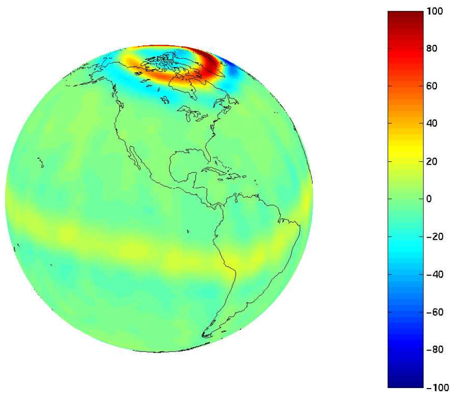

FIGURE 3. Radial Current Distribution from MAGSAT data, calculated via Wavelet-Mie-Representation $[nA/m^2]$

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El Niño and La Niña

Erwin Groten

El Niño is one of those phenomena where the mathematical detailed treatment still needs to overcome a variety of difficulties. With respect to the associated pressure and temperature variations, we even do not know whether El Niño is the source or simply the consequence of observed associated phenomena. Present mathematical treatment in terms of simplified Navier-Stokes equations is far from satisfying. It is by no means clear why a quasi-periodic phenomenon, such as El Niño, appears with a period of about 7 years around Christmas (=El Niño) at the Peruvian coast. Also unknown is its relationship to La Niña, which is now known since about 30 years. Attempts were made to explain the initial energy of the phenomenon by seismic events at the ocean bottom. The energy exchange between atmosphere and ocean (in one or the opposite direction) cannot be handled by traditional oceanic or meteorological equation systems in a satisfactory way. The phenomenon is clearly ill-posed, as minor input variations obviously generate substantial output changes. To explain El Niño as a subset of the global warming problem, as is often done to identify the origin of recently increased El Niño frequencies and strong variations in its intensity, is by no means justified.

Using precise geodetic satellite altimetry and ocean temperature and air pressure variation observations together with in-situ measurements and surveys led to a lot of clarification. But still the complicated non-linear problems of El Niño have not yet been solved. Mainly a mathematical theory is lacking which would include the sophisticated processes along the ocean-atmosphere boundary. It seems that, moreover, the superposition of several ocean circulation processes, not only within the Pacific Ocean, may have an influence on the variable El Niño phenomenon.

Influences may incorporate processes down to the ocean bottom where pressure variations may play a role. Coherence and cross-correlation studies, using pressure, temperature and El Niño-associated mean sea level variation data indicate such interrelations with deeper parts of the ocean. Also PCA- (=Prime Component analysis) studies carried out by us indicate such possibilities; for details see [1, 2].

Still the main problem is the "hen-egg" problem, where we wonder whether the hen or the egg was first. By considering the significant pressure, temperature and sea level variations it still remains open from where the energy transfer originates. It is clear that El Niño is not simply a steric phenomenon where sea level is going up in a static way as a consequence of temperature rise and expansion of the water. Nevertheless, this phenomenon plays a role as is demonstrated from low cross-covariances between polar motion (and related changes of moments of inertia of the Earth) and El Niño.

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Inverse Eigenvalue Problems: The Quadratic and Singular Cases

William Rundell

We consider two generalisations of the classical inverse Sturm-Liouville-problem. First, when a singular term of the form $l(l+1)/x^2$ is included and second, when there is a quadratic dependence on the eigenvalue parameter. We show local uniqueness, that certain spectral sets can uniquely determine the unknown potential(s), provided they are in a sufficiently small ball around the origin. Our motivation for these problems was provided by helioseismological applications – determining the sound speed and density of the interior of the sun. There are clearly equivalent geophysical issues, and indeed the singular term arises in any situation where we consider radially-symmetric solutions of the 3-d wave equation.

Topological Spaces of Harmonic Functions and Geodetic Boundary-value Problems

Fausto Sacerdote and Fernando Sanso

The theory of boundary-value elliptic problems in Hilbert spaces has been extensively illustrated some decades ago by J.L.Lions and E.Magenes for very general differential operators, with coefficients, right-hand sides of the equations and boundary conditions belonging to irregular function or distribution spaces; consequently solutions too are defined in some generalized sense and belong in general to distribution spaces. In Laplace equation, on the contrary, with constant coefficients and zero right-hand side, maximal regularity properties are met inside the domain of harmonicity. It is therefore interesting, along with the usual regularization procedures of the general theory, to develop an autonomous scheme that, making use of these regularity properties, allows to define a general topological structure for the space of the solutions of the Laplace equation in an open set. It is shown that they can be classified according to the regularity of their boundary conditions, formulating suitable trace theorems. New Hilbert spaces of harmonic functions are then defined, which are different and in a sense complementary to the spaces described by Lions and Magenes. The results can be very simply proved and illustrated in the case of a spherical boundary, for which it is possible to use explicit spherical harmonic representations, but can be generalized to the case of an arbitrary regular boundary. As a matter of fact one can see that the space $\mathbf{H}(\Omega)$ of all the harmonic functions in an open simply connected smooth set Ω (internal or external to its boundary, that is assumed to be a bounded connected surface) can be endowed with a topological structure of Fréchet space, and that its dual space can be represented itself by a space of harmonic functions, with a coupling that can be expressed as an integral over a surface internal to the harmonicity domain arbitrarily close to the boundary. Making use of this result, it is possible to formulate trace theorems for $\mathbf{H}(\Omega)$, and consequently to give a meaning and to obtain existence and uniqueness results for the Dirichlet problem in this space. These results can be extended to Neumann and oblique derivative problems, in a comprehensive theory of boundary-value problems for the Laplace operator.

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Mathematical Methods in Oceanography – Approximation Methods in Ocean Modeling –

Dominik Michel

In recent years, climatic phenomena have occured with increasing number and intensity, hitting humanity frequently and causing damage with serious consequences. This gives reason to demand a better understanding of the system Earth. The oceans, used for example as resource spots, for further industrial purpose or just for recreation, cover two thirds of the Earth's surface. They have a great influence on the weather and the climate itself. Possible shifts of the Gulf stream for example are often assumed to have a huge impact on the European climate. Thus, understanding the climate depends essentially on a better unterstanding and accurate modeling of the complex system ocean.

Fortunately, the Earth is not yet a water planet, i.e. the oceans do not cover the whole sphere. But unfortunately, this implies the handling of a bounded region on a spherical domain. Common approximation methods for scalar functions on the sphere Ω use Fourier expansions of spherical harmonics $\{Y_{n,k}\}_{n\in\mathbb{N}_0,k=1,\dots,2n+1}$, being an orthonormal basis within the space of all square-integrable spherical functions $\mathcal{L}^2(\Omega)$. I.e. for all $F \in \mathcal{L}^2(\Omega)$ we have

$$F = \lim_{N\to\infty} \sum_{n=0}^{N} (F, Y_{n,k})_{\mathcal{L}^2(\Omega)} Y_{n,k} \quad ,$$

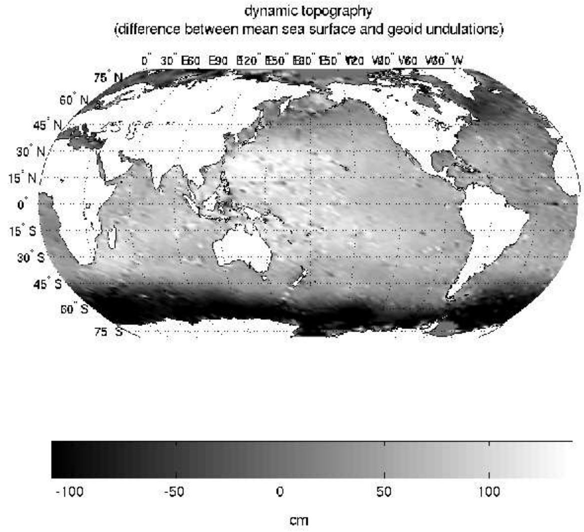
where the convergence is understood in the $\mathcal{L}^2(\Omega)$-sense. But since these basis functions, being homogeneous harmonic polynomials restricted to the sphere, have a global support, they are not well-suited for the problem under consideration. Thus, new methods of constructive approximation have to be used, introducing on the one hand radial basis functions with locally compact support, for example the so-called Haar function

$$L_h^{(k)}(t) = \begin{cases} 0\,, \text{if } t \in [-1,h) \\ \frac{k+1}{2\pi}\frac{(t-h)^k}{(1-h)^{k+1}}\,, \text{if } t \in [h,1] \end{cases} \quad ,$$

and leaving on the other hand the spectral ansatz of such a Fourier approach by spherical harmonics. In the last two decades, spherical spline- and wavelet-techniques have been developed by the Geomathematics Group in Kaiserslautern (see e.g. [2]). For this task, lately applied space localizing basis functions will be combined with a spherical wavelet approach. Equipped with a numerical integration rule to discretize the $\mathcal{L}^2$-scalar product this results in an analogous series expansion for $F \in \mathcal{L}^2(\Omega)$, i.e.

$$F = \lim_{\substack{h\to 1,\\ h<1}} \sum_{i=1}^{N_h} c_i^h(F)\; L_h^{(k)}\left(\cdot\eta_i^h\right) \quad ,$$

converging also in the $\mathcal{L}^2(\Omega)$-sense. Besides the spatial part of the position-based function system $\{L_h^{(k)}\left(\cdot\eta_i^h\right)\}_{i=1,\dots,N_h}$ one may introduce a multiscale approximation by discretizing h into $h_J = 1 - 2^{-J}$, $J \in \mathbb{N}_0$. These scales provide low-pass-filtered versions of the function under consideration, separating between high- and low-frequent phenomena. In addition to the non-spectral ansatz, this method recognizes local structures with more detail than global methods and is able to include position based error models into its spatial approach. Whereas the spherical harmonics contain a global averaging of the data, which is certainly helpful for the representation of a global trend, the Haar scaling function is only averaging the data within a small subset of the domain due to its local support, which is advantageous with respect to efficiency of the computation and local evaluation and adaptation of the model.



As an application, the modeling of the mean dynamic topography and the geostrophic flow was chosen. The system of differential equations designed for fluid dynamics is canonically the system of Navier-Stokes equations. Under the assumptions of having stationary flow, frictionless motion (i.e. being away from coasts, ocean surfaces and bed), considering only an homogeneous and incrompressible fluid and neglecting turbulent flows and vertical velocities, the large-scale currents of oceanic circulations can be approximated by the so-called geostrophic flow (see e.g. [1, 5]).
The equation of motion reduces to the balance between the Coriolis force and the horizontal pressure gradient, such that this first approximation of oceanic currents depends directly on the mean dynamic topography. For an ocean at rest, the geoid (equipotential surface of the gravitational potential) would be equal to the sea surface height. Altimetric measurements, for example from current, past and future satellite missions like CHAMP and GRACE but also ERS and TOPEX/POSEIDON and the oncoming GOCE, provide the actual sea surface height. The difference between this sea surface height and the geoid (the height it should have at rest) is the so-called dynamic topography. Averaged over several years we obtain the mean dynamic topography. As data sets the French CLS01 data are used for the mean sea surface topography and are compared to the EGM96 geoid. Here, this will be used to model major parts of the Gulf stream by geostrophic assumptions. The approximation at scale eight is presented below. Model errors around islands, here the Canary islands, occur in higher scales, being high-frequent phenomena. Although located close to the boundary, they are smoothed out by this low-pass filter.

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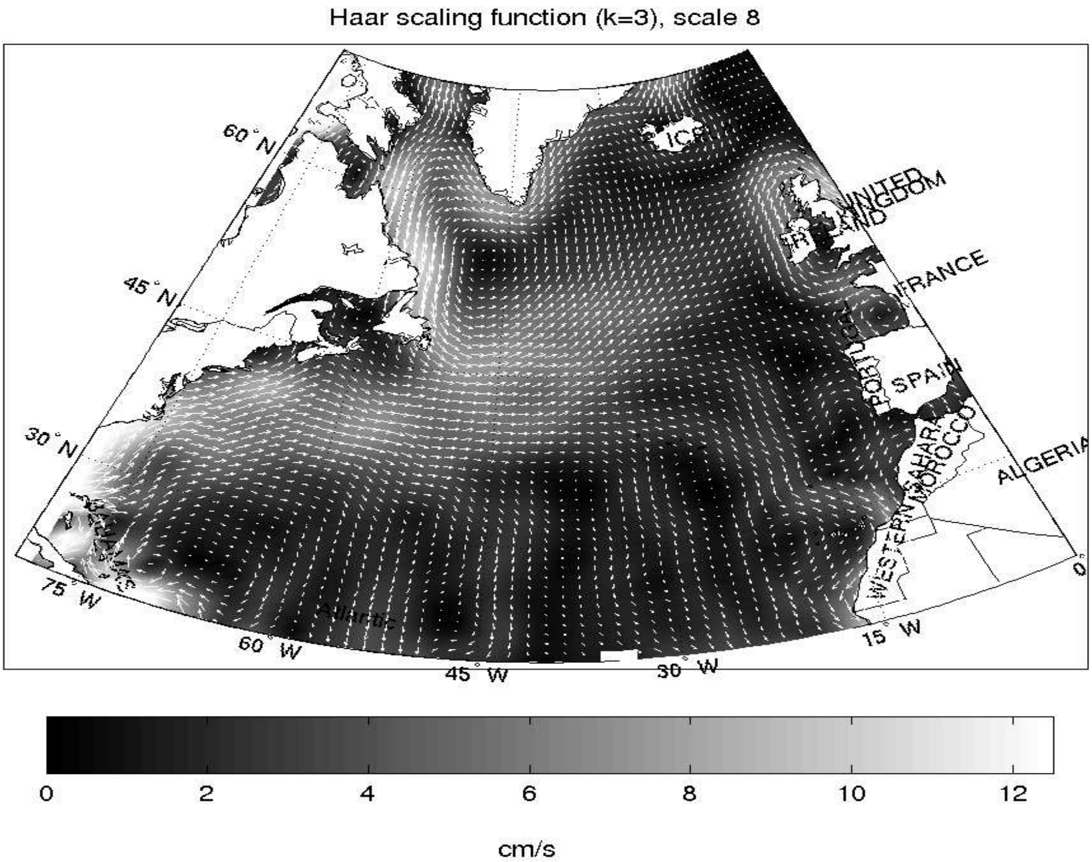

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A Fast Nonlinear Galerkin Scheme Involving Vector and Tensor Spherical Harmonics for Solving the Incompressible Navier-Stokes Equation on the Sphere

Martin J. Fengler

The full set of coupled partial differential equations for forecasting an tangential incompressible atmospherical flow can be reduced in the weak sense to

$$
\begin{aligned}
&(37) \qquad & \frac{\partial u}{\partial t} &= -(u \cdot \nabla^*)u - 2\omega \wedge u + \nu \Delta^* u + f \\
&(38) \qquad & \nabla^* \cdot u &= 0 \\
& & u(0) &= u_0,
\end{aligned}
$$

where u denotes the velocity field of the considered flow, and ω the rotational axes coinciding with the z-axis of an Earth-fixed reference frame. Moreover, we consider an inhomogeneous flow, by letting f be a time depending external flow driving force. For more details on the considered function spaces, operators and further conditions we refer to [4]. Existence and uniqueness of a generalized weak

solution of 37 is provided by [7].
Freeden [6] introduces orthonormal type 3 vector spherical harmonics given by $y^{(3)}_{n,k}(\eta) = \frac{1}{\sqrt{n(n+1)}}(\eta \wedge \nabla^*_\eta)Y_{n,k}(\eta)$, which satisfy Eq. (38). The span of all tangential and surface divergence free vector spherical harmonics up to degree N forms the finite dimensional Hilbert space $\text{harm}^{(3)}_{1,\ldots,N}$, equipped with the canonical inner product on $l^2(\Omega)$. The nonlinear Galerkin approximation

$$u_N = \sum_{n=1}^{N} \sum_{k=-n}^{n} u_{n,k} y^{(3)}_{n,k}$$

is obtained by solving

$$\begin{aligned}
\frac{\partial}{\partial t}\langle u_N, v\rangle_{l^2(\Omega)} - \nu\langle \Delta^* u_N, v\rangle_{l^2(\Omega)} + 2\langle \omega \wedge u_N, v\rangle_{l^2(\Omega)} + \langle u_N \cdot \nabla^* u_N, v\rangle_{l^2(\Omega)} \\
+ \langle z_N \cdot \nabla^* u_N, v\rangle_{l^2(\Omega)} + \langle u_N \cdot \nabla^* z_N, v\rangle_{l^2(\Omega)} = \langle f, v\rangle_{l^2(\Omega)}, \\
-\nu\langle \Delta^* z_N, \tilde{v}\rangle_{l^2(\Omega)} + 2\langle \omega \wedge z_N, \tilde{v}\rangle_{l^2(\Omega)} + \langle u_N \cdot \nabla^* u_N, \tilde{v}\rangle_{l^2(\Omega)} = \langle f, \tilde{v}\rangle_{l^2(\Omega)},
\end{aligned}$$

for all $v \in \text{harm}^{(3)}_{1,\ldots,N}$ and $\tilde{v} \in \text{harm}^{(3)}_{N+1,\ldots,2N}$, together with

$$u_N(0) = u_0|_{\text{harm}^{(3)}_{1,\ldots,N}},$$

where $z_N = \sum_{n=N+1}^{2N} \sum_{k=-n}^{n} z_{n,k} y^{(3)}_{n,k}$ can be interpreted as a high-frequent perturbation of the flow. Above noted nonlinear Galerkin scheme yields an ordinary differential equation in u_N, which obeys a unique solution with domain of convergence $[0, +\infty)$. Moreover, a proof of convergence for the limit $N \to +\infty$, such that $u_N \to u$ converges in a strong topology is given by [5, 8]. The arising coupling terms of vector and tensor spherical harmonics can be stated explicitly in terms of Wigner-3j coefficients.

Theorem: Calculation of the Coriolis Term:
Let be $k, l, n, j, r, s \in \mathbb{N}$ with $k > 0, r > 0$, and $n > 0$. Then,

$$\int_\Omega (\omega \wedge y^{(3)}_{k,l}(\eta)) \cdot y^{(3)*}_{n,j}(\eta) dS(\eta) = |\omega| i \frac{-l}{n(n+1)} \delta_{kn}\delta_{lj} \tag{39}$$

Theorem: Calculation of the Advection Term:
Let be $k, l, n, j, r, s \in \mathbb{N}$ with $k > 0, r > 0$, and $n > 0$. Then,

$$\int_\Omega [(y^{(3)}_{k,l}(\eta) \cdot \nabla^*_\eta) y^{(3)}_{r,s}(\eta)] \cdot y^{(3)*}_{n,j}(\eta) dS(\eta) = T(n, j, r, s, k, l) \tag{40}$$

with

$$\begin{aligned}
T(n,j,r,s,k,l) &= (-1)^{j+1} i \frac{1}{\sqrt{4\pi}} \frac{n(n+1) + r(r+1) - k(k+1)}{4\sqrt{r(r+1)n(n+1)k(k+1)}} \\
&\quad \times \begin{pmatrix} r & n & k \\ s & -j & l \end{pmatrix} \sqrt{(2r+1)(2n+1)(2k+1)} \begin{pmatrix} r-1 & n & k \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad \times \sqrt{(n+k+r+1)(k+r-n)(n-k+r)(n+k-r+1)}.
\end{aligned}$$

Although one can exploit Wigner-$3j$ selection rules, a full spectral code based on the results of Eq. (39) and Eq. (40) scales with $O(N^5)$, if N denotes the maximal spherical harmonic degree to be resolved. The reason is obviously the "quadratic" (nonlinear) advection term.
To improve algorithmic efficiency we write for the advection $(u\cdot\nabla^*)u = (\nabla^*\otimes u)^T u$, and generalize the concept of pseudo spectral algorithms to vector spherical and tensor spherical harmonics of type defined by [6]. The idea is to express the velocity field u as well as the tensorial part $(\nabla^*\otimes u)^T$ in terms of a local coordinate system. In doing so, we can separate latitudinal scalar quantities from longitudinal terms. This looks, for example, like

$$\mathbf{u}^{(2,3)} = {}^{\varepsilon^\phi\otimes\varepsilon^\phi}u^{(2,3)}\varepsilon^\phi\otimes\varepsilon^\phi + {}^{\varepsilon^\phi\otimes\varepsilon^t}u^{(2,3)}\varepsilon^\phi\otimes\varepsilon^t + {}^{\varepsilon^t\otimes\varepsilon^\phi}u^{(2,3)}\varepsilon^t\otimes\varepsilon^\phi + {}^{\varepsilon^t\otimes\frac{\partial}{\partial t}\varepsilon^t}u^{(2,3)}\varepsilon^t\otimes\frac{\partial}{\partial t}\varepsilon^t + {}^{\varepsilon^\phi\otimes\frac{\partial}{\partial\phi}\varepsilon^\phi}u^{(2,3)}\varepsilon^\phi\otimes\frac{\partial}{\partial\phi}\varepsilon^\phi + {}^{\varepsilon^t\otimes\varepsilon^t}u^{(2,3)}\varepsilon^t\otimes\varepsilon^t.$$

The latter representation allows us to apply modified Gauss-Legendre transforms and subsequent FFTs [1, 9] to reconstruct the vector and tensor on a Gauss-Legendre integration grid in the space domain. There we expand the scalar quantities to vector, resp. tensors, compute in the space domain the integrand of the advection term. By performing subsequently a scalar reduction of the considered vector field, we reconstruct the type 3 Fourier coefficients by FFTs and modified inverse Gauss-Legendre transforms. It is interesting to note, that this technique is also able to separate vector fields of mixed type since it is based on a polynomial exact integration. Moreover our proposed algorithm profits from the semi-linear scaling of the FFTs. The Gauss-Legendre transforms, which are realized by multiple matrix-vector products yields finally an overall $O(N^3)$ method. An extensive numerical realization has been also presented, see Fig. 4 and [4]. It is interesting to note that we observe a power-law decay given by N^{-4} in the inertial range, and an exponential decay in the dissipative range of the energy spectrum illustrated in Fig. 5. This has been also reported by Debussche et al. [2] in a two-dimensional periodic domain. A detailed description of our proposed method, and the results noted above can be found in [5] and [4], whereas the latter includes also an extension to spherical vector wavelets.

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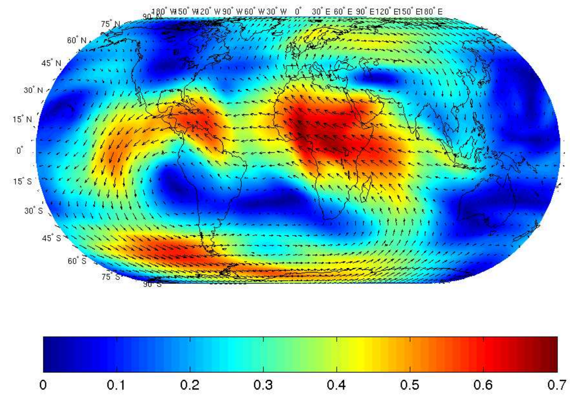

FIGURE 4. Flow at t=20.

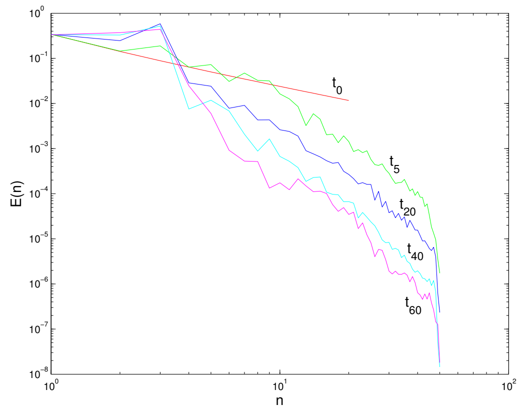

FIGURE 5. Energy Spectrum.

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Best conformal map projections

JESÚS OTERO

Most of the large-scale national topographic maps (what entails a division of the country in a great number of small rectangular regions) are based on conformal map projections, specifically on the transverse Mercator or Gauss-Krüger projection. The reason is cleverly pointed out by C.F.Gauss who on 11 December, 1825, in a letter to Hansen, writes: "You are quite right that the essential condition in every map-projection is the **infinitesimal similarity**, a condition which should be neglected only in very special cases of need" (see [2]). In fact, if $p : \Omega \to \mathbb{R}^2$ is a conformal map projection defined on an open subset Ω of the terrestrial ellipsoid Σ (an oblate ellipsoid of revolution), then p has a well defined *infinitesimal-scale* $\sigma(x)$ at each point x of Ω,

$$\sigma(x) = \lim_{y \to x} \frac{|p(y) - p(x)|}{d_\Sigma(x, y)}$$

where $d_\Sigma(x, y)$ denote the geodesic distance between the two points x and y (see [6]). It is therefore both of practical and theoretical interest to find a "best possible" conformal map projection.

Let $(u,v) \in \Omega_b = p_b(\Omega)$ denote the rectangular coordinates of the projected points according to some map projection p_b on Ω. Using (u,v) as coordinates on Σ, then $ds2 = Edu^2 + 2Fdudv + Gdv^2$ and the logarithm of the infinitesimal-scale function associated with a conformal map projection p on Ω satisfies the partial differential equation

$$L(\log\sigma) := \mathrm{Div}(\mathbf{A}\nabla\log\sigma) = HK > 0 \quad \text{in } \Omega_b. \tag{41}$$

Here K is the Gaussian curvature of Σ and

$$\mathbf{A} = \frac{1}{H}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix}, \quad H = \sqrt{EG - F^2}.$$

We shall refer to p_b as the *base projection* and to Ω_b as a *first map* of Ω (compare [4]). The problem of finding a "best" conformal map projection on Ω has two parts and this report is partly devoted to the first of them: (i) Find an element $g_0 \in M := \{g \in C^2(\Omega_b) : Lg = HK \text{ in } \Omega_b\}$ that "deviates least from zero", or in general from a constant function. (ii) Find, if possible, a conformal map projection p_0 on Ω such that $\sigma_0 = \exp(g_0)$. If the base projection is conformal, then $E = G = \sigma_b^{-2}$ and $F = 0$. Hence $A = I_2$ and the partial differential equation (41) becomes $\Delta\log\sigma = \sigma_b^{-2}K = \Delta\log\sigma_b$. The function $U = \log\big(\sigma/\sigma_b\big)$ is harmonic in Ω_b, and the first question can be reformulated as a problem of best harmonic approximation: Find a harmonic function U_0 in Ω_b that "deviates least from the *superharmonic* function $h := \log\sigma_b^{-1}$".

Chebyshev ([1]) studied conformal map projections, using the oscillation of the function $g := \log\sigma$

$$\delta(g) := \sup_{\Omega_b} g - \inf_{\Omega_b} g.$$

as a measure of distortion.

Theorem 3 (Chebyshev-Milnor)**.** Let $g_0 \in M$ be such that $g_0 = 0$ on $\partial\Omega_b$. Then, $\delta(g_0) \le \delta(g)$ for all $g \in M$, where equality holds if and only if $g = g_0 + \textit{constant}$.

In words of Hill (see [5]) this theorem says (*Chebyshev's principle*): "that conformal map is best in which the scale is constant along its boundary". The proof of Theorem 3 is based on the strong maximum principle (see [6]). The arbitrary constant can be fixed using the uniform norm: the function $g_0 + c_0$, where $c_0 = \delta(g_0)/2$ is the best *uniform* approximant to the zero function from M, that is

$$\sup_{\Omega_b} |g_0 + c_0| \le \sup_{\Omega_b} |g| ,$$

for all $g \in M$. We note that $\delta_0(\Omega) := \delta(g_0) = -\inf_{\Omega_b} g_0$ (*minimum conformal distortion* associated with Ω). If the base projection is conformal, we have alternately that the solution of the Dirichlet boundary problem

$$\Delta U_0 = 0 \quad \text{in } \Omega_b\,, \quad U_0 = h \quad \text{on } \partial\Omega_b, \tag{42}$$

is a best harmonic Chebyshev approximant to h, that is $\delta(U_0 - h) \le \delta(U - h)$ for all U harmonic in Ω_b, and the equality holds if and only if $U = U_0 + \textit{constant}$ (see [7]). This is the original approach followed by Chebyshev with the Mercator projection as base projection, and therefore $h(v) = \log(\operatorname{sech} v)$ if the surface is a sphere where v is the *isometric* latitude.

Some examples to illustrate these ideas.

(i) Region bounded by two parallels. We have the following.

Theorem 4 ([8]). The best Chebyshev conformal map projections for a region bounded by two parallels are conformal conic projections.

To prove this theorem it is convenient to choose as base projection any azimuthal conformal projection. Then Ω_b is a plane annulus and $h = \log \sigma_b^{-1}$ depends only on $r = (u^2+v^2)^{1/2}$. The solution of the Dirichlet problem (42) is $U_0(r) = a_0 \log r + a_1$ which corresponds to a conformal conic projection. This is a beautiful example of a case where the best Chebyshev map projection "although locally well behaved, may not be one-to-one in the large" (see [6, p.1112]).

(ii) Rectangular regions (see [5]). Let Ω be the region of the ellipsoid contained between portions of two parallels and two meridians. If the base projection is the Mercator projection, and we count the longitudes from the middle meridian, the first map of Ω is the rectangular region $\Omega_b = [-u_0, u_0] \times [v_s, v_n]$, where $\pm u_0$ are the limiting values of the longitude, and v_s, v_n are the lower and upper limits of the isometric latitude v. The following particular case is considered: $u_0 = 5^o$, $\varphi_s = 40^o$ and $\varphi_n = 50^o$, where φ denotes geodetic latitude. For a sphere, we have solved using Matlab (Finite Element Method) the boundary value problem

$$\Delta g_0 = \Delta \log(\cosh v) = \operatorname{sech}^2 v \quad \text{in } \Omega_b\,, \quad g_0 = 0 \quad \text{on } \partial\Omega_b.$$

We get $\delta_0(\Omega) = 1.4822 \times 10^{-3}$. For the transverse Mercator projection and the best Chebyshev conformal conic projection for this region we have respectively: $\delta_{tm}(\Omega) = 2.2338 \times 10^{-3}$, $\delta_l = 3.8158 \times 10^{-3}$.

(ii) Regions bounded by two meridians. F.Eisenlohr (see [3] and [9]) regards as a measure of the accuracy of a conformal map the distortion of the geodesic lines of the surface. If the base projection is conformal, this leads to the following problem: Find a harmonic function U_0 in Ω_b such that

$$\|\nabla(U_0 - h)\|_{L^2(\Omega_b)} \leq \|\nabla(U - h)\|_{L^2(\Omega_b)} \tag{43}$$

for all U harmonic in Ω_b. Eisenlohr proves that if U_0 satisfies (43) then $U_0 = h + \textit{constant}$ on $\partial\Omega_b$ (and then $\log \sigma_0$ is constant on the boundary!). Later Whittemore ([9]) specifies that if Ω_b is simply connected this statement is right. In addition, Eisenlohr considers the case of a region bounded by two meridians. Under Mercator projection, this region is transformed into the strip $S(b) = [-b/2, b/2] \times \mathbb{R}$ where $b \in (0,2\pi]$. For a sphere ($h = \log(\operatorname{sech} v)$), Eisenlohr states that the function $U_0 = \operatorname{Re}(w(z))$, where $z = u + iv$ and

$$w(z) = -\frac{1}{b} \int_{-\infty}^{\infty} \operatorname{sech}\big[(iz + t)\frac{\pi}{b}\big] h(t)\, dt,$$

solves the Dirichlet problem $\Delta U_0 = 0$ in $S(b)$ and $U_0 = h$ on $\partial S(b)$.

Conjecture: U_0 is a best Chebyshev harmonic approximant to h in the unbounded domain $S(b)$. We note that if $b = \pi$, then (see [3, p.148])

$$w(z) = -2\log\bigl(\cosh\frac{iz}{2}\bigr) \Rightarrow U_0 = \log\Bigl(\frac{2}{\cos u + \cosh v}\Bigr).$$

This function U_0 is the harmonic function associated with the transverse stereographic projection. This projection is a best Chebyshev conformal map projection for $S(\pi)$.

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Dirichlet Forms and (Stochastic) Partial Differential Equations

Martin Grothaus

In the last decades the theory of Dirichlet forms was approved as a useful tool of modern mathematics. Applications can be found in research areas like Partial Differential Equations, Mathematical Physics (Quantum (Field) Theory, Statistical Physics), Stochastic (Partial) Differential Equations and Stochastic Analysis. In the first part of my talk I illustrated along elementary examples the concepts of this theory. E.g., Dirichlet forms corresponding to the heat equation or SDEs solved by Ornstein–Uhlenbeck processes I presented. Then I explained the general strategy for solving a given PDE or SPDE. Existence of the solutions and an analysis of their properties then follows along the monographs [2], [6]. In the second part I spoke about applications to, e.g., the Langevin equations of Statistical Mechanics or birth and death processes. Finally, I gave a review of my current research which is about limit theorems like scaling limits, hydrodynamic Al limits and finite dimensional approximations, see [3], [5], [1], [4]. For deriving and analyzing such limits Dirichlet forms turned out to be a very useful tool.

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Analysis on the Unit Sphere and on the Unit Ball

Yuan Xu

There is a close relation between the analytic structures on the unit sphere and on the unit ball. Indeed, the simple mapping of $x \mapsto (x, \sqrt{1-\|x\|^2})$ and the symmetry between upper and lower hemisphere turns out to preserves the orthogonality; more specifically, the orthogonality on the sphere with respect to a weight function $H(x)$ is equivalent to orthogonality on the unit ball with respect to a pair of weight functions, $H(x)/\sqrt{1-\|x\|^2}$ and $H(x)\sqrt{1-\|x\|^2}$, on the unit ball. This relation has lead to a compact formula for the reproducing kernel on the unit ball. It turns out that many results in approximation theory on the ball takes the same form as those on the sphere; this includes the definition of modulus of smoothness and K-functional, direct and inverse type theorems. Similar relations hold between cubature formulas and interpolation between these two domains.

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Oblique Boundary Value Problems and Domain Decomposition Methods

Martin Gutting

The exterior oblique boundary value problem on surfaces like the Earth's real surface demands approximation methods like interpolating harmonic splines to obtain a solution in the realistic case of discrete data. That solution fulfills the

Laplace equation in the exterior domain and fits the boundary values by construction. However, in order to obtain the essential spline coefficients large full systems of linear equations need to be solved. This is performed by splitting the domain corresponding to the data set into several subdomains where the splines can be calculated easily. These 'subsplines' have to be fit together nicely which is done iteratively by a Schwarz alternating algorithm.
For the mathematical formulation of the discrete boundary value problem we require Σ to be a regular surface, i.e Σ separates $\mathbb{R}^3$ into a bounded interior and an unbounded exterior region (where the interior part contains the origin), the surface is closed, it has no double points and possesses a local $C^{(2,\lambda)}$-parametrization. We are given N points $x_1, \ldots, x_N \in \Sigma$ and N directions $v_1, \ldots, v_N \in \mathbb{R}$ with $\|v_i\| = 1$ and $v_i \cdot n(x_i) > 0$ for $i = 1, \ldots, N$ where n denotes the normal vector field of Σ. As boundary values the discrete data $\frac{\partial U}{\partial v_i}(x_i)$ of the desired potential $U \in Pot^{(1,\lambda)}(\overline{\Sigma^{ext}})$, i.e. U is harmonic in the exterior space of Σ, $C^{(1,\lambda)}$ at the boundary and decays to zero at infinity. (More details can be found in [4], [3].)
The solution is approximated in a Sobolev space $\mathcal{H} = \mathcal{H}\left(\{A_n\}; \overline{\Omega_R^{ext}}\right)$ where Ω_R denotes an inner Runge sphere (also called Bjerhammer sphere) and the sequence of symbols A_n fulfills the following summability condition:

$$\sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \frac{2n+1}{4\pi} \frac{1}{A_n^2} < \infty.$$

The weighted outer harmonics $\frac{1}{A_n} H_{n,k}(R; \cdot)$, $n = 0, 1, \ldots,$ $k = 1, \ldots, 2n+1$ form a basis system of this Sobolev space where a convolution is defined by

$$F *_{\mathcal{H}} G = (F, G)_{\mathcal{H}} = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} A_n^2 F_R^{\wedge}(n,k) G_R^{\wedge}(n,k)$$

with the Fourier coefficients $F_R^{\wedge}(n,k)$ and $G_R^{\wedge}(n,k)$. Due to summability the space possesses a reproducing kernel $K_{\mathcal{H}}$ that takes the form (with $x, y \in \overline{\Omega_R^{ext}}$):

$$\begin{aligned} K_{\mathcal{H}}(x,y) &= \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \frac{1}{A_n} H_{n,k}(R;x) \frac{1}{A_n} H_{n,k}(R;y) \\ &= \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \frac{1}{A_n^2} \frac{2n+1}{4\pi R^2} \left(\frac{R^2}{|x||y|} \right)^{n+1} P_n \left(\frac{x}{|x|} \cdot \frac{y}{|y|} \right), \end{aligned}$$

where we applied the addition theorem and P_n stands for the Legendre polynomial of degree n. For the special choice $A_n = h^{-n/2}$ with $h \in (0,1)$ the repro-kernel becomes the Abel-Poisson kernel and can be expressed by an elementary function:

$$K_{\mathcal{H}}(x,y) = \frac{1}{4\pi} \frac{|x|^2|y|^2 - h^2R^4}{\left(|x|^2|y|^2 + h^2R^4 - 2hR^2(x \cdot y)\right)^{3/2}}$$

Therefore, it can be computed exactly (no truncation of the infinite series) and very quickly.
Due to Runge's theorem (cf. [6]) we can use such basis functions that possess a

larger domain of harmonicity in order to approximate the desired potential which is harmonic only outside Σ.

Splines relative to a set of bounded linear functionals $\mathcal{L}_i$ (like in this case the directional derivatives) are denoted by $S = \sum_{i=1}^{N} \alpha_i \mathcal{L}_i K_{\mathcal{H}}(\cdot,\cdot)$. They can be forced to fulfill the interpolation conditions $\mathcal{L}_i F = \mathcal{L}_i S^F$ by calculating the spline coefficients from the corresponding system of linear equations:

$$\sum_{i=1}^{N} \alpha_i \mathcal{L}_i \mathcal{L}_j K_{\mathcal{H}}(\cdot,\cdot) = \mathcal{L}_j F \quad \text{for } j = 1,\ldots,N.$$

More about the so-called $\mathcal{H}$-splines (further properties, convergence theorems) can be found e.g. in [2], [4] or [5]. The high space localization of the reproducing kernels allows to use these splines for local, i.e. only a part of the whole surface is considered, as well as for global problems.

Since these systems lead to large matrices an algorithm has to be found to split the spline into smaller pieces that can be computed easier. Those smaller 'subsplines' are then put together iteratively by a Multiplicative Schwarz Alternating Algorithm (MSAA): Splines that are already calculated are subtracted from the original right hand side and only the difference is interpolated. All splines are added to get the final approximation. The algorithm works as follows:

Algorithm (MSAA): *Let n count the number of iterations and let k be the number of subdomains. $r = 1,\ldots,k$ denotes the current subdomain. The starting values for the residual are the data: $F_0 = F$ and for the spline we take $S_0^F = 0$. The iteration is performed the following way:*

$$\begin{aligned} F_{nk+r} &= F_{nk+(r-1)} - P_r F_{nk+(r-1)} \\ S^F_{nk+r} &= S^F_{nk+(r-1)} + P_r F_{nk+(r-1)}, \end{aligned}$$

with projectors $P_r : \mathcal{H} \longrightarrow \mathcal{H}$, $G \mapsto P_r G = S_r^G$ which denote the spline interpolation within the subdomains.

For a more detailled description of the algorithm see [4] or [5]; in [1] the algorithm is formulated with regard to radial basis function interpolation. The iteration converges (cf. [4] or [5] for a proof):

$$\left(S^F_{nk}\right)_n \longrightarrow PF = S^F \quad \text{for } n \longrightarrow \infty.$$

In practice, the iteration is stopped when a prescibed accuracy is achieved. Since the set of interpolated functionals which is split for the MSAA is related to a set of points on Σ the splitting can be performed in such a way that it corresponds to the decomposition of the computational domain. In our implementation we go back to the sphere which we divide into polar caps and rectangles in the φ-θ-parameter plane. (For details see [4], [5].) As a numerical example we present a local calculation for Asia. The Terrainbase model is used to generate the Earth's surface and the directions are given by the direction of the gradients of the gravitational potential in the points. To reduce the boundary effects the evaluation area is chosen smaller than the area where data was given. The aim of this example was the reconstruction of a spherical harmonics model of degrees 16-200 (EGM96) where 21643 points in this local area have been given. The mean absolute error amounts to $0.308 m^2/s^2$. (Further local and global results as well as more details can be found in [4].) Furthermore, it should be noted that this technique can be

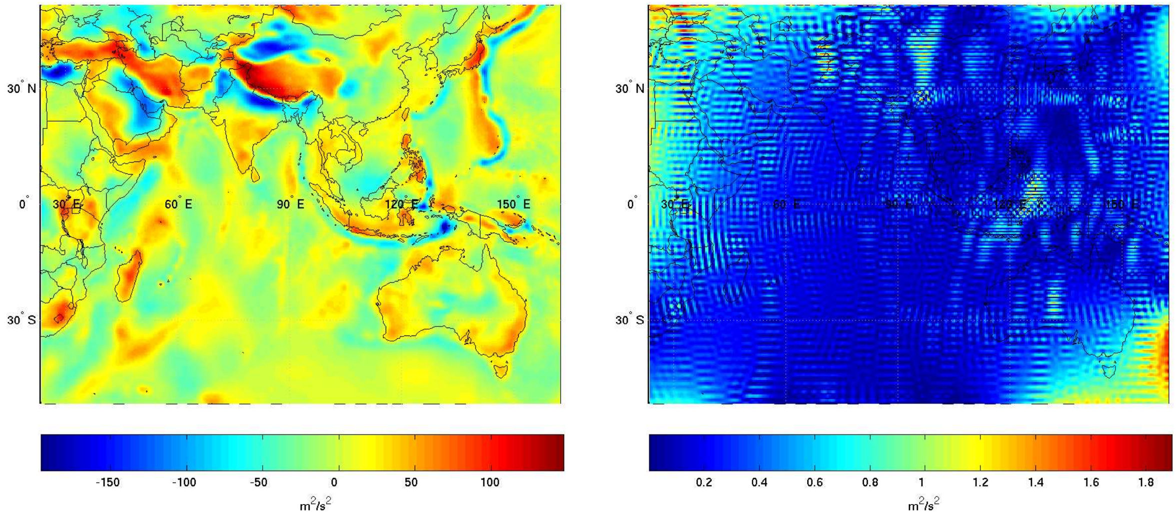

Figure 6. *The interpolating spline (left) and the absolute error (right).*

easily combined with $\mathcal{H}$-scaling functions and $\mathcal{H}$-wavelets (see [1] or [3]) to obtain a multiscale representation of the approximation (cf. [4] for an implementation).

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Determination of the Geopotential Field out of Oblique Derivatives, an Alternate Approach

Frank Bauer

In the field of gravity determination a special kind of boundary value problem respectively ill-posed satellite problem occurs; the data and hence side condition of our PDE are oblique derivatives of the gravitational potential.

In mathematical terms this means that our gravitational potential V fulfills $\Delta V = (\partial_1\partial_1 + \partial_2\partial_2 + \partial_3\partial_3)V = 0$ in the exterior space of the Earth Σ_{ext} and

$\mathscr{D}V = F$ on the discrete data location Σ_D which is on the Earth's surface Σ for terrestrial measurements and on a satellite track in Σ_{ext} for spaceborne measurement campaigns. $\mathscr{D}$ is a first order derivative for methods like geometric astronomic levelling and satellite-to-satellite tracking (e.g. CHAMP); it is a second order derivative for other methods like terrestrial gradiometry and satellite gravity gradiometry (e.g. GOCE).

Classically one can handle first order side conditions which are not tangential to the Σ and second derivatives pointing in the radial direction employing integral and pseudo differential equation methods. We will present a different approach: We classify all first and purely second order operators $\mathscr{D}$ which allow us to solve the problem with oblique side conditions as if we had ordinary i.e. non-derived side conditions. The only additional work which has to be done is an inversion of $\mathscr{D}$, i.e. integration.

Split Operators

We consider the following more general problem which is our oblique derivative problem in the geoscientifical case if we set Σ_{Ext} to the exterior of the Earth, Σ_D to the data location, $\mathfrak{U} = \Delta$ and $\mathfrak{D} = \mathscr{D}$ to an oblique derivative at Σ_D.

Problem *Let $\mathcal{S}$, $\mathcal{T}_1$ and $\mathcal{T}_2$ be separable normed linear function spaces defined on a domain Σ_{ext} and assume $\Sigma_D \subset \Sigma_{ext} \subset \mathbb{R}^n$. Let $\mathfrak{U} : \mathcal{S} \to \mathcal{T}_1$ and $\mathfrak{D} : \mathcal{S} \to \mathcal{T}_2$ be linear operators. Assume furthermore $T_2 \in \mathcal{T}_2$.*
We search all $V \in \mathcal{S}$ fulfilling

$$\mathfrak{U}V = 0$$
$$(\mathfrak{D}V)|_{\Sigma_D} = T_2|_{\Sigma_D}$$

The operator $\mathfrak{D}$ is globally defined, so the following definition makes sense:

Definition (Split Operator) $\mathfrak{U}_{\mathfrak{D}} : \mathcal{T}_2 \to \mathcal{T}_1$ *is called* split operator *for* $\mathfrak{U}$ *with respect to* $\mathfrak{D}$ *if it fulfills the following property:*

$$\mathfrak{U}V = 0 \Rightarrow \mathfrak{U}_{\mathfrak{D}}\mathfrak{D}V = 0 \qquad \textit{for all } V \in \mathcal{S}$$

Please observe that neither existence nor uniqueness of the split operator is assured. In particular is the 0 operator a split operator, however not a sensible one.

Using a split operator we can "split" our problem in two parts, one standard Dirichlet type problem, i.e. with standard boundary conditions and a comparably simple integration problem.

Lemma(Split Lemma) *Let $\mathfrak{U}_{\mathfrak{D}}$ be a split operator for $\mathfrak{U}$ with respect to $\mathfrak{D}$. If V is a solution of the problem described beforehand then it is also a solution of the problem*

$$\mathfrak{U}_{\mathfrak{D}}V_{\mathfrak{D}} = 0$$
$$V_{\mathfrak{D}}|_{\Sigma_D} = T_2|_{\Sigma_D}$$
$$\mathfrak{D}V = V_{\mathfrak{D}}$$

The split operator has two further particularly nice properties, namely:

- *Composition*: Assume $\mathfrak{D} = \mathfrak{D}_2\mathfrak{D}_1$ and let $\mathfrak{U}_{\mathfrak{D}_2}$ and $(\mathfrak{U}_{\mathfrak{D}_2})_{\mathfrak{D}_1}$ be the corresponding split operators. Then it holds
$$(\mathfrak{U}_{\mathfrak{D}_2})_{\mathfrak{D}_1} = \mathfrak{U}_{(\mathfrak{D}_2\mathfrak{D}_1)} = \mathfrak{U}_{\mathfrak{D}}$$
- *Linearity*: Assume $\mathfrak{U}_{\mathfrak{D}_1} = \mathfrak{U}_{\mathfrak{D}_2}$ are split operators with respect to $\mathfrak{D}_1$ and $\mathfrak{D}_2$ respectively. Assuming
$$\mathfrak{D} = \alpha_1\mathfrak{D}_1 + \alpha_2\mathfrak{D}_2 \qquad \text{where} \quad \alpha_1, \alpha_2 \in \mathbb{R}$$
we have $\mathfrak{U}_{\mathfrak{D}} := \mathfrak{U}_{\mathfrak{D}_1}$ is a split operator with respect to the operator $\mathfrak{U}$.

Split Operators for Δ

Now we want to classify possible first order derivatives $\mathscr{D} = \sum_{i=1}^{3} D_i\partial_i + D$ (where D_i and D are smooth functions) and corresponding split operators $\Delta_{\mathscr{D}}$ for Δ systematically. Most results presented also hold for more general second order (elliptic) differential operators than Δ.

Lemma *There does not exist a nontrivial split operator in the form* $\Delta_{\mathscr{D}} = \sum_{i=1}^{3} B_i\partial_i + B$ *where the* B_i *and* B *denote smooth functions.*

So we try as a candidate for our split operator (B_{ij}, B_i and B are smooth functions):

$$\Delta_{\mathscr{D}} = \sum_{1\le i\le j\le 3} B_{ij}\partial_i\partial_j + \sum_{i=1}^{3} B_i\partial_i + B$$

We are getting a number of different compatibility conditions which need to hold, namely:

$$\Delta_{\mathscr{D}} = \Delta + \sum_{i=1}^{3} B_i\partial_i + B$$

and

$$\begin{aligned}
0 =& \Delta_{\mathscr{D}} D_i + 2\partial_i D + B_i D && \text{for all } i\\
0 =& 2\partial_i D_j + B_i D_j + 2\partial_j D_i + B_j D_i && \text{for all } i \neq j\\
0 =& 2\partial_i D_i + B_i D_i - 2\partial_j D_j - B_j D_j && \text{for all } i \neq j\\
0 =& \Delta_{\mathscr{D}} D
\end{aligned}$$

Solutions

We need to solve this system of PDE's symbolically. We therefore transferred it to the language of (non-)commutative algebra. There we were just able to succeed when we assumed that the B_i and B commute with the ∂_j, i.e. B_i, B are constant.

Now we will just show the different prototype solutions, the rest can be obtained by summing up multiples these:

$$\begin{aligned}
\mathscr{D} &= \boldsymbol{e}^{-\frac{1}{2}(B_1x_1+B_2x_2+B_3x_3)}(\ 1\ ) \\
\mathscr{D} &= \boldsymbol{e}^{-\frac{1}{2}(B_1x_1+B_2x_2+B_3x_3)}(\ \partial_i\ ) && \text{for all } i \\
\mathscr{D} &= \boldsymbol{e}^{-\frac{1}{2}(B_1x_1+B_2x_2+B_3x_3)}(\ x_j\partial_i - x_i\partial_j\ ) && \text{for all } i \neq j \\
\mathscr{D} &= \boldsymbol{e}^{-\frac{1}{2}(B_1x_1+B_2x_2+B_3x_3)}(\ x_1\partial_1 + x_2\partial_2 + x_3\partial_3\ ) \\
\mathscr{D} &= \boldsymbol{e}^{-\frac{1}{2}(B_1x_1+B_2x_2+B_3x_3)}(\ x_i^2\partial_i + x_i + \sum_{j\in\{1,2,3\}\setminus\{i\}} -x_j^2\partial_i + x_ix_j\partial_j\ ) && \text{for all } i
\end{aligned}$$

with split operator

$$\Delta_{\mathscr{D}} = \Delta + \sum_{i=1}^{3} B_i\partial_i + \frac{1}{4}\sum_{i=1}^{3} B_i^2 \qquad \text{where } B_i \in \mathbb{R} \text{ for all } i$$

As we are mainly interested in the direction of the derivatives and not of the (common) scaling factor $\boldsymbol{e}^{-\frac{1}{2}(B_1x_1+B_2x_2+B_3x_3)}$ we can set $B_1 = B_2 = B_3 = 0$ and so can drop it. The split operator gets $\Delta_{\mathscr{D}} = \Delta$.

Please note that these operators $\mathscr{D}$ are not depending on the particular shape of the surface Σ_D where the boundary data are.

Further Remarks

A similar strategy can be applied to purely second order operators $\mathscr{D}$, surprisingly we just get composed solutions from the first order case.

Another important fact is that these operator map spherical harmonics of equal degree again to spherical harmonics of equal degree. The occurring transformation matrices have band structure and hence are numerically rather easy to treat.

When we consider the satellite problem this approach enables us to separate solving the boundary value type problem (BVTP) for the outer space outside the satellite track, the downward continuation (i.e. analytic continuation to the Earth) and the inversion of $\mathscr{D}$. The solution of the BVTP has to be done in the first place, but for the two others we are free to choose the order.

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Experiences in mathematical geodesy education

Martin Vermeer

Having been a researcher at the Finnish Geodetic Institute for two decades, I moved on to HUT to a chair with major educational duties. I describe here some of my experiences in trying to teach geodesy as an exact discipline to the next generation.

Firstly I report on the use of Matlab™ exercises in the context of several geodesy courses.

In the Geometrical Geodesy course, we developed

(1) an exercise requiring the solution of the geodetic forward problem on the ellipsoid by numerically integrating the differential equations for the geodesic: instead of the traditional series expansions in powers of e^2, a Matlab ODE routine was used.

(2) an excercise about solving for the coefficients of a complex polynomial describing the GaussKrüger projection used on Finnish territory, using as boundary conditions an array of points on the principal meridian. The objective was to achieve mm accuracy over the whole Finnish territory with minimal coefficients.

The projection was modelled as a two-stage operation, the first of which was a tradional Mercator. Then, the mapping from Mercator plane to GaussKrüger plane was modelled as described above. Mercator has the virtue of being closed formulae also on the ellipsoid, except for the evaluation of an integral called "isometric latitude", which also appears in the stereographic and Lambert conformal projection formulae. This integral was evaluated, instead of by the traditional series expansion, by a standard Matlab integration routine.

Also in the Fundamental Geodesy course we experimented with simple Matlab exercises: one was about computing DOP (Dilution of Precision, a measure of strength of the satellite geometry) quantities and optimizing it by manually shifting the positions of five satellites on the celestial sphere (and realizing that only above-horizon positions are physically realistic). This experiment was didactically somewhat risky, as both matrix algebra skills and Matlab familiarity were still very thin and fresh at that stage.

All these exercises were carefully designed to not require any 'deep' coding. In the geodesic problem, the students were required to fill in the formula code representing the differential equations, though.

From the scant feedback received from these exercises, it appears the students appreciate especially the problem solving component, figuring out something (more or less) for yourself, rather than dumb computation. This is of course as it should be: even while most of these students will not continue into a scientific career, it is clear that their future employers want to especially see robust problem-solving skills.

In the Geometical Geodesy course, which I lectured for one semester replacing a colleague on sick leave, I also experimented with teaching simple modern tensor calculus and differential geometry (metric tensor, Christoffel symbols, curvature, geodesic, etc.) as a way to describe the Earth's curved surface as well as to elucidate map projection theory – Tissot's indicatrix being a metric tensor. I also led the classroom exercises on this, and was left with the impression that it was found more difficult, but also more rewarding, than the traditional way of teaching geometrical geodesy.

More details on this and other things, unfortunately mostly in Finnish, can be found on my Web site: `http://www.hut.fi/~mvermeer`.

Determination of the Earths Gravity Field by means of the new GPS-tracked Satellite Missions CHAMP, GRACE and GOCE - a new algorithm for orbit analysis based on accelerations

Tilo Reubelt
(joint work with Erik Grafarend)

The most important aim of the three new satellite missions CHAMP, GRACE and GOCE is the accuracy improvement of the global gravity field model. Different measurement principles are applied in these missions to determine a certain resolution of the gravity field. In the already launched mission CHAMP, the orbit analysis of a single satellite is used that is sensitive to long wavelengths (¿ 500 km) of the gravity field. The main principle of the twin satellite mission GRACE is the measurement and analysis of the intersatellite distance between both satellites which allow for determination of medium wavelengths (500 km 250 km) of the gravity field. The gravity field recovery will be completed by the planned GOCE mission where gradiometry should guarantee resolution of short wavelengths (250 km - 130 km) of the gravity field.

The common measurement principle in these three satellite missions is the orbit analysis by means of GPS-tracking and acceleration (measurements of non-gravitational disturbing accelerations) data. There are different algorithms for the orbit analysis that are mainly based on integration of the equations of motions of the satellites (variational approach) or the energy balance principle. The former method suffers from a large computational effort resulting from numerical integration and iteration due to a nonlinear system of equations and from the influence of low frequency noise due to integration. The main disadvantage of the second method is that noisy observables (velocities determined by numerical differentiation of positions) must be squared which leads again to the noise magnification. To circumvent these problems, we have developed and investigated a third method based on satellite accelerations. This new method is fast and suppresses the low-frequency noise. However, the high-frequency noise still can be amplified due to numerical differentiation.

The method works as follows: First, by means of numerical differentiation, satellite accelerations are determined from the GPS-tracked positions of the satellite in the quasi-inertial system. Subsequently, these accelerations are reduced for satellite surface forces measured by an on-board accelerometer and tidal accelerations computed from models. After rotation of the reduced accelerations into the local system, they are balanced by the gravity vector expressed in spherical harmonic. Finally, spherical harmonic geopotential coefficients (up to 10,000 unknowns) are determined from the resulting system of linear equations. In order to deal with the restricted computer memory, the iterative method of preconditioned conjugate gradients is applied with a convergence achieved within 10 iterations.

The crucial point for the accuracy of results is the noise amplification during numerical differentiation. Thus, different numerical differentiation schemes (Newton interpolation, Splines) as well as smoothing methods (regression polynomials, smoothing Splines) are tested. Simulations show that the Newton and Spline interpolations are superior to smoothing methods.

From various simulations as well as the analysis of short real CHAMP data sets we conclude that the new method can determine the Earths gravity field with good accuracy. Nevertheless, it can only be found out by comparison which algorithm is the best. For the future, the focus will be on the analysis of longer (1 year) real CHAMP data sets. The algorithm can be classified as fast, since the system of linear equations is linear and no integration is necessary to obtain the elements of the normal matrix.

Wavelets on Regular Surfaces Generated by Layer Potentials

Carsten Mayer

By means of the classical limit and jump relations of potential theory the framework of a scalar as well as a vectorial wavelet approach on a regular surface is established. The setup of a multiresolution analysis is defined by interpreting the kernel functions of the limit and jump integral operators as scaling functions on regular surfaces. The distance of the parallel surface to the surface under consideration thereby represents the scale level in the scaling function. This procedure results in scalar kernel functions which lead to vectorial ones by applying the surface operators with respect to the Helmholtz decomposition.

Scaling functions and wavelets show space localizing properties. Thus, they can be used to represent scalar and vector fields locally on a regular surface. This fact will be demonstrated by an approximation of the (scalar) gravity potential on the Earth's reference ellipsoid and of a (vectorial) deformation field on the regular Earth's surface given by the TerrainBase model.

1. Construction of Wavelets

The starting point of our construction principle of wavelets on regular surfaces is an elliptic boundary value problem and its fundamental solution.

BVP Let $\Sigma \subset \mathbb{R}^3$ be a regular surface, i.e. a closed and compact surface containing the origin, free of double points and possessing a continuous normal field pointing into the outer space Σ_{ext}. Let F on Σ be given and $\diamond$ be an elliptic partial differential operator with constant coefficients. Find U sufficiently smooth such that

$$\begin{gathered} \diamond U = 0 \quad \text{in } \Sigma_{int} \quad (\text{resp. } \Sigma_{ext}) \\ U|_\Sigma = F \quad \text{or} \quad \left.\left(\frac{\partial}{\partial \nu}\right)\right|_\Sigma = F \quad \text{on}\Sigma \qquad (44) \\ U \text{ is regular at infinity.} \end{gathered}$$

The bivariate function G is called fundamental solution of the boundary value problem if

$$\diamond_x G(x,y) = \delta(x-y), \quad x,y \in \mathbb{R}^3 \setminus \{\Sigma\}.$$

For the case $\diamond = \Delta$ the fundamental solution is given by $G(x,y) = \frac{1}{|x-y|}$, which has been discussed in [3]. For the case of the Helmholtz equation we have $\diamond = \Delta + k^2 Id$ and $G(x,y) = \frac{\exp(ik|x-y|)}{|x-y|}$. This case has been studied in [4]. In [1] the Cauchy-Navier equation has been investigated, where $\diamond = \mu\Delta + (\lambda+\mu)\nabla\nabla$, μ, λ Lame constants. The fundamental solution is, in this case, a tensorial function which is explicitly known and which can be found in [5].
Based on the fundamental solution we now define, for $\tau, \sigma > 0$, the potential operator $P(\tau,\sigma) : L^2(\Sigma) \to L^2(\Sigma)$ by

$$P(\tau,\sigma)F(x) = \int_\Sigma F(y) G(x+\tau\nu(x), y+\sigma\nu(y))\, d\omega(y), \quad F \in L^2(\Sigma).$$

The operator $P(\tau,0)$ is called the operator of the single layer potential and the operator $P(\tau,0)_{|\sigma}F(x) = \left(\frac{\partial}{\partial\sigma} P(\tau,\sigma)F(x)\right)_{\sigma=0}$, is called operator of the double layer. Using these operators we can formulate the well-known jump relations of potential theory (see [6],[7]). Here, we just present the jump relation for the double layer potential, i.e.

$$(P_{|\sigma}(\tau,0) - P_{|\sigma}(-\tau,0))F \stackrel{\tau\to 0}{\longrightarrow} 4\pi F \qquad (45)$$

holds in the sense of the $||\cdot||_\infty$−norm and of the $||\cdot||_2$−norm. Writing out (45) explicitly we get

$$\lim_{\substack{\tau\to 0\\ \tau>0}} (F * \Phi_\tau)(x) = \lim_{\substack{\tau\to 0\\ \tau>0}} \int_\Sigma F(y)\Phi_\tau(x,y)\, d\omega(y) = F(x), \quad x \in \Sigma, \quad F \in C(\Sigma),$$

with the kernel functions $\Phi_\tau(x,y)$, $\tau \in (0,\infty)$, given by

$$\Phi_\tau(x,y) = \frac{1}{4\pi}\frac{\partial}{\partial\nu(y)}\Big(G\big(x+\tau\nu(x), y+\sigma\nu(y)\big) - G\big(x-\tau\nu(x), y+\sigma\nu(y)\big)\Big)_{|\sigma=0}.$$

The family of functions Φ_τ, $\tau \in (0,\infty)$, are called $\Sigma-$scaling functions. Thus, for $\tau \in (0,\infty)$ fixed, the family of $\Sigma-$wavelet functions, Ψ_τ, is defined by

$$\Psi_\tau(x,y) = -\tau \ \frac{d}{d\tau}\Phi_\tau(x,y), \quad x,y \in \Sigma.$$

Using these scaling functions and wavelets we can define filter operators $P_\tau = F * \Phi_\tau$, called low-pass filter and $R_\tau = F * \Psi_\tau$, called band-pass filter. The corresponding image spaces of $L^2(\Sigma)$ under these operators are denoted by $V_\tau(\Sigma)$, called scale space, and $W_\tau(\Sigma)$, called detail space. The scale spaces fulfill the fundamental properties of a multiresolution analysis, i.e. $\lim_{\tau\to 0,\tau>0} V_\tau(\Sigma)$ is dense in $L^2(\Sigma)$ and $\{0\} \subset V_\tau(\Sigma) \subset V_{\tau'}(\Sigma) \subset L^2(\Sigma)$ if $0 < \tau' < \tau < \infty$. The first property is clear and the second one has been shown for the case of sphere in [3].
In order to construct vectorial kernel functions out of the scalar ones we use the Helmholtz surface theorem (see e.g. [2]).
Helmholtz Theorem. Let $f \in c(\Sigma)$ be a vector field on Σ, then there exist sufficiently smooth scalar fields F_1, F_2, F_3, such that

$$f = o_\Sigma^{(1)} F_1 + o_\Sigma^{(2)} F_2 + o_\Sigma^{(3)} F_3 = \nu F_1 + \nabla_\Sigma F_2 + (\nu \wedge \nabla_\Sigma)F_3. \tag{46}$$

By use of the operators $o_\Sigma^{(i)}$ in (46) we can define vector kernels $\phi_\tau^{(i)}$, for $\tau \in (0,\infty)$, $x,y \in \Sigma$, by

$$\phi_\tau^{(1)}(x,y) = o_\Sigma^{(1)}\Phi_\tau(x,y), \quad \phi_\tau^{(2)}(x,y) = o_\Sigma^{(2)}\Phi_\tau(x,y), \quad \phi_\tau^{(3)}(x,y) = o_\Sigma^{(3)}\Phi_\tau(x,y).$$

Let now $\{y_j\}_{j=1,\dots,M} \subset \Sigma$ be a set of equidistributed points on the regular surface Σ, e.g. integration knots of a suitable integration rule on Σ. In order to approximate f on Σ the task is to find coefficients $a_j^{(i)}$, $j = 1,\dots,M$, $i = 1,2,3$, such that

$$f(x) = \sum_{j=1}^{M} (a_j^{(1)}\phi_\tau^{(1)}(x,y_j) + a_j^{(2)}\phi_\tau^{(2)}(x,y_j) + a_j^{(3)}\phi_\tau^{(3)}(x,y_j)), \quad x \in \Sigma. \tag{47}$$

These coefficients can be obtained by appropriate methods such as collocation, Galerkin procedure or least square approximation.

2. Applications

First, we show in Figure 7 a local wavelet reconstruction of the Earth's gravitational potential over Italy. The reconstruction is performed using a discrete $\Sigma-$wavelet at scale $\tau = 2^{-8}$ on the Earth's surface given by the TerrainBase model (see [9]). The necessary convolution is discretized by a polynomial exact integration rule as for example the Gauss-Legendre rule.
As a second application we present in Figure 7 a scale approximation of a vectorial deformation field over Turkey. The underlying data, which is plotted in white, is due to [8] and shows yearly means of the displacement of several GPS stations in the Greece-Turkey area. The reconstruction is obtained by collocation according to Equation 47. The scale of the corresponding vector kernel in (47) thereby is chosen such that the error of the approximation is minimized.
Future applications are the solution of boundary integral equations over regular

surfaces which occur from layer potential approaches for the boundary value problems discussed in (44). Open problems in this field are, for example,

- suitable integration rules for regular surfaces,
- equidistributed point-sets on regular surfaces,
- computation of the normal field for discretely given surfaces.

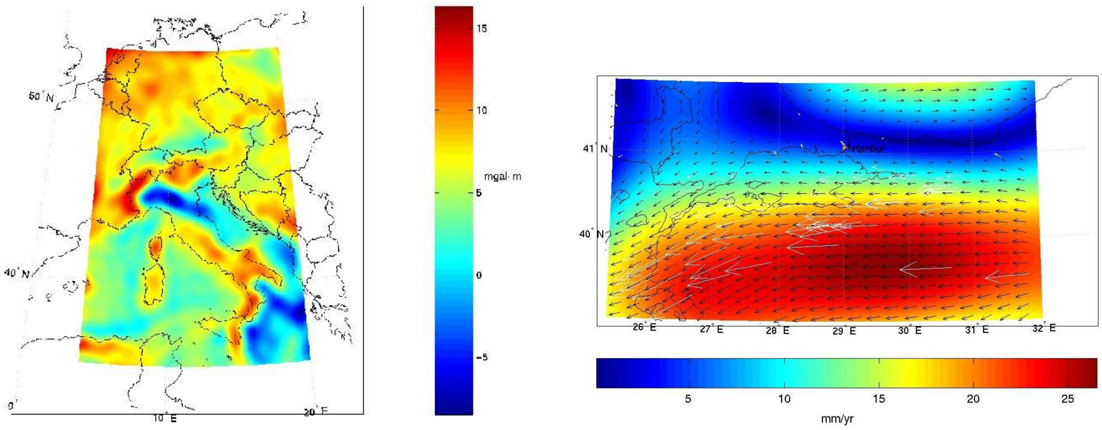

FIGURE 7. Left: Wavelet reconstruction at scale $\tau = 2^{-8}$ of the Earth's gravitational potential over Italy. Right: Scaling function reconstruction of a deformation field over Turkey. Black arrows indicate the direction while color indicates the strength of the approximated displacement.

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