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Buildings and Curvature

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Introduction by the Organisers

It was the aim of the meeting to bring together international experts from the theory of buildings, differential geometry and geometric group theory.

Buildings are combinatorial structures (simplicial complexes) which can be seen as simultaneously generalizing projective spaces and trees. Already from these examples it is clear that there will be interesting groups acting on buildings. Conversely, groups can be studied using their actions on given buildings. Groups coming up in this context are in particular groups having a BN-pair. Examples of such groups include the classical groups, simple Lie groups and algebraic groups (also over local fields), Kac-Moody groups and loop groups. This already indicates that these groups play an important role in many different areas of mathematics such as algebra, geometry, number theory, physics and analysis. Kac-Moody groups correspond to so-called twin buildings, a particularly active area in the theory of buildings.

Geometric group theory is concerned with the investigation of group actions on metric spaces using the interplay of group theoretic properties and metric properties like curvature in the sense of Alexandrov, or CAT(0)-spaces. The geometric realization of a building is a metric space with interesting curvature properties on which the above mentioned groups as well as their subgroups like uniform lattices or arithmetic groups act in a natural way by isometries. In this respect there are

a number of canonical connections between the theory of buildings and geometric group theory. One of the current problems concerns the characterization of buildings as metric spaces.

In differential geometry these aspects also play an important role, e.g. in connection with Hadamard manifolds, (simply connected Riemannian manifolds of nonpositive curvature). A special role is played by the Riemannian symmetric spaces and their quotients of finite volume which one wants to characterize geometrically. By considering the fundamental groups, one obtains discrete group actions also studied in geometric group theory. Buildings come up in differential geometry as the compactifications of Riemannian symmetric spaces yielding examples of topological buildings. Asymptotic cones (and ultrapowers) of symmetric spaces present non-discrete affine buildings and create new and interesting relations to model theory. These constructions are important in new proofs of differential geometric rigidity theorems, like Mostow Rigidity and the Margulis Conjecture.

This shows that there are close connections between the areas, and this meeting was the first in a number of years in Oberwolfach having these connections as its topic. Geometric group theory has recently introduced interesting aspects into the theory of buildings, in particular the hyperbolic buildings. Conversely, new developments in the theory of buildings, e.g. the twin buildings have interesting group theoretic applications, for example in the theory of S-arithmetic groups or in the theory of Kac-Moody groups. All these aspects played an important part in this meeting and the interaction between the participants from different areas was very lively.

Workshop on Buildings and Curvature

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Abstracts

Introduction to buildings

Kenneth S. Brown

In this talk I reviewed three ways of thinking about Coxeter complexes and buildings. The emphasis was on giving an intuitive understanding, with the aid of pictures (which are not reproduced here).

1. Geometric realizations of Coxeter groups

1.1. **The classical (simplicial) Coxeter complex.** Let (W, S) be a Coxeter system with S finite. The classical geometric realization of (W, S) is Tits's Coxeter complex $\Sigma = \Sigma(W, S)$. See [1] for a detailed exposition. Σ is a chamber complex on which W acts with a strict fundamental domain consisting of a single closed chamber. The stabilizers along the fundamental domain are the standard parabolic subgroups $W' = \langle S' \rangle$, where $S' \subseteq S$. [Convention: We include the empty simplex, whose stabilizer is W.] The simplices of Σ correspond to standard cosets wW', and the face relation on simplices corresponds to the opposite of the inclusion relation on cosets.

Topologically, Σ is either contractible or a sphere. The latter occurs if and only if W is finite, in which case there is a canonical way to realize Σ as the boundary of a convex polytope. [This is a general fact about the cell complex associated with an arbitrary central hyperplane arrangement.]

The Coxeter complexes $\Sigma(W', S')$ associated to standard parabolic subgroups occur naturally as subcomplexes of $\Sigma(W, S)$; they are the links of the simplices in the fundamental domain.

An important fact about Σ is that the vertices naturally fall into types, with one type for each $s \in S$. One can therefore refine the adjacency relation on chambers by declaring two chambers to be s-adjacent if they have the same panel of cotype s.

1.2. **The underlying chamber system.** Let's forget about all simplices except the chambers, and remember only that they have a family of s-adjacency relations. The structure now is a graph with colored edges. There is one vertex for each chamber, and an edge between two adjacent chambers. The edge is colored to reflect the type of adjacency. Thus there is one color for each $s \in S$. This graph is in fact nothing but the Cayley graph of (W, S), so it gives a quite natural geometric representation of the Coxeter group. One can reconstruct the entire Coxeter complex by taking "residues". For the sake of intuition, it is useful to draw the chamber graph superimposed on a picture of the Coxeter complex. There is a vertex in each open chamber and a (colored) edge cutting across each panel.

1.3. **Davis's dual Coxeter complex.** When one looks at the picture of the chamber graph superimposed on the Coxeter complex, one sees spherical configurations, one for each simplex with finite (hence spherical) link. Davis cones off these spheres to get cells. See [3], for example. If W is infinite, the resulting cell complex $\Sigma_d = \Sigma_d(W, S)$ sits inside the geometric realization $|\Sigma|$. It is obtained, roughly speaking, by deleting each vertex at which Σ fails to be locally finite, and it is still contractible. If W is finite, on the other hand, then Σ itself is one of the spherical links that gets coned off. [It is the link of the empty simplex.] Thus Σ_d is a topological ball in this case, hence again contractible. It can be realized as the convex polytope polar to the one mentioned in Section 1.1. This sort of polytope associated with a central hyperplane arrangement is called a *zonotope*. The permutahedron is a famous example.

A more precise description of Σ_d is that it is a regular cell complex with one nonempty cell for each finite standard coset, with the face relation now corresponding to inclusion rather than the opposite of inclusion. The closed cells are themselves isomorphic to complexes $\Sigma_d(W', S')$ associated to finite standard subgroups, so they can be viewed as convex polytopes. This gives Σ_d a piecewise Euclidean structure. Note that it is locally finite and that the W-action is proper. Moussong [4] proved:

Theorem 1. *Σ_d, with its piecewise Euclidean path metric, is a complete CAT(0)-space.*

For some purposes it is important that Σ_d has a canonical cubical subdivision. [Warning: The cells in the subdivision are combinatorial cubes but not metric cubes except in the special case of right-angled Coxeter groups.] Some such subdivision is needed, for example, if one wants to describe combinatorially a fundamental domain for the action of W on Σ_d. It is also needed when one glues Coxeter complexes together to form buildings (Section 2.3).

2. Buildings

There are three ways of thinking about buildings, corresponding to the three approaches to realizing Coxeter complexes. We review them briefly.

2.1. **The simplicial approach.** This is also sometimes called the old-fashioned appproach. Here a building of type (W, S) is a simplicial complex Δ that has a system of subcomplexes called *apartments*, each of which is isomorphic to $\Sigma(W, S)$. One assumes (a) any two simplices are contained in an apartment and (b) for any two apartments, there is an isomorphism between them fixing their intersection.

2.2. **The Chamber system approach.** Buildings, like Coxeter complexes, admit type functions. Once again, there is one type of vertex for each $s \in S$. So we can form a graph with colored edges, as in Section 1.2. It is still true that one can recover all the simplices from the chamber system. There are various ways to axiomatize buildings from this point of view. See, for instance, Weiss [5]. An interesting variant of this approach views the set of chambers as a "metric

space" in which the metric takes values in W. A statement of the axioms can be found in [2], along with a sketch of a proof that this approach is equivalent to the simplicial definition.

2.3. **The Davis realization.** To get the Davis realization Δ_d of Δ, replace each apartment Σ by Davis's Σ_d. See Davis [3] for a more precise statement and a proof of the following result, which Davis says was also known to Moussong:

Theorem 2. *The Davis realization Δ_d admits a metric consistent with the piecewise Euclidean metric on each apartment. With this metric it is a complete CAT(0)-space.*

The proof is similar to the proof in [1] for Euclidean buildings. The significance is that results such as the Bruhat–Tits fixed-point theorem can be applied to arbitrary buildings.

References

[1] K. S. Brown, *Buildings*, Springer-Verlag, New York, 1989.

[2] ———, *Five lectures on buildings*, Group theory from a geometrical viewpoint (Trieste, 1990), World Sci. Publishing, River Edge, NJ, 1991, pp. 254–295.

[3] M. W. Davis, *Buildings are* CAT(0), Geometry and cohomology in group theory (Durham, 1994), London Math. Soc. Lecture Note Ser., vol. 252, Cambridge Univ. Press, Cambridge, 1998, pp. 108–123.

[4] G. Moussong, *Hyperbolic coxeter groups*, Ph.D. thesis, Ohio State University, 1988.

[5] R. M. Weiss, *The structure of spherical buildings*, Princeton University Press, Princeton, NJ, 2003.

Weighted L^2-cohomology of Coxeter groups

J. Dymara and T. Januszkiewicz

(joint work with M.W. Davis and B. Okun)

Suppose (W, S) is a Coxeter system. Let $i : S \to I$ be a function to some index set I so that $i(s) = i(s')$ whenever s and s' are conjugate. Given an I-tuple $\mathbf{q} = (q_i)_{i\in I}$ of positive real numbers, there is a certain deformation of the group algebra of W called the "Hecke algebra" of W. We denote it by $\mathbb{R}$.

Also associated to $\mathbf{q}$, there is an inner product $\langle\ ,\ \rangle_{\mathbf{q}}$ on $\mathbf{R}^{(W)}$ defined by $\langle e_w, e_{w'}\rangle_{\mathbf{q}} = q_w\delta_{ww'}$, where $\delta_{ww'}$ is the Kronecker delta. The completion of $\mathbf{R}^{(W)}$ with respect to this inner product is denoted $L^2_{\mathbf{q}}(W)$ or simply $L^2_{\mathbf{q}}$ when W is understood. $L^2_{\mathbf{q}}$ is an $\mathbb{R}$-bimodule. There is an anti-involution on $\mathbb{R}$, denoted by $x \to x^*$ and defined by $(e_w)^* := e_{w^{-1}}$. As is explained in [9], this makes $\mathbb{R}$ into a "Hilbert algebra" in the sense of Dixmier. It follows that there is an associated von Neumann algebra $\mathcal{N}_{\mathbf{q}}$ acting on $L^2_{\mathbf{q}}$ from the right.

As in the case of a von Neumann algebra associated to a group algebra, $\mathcal{N}_{\mathbf{q}}$ is equipped with a trace and one can use this trace to define the "dimension" of any $\mathbb{R}$-stable closed subspace V of a finite direct sum of copies of $L^2_{\mathbf{q}}$.

Suppose W acts as a reflection group on a some CW complex $\mathcal{U}$ with a strict fundamental domain Z. Assume further that for each $s \in S$ there is a subcomplex $Z_s \subseteq Z$, called a "mirror" of Z, so that s acts on $\mathcal{U}$ as a reflection across Z_s. Then $\mathcal{U}$ is formed by gluing together copies of Z, one for each element of W. In other words, $\mathcal{U} \cong (W \times Z)/\sim$, where the equivalence relation $\sim$ is defined in an obvious fashion.

The second author [9] has defined "weighted L^2-cohomology spaces, " denoted $L^2_{\mathbf{q}}\mathcal{H}^i(\mathcal{U})$. The weighted L^2-cochain complex, $L^2_{\mathbf{q}}C^*(\mathcal{U})$, is a subcomplex of the complex $C^*(\mathcal{U};\mathbf{R})$ of ordinary cellular cochains. The subcomplex $L^2_{\mathbf{q}}C^*(\mathcal{U})$ consists of those cochains which are square summable with respect to an inner product defined via a weight function depending on the multiparameter $\mathbf{q}$.

To each of the Hilbert spaces $L^2_{\mathbf{q}}\mathcal{H}^i(\mathcal{U})$ one can attach a "von Neumann dimension." It is a nonnegative real number, denoted by $b^i_{\mathbf{q}}(\mathcal{U})$ and called the i^{th} *$L^2_{\mathbf{q}}$-Betti number of $\mathcal{U}$.*

Our principal interest in the weighted L^2-cohomology comes from the fact that it computes the L^2-cohomology of buildings of type (W, S). Here $\mathbf{q}$ is a certain I-tuple of positive integers called the "thickness vector" of the building. In other words, for buildings only $\mathbf{q}$ with integral components can occur.

Non-integral weighted cohomology groups seems to be interesting on their own right as a rich source of numerical invariants of Coxeter groups which generalize growth functions. Indeed the theory of the weighted L^2-cohomology of Σ is closely tied to several other topics: growth series of Coxeter groups, decompositions of "Hecke - von Neumann algebras" and the Singer Conjecture. Moreover, as $|\mathbf{q}|$ goes from 0 to ∞, $L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$ interpolates between ordinary cohomology and cohomology with compact supports. For these reasons, we believe that the study of weighted L^2-cohomology of Coxeter groups has intrinsic interest, independent of its connection to buildings. On the technical side, our strategy of computations is to deal first with the case of small, nonintegral $|\mathbf{q}|$, then derive consequences for large $|\mathbf{q}|$.

Here is the statement of our calculation of $L^2_{\mathbf{q}}$-cohomology.

The Main Theorem. [(a)]

(1) If $\mathbf{q} \in \overline{\mathcal{R}}$, then

$$L^2_{\mathbf{q}}\mathcal{H}^*(\mathcal{U}) \cong \bigoplus_{T\in\mathcal{S}} H^*(Z, Z^T) \otimes D_T.$$

(2) If $\mathbf{q} \in \overline{\mathcal{R}^{-1}}$, then

$$L^2_{\mathbf{q}}\mathcal{H}^*(\mathcal{U}) \cong \bigoplus_{T\in\mathcal{S}} H^*(Z, Z^{S-T}) \otimes D_{S-T}.$$

(We note that for $\mathbf{q} \in \mathcal{R}$, $(e_w h_T a_{S-T})_{w\in W^T}$ spans a dense subspace of D_T; while for $\mathbf{q} \in \mathcal{R}^{-1}$, $(e_w h_{S-T} a_T)_{w\in W^T}$ spans a dense subspace of D_{S-T}.)

The proof of the Main Theorem depends on the following result.

The Decomposition Theorem. [(a)]

(1) If $\mathbf{q} \in \overline{\mathcal{R}}$, then

$$\sum_{T \in \mathcal{S}} D_T$$

is a direct sum decomposition and a dense subspace of $L^2_{\mathbf{q}}$.

(2) If $\mathbf{q} \in \overline{\mathcal{R}^{-1}}$, then

$$\sum_{T \in \mathcal{S}} D_{S-T}$$

is a direct sum decomposition and a dense subspace of $L^2_{\mathbf{q}}$.

In the case when W is finite and $\mathbf{q} = \mathbf{1}$ (i.e., when the Hecke algebra is the group algebra) a similar result was proved by Solomon in 1968. The Decomposition Theorem is also compatible with the theory of representations of Hecke algebras developed by Kazhdan–Lusztig.

The results of this paper raise more questions than they answer.

- The Main Theorem gives a complete calculation of $L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$. On the other hand, our knowledge about what happens for $\mathbf{q} \notin \overline{\mathcal{R}} \cup \overline{\mathcal{R}^{-1}}$ is fragmentary.
- Is there a version of this theory for weighted differential forms?
- Is there a version of this for groups other than Coxeter groups?

(the short answer to two last questions is "yes.")

References

[1] N. Bourbaki, *Lie Groups and Lie Algebras, Chapters 4–6*, Springer-Verlag, 2002.

[2] M.W.Davis, *The homology of a space on which a reflection group acts*, Duke Math. J. **55** (1987), 97–104.

[3] ———— *The cohomology of a Coxeter group with group ring coefficents*, Duke Math. J. **91** (1998), 297–314.

[4] ———— *Nonpositive curvature and reflection groups* in *Handbook of Geometric Topology* (eds., R.J. Daverman and R.B. Sher), pp. 373-422, Elsevier, Amsterdam, 2002.

[5] M.W. Davis, J. Dymara, T. Januszkiewicz and B. Okun, *Weighted L^2-cohomology of Coxeter groups*, submitted, math.GT/0402377.

[6] M.W. Davis and B. Okun, *Vanishing theorems and conjectures for the ℓ^2-homology of right-angled Coxeter groups*, Geometry & Topology **5** (2001), 7–74.

[7] ———— *ℓ^2-homology of right-angled Coxeter groups associated to barycentric subdivisions*, Topology and Its Applications (2004).

[8] J. Dymara, *L^2-cohomolgy of buildings with fundamental class*, to appear in Proc. Amer. Math. Soc.

[9] ———— *Thin buildings*, preprint.

[10] J. Dymara and T. Januszkiewicz, *Cohomology of buildings and their automorphism groups*, Invent. Math. **150** (2002), 579–627.

[11] B. Eckmann, *Introduction to ℓ_2-methods in topology: reduced ℓ_2-homology, harmonic chains, ℓ_2-Betti numbers*, Israel J. Math. **117** (2000), 183–219.

[12] L. Solomon, *A decomposition of the group algebra of a finite Coxeter group*, J. of Algebra **9** (1968), 220–239.

Building groups are automatic

Jacek Świątkowski

Let G be a group acting on a building Δ. Suppose that
(1) Δ is of finite thickness;
(2) G has finitely many orbits on the set of chambers of Δ;
(3) stabilizers of chambers are finite.
Equivalently, G acts properly discontinuously and cocompactly on the Davis' realization of Δ.

Main result. *Every group G as above is automatic.*

This result was known previously with the stronger conclusion that G is biautomatic, in the cases when the associated Coxeter system is

- Gromov hyperbolic (Cartwright-Shapiro, 1995)
- right-angled (Niblo-Reeves, 1998)
- affine, with additional assumption that the action of G is free and type-preserving (Noskov, 2000).

The idea of my proof is to "lift" to buildings, in certain precise sense, the following result due to Brink-Howlett 1993:
Every Coxeter group admits a geodesic automatic structure.

The proof uses also a new tool, sort of a "finite state orbiautomaton", that I have invented for proving automaticity or biautomaticity of groups acting on various spaces.

An extension of Bruhat-Tits buildings

Helmut Behr

For some applications, e.g. the proof of finiteness properties of S-arithmetic groups over function fields, Bruhat–Tits buildings seem to be too small. This can be demonstrated in comparing them with the use of symmetric spaces for arithmetic groups over number fields.

1. Number fields: The Iwasawa decomposition $GL_n(\mathbb{R}) = KAN$ is unique and defines the symmetric space $X = K \setminus GL_n(\mathbb{R})$ whose elements may be interpreted as lattices, i.e. $\mathbb{Z}$-modules of rank n with an inner product. These lattices L admit a unique HN-filtration $\{0\} \subset L_1 \subset \ldots \subset L_k \subset L$ by sublattices which corresponds to a flag $V_1 \subset \ldots \subset V_k$ of proper subspaces of $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$; lattices with trivial filtration $\{0\} \subset L$ are called semi-stable. The flags can be viewed as simplices in the spherical Tits building X_0, their stabilizers in $GL_n(\mathbb{Q})$ are proper parabolic subgroups P, thus each unstable $x \in X$ determines a canonical parabolic group $P = \pi(x)$. The unstable region $X' \subset X$ has a cover by contractible sets $X'_P = \{x \in X' \mid \pi(x) \supseteq P\}$, whose nerve is the simplicial complex X_0, which implies that X' is $(n-2)$-spherical as is X_0 itself.
X' can be retracted along geodesic lines to $\partial X' = \overline{X'} \cap (\overline{X \setminus X'})$, and the same

process also shows that $X_{ss} = X \setminus X'$ is a strong deformation retract of X, so X_{ss} is contractible. By normalizing the (co-) volumes of lattices or equivalently passing from GL_n to SL_n, one obtains that $[X_{ss}]$ is compact modulo the arithmetic subgroup $\Gamma = SL_n(\mathbb{Z})$. All these ideas can be generalized to arbitrary number fields and semi-simple groups and provide an alternate proof of Borel–Serre's theorem, that all arithmetic groups over number fields are of type F_∞ (a group Γ is of type F_n if it admits a $K(\Gamma, 1)$-complex with finite n-skeleton, and it is F_∞ if it is F_n for all n). The results of this section are due to U. Stuhler and D. Grayson (cf. [G2]).

2. Function fields: Consider fields F with $[F : \mathbb{F}_q(t)] < \infty$ and a valuation v, valuation ring R, prime element π and the ring $\mathcal{O}$ of S-integers in F, defined by $S = \{v\}$. Denote by X the Bruhat–Tits building of $(SL_n(F), v)$ whose simplices can be described by chains of R-lattices of rank n, so dim X is only $(n-1)$. The Iwasawa decomposition $SL_n(F) = KAN$ with $K \cong SL_n(R)$, $A = \{\mathrm{diag}\,(\pi^{k_i})\}$, $N = U_n(F)$ is not unique because there exist shifts between K and N, depending on the middle term.
Once more there exist filtrations by sublattices which are unique. This was proved in the language of vector bundles over a projective curve by Serre for trees, and in general by Harder–Narasimhan, and used by Quillen–Grayson (see [G1]) to define an unstable region X' with a cover by sets X'_P, P parabolic, whose nerve is again the Tits building X_0, thus proving that X' is $(n-2)$-spherical and also that the semi-stable part $X_{ss} = X \setminus X'$ is modulo the arithmetic group Γ (with coefficients in $\mathcal{O}$) a finite complex. The existence of a canonical parabolic group can be shown for arbitrary reductive groups (cf. [B] and [St]).
But in the function field case, it is not possible to retract X' to its boundary $\partial X'$, and X_{ss} is not contractible which can easily be seen for trees. In this situation it seems to be natural to define an extension $\widetilde{X}$ of X' in such a way that

(i) $\widetilde{X}$ has the same homotopy type as X',
(ii) $\widetilde{X}$ has a strong deformation retract $\partial\widetilde{X}$ which is finite modulo Γ,

and by the way restoring the uniqueness of the Iwasawa decomposition, finally $\widetilde{X}$ has the same dimension as the corresponding (real) symmetric space.

3. Definition and properties of an extension $\widetilde{\mathbf{X}}$ of X: To a given filtration $\{0\} \subset L_1 \subset \ldots \subset L_k \subset L$ we associate complementary lattices L'_i with $L = L_i \oplus L'_i$ which constitute an "opposite flag". Parabolic groups P and P' are called opposite ($P \,\mathrm{op}\, P'$) if $P \cap P'$ is a Levi subgroup of both. Their pairs define a simplicial complex $\mathrm{Opp}\, X_0 = \{(P, P') \in X_0 \times X_0 \mid P \,\mathrm{op}\, P'\}$, sometimes called a "split building" which has the same homotopy type as X_0 itself (for a general proof see [H]).

A pair (B, B') of opposite Borel groups determines a (split) maximal torus $T = T(B, B')$ and also an apartment A_T of X — we always think of X_0 as the building X_∞ at infinity of X. Set $\mathcal{A}_{P,P'} = \bigcup\{A_T \in X \mid T = T(B, B'),\ (B, B') \subseteq (P, P')\}$. For a fixed pair (P_0, P'_0) there is a bijection between $\{P' \mid P' \,\mathrm{op}\, P_0\}$ and

$U_{P_0}(F)$ (the unipotent radical of P_0), given by conjugation. Choose now an origin $x_0 \in A_T$ and acting by $g \in KAN$ from the right, we have: $x = x_0(kan) \in \mathcal{A}_{P,P'}$ iff $n \in N_1$ with $N_1 = \{n \in N \mid ana^{-1} \in K\}$ — this motivates the following

Definition:
$\widetilde{X} = \{(x, y) \in X' \times X_0 \mid \pi(x) = P \operatorname{op} P',\ y \in \overline{P'}\ ,\ x \in \mathcal{A}_{P,P'}\}$ ($\pi(x)$ denotes the canonical parabolic subgroup to $x \in X'$ and $\overline{P}' = \{Q' \text{ parabolic} \mid Q' \supseteq P'\}$, understood as the closure of the simplex P' in $X_\infty \cong X_0$). The elements of $\widetilde{X}$ may be viewed as closed cones (or sectors) with vertex X' and "basis" $\overline{\operatorname{star} P'}$ at infinity. The topology on $\widetilde{X}$ is induced by the euclidean topology on X' and the v-analytic one on X_0, refining the simplicial topology. This definition is valid for a split reductive group G. Again there is a cover of $\widetilde{X}$ by contractible sets $\widetilde{X}_{P,P'}$ whose nerve is $\operatorname{Opp} X_0$, and there exists now a retraction of $\widetilde{X}$ to $\partial\widetilde{X} = \widetilde{X} \cap (\partial X' \times X_0)$ since the cones provide X' with directions — thus $\partial\widetilde{X}$ is $(r-1)$-spherical, $r = \operatorname{rank}_F G$. Unfortunately $\partial\widetilde{X}$ is not a finite complex modulo Γ, so we have to restrict our definitions: For $\operatorname{Opp} X_0$ we have to consider pairs of opposite $\mathcal{O}$-modules instead of subspaces (for $G = SL_n$), described locally by $U_P(\mathcal{O})$ instead of $U_P(F)$, denote this subcomplex by $\operatorname{Opp}_\Gamma X_0$ and the corresponding subcomplex of $\widetilde{X}$ by $\widetilde{X}_\Gamma$ — and now we can show that $\widetilde{X}_\Gamma \bmod \Gamma$ is finite. For $G = SL_n$ also $\operatorname{Opp}_\Gamma X_0$ is $(n-2)$-spherical, in general no direct proof is known, but $\widetilde{X}_\Gamma$ is a retract of $\widetilde{X}$ (which is not true for the pair $\operatorname{Opp}_\Gamma X_0$ and $\operatorname{Opp} X_0$!).

4. Application: The techniques described in section 3 allow us to prove that for an almost simple Chevalley group G over F the arithmetic subgroups Γ, defined by one valuation of F, are of type F_{r-1}, $r = \operatorname{rank}_F G$ — it remains open if they are not of type F_r. We hope to treat also S-arithmetic groups for $|S| > 1$ and non-split groups: for the state of this problem cf. [B].

5. Literature:

[B] H. Behr: *Higher finiteness properties of S-arithmetic groups in the function field case I,* in: Groups, Topological, Combinatorial and Arithmetic Aspects (ed. T.W. Müller), London MS Lect. Notes 311 (2004), 27 – 42

[G1] D. Grayson: *Finite generation of K-groups of a curve over a finite field,* Springer Lect. Notes 996 (1980), 69 – 90

[G2] D. Grayson: *Reduction theory using semi-stability, I, II,* Comm. Math. Helv. 59 (1984), 600 – 634 and 61 (1986), 661 – 676

[H] A. v. Heydebreck: *Homotopy properties of certain complexes associated to spherical buildings,* Israel J. Math. 133 (2003), 369 – 374

[St] U. Stuhler: *Canonical parabolic subgroups of Arakelov group schemes in the function field case,* Preprint Göttingen 2002

Compactifications of Bruhat-Tits buildings

Annette Werner

Let K be an non-archimedean local field, and let G be a reductive group over K. By $X(G)$ we denote the Bruhat-Tits building associated to G. It is a complete metric space with an isometric $G(K)$-action and a poly-simplicial structure. There are several ways of compactifiying $X(G)$, e.g. the Borel-Serre compactification (see [1]) or the polyhedral compactification due to Landvogt (see [2]).

In [4] I have constructed and investigated another compactification of $X(G)$ for the group $G = PGL(V)$, where V is a finite dimensional K-vector space. As boundary components all Bruhat-Tits buildings corresponding to groups $PGL(W)$ appear, where W runs over the non-trivial linear subspaces of V. In order to prove that this space is compact and carries a continuous action by the group $PGL(V)$ one has to show a mixed Bruhat decomposition theorem involving the subgroups of $PGL(V)$ appearing as stabilizers of the boundary points. A related compactification of the vertex set of $X(PGL(V))$ was previously sketched in [3].

By work of Goldman and Iwahori, the building $X(PGL(V))$ can be identified with the space of norms on V up to scaling. In [5] I have investigated a dual version $\overline{X(PGL(V))}$ of the compactification previously described, where the boundary components are the Bruhat-Tits buildings corresponding to the groups $PGL(W)$ for the quotient spaces W of V. It turns out that $\overline{X(PGL(V))}$ has a natural topological and $PGL(V)$-equivariant identification with the space of seminorms on V up to scaling.

Besides, it is shown in [5] that in the world of p-adic analytic Berkovich spaces, the reduction map from Drinfeld's p-adic upper half-plane to $X(PGL(V))$ has a natural extension to a map from the whole projective space to the compactification $\overline{X(PGL(V))}$. Moreover, this map identifies $\overline{X(PGL(V))}$ topologically with a closed subset of the projective Berkovich space.

In my talk I described a generalization of these results to Bruhat-Tits buildings associated with arbitrary split semisimple groups G. (The non-split case will be worked in later on.) Namely, for every irreducible algebraic representation $\rho : G \to GL(V)$ one can define a compactification $\overline{X(G)}^{\rho}$ of $X(G)$ using the combinatorics of the weights of ρ. Roughly, this is constructed as follows. Let A be the appartment corresponding to a maximal torus in G. Using the action of the weights of ρ on A, I define a cone decomposition of A, which gives rise to a compactification $\overline{A}$ of A carrying a natural action by the normalizer N of the torus. Then I define for all $x \in \overline{A}$ a certain subgroup P_x of $G(K)$ which later on turns out to be the stabilizer of x in the compactified building $\overline{X(G)}^{\rho}$. With these data one can imitate the definition of the Bruhat-Tits building in the following way: The compactification $\overline{X(G)}^{\rho}$ is defined as the quotient of $G(K) \times \overline{A}$ by the following equivalence relation

$$(g,x) \sim (h,y) \quad \text{iff there exists an element } n \in N \text{ such that } nx = y \text{ and } g^{-1}hn \in P_x.$$

In fact, the space $\overline{X(G)}^{\rho}$ depends only on the Weyl chamber face containig the highest weight of ρ. Hence we get a finite zoo of compactifications for each $X(G)$. In special cases, we rediscover the compactification for $G = PGL(V)$ discussed previously and also Landvogt's polyhedral compactification from [2].

Besides, I explained that this construction can be regarded as an analogon of Satake's compactifications of symmetric spaces. It is to be expected that $\overline{X(G)}^{\rho}$ can also be identified with a closed subset of some homogeneous Berkovich analytic space.

References

[1] A. Borel, J.-P. Serre: Cohomologie d'immeubles et de groupes S-arithmétiques. Topology 15 (1976), 211-232.

[2] E. Landvogt: A compactification of the Bruhat-Tits building. Lecture Notes in Mathematics 1619, Springer 1996.

[3] G.A. Mustafin: Nonarchimedean Uniformization. Math. USSR Sbornik 34 (1978), 187-214.

[4] A. Werner: Compactification of the Bruhat-Tits building of PGL by lattices of smaller rank. Doc. Math. 6 (2001), 315-342.

[5] A. Werner: Compactification of the Bruhat-Tits building of PGL by seminorms. To appear in Math. Zeitschrift.

Geometry of linear groups, invariant metrics on reductive groups

Herbert Abels

There are several natural pseudometrics on $GL(n, \mathbb{R})$. They have a geometric property in common, which we call coarsely geodesic. The main result is that every such pseudometric is normlike which means that it is determind by a norm on the vector space of diagonal matrices. This is the main result of the joint work [AM] with G.A. Margulis. It holds for reductive groups over local fields and more generally for cocompact proper isometric actions of such groups on pseudometric spaces.

We start out by giving the examples of pseudometrics we are interested in. We then define the notion of a normlike pseudometric. After stating the main result the notion of a coarsely geodesic pseudometric space will be explained.

A *pseudometric* $d : X \times X \to \mathbb{R}_+$ is a map which has all the properties of a metric except that $d(x, y) = 0$ does not imply $x = y$. A pseudometric d on a group is called *left invariant*, if $d(g_1 g, g_1 h) = d(g, h)$ for g_1, g, h in G. Then $d(g, h) = d(e, g^{-1}h)$.

An important left invariant metric on a group is the *word metric*. If Γ is a discrete group with a finite set A of generators then the word length of an element $\gamma \in \Gamma$ with respect to A is defined by

$$\ell_A(\gamma) = \min\{q; \gamma = a_1^{\varepsilon_1} \cdots a_q^{\varepsilon_q} \ , \ a_i \in A \ , \ \varepsilon_i \in \{+1, -1\}\}.$$

Then $d_{\text{word}}(\gamma_1, \gamma_2) = \ell_A(\gamma_1^{-1}\gamma_2)$ is a left invariant metric on Γ. The same definition works for a locally compact topological group G having a compact set A

of generators. To describe the dependence of these metrics on A the following definitions are useful.

Let (X, d) and (X', d') be pseudometric spaces. A map $f : X \to X'$ is called a *quasiisometry* if there are constants $C_1 \geq 0$ and $C_2 > 0$ such that

$$-C_1 + C_2^{-1} d(x, y) \leq d'(f(x), f(y)) \leq C_2 d(x, y) + C_1$$

for every x, y in X and the image $f(X)$ is C_1–dense in X', i.e., for every $x' \in X'$ there is an $x \in X$ such that $d(f(x), x') \leq C_1$. The map f is a *coarse isometry* if one can choose $C_2 = 1$, it is an *isometry* if $C_2 = 1$ and $C_1 = 0$.

Two pseudometrics on the same set X are called quasiisometric (coarsely isometric) if the identity map is a quasiisometry (coarse isometry).

Note that any two vector space norms on $\mathbb{R}^n$ are quasiisometric (even Lipschitz equivalent, i.e., we can choose $C_1 = 0$) but coarsely isometric only if they are equal.

Coming back to the word metric, it is easy to see that the word metrics on a finitely generated discrete group Γ for any two finite generating sets are quasiisometric, even Lipschitz–equivalent, and similarly on a compactly generated locally compact group, by a Baire category argument.

A second type of metrics comes from actions on metric spaces. Let the group G act by isometries on the pseudometric space (X, d). Pick a point $x_0 \in X$. Define the G–invariant pseudometric

$$d_{\mathrm{geom}}(g, h) := d(gx_0, hx_0) = d(g^{-1}hx_0, x_0)$$

on G. Examples to think of are the action of a semisimple real Lie group G on its symmetric space $X = G/K$ endowed with a G–invariant Riemannian metric or the action of a semisimple group G over a non–archimedian local field on its Bruhat–Tits building endowed with an affine metric. In both cases the pseudometric d_{geom} is not a metric, because every element of the isotropy group of x_0 has distance zero from the identity.

A third type of pseudometrics can be defined for a subgroup G of the general linear group $GL(V)$ where V is a finite dimensional vector space over a local field. Choose a vector space norm $\|\cdot\|$ on V. Then define the G–invariant pseudometric

$$d_{\mathrm{norm}}(g, h) = \sup\{|\log \|g^{-1}h\||, |\log \|h^{-1}g\||\}$$

on G. Since any two norms on V are Lipschitz equivalent, any two d_{norm} are coarsely isometric. This construction can be varied by taking for a given group G a representation $\rho : G \to GL(V)$.

For the general linear group $G = GL(n, k)$ over a local field k let $T(n, k)$ be the subgroup of invertible diagonal matrices. $T(n, k)$ contains a cocompact discrete subgroup D isomorphic to $\mathbb{Z}^n$. We call a G–invariant pseudometric d on G *normlike* if there is a vector space norm $\|\cdot\|$ on $D_{\mathbb{R}} := D \bigotimes_{\mathbb{Z}} \mathbb{R}$, such that $d \mid D \times D$ and the metric on D given by the norm are coarsely isometric, equivalently if the function

$$|d(e, d) - \|d\||$$

is bounded on D.

Note that if such a norm $\|\cdot\|$ on $D_{\mathbb{R}}$ exists, it is unique by the remark above. If follows that then $\|\cdot\|$ is invariant against permutation of the coordinates, since any inner automorphism of G is a coarse isometry (for any left invariant pseudometric on G). Note also that d is uniquely determined by $\|\cdot\|$ up to coarse isometry, by the Cartan decomposition $G = K \cdot D \cdot K$, where K is a compact subset of G.

It is easy to see that d_{geom} for the symmetric space and the Bruhat Tits building are normlike — here $\|\cdot\|$ is the Euclidean norm — and also d_{norm} with $\|\cdot\|$ the ℓ^∞–norm. This is also true for the word metric.

Theorem 3. (Abels–Margulis) *Let k be a local field, e.g. $k = \mathbb{R}, \mathbb{Q}_p$ or $\mathbb{F}_q(t)$. The word metric on $G = GL(n,k)$ with respect to a compact set of generators of G is normlike.*

This is a special case of the main result of [2]:

Theorem 4. (Abels–Margulis) *Let G be a group with a left invariant pseudometric d. Suppose G has a weak Cartan decomposition. If d is coarsely geodesic and satisfies the properness condition (P) then it is normlike.*

The concept of weak Cartan decomposition is quite technical and cannot be explained here. It involves a certain subgroup D playing a similar role as the group D in $GL(V)$ above. Every reductive group over a local field has a Cartan decomposition and hence a weak one. But also $G = \mathbb{Z}^n$ is admitted. The concept of a normlike pseudometric is then defined exactly as above.

A parametrized curve $c : [0,a] \to X$ in a pseudometric space (X,d) is called a *C–coarse geodesic* if $d(c(s),c(t)) =_C |s-t|$ for any two $s,t \in [0,a]$. The notation $a =_C b$ means $|a-b| \le C$. The space (X,d) is called *C–coarsely geodesic* if any two points in X can be joined by a C–coarse geodesic, i.e., for any x,y in X there is a curve $c : [0,a] \to X$ with $c(0) = x$, $c(a) = y$ and $d(c(s),c(t)) =_C |s-t|$ for any $s,t \in [0,a]$. In particular $a =_C d(x,y)$. The space is called *coarsely geodesic* if it is C–coarsely geodesic for some $C \ge 0$. Examples of coarsely geodesic pseudometric spaces are Riemannian and Finsler manifolds and a group G with a word metric. If two pseudometric spaces are coarsely isometric and one of them is coarsely geodesic then so is the other one. It follows that if a group G acts isometrically on a coarsely geodesic pseudometric space with a C–dense orbit, then G with the metric d_{geom} above is also coarsely geodesic. We thus obtain as a special case of our theorem Burago's result [1], namely that every $\mathbb{Z}^n$–invariant Riemannian metric on $\mathbb{R}^n$ is of bounded distance from a norm on $\mathbb{R}^n$.

References

[1] Abels, H., *Reductive groups as metric spaces*, proceedings of the Bielefeld conference 1999, Groups: topological, combinatorial and arithmetic aspects, London Mathematical Society Lecture Notes 311, pages 1–20, 2004. Edited by T.W. Müller.

[2] Abels, H. and G.A. Margulis *Coarsely geodesic metrics on reductive groups*, Preprint 04–001 of the Forschergruppe "Spectral analysis, asymptotic distributions and stochastic dynamics", Bielefeld.

[3] Burago, D., *Periodic metrics*, Adv. in Soviet Math. 9 (1992) 205–210.

Lie groups from afar

Katrin Tent
(joint work with Linus Kramer)

We use a generalization of the definition of asymptotic cones due to van den Dries and Wilkie to prove the following results.

Theorem 5. *([4]) If R is a real closed field, G is a semisimple R-isotropic algebraic group defined over R and $G(R)$ is equipped with a left-invariant norm-like metric, then the layers of $G(R)$ are affine Λ-buildings of the form $G(R^\alpha)/G(O)$ where R^α is a real closed field, $O \subseteq R^\alpha$ is a convex valuation ring and $\Lambda \cong R^{\alpha *}/O^*$ is an archimedean ordered abelian group.*

In particular, the asymptotic cone of a semisimple real Lie group $G(\mathbb{R})$ is of the form $G({}^\rho\mathbb{R}_\mu)/G(O)$ where ${}^\rho\mathbb{R}_\mu$ is Robinson's real closed valued field constructed from $\mathbb{R}$ using the ultrafilter μ used to define the asymptotic cone.

We apply this to prove

Theorem 6. *([4]) If R is a real closed field, G and H are semisimple R-isotropic algebraic groups defined over R and $G(R)$ and $H(R)$ are equipped with left-invariant norm-like metrics such that $f : G(R) \longrightarrow H(R)$ is a quasi-isometry (with respect to R), then G and H are isomorphic as algebraic groups. Furthermore, if $\bar{R}$ is the total competion of R, then there is an $\bar{R}$-rational isomorphism $g : G(\bar{R}) \longrightarrow H(\bar{R})$ which has R-bounded distance from f on $G(R)$.*

This generalizes results of Kleiner and Leeb [1] on quasi-isometries between Riemannian symmetric spaces and the Margulis Conjecture.

As the asymptotic cones are defined with respect to an ultrafilter μ, Gromov asked whether there are finitely presented groups whose asymptotic cone depends on μ. If Γ is a uniform lattice in $G(\mathbb{R})$, then Γ is finitely presented and $Cone(\Gamma) = Cone(G(\mathbb{R}))$. It follows from our description of $Cone(G(\mathbb{R}))$ that $Cone_\mu(G(\mathbb{R})) \cong Cone_{\mu'}(G(\mathbb{R}))$ if and only if ${}^\rho\mathbb{R}_\mu \cong {}^\rho\mathbb{R}_{\mu'}$. In joint work with S. Thomas and S. Shelah we show

Theorem 7. *([2]) The existence of ultrafilters μ, μ' with ${}^\rho\mathbb{R}_\mu \ncong {}^\rho\mathbb{R}_{\mu'}$ is equivalent to the negation of the Continuum Hypothesis (i.e., is equivalent to the statement $2^{\aleph_0} > \aleph_1$). Furthermore, if the Continuum Hypothesis holds (i.e., if $2^{\aleph_0} = \aleph_1$), then any finitely generated group has at most $2^{\aleph_0}$-many cones up to homeomorphism.*

References

[1] B. Kleiner, B. Leeb, *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings.* Inst. Hautes Études Sci. Publ. Math. No. 86 (1997) 115–197.

[2] L. Kramer, S. Shelah, K. Tent, S. Thomas, *Asymptotic cones of finitely presented groups*, to appear in Advances in Mathematics.

[3] L. Kramer, K. Tent, *Asymptotic cones and ultrapowers of Lie groups*, to appear in: Bulletin of Symbolic Logic.

[4] L. Kramer, K. Tent, *Quasi-isometries of real algebraic groups and affine* Λ*-buildings*, preprint.

Rigidity theorems for symmetric spaces

J.-H. Eschenburg

A *symmetric space* is a Riemannian manifold X with an isometric point reflection s_p at any point $p \in X$, i.e. $s_p \in G = I(X)$ (isometry group) with $s_p(p) = p$ and $(ds_p)_p = -I$. This notion has various aspects (cf. [3]) leading to different characterizations of symmetric spaces. These so called "rigidity theorems" state that certain geometric properties are fulfilled only by symmetric spaces. A common feature of all such theorems is that a certain dimension must be bounded from below in order to give enough room for the constructions. In some sense the first example was Desargues' theorem: Desargues' configuration holds in any projective space of dimension ≥ 3 causing it to be a projectivized vector space over some (skew) field; however this property may fail in dimension 2 (projective planes). A vast extension was given by Burns and Spatzier [2]; we will call it Theorem (A): Spherical buildings of rank ≥ 3 with a decent topology are always associated to symmetric spaces; for rank 2 this is wrong as the generalized polygons show, e.g. the generalized quadrangles corresponding to inhomogeneous isoparametric hypersurfaces (cf [9], [6]). A quite different rigidity theorem (B) was proved by Berger [1] and Simons [5] with a beautiful new proof of Olmos [10]. It characterizes an irreducible symmetric space of rank ≥ 2 by "small holonomy" where "small" means that the holonomy group does not act transitively on the unit sphere. Recall that the *holonomy group* at some point p of a Riemannian manifold consists of the parallel displacements along all loops starting and ending at p; it is a subgroup of the orthogonal group of the tangent space at p and measures the path dependence of the Riemannian parallel displacement. Again the dimension restriction is essential: The theorem holds if the codimension of the holonomy orbits is ≥ 2 but fails for codimension 1. There are two other classes of rigidity theorems which are all relying on either (A) or (B). In the first class (C) one assumes in particular that the manifold has rank ≥ 2, i.e. any geodesic lies in a totally geodesic flat subspace of dimension ≥ 2; again the dimension restriction is necessary. The second class (D) does not characterize symmetric spaces themselves by their geometric properties, but instead the principal orbits of their isotropy representations; these are the so called *isoparametric submanifolds*, and indeed any such submanifold is an isotropy orbit of a symmetric spaces provided that the codimension is ≥ 3; for codimension 2 the above mentioned inhomogeneous isoparametric hypersurfaces in spheres are counterexamples. For results of type (C) and (D) and references see [7] and [8].

The results mentioned so far could be called "absolute rigidity theorems" since the symmetry group has to be constructed out of the nowhere, using the geometric assumptions. I would like to finish this report with a new result which can be considered as a "relative rigidity theorem": The group is already given,

one only has to show invariance under this group. The theorem characterizes a certain subclass of symmetric spaces, the *extrinsic symmetric spaces*. A compact submanifold $M \subset V$ of a euclidean vector space V is called *extrinsic symmetric* if it is preserved by the reflection s_p at any of its normal spaces, i.e. s_p is the affine isometry s_p fixing p with $ds_p = I$ on the normal space N_pM and $ds_p = -I$ on the tangent space T_pM. Clearly such a space is symmetric when viewed as a Riemannian manifold with the metric induced from the ambient space. In fact most but not all symmetric spaces allow such an embedding, e.g. the Lie group $U(n) \subset \mathbb{C}^{n\times n}$ does, but $SU(n)$ does not. By a theorem of D. Ferus ([5], [4]), these spaces are known to be certain orbits of the isotropy representation of another symmetric space (of noncompact type) and hence they also allow an effective action of a noncompact Lie group containing the isometry group. The easiest example is the sphere $S^n \subset \mathbb{R}^{n+1}$ where the noncompact group is the conformal (Moebius) group. But also the projective spaces and more generally the Grassmannians are embedded as extrinsic symmetric spaces: just assign to each k-plane $E \subset \mathbb{R}^n$ the reflection at E which is an element of the vector space of symmetric matrices; the obvious noncompact group acting on Grassmannians is $PGL(n, \mathbb{R})$.

Theorem 8. *Let V be a euclidean vector space containing an irreducible extrinsic symmetric space M_o which is full, i.e. contained in no proper affine subspace of V. Let $M \subset V$ be another full submanifold of the same dimension such that each tangent space of M is also a tangent space of M_o:*

$$\{T_xM;\ x \in M\} \subset \{T_pM_o;\ p \in M_o\}.$$

If the codimension of M is ≥ 2, then M is an open subset of M_o (up to motions and rescaling).

Obviously the restriction of the codimension is necessary: If M is an arbitrary hypersurface (codimension 1), then each tangent space T_xM is also a tangent space of the sphere $M_o = S^n$, hence our assumption does not give any restriction in this case.

The proof uses the Lie triple product R on V given by Ferus' theorem. Since tangent and normal spaces are Lie subtriples, we can show $\nabla R = 0$ where ∇ denotes the Levi-Civita derivative for tangent and normal vectors of M. Hence the second fundamental form $L = \nabla - \partial$ satisfies $L_vR = 0$ for any tangent vector v, in other words, L_v is a derivation of R. This is the first step to show that the second fundamental forms of M and M_o agree, and the result follows from the congruence theorem for submanifolds in euclidean space.

References

[1] M. Berger, Bull. Soc. Math. France **83** (1955), 279 - 330.
[2] K. Burns, R. Spatzier, Publ. IHES **65** (1987), 5 - 34.
[3] J.-H. Eschenburg: *Lecture Notes on Symmetric Spaces*, Preprint.
[4] J.-H. Eschenburg, E. Heintze, manuscripta math. **88** (1995), 517 - 524, **92** (1997), 408.
[5] D. Ferus, Math. Ann. **246** (1980), 81 - 93.
[6] D. Ferus, H. Karcher, H.F. Münzner, Math. Z. 177 (1981), 479 - 502.
[7] J. Heber, invent. math. **112** (1993), 151 - 170.

[8] E. Heintze, X. Liu, Ann. of Math. **149** (1999), 149 - 181.
[9] S. Immervoll, J. reine angew. Math. **554** (2003), 1 - 17.
[10] C. Olmos: *On the geometry of holonomy systems*, Preprint.
[11] J. Simons, Ann. of Math. **76** (1962), 213 - 234.

Hyperbolic rank of euclidean buildings

Viktor Schroeder
(joint work with A. Dranishnikov)

We consider a finitely generated right angled Coxeter group Γ , i.e. a group Γ together with a finite set of generators S, such that every element of S has order two and that all relations in Γ are consequences of relations of the form $st = ts$, where $s, t \in S$.

We prove embedding results of the Cayley graph $C(\Gamma, S)$ into products of trees. On graphs and trees we consider always the simplicial metric, hence every edge has length 1. On a product of trees we consider the l_1-product metric, i.e. the distance is equal to the sum of the distances in the factors.

In [3] it was shown that the Cayley graph of a Coxeter group admits an equivariant isometric embedding into a finite product of locally infinite trees. Here we give a better estimate on the number of factors in the right-angled case. The estimate is given in terms of the chromatic number. Consider therefore *colourings* $c : S \to \{1, \dots, n\}$ with the property that for different $s, t \in S$ with $st = ts$ we have $c(s) \neq c(t)$. The minimal number n of colours needed is called the chromatic number of Γ.

Theorem 9. *Suppose that the chromatic number of a right-angled Coxeter group Γ is n. Then the Cayley graph $C(\Gamma, S)$ admits an equivariant isometric embedding into the product of n simplicial trees.*

Besides of trivial cases, these trees are locally infinite. However we are able to embed the Cayley graph bilipschitz into a product of locally compact trees.

Definition. A pointed simplicial tree (T, t_0) is called exponentially branching, if there exists a number $\sigma > 0$ such that every vertex $t \in T$ has more than $e^{\sigma d(t,t_0)}$ neighbours where d is the metric on T.

Theorem 10. *Let Γ be a right-angled Coxeter group with chromatic number n, let T be an exponentially branching locally compact simplicial tree, and let $r > 0$ be a number. Then there exists bilipschitz embedding $\psi : C(\Gamma, S) \to T \times \cdots \times T$ (n-factors), such that ψ restricted to every ball of radius r is isometric.*

It is an interesting open problem, if a corresponding embedding result holds for trees with bounded valence.

We can apply Theorem 10 for a special Coxeter group operating on the hyperbolic plane $\mathbb{H}^2$ and obtain:

Corollary. *For every exponentrially branching tree T there exists a bilipschitz embedding $\phi : \mathbb{H}^2 \to T \times T$.*

Combining this with a result of Brady and Farb we get the following higher dimensional version:

Corollary. *For every exponentrially branching tree T there exists a bilipschitz embedding $\psi : \mathbb{H}^n \to T \times \cdots \times T$ of the hyperbolic space $\mathbb{H}^n$ into the $2(n-1)$ fold product of T.*

It is an open question, if for $n \geq 3$ there is a bilipschitz embedding of $\mathbb{H}^n$ into the n-fold product of locally compact trees or more general for euclidean buildings. There are two partial results in this direction. In [2] it is show that there exists a quasiisometric embedding of $\mathbb{H}^n$ into an n-fold product of locally infinite trees. On the other hand a recent construction of Januszkiewicz and Swiatkowski [4] shows for every n the existence of a right angled Gromov hyperbolic Coxeter group with virtual cohomological dimension and colouring number equal to n. Combining Theorem 10 with that result we obtain:

Corollary. *For every exponentially branching tree T and any given number n there exits a Gromov hyperbolic group Γ_n with virtual cohomological dimension n and a bilipschitz embedding of the Cayley graph of Γ_n into the product $T \times \cdots \times T$ (n-factors).*

Corollary can be used to determine the hyperbolic rank (compare [1]) of a product of trees:

Corollary. *The hyperbolic rank of the product of n trees with exponential branching is $(n-1)$.*

References

[1] S. Buyalo, V. Schroeder, *Hyperbolic rank and subexponential corank of metric spaces*, GAFA 12 (2002), 293-306.

[2] S. Buyalo, V. Schroeder, *Embedding of hyperbolic spaces in the product of trees*, arXive:math.GT/03115242003.

[3] A. Dranishnikov, T. Januszkiewicz, *Every Coxeter group acts amenably on a compact space* Proceedings of the 1999 Topology and Dynamics Conference (Salt Lake City, UT), p. 135-141, 1999.

[4] T. Januszkiewicz , J. Swiatkowski, *Hyperbolic Coxeter groups*, Commentarii Math. Helvetici, to appear.

Two-step nilpotent Lie algebras of type (p, q)

Patrick Eberlein

Definition and examples

A Lie algebra $\mathfrak{N}$ is *$2-$step nilpotent* if the commutator ideal $[\mathfrak{N}, \mathfrak{N}]$ lies in the center of $\mathfrak{N}$. A 2-step nilpotent Lie algebraLet $\mathfrak{N}$ is of *type* (p, q) if $[\mathfrak{N}, \mathfrak{N}]$ has dimension p and codimension q in $\mathfrak{N}$. Let N(p,q) denote the space of 2-step nilpotent real Lie algebras of type (p,q). We first describe a simple class of examples in N(p,q).

Let $\langle\ ,\ \rangle$ denote the standard inner product on $\mathbb{R}^q$, and let $\langle\ ,\ \rangle^*$ denote the inner product on $\mathfrak{so}(q,\mathbb{R})$ given by $\langle\ Z,\ Z'\ \rangle = -$ trace ZZ'. Every 2-step nilpotent Lie algebra $\mathfrak{N}$ in N(p,q) can be expressed (up to Lie algebra isomorphism) as a vector space $\mathfrak{N} = \mathbb{R}^q \oplus W$, where W is a p-dimensional subspace of $\mathfrak{so}(q,\mathbb{R})$, together with a bracket structure defined by the conditions

1) W lies in the center of $\mathfrak{N}$

2) For each X,Y in $\mathbb{R}^q$, [X,Y] is the unique element of W such that $\langle [X,Y], Z\rangle^* = \langle Z(X), Y\rangle$ for all Z in W.

Remarks 1)It is easy to see that $[\mathfrak{N},\mathfrak{N}] = W$ in the examples above. Moreover, $W = \mathfrak{Z}$, the center of $\mathfrak{N}$, $\iff \{0\} = \text{Ker } W = \{u \in \mathbb{R}^q : Z(u) = 0 \text{ for all } Z \in W\}$.

2) The most interesting examples occur when W is a special subspace of $\mathfrak{so}(q,\mathbb{R})$. One interesting class (Heisenberg type) arises from representations $j : C\ell(p) \to \text{End}(\mathbb{R}^q)$, where $C\ell(p)$ denotes the classical negative definite Clifford algebra defined by $\mathbb{R}^p$ with its natural inner product. In this case there exists an inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^q$ such that $W = j(\mathbb{R}^p) \subset j(C\ell(p))$ is a p-dimensional subspace of $\mathfrak{so}(q,\mathbb{R})$ with the property that $j(Z)^2$ is a negative multiple of the identity for any nonzero element Z of $\mathbb{R}^p$. Conversely, if W is a p-dimensional subspace of $\mathfrak{so}(q,\mathbb{R})$ with the property that Z^2 is a negative multiple of the identity for every nonzero element Z of W, then W arises as above from a representation $j : C\ell(p) \to \text{End}(\mathbb{R}^q)$ of the Clifford algebra $C\ell(p)$.

Another interesting class of subspaces W of $\mathfrak{so}(q,\mathbb{R})$ occurs when W is a Lie subalgebra of $\mathfrak{so}(q,\mathbb{R})$. These arise from representations $\rho : G \to GL(\mathbb{R}^q)$, where G is a compact, connected Lie group. If $\mathbb{R}^q$ is equipped with a $\rho(G)$-invariant inner product $\langle\ ,\ \rangle$, then W, the Lie algebra of $\rho(G)$, is a subalgebra of $\mathfrak{so}(q,\mathbb{R})$.

A subspace W of $\mathfrak{so}(q,\mathbb{R})$ is called a *Lie triple system* if $[W,[W,W]] \subseteq W$. Clearly every subalgebra W of $\mathfrak{so}(q,\mathbb{R}$ is a Lie triple system, but the converse is false. For example, the subspace W arising as above from a Clifford algebra representation $j : C\ell(p) \to \text{End}(\mathbb{R}^q)$ is a Lie triple system in $\mathfrak{so}(q,\mathbb{R})$ but not a subalgebra of $\mathfrak{so}(q,\mathbb{R})$.

The space $X(p,q)$ of isomorphism classes in $N(p,q)$ ([E3])

Let X(p,q) denote the space of isomorphism classes in N(p,q). Using the notation above, let $\mathfrak{N}_1 = \mathbb{R}^q \oplus W_1$ and $\mathfrak{N}_2 = \mathbb{R}^q \oplus W_2$ be two elements of N(p,q), where W_1 and W_2 are p-dimensional subspaces of $\mathfrak{so}(q,\mathbb{R})$. One can show that $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are isomorphic $\iff W_2 = gW_2g^t$ for some element g in $GL(q,\mathbb{R})$. It follows that X(p,q) can be identified with the compact coset space $G(p,\mathfrak{so}(q,\mathbb{R}))$ / $GL(q,\mathbb{R})$, where $GL(q,\mathbb{R})$ acts on $\mathfrak{so}(q,\mathbb{R})$ by $g(Z) = gZg^t$ for $g \in GL(q,\mathbb{R})$ and $Z \in \mathfrak{so}(q,\mathbb{R})$.

We define the dimension of the coset space $X(p,q) = G(p,\mathfrak{so}(q,\mathbb{R}))$ / $GL(q,\mathbb{R})$ to be the codimension of a generic $GL(q,\mathbb{R})$ orbit in $G(p,\mathfrak{so}(q,\mathbb{R}))$. The pairs (p,q) where X(p,q) has dimension zero (i.e. where $GL(q,\mathbb{R})$ has an open orbit in $G(p,\mathfrak{so}(q,\mathbb{R}))$) are of particular interest. The following is a complete list of such pairs, including a "duality" which follows from the fact that X(p,q) and X(D-p,q) are always homeomorphic, where D = (1/2)q(q-1).

Zero dimensional examples

[1] (1, q) and (D−1,q) $q \geq 2$

[2] (D, q) $q \geq 2$ (free 2-step nilpotent Lie algebras)

[3] (2, 2k+1) and (D−2,2k+1), $k \geq 1$

[4] (2, 4) and (4,4)

[5] (2, 6) and (13,6)

[6] (3, 4) (self dual)

[7] (3, 5) and (7,5)

[8] (4, 5) and (6,5)

We complete the description of the dimension of X(p,q) : dim X(3,6) = dimX(12, 6) = 2 ; dim X(2, 2k) = dim X($2k^2 - k - 2$, 2k) = k − 3 for $k \geq 4$ and dim X(p,q) = p(D−p) + 1 − $q^2 > 0$ for all remaining pairs (p,q).

Lattices and the Mal′cev criterion ([E2])

Let N be a simply connected nilpotent Lie group, and let $\mathfrak{N}$ denote the Lie algebra of N. Recall that exp : $\mathfrak{N} \to$ N is a diffeomorphism, and log : N $\to \mathfrak{N}$ denotes the inverse of exp. A *lattice* in N is a discrete cocompact subgroup Γ. A basis $\mathfrak{B}$ of $\mathfrak{N}$ defines a *rational structure* on $\mathfrak{N}$ if the structure constants of $\mathfrak{B}$ are rational numbers, or equivalently, if $\mathfrak{N}_{\mathbf{q}} = \mathbf{q}$-span$\{\mathfrak{B}\}$ is a Lie algebra over $\mathbf{q}$. A result of Mal'cev states that N admits a lattice Γ $\iff$ $\mathfrak{N}$ admits a rational structure. If Γ is a lattice in N, then $\mathfrak{N}_{\mathbf{q}} = \mathbf{q}$-span(log Γ) is a rational structure in $\mathfrak{N}$. There is a one-one correspondence between rational structures in $\mathfrak{N}$ and commensurability classes of lattices in N.

If $\mathfrak{N} = \mathbb{R}^q \oplus$ W, where W $\subset \mathfrak{so}$(q,$\mathbb{R}$) is a Lie triple system with compact center, then $\mathfrak{N}$ admits a rational structure. See [E4].

Since the number of matrices with rational entries that describe structure constants is countable it follows that only countably many elements of X(p,q) admit a rational structure. In particular, if X(p,q) has positive dimension, then a generic element $\mathfrak{N}$ of N(p,q) determines an isomorphism class [$\mathfrak{N}$] in X(p,q) with no rational structure. On the other hand, if X(p,q) has dimension zero, then a generic element $\mathfrak{N}$ of N(p,q) determines an isomorphism class [$\mathfrak{N}$] in X(p,q) with a rational structure.

Equivalence classes of rational structures ([E1], section 5)

A subspace W of $\mathfrak{so}$(q,$\mathbb{R}$) is said to be a *standard rational* subspace if W has a basis $\mathfrak{B}$' of matrices with rational entries ; that is, a basis $\mathfrak{B}$' in $\mathfrak{so}$(q,$\mathbf{q}$) It is easy to check that the $\mathfrak{N} = \mathbb{R}^q \oplus$ W as defined above has a rational structure $\mathfrak{N}_{\mathbf{q}} = \mathbf{q}$-span$\{\mathfrak{B}\}$, where $\mathfrak{B}$ is the union of $\mathfrak{B}$' and the natural basis $\{e_1, \ldots e_q\}$ of $\mathbb{R}^q$. Conversely, one may show that if $\mathfrak{N}$ is a 2-step nilpotent Lie algebra of type (p,q) that admits a rational structure, then $\mathfrak{N}$ is isomorphic to a Lie algebra $\mathbb{R}^q \oplus$ W for some standard rational subspace W of $\mathfrak{so}$(q,$\mathbb{R}$).

Let $\mathfrak{N}$ be a 2-step nilpotent Lie algebra of type (p,q) that admits a rational structure, and let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be bases of $\mathfrak{N}$ such that $\mathfrak{N}_{1,\mathbf{q}} = \mathbf{q}$-span$\{\mathfrak{B}_1\}$ and $\mathfrak{N}_{2,\mathbf{q}} = \mathbf{q}$-span$\{\mathfrak{B}_2\}$ are rational structures for $\mathfrak{N}$. We say that $\mathfrak{N}_{1,\mathbf{q}}$ and $\mathfrak{N}_{2,\mathbf{q}}$ are

equivalent rational structures if there exists an automorphism φ of $\mathfrak{N}$ such that $\varphi(\mathfrak{N}_{1,\mathbf{q}}) = \mathfrak{N}_{2,\mathbf{q}}$.

To determine the space of equivalent rational structures on $\mathfrak{N}$ we may assume without loss of generality that $\mathfrak{N} = \mathbb{R}^q \oplus$ W, where W is a standard rational subspace of $\mathfrak{so}$(q,$\mathbb{R}$). Let G denote GL(q,$\mathbb{R}$) and let $G_{\mathbf{q}}$(W) denote those subspaces in G(W) = {gWg^t : g $\in$ G} that are standard rational. Clearly GL(q,$\mathbf{q}$) leaves $G_{\mathbf{q}}$(W) invariant.

Proposition Let $\mathfrak{N} = \mathbb{R}^q \oplus$ W be a 2-step nilpotent Lie algebra of type (p,q), where W is a p-dimensional standard rational subspace of $\mathfrak{so}$(q,$\mathbb{R}$). Then the space of rational structures on $\mathfrak{N}$ may be identified with the coset space $G_{\mathbf{q}}$(W) / GL(q,$\mathbf{q}$).

Ricci tensor ([E1],[E2])

Let $\mathfrak{N}$ be a 2-step nilpotent Lie algebra of type (p,q), and let $\langle\ ,\ \rangle$ be an inner product on $\mathfrak{N}$. This defines a unique left invariant inner product on N, the simply connected Lie group with Lie algebra $\mathfrak{N}$. Let $\mathfrak{Z}$ denote the center of $\mathfrak{N}$, and let $\mathcal{V}$ denote the orthogonal complement of $\mathfrak{Z}$ in $\mathfrak{N}$. It is known that $\mathcal{V}$ and $\mathfrak{Z}$ are orthogonal with respect to the Ricci tensor Ric. Moreover, Ric is negative definite on $\mathbb{R}^q$ and positive semidefinite on W. It is of interest to find inner products $\langle\ ,\ \rangle$ on $\mathfrak{N}$ such that Ric has special properties. In particular we say that Ric is *optimal* if Ric is a negative multiple of the identity on $\mathcal{V}$ and a positive multiple of the identity on $\mathfrak{Z}$. We say that Ric is *geodesic flow invariant* if the Ricci curvature in TN is constant along orbits of the geodesic flow in TN.

Existence of elements of $X(p,q)$ with special Ricci tensors 9 ([E1] , section 7)

Let $\mathfrak{N}$ be a 2-step nilpotent Lie algebra of type (p,q). A basis

$$\mathfrak{B} = \{v_1,\ \ldots\ , v_q\ ;\ Z_1,\ \ldots\ , Z_p\}$$

of $\mathfrak{N}$ is said to be an *adapted* basis of $\mathfrak{N}$ if $\{Z_1,\ \ldots\ , Z_p\}$ is a basis of $[\mathfrak{N},\mathfrak{N}]$. Let $\{C^1,\ \ldots\ , C^p\} \subset \mathfrak{so}$(q,$\mathbb{R}$) be the structure matrices defined by the bracket relations

$$[\mathrm{v}_i,\mathrm{v}_j] = \textstyle\sum_{k=1}^{p} C_{ij}^k Z_k$$

Define $C_{\mathfrak{B}} = (C^1,\ \ldots\ , C^p) \in \mathfrak{so}(\mathrm{q},\mathbb{R})^p = \mathfrak{so}$(q,$\mathbb{R}$) x ... x $\mathfrak{so}$(q,$\mathbb{R}$) (p-times). Extend the action of SL(q,$\mathbb{R}$) on $\mathfrak{so}$(q,$\mathbb{R}$) to the diagonal action on $\mathfrak{so}$(q,$\mathbb{R}$)^p. Recall that $\mathfrak{so}$(q,$\mathbb{R}$)^p is isomorphic to $\mathfrak{so}(q,\mathbb{R}) \otimes \mathbb{R}^p$ under the map $(C^1,\ \ldots\ , C^p) \to \sum_{k=1}^{p} C^k \otimes v_k$, where $\{v_1,\ \ldots\ , v_p\}$ is any basis of $\mathbb{R}^p$. We obtain an action of SL(q,$\mathbb{R}$) x SL(p,$\mathbb{R}$) on $\mathfrak{so}$(q,$\mathbb{R}$) $\otimes\mathbb{R}^p$ such that (g,h)$(C \otimes v)$ = (g(C) $\otimes$ h(v)) for all (g,h) $\in$ SL(q,$\mathbb{R}$) x SL(p,$\mathbb{R}$), $C \in \mathfrak{so}$(q,$\mathbb{R}$) and v $\in \mathbb{R}^p$.

We now relate the existence of inner products $\langle\ ,\ \rangle$ on $\mathfrak{N}$ with special Ricci tensors to closed orbits of the groups SL(q,$\mathbb{R}$) and SL(q,$\mathbb{R}$) x SL(p,$\mathbb{R}$) acting on $\mathfrak{so}(q,\mathbb{R})^p$.

Proposition Let $\mathfrak{N}$ be a 2-step nilpotent Lie algebra of type (p,q), and let $\mathfrak{Z}$ denote the center of $\mathfrak{N}$. Let $\mathfrak{B}$ be an adapted basis of $\mathfrak{N}$ and let $C_{\mathfrak{B}} \in \mathfrak{so}(q,\mathbb{R})^p$ be as above. Then the following statements are equivalent :

1) $[\mathfrak{N},\mathfrak{N}]=\mathfrak{Z}$ and $\mathfrak{N}$ admits an inner product $\langle\ ,\ \rangle$ whose Ricci tensor is geodesic flow invariant.

2) The SL(q,$\mathbb{R}$) orbit of $C_{\mathfrak{B}}$ in $\mathfrak{so}(q,\mathbb{R})^p$ is closed in $\mathfrak{so}(q,\mathbb{R})^p$.

Proposition Let $\mathfrak{N}$ be a 2-step nilpotent Lie algebra of type (p,q), and let $\mathfrak{Z}$ denote the center of $\mathfrak{N}$. Let $\mathfrak{B}$ be an adapted basis of $\mathfrak{N}$ and let $C_{\mathfrak{B}}\in\mathfrak{so}(q,\mathbb{R})^p$ be as above. Then the following statements are equivalent :

1) $[\mathfrak{N},\mathfrak{N}]=\mathfrak{Z}$ and $\mathfrak{N}$ admits an inner product $\langle\ ,\ \rangle$ whose Ricci tensor is optimal.

2) The SL(q,$\mathbb{R}$) x SL(q,$\mathbb{R}$) orbit of $C_{\mathfrak{B}}$ in $\mathfrak{so}(q,\mathbb{R})^p$ is closed in $\mathfrak{so}(q,\mathbb{R})^p$.

References

[E1] P. Eberlein, " Geometry of 2-step nilpotent Lie groups ", accepted for publication, available on author's website.

[E2] ———, " The geometry of 2-step nilpotent Lie groups with a left invariant metric ", Ann. Ecole Normale Sup. 27 (1994), 611-660.

[E3] ———, " The moduli space of 2-step nilpotent Lie algebras of type (p,q) ", Contemp. Math 332 (2003), 37-72.

[E4] ———, " Riemannian submersions and lattices in 2-step nilpotent Lie groups ", Commun. in Analysis and Geom.11 (2003),411-488.

Spherical buildings, submetries and isoparametric foliations

Alexander Lytchak

We discuss a rigidity result for spherical building that is motivated by a theorem of Leeb ([6]) and a theorem of Eberlein ([3]), closely related to the rank rigidity. The theorem of Leeb describes the structure of a Hadamard space if the Tits geometry of the boundary at infinity is a spherical building and the result of Eberlein can be interpreted in a similar way.

Our theorems describe the structure of surjective 1-Lipschitz maps of spherical buildings onto geodesically complete $CAT(1)$ spaces. The connections with the theorems mentioned above is provided by the fact, that there are canonical (logarithmic) map from the boundary at infinity of a Hadamard spaces to the links and that these maps are 1-Lipschitz and surjective if the Hadamard space is geodesically complete. This work is a continuation of [7], where it was shown that if $f:G\to X$ is a surjective 1-Lipschitz map of a spherical building G of dimension ≥ 1 and X is a geodesically complete $CAT(1)$ space of finite dimension ([5]), then X is a spherical join or a building. We prove the following results, that also give a precise description of the map f:

Theorem 11. *Let G be an irreducible building, $f:G\to X$ a surjective 1-Lipschitz map onto a geodesically complete finite dimensional $CAT(1)$ space X. Then f splits as $f=\hat{f}\circ\tilde{f}$, where $\hat{f}:G\to G^f$ is a submetry onto a simplicial space of the same dimension as G and $\tilde{f}:G^f\to X$ is bijective and 1-Lipschitz. The space G^f is a building unless G is a generalized $6k$-gon and X is the Euclidean sphere S^{13}.*

Theorem 12. *Let $f : G \to X$ be a bijective 1-Lipschitz map. If G is an irreducible building of dimension ≥ 1 and X is a geodesically complete finite dimensional $CAT(1)$ space, then either f is an isometry or X is a Euclidean sphere S^l and f corresponds to an isoparametric foliation.*

The two types of surjective 1-Lipschitz maps that appear in the theorems above (folding maps between buildings of the same dimension and bijective maps induced by isoparametric foliations) correspond precisely to the logarithmic maps of the boundary at infinity onto a link in an affine building resp. in a symmetric space.

The proof of the first theorem resembles and is connected to the theorem of Thorbergsson ([8]) and uses essentially the results of [4] and [2]. The proof of the second theorem is established by finding relations between surjective 1-Lipschitz maps as above, special submetries of spheres and isoparametric foliations and by the observation that submetries of spheres are very rigid. The second theorem can be considered as a metric version of the topological result of [1].

References

[1] K. Burns and R. Spatzier. On topological Tits buildings and their classification. *Inst. Hautes tudes Sci. Publ. Math*, 65:5–34, 1987).

[2] R. Charney and A. Lytchak. Metric characterizations of spherical and Euclidean buildings. *Geom. Topol.*, 5:521–550, 2001.

[3] P. Eberlein. Symmetry diffeomorphism group of a manifold of nonpositive curvature. ii. *Indiana Univ. Math. J.*, 37:735–752, 1988.

[4] S. Immervoll. Isoparametric hypersurfaces and smooth generalized quadrangles. *J. Reine Angew. Math.*, 2003.

[5] B. Kleiner. The local structure of length spaces with curvature bounded above. *Math. Z.*, 231:409–456, 1999.

[6] B. Leeb. A characterization of irreducible symmetric spaces and Euclidean buildings of higher rank by their asymptotic geometry. Habilitation, 1997.

[7] A. Lytchak. Rigidity of spherical buildings and joins. Preprint.

[8] G. Thorbergsson. Isoparametric foliations and their buildings. *Ann. of Math*, 1991.

Randomly generated subgroups of $\mathrm{Aut}(T)$

Yair Glasner

(joint work with Miklós Abért)

Randomly generated groups were studied extensively in the setting of finite and pro-finite groups. In particular random generation was studied in the automorphism of rooted trees (see for example [1] [2]).

Let T be a (bi-)regular tree and $A = \mathrm{Aut}(T)$ its automorphism group. We investigate properties of randomly generated subgroups of A. Our sample space, after fixing the number of generators to be n, is the group A^n with its (infinite)

Haar measure μ. One says that a group property is generic if it holds for the group $\langle \mathbf{a} \rangle = \langle a_1, a_2, \ldots, a_n \rangle$ for almost all $\mathbf{a} \in A^n$. [1]

Theorem 13. *A randomly generated subgroup of* $\mathrm{Aut}(T)$ *will generically:*

(1) *be a non-Abelian free group.*
(2) *have one of the following as its closure:*
   - *A discrete group.*
   - *The whole group A, or its index* 2 *subgroup.*
   - *A compact group.*
(3) *Act almost freely on the vertices.*

Definition. A group acts *almost freely* on a set if every non trivial element fixes only a finite number of points. In our case this is the same as saying that every elliptic element fixes no points on the boundary ∂T.

The structure of these groups.
Discrete free groups. These are fundamental groups of regular graphs. The graphs will typically be infinite but they will admit a strong deformation retract to a finite graph.
Dense groups. By Bass-Serre theory a dense free subgroup in $\mathrm{Aut}^0(T)$ decomposes as an amalgamated free product $F_n = A *_C B$, where $A, B, C < F$ are all countably generated free groups. All the groups A, B, C are proper subgroups of F but they map onto every proper image of F.
Pre-compact subgroups. These fix a vertex (or a geometric edge) so one can think of them as random subgroups acting on a rooted tree. The closure of these groups will never be the full automorphism group of the rooted tree because the later is not finitely generated as a topological group. One should consult [1] and the references therein for what is known about randomly generated subgroups of rooted trees. I mention just a two results from that paper:

- A random subgroup admits a maximal Hausdorff dimension.
- The quotient of the tree by the action of a random subgroup will be a tree with finitely many ends.

We can generalize the second result and show that the quotient of the tree by a non-cyclic subgroup of a randomly generated subgroup will have finitely many ends.
Free action The key point of the proof. Here is an example of the arguments involved. Choose a word in the free group, say $\omega = aab \in F(a, b)$ and a sequence of different vertices (v_0, v_1, v_2). Consider

$$(s,t)C = \{(a,b) \in A^2 | v_0 \xrightarrow{b} v_1 \xrightarrow{a} v_2 \xrightarrow{a} v_0\}.$$

The notation $(s,t)C$ signifies a coset of the subgroup

$$C = \mathrm{Stab}(\{v_1, v_2\}) \times \mathrm{Stab}(\{v_0\}) < A^2 \,.$$

[1] Another possible interpretation of randomness is the purely topological one, say that a group property is generic if it holds for all but a set of the first category in A^n. All of the theorems that I will state hold also in the topological setting.

Fix (s,t) once and for all thus pushing all of the randomness into our choice of elements $(x,y) \in C$. Our goal is to show that there is only a null set of elements $(x,y) \in C$ such that $\omega(sx,ty)$ has infinitely many fixed points.

After some rearrangement one can write

$$\omega(sx,ty) = sxsxty = (sst)(x^{t^{-1}s^{-1}})(x^{t^{-1}})(y) = \alpha_0 x^{\alpha_1} x^{\alpha_2} y$$

as a product of elements fixing the point v_0. By induction on the length of the word we can assume that if $d(v_0,w_0)$ is big enough then the points $w_0 \stackrel{\alpha_0}{\to} w_1 \stackrel{x^{\alpha_1}}{\to} w_2 \stackrel{x^{\alpha_2}}{\to} w_3$ will all be distinct. Consider the corresponding maps induced on the "shadows":

$$\mathrm{Sh}(v_0,w_0) \stackrel{\alpha_0}{\to} \mathrm{Sh}(v_0,w_1) \stackrel{x^{\alpha_1}}{\to} \mathrm{Sh}(v_0,w_2) \stackrel{x^{\alpha_2}}{\to} \mathrm{Sh}(v_0,w_3) \stackrel{y}{\to} \mathrm{Sh}(v_0,w_0).$$

Here $\mathrm{Sh}(v_0,w) = \{v | w \in [v,v_0]\}$. The map y is independent of all the other maps so the product $\omega(sx,ty)$ is a random element when restricted to each one of these shadows. This argument concludes the proof for this example. For more general words though we can not assume that one of the maps is independent of all the others. The maps $x^{\alpha_1}, x^{\alpha_2}$ for example need not be independent, in fact one of them might even be a function of the other. Even if one can not establish independence one can always find one of the words that is independent of the others when restricted to the l'th level of the shadow for each l. As it turns out this is enough to finish the proof.

Classification of closures: Consider a free group generated by 2 random elements (a,b). There are 3 different possibilities: both are elliptic, both are hyperbolic, one is elliptic and one is hyperbolic. The most interesting part of the theorem is to prove that an elliptic and a hyperbolic element almost surly generate a dense subgroup. After establishing this then more or less standard arguments show that:

- If both (a,b) are elliptic but don't have a common fixed point then (a,ab) is (elliptic,hyperbolic).
- If both (a,b) are hyperbolic but don't generate a discrete free group then after applying a sequence of Nielsen transformations one can reduce to the (elliptic,hyperbolic) case.

References

[1] M. Abért and B Virág, *Dimension theory of pro-p groups acting on rooted trees*, Preprint, 2002.

[2] Meenaxi Bhattacharjee, *The ubiquity of free subgroups in certain inverse limits of groups*, J. Algebra **172** (1995), no. 1, 134–146. MR **96c:**20044

Limit groups and free actions on $\mathbb{R}^n$-trees

Vincent Guirardel

In his first paper about the Tarski problem, [13], Sela introduced the notion of limit group. These groups appeared to coincide with the long-studied class of

finitely generated fully residually free groups (see [2], [1], [8, 9], [4] and references). In a joint work with Champetier ([3]), we interpret the set of limit groups as a compactification of the set of marked free group in the compact set of marked groups.

One can give several other equivalent characterizations of limit groups ([11]): limit groups are the finitely generated subgroups of a non-standard free group, and are the finitely generated groups having the same universal theory as a free group.

One major result about limit groups is the fact that they are finitely presented ([8, 9, 13]). More precisely, one can prove that for every limit group G there is a complexity $C(G) \in \mathbb{N}$ such that

(1) limit groups of complexity 0 are free products of free abelian groups and surface groups
(2) if $C(G) > 0$, then G can be written as the fundamental group of a graph of groups with trivial or cyclic edge groups, and whose vertex groups are limit groups of lower complexity.

In particular, G has a finite classifying space.

We give a new proof of this result using the fact that a limit group has a free action on an $\mathbb{R}^n$-tree ([10]).

Theorem 14. *([7],[5]) Any finitely generated group having a free action on an $\mathbb{R}^n$-tree ($n \geq 2$) can be written as the fundamental group of a graph of groups with trivial or cyclic edge groups, and whose vertex groups are finitely generated and have a free action on an $\mathbb{R}^{n-1}$-tree.*

The proof is based on Sela's version of Rips Theory for finitely generated groups acting on $\mathbb{R}$-trees.

References

[1] Gilbert Baumslag. On generalised free products. *Math. Z.*, 78:423–438, 1962.
[2] Benjamin Baumslag. Residually free groups. *Proc. London Math. Soc. (3)*, 17:402–418, 1967.
[3] Christophe Champetier and Vincent Guirardel. Limit groups as limits of free groups: compactifying the set of free groups. To appear in Israel Journal of Math.
[4] Ian Chiswell. *Introduction to Λ-trees*. World Scientific Publishing Co. Inc., River Edge, NJ, 2001.
[5] Shalom Gross. *Group actions on Λ-trees*. PhD thesis, Hebrew University, Jerusalem, 1998.
[6] Vincent Guirardel. Rips theory for actions of finitely generated groups on $\mathbb{R}$-trees. In preparation.
[7] Vincent Guirardel. Limit groups and groups acting freely on $\mathbb{R}^n$-trees. Preprint.
[8] O. Kharlampovich and A. Myasnikov. Irreducible affine varieties over a free group. I. Irreducibility of quadratic equations and Nullstellensatz. *J. Algebra*, 200(2):472–516, 1998.
[9] O. Kharlampovich and A. Myasnikov. Irreducible affine varieties over a free group. II. Systems in triangular quasi-quadratic form and description of residually free groups. *J. Algebra*, 200(2):517–570, 1998.
[10] V. N. Remeslennikov. $\exists$-free groups. *Siberian Math. J.*, 30(6):998–1001, 1989. translation (1990) from Sibirsk. Mat. Zh. **30** (1989), 193–197.
[11] V. N. Remeslennikov. $\exists$-free groups as groups with a length function. *Ukrainian Math. J.*, 44(6):733–738, 1992. Translation (1993) from Ukraïn. Mat. Zh. **44**(6):813-818, 1992.

[12] Z. Sela. Acylindrical accessibility for groups. *Invent. Math.*, 129(3):527–565, 1997.
[13] Zlil Sela. Diophantine geometry over groups. I. Makanin-Razborov diagrams. *Publ. Math. Inst. Hautes Études Sci.*, 93:31–105, 2001.

Some results on groups acting on trees and Moufang polygons

Richard Weiss

We discuss various connections between Moufang polygons and the action of groups on trees, for example:

Theorem 15. *([2]). Let Γ be an arbitrary graph (in particular, Γ may be a tree), let G be a subgroup of $\mathrm{Aut}(\Gamma)$ and let n be an integer greater than two. Suppose that for every path $(x_0, x_1, \ldots, x_n)$ in Γ,*
(i) $G^{[1]}_{x_1,\ldots,x_{n-1}}$ acts transitively on $\Gamma_{x_n}\backslash\{x_{n-1}\}$ and
(ii) $G_{x_0,x_1} \cap G_{x_0,\ldots,x_n} = 1$

(where $G^{[1]}_x$ denotes the pointwise stabilizer of $\{x\} \cup \Gamma_x$ and $G^{[1]}_{x,\ldots,y} = G^{[1]}_x \cap \cdots \cap G^{[1]}_y$ for all vertices $x, \ldots, y$ of Γ). Then there is a G-invariant equivalence relation $\equiv$ on the vertex set $V(\Gamma)$ of Γ such that $\Gamma/\equiv$ is a generalized polygon (where the two equivalence classes are joined by an edge in $\Gamma/\equiv$ whenever there is some edge of Γ joining a vertex in the one equivalence class with a vertex in the other) and the map from Γ to $\Gamma/\equiv$ is a local isomorphism.

This means that every such triple (Γ, G, n), where now Γ is assumed to be a tree, arises as follows. Let Δ be a Moufang n-gon, let D be a subgroup of $\mathrm{Aut}(D)$ containing all the root groups of Δ and let $\{u, v\}$ be an edge of Δ. Then set G equal to the free amalgamated product of D_u and D_v over their intersection $D_{u,v}$ and set Γ equal to the corresponding tree associated with this free amalgamated product. Moufang polygons were classified in [1].

References

[1] J. Tits and R. Weiss. *Moufang Polygons.* Springer, 2002.
[2] R. Weiss. Moufang trees and generalized octagons. *Duke Math.J.* **88** (1997), 449–464.

Lattices in product of trees

Shahar Mozes

The talk described an ongoing study of lattices in the automorphisms groups of products of trees. We refer to [BM97], [BM00a], [BM00b], [Moz98], see also [Gla03], [BG02], [Rat04]. The talk concerned a joint work with Marc Burger and Bob Zimmer on the interplay between the linear representation theory and the structure of these lattices, see [BMZ04]. Let T_1 and T_2 be locally finite regular trees. We are interested in cocompact lattices $\Gamma < \mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$. Such a lattice is called reducible when both projections $\mathrm{pr}_i(\Gamma)$ are discrete. Considering

an irreducible lattice $\Gamma < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ let us denote by $H_i = \overline{\operatorname{pr}_i(\Gamma)}$ the closures of the projections in each factor.

Basic Question. *Which groups arise as closures of projections of cocompact lattices in* $\operatorname{Aut} T_1 \times \operatorname{Aut} T_2$*?*

A main theme in [BM97], [BM00a], [BM00b] was that certain local properties of the subgroups $H_i < \operatorname{Aut} T_i$ have far reaching consequences for the structure of Γ. For a regular tree T we shall say that a subgroup $H < \operatorname{Aut} T$ is locally quasiprimitive (resp. primitive) if for each vertex $x \in T$ its stabilizer in H acts on the neighbouring edges as a quasiprimitive (resp. primitive) permutation group. To state our results we need the following definitions from [BM00a]:

$$H^{(\infty)} = \cap_{L<H} L$$

where the intersection is taken over all open finite index subgroups. Let

$$\mathrm{QZ}(H) = \{h \in H \ : \ Z_H(h) \text{ is open } \}$$

be the quasi-center of H. Both are topologically characteristic subgroups of H. The subgroup $H^{(\infty)}$ is closed, and any normal discrete subgroup of H is contained in $\mathrm{QZ}(H)$. We recall next a few basic results established in [BM00a] concerning the structure of these subgroups.

Theorem 16. ([BM00a] **Prop. 1.2.1)** *Let $H < \operatorname{Aut} T$ be a closed non discrete locally quasiprimitive group. Then*

(1) *$H/H^{(\infty)}$ is compact.*
(2) *$\mathrm{QZ}(H)$ is a discrete not cocompact subgroup of H.*
(3) *Any closed normal subgroup of H either contains $H^{(\infty)}$ or is contained in $\mathrm{QZ}(H)$.*

We turn now to the results reported in the talk:

Proposition 17. *Let $H < \operatorname{Aut} T$ be a closed non discrete locally quasiprimitive group. Assume that it admits a $\mathbb{Q}_p$-analytic structure. Let $\mathcal{H}$ denote the Lie algebra of H, let $\mathbb{G} = \operatorname{Aut}(\mathcal{H} \otimes \overline{\mathbb{Q}_p})$ a linear algebraic group defined over $\mathbb{Q}_p$ and let $\operatorname{Ad} : H \to \mathbb{G}(\mathbb{Q}_p)$ be the adjoint representation. Then*

(1) *$\mathbb{G}$ is adjoint and semisimple.*
(2) *$\ker \operatorname{Ad} = \mathrm{QZ}(H)$.*
(3) *$\operatorname{Ad}(H) \supset \mathbb{G}^+$.*

Our main result is:

Theorem 18. *Let T_1, T_2 be locally finite trees. Let $\Gamma < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ be a cocompact lattice. Assume*

(1) *$H^{(\infty)} < \overline{\operatorname{pr}_i(\Gamma)} < H_i$, where $H_i < \operatorname{Aut} T_i$ is a closed non discrete, locally quasiprimitive subgroup.*
(2) *There is a linear representation $\pi : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ with infinite image.*

Then there are prime numbers p_1, p_2 such that H_i is $\mathbb{Q}_{p_i}$-analytic and we have an exact sequence

$$1 \to \Lambda_1 \times \Lambda_2 \to \Gamma \to (\operatorname{Ad}_1 \times \operatorname{Ad}_2)(\Gamma) \to 1$$

where

- $\Lambda_i := \Gamma \cap H_i$ *is of finite index in* $\mathrm{QZ}(H_i) = \ker \mathrm{Ad}_i$.
- $(\mathrm{Ad}_1 \times \mathrm{Ad}_2)(\Gamma)$ *is an arithmetic lattice in* $\mathbb{G}_1(\mathbb{Q}_{p_1}) \times \mathbb{G}_2(\mathbb{Q}_{p_2})$, *where* $\mathbb{G}_i$ *is the* $\mathbb{Q}_{p_i}$*-semisimple group given by Proposition 17.*

Using the above result we can now characterize the "classical"situation:

Corollary 19. *Let* T_1, T_2 *be locally finite trees. Let* $\Gamma < \mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ *be a cocompact lattice. Assume*

(1) ${H_i}^{(\infty)} < \overline{\mathrm{pr}_i(\Gamma)} < H_i$, *where* $H_i < \mathrm{Aut}\, T_i$ *is a closed non discrete, locally primitive subgroup.*
(2) *There is a linear representation* $\pi : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ *with infinite image.*

Then the following are equivalent:

(1) Γ *is linear over* $\mathbb{C}$.
(2) Γ *is residually finite.*
(3) $\mathrm{rank}_{\mathbb{Q}_{p_i}}(\mathbb{G}_i) = 1$ *for both* $i = 1, 2$.

In this case the geometric realization $|T_i|$ *is isometric to the Bruhat-Tits tree associated to* $\mathbb{G}_i$.

References

[BG02] Nadia Benakli and Yair Glasner, *Automorphism groups of trees acting locally with affine permutations*, Geom. Dedicata **89** (2002), 1–24. MR **2003a:**20043

[BM97] Marc Burger and Shahar Mozes, *Finitely presented simple groups and products of trees*, C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), no. 7, 747–752. MR **98g:**20041

[BM00a] ———, *Groups acting on trees: from local to global structure*, Inst. Hautes Études Sci. Publ. Math. (2000), no. 92, 113–150 (2001). MR **2002i:**20041

[BM00b] ———, *Lattices in product of trees*, Inst. Hautes Études Sci. Publ. Math. (2000), no. 92, 151–194 (2001). MR **2002i:**20042

[BMZ04] Marc Burger, Shahar Mozes, and Robert J. Zimmer, *Linear representations and arithmeticity for lattices in products of trees*, preprint, 2004.

[Gla03] Yair Glasner, *A two-dimensional version of the Goldschmidt-Sims conjecture*, J. Algebra **269** (2003), no. 2, 381–401. MR 2 015 283

[Moz98] Shahar Mozes, *Products of trees, lattices and simple groups*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), no. Extra Vol. II, 1998, pp. 571–582 (electronic). MR **2000a:**20056

[Rat04] Diego Rattaggi, *Computations in groups acting on a product of trees: normal subgroup structure and quaternion lattices*, Ph.D. thesis, ETH Zürich, 2004.

The normal subgroup property according to Bader-Shalom; application to Kac-Moody groups

Bertrand Rémy

We introduce the normal subgroup property, a purely group-theoretic property which was first proved by G.A. Margulis for (irreducible) lattices in higher-rank (semi)simple Lie groups. We quote a recent theorem by U. Bader and Y. Shalom generalizing this result to lattices of topological groups. We explain why it fits

particularly well to topological groups obtained as closed automorphism groups of buildings. We recall then why Kac-Moody groups over finite fields belong to this framework. We announce finally that the main technical assumption to apply Bader-Shalom's theorem is fulfilled; this is a square-integrability condition which not only proves the normal subgroup property for Kac-Moody lattices, but may also have other applications in rigidity theory.

3. Normal subgroup property; amenable and Kazhdan groups

3.1. **Normal subgroup property.** We start by recalling the following result, which is due to G.A. Margulis and which covers the case of lattices in Lie groups, [4], §IV, and [9], §8.

Theorem 20. *Let G be a connected (semi)simple Lie group of rank at least 2 with finite center. Let Γ be an (irreducible) lattice in G. If $N \triangleleft \Gamma$, then either $N < Z(G)$ or Γ/N is finite.*

We henceforth say that a group Γ has the *normal subgroup property* (NSP) if any normal subgroup of Γ either is finite and central, or has finite index in Γ.

3.2. **Amenability and Kazhdan property.** Each of these two properties has a lot of equivalent definitions, and the properties themselves are complementary to one another [3].

Definition. (i) Let $\rho : G \to U(\mathcal{H})$ be a unitary representation. We say that ρ almost has invariant vectors if for any $\varepsilon > 0$ and any compact subset $C \subset G$, there is a unit vector v such that $\sup_{g \in C} \| \rho(g).v - v \| < \varepsilon$.

(ii) A locally compact group G is called amenable if its regular representation $L^2(G)$ almost has invariant vectors.

(iii) A locally compact group G is called Kazhdan if any unitary representation of G which almost has invariant vectors actually has non-trivial invariant vectors. Then G is also said to have property (T).

The properties are complementary since a locally compact group G which is both amenable and Kazhdan is such that the constant functions lie in $L^2(G)$, which implies its compactness. The main idea in Margulis' normal subgroup theorem (Theorem 20) is to prove that a factor group Γ/N, when N is not central, is both amenable and Kazhdan for the discrete topology.

3.3. **Bader-Shalom's result.** This strategy is also the starting point of the theorem below ([1], Theorem 1.1), which deals with the case of quite arbitrary products of locally compact groups as ambient groups of irreducible lattices.

Theorem 21. *Let G_1, G_2 be two locally compact, non-discrete, compactly generated groups, not both isomorphic to $(\mathbf{R}, +)$. Let $\Gamma < G_1 \times G_2$ be an irreducible cocompact lattice. If every non-trivial closed normal subgroup of G_1 or G_2 is cocompact, then every proper quotient of Γ is finite.*

Recall that in this general context, a lattice Γ in a product of topological groups is called *irreducible* if the projections of Γ to each factor is dense.

4. Building automorphisms

4.1. **Strong transitivity and normal subgroups.** The structure of buildings is well-adapted to the normal subgroup property since there is also a sharp dichotomy on the size of normal subgroups of automorphism groups of buildings.

Proposition 22. *Let X be an irreducible, thick building. Let G be a group acting faithfully and strongly transitively on it, that is transitively on the inclusions of a chamber in an apartment. Then any normal subgroup of G acts transitively on the chambers of X.*

This result is the combination of [2], IV.2.7, Lemme 2, and of the well-known fact that a group acting on a building as above has a BN-pair. Note that if X is furthermore locally finite, then $\mathrm{Aut}(X)$ is naturally a locally compact group, and its closed normal strongly transitive subgroups are cocompact (in particular amenable).

4.2. **Amenable quotients.** As already mentioned (Section 1.2), the proof of the normal subgroup property splits into proving amenability and property (T) for quotient groups. We can state a more precise result due to U. Bader and Y. Shalom, only dealing with amenability but not requiring cocompactness of the irreducible lattice [1], Theorem 1.3.

Theorem 23. *Let G_1, G_2 be two locally compact groups. Let $\Gamma < G_1 \times G_2$ be an irreducible lattice. Let N be a normal subgroup in Γ. Then Γ/N is amenable if and only if both $G_i/\overline{\mathrm{pr}_i(N)}$ are.*

Here pr_i denotes the projection on the factor G_i. The proof of this theorem makes heavy use of probability theory on topological groups, more precisely of Poisson and Furstenberg boundaries for such groups. Together with the ideas presented in 2.A, this leads U. Bader and Y. Shalom to show that if Γ is a cocompact lattice of a product of irreducible buildings with strongly transitive actions on simple factors, then Γ has no infinite proper quotient.

5. Exotic things

5.1. **Kac-Moody groups over finite fields.** The analogy between Kac-Moody groups and S-arithmetic lattices in positive characteristic is supported by many arguments. We refer to [6] instead of going into details, but we note that for results on lattices in general topological groups, Kac-Moody theory provides a wide family of groups which are new with respect to the classical algebraic group case. An infinite Kac-Moody group Λ (over a finite field) acts diagonally on the product $X_- \times X_+$ of its twinned (locally finite) buildings. The Λ-action on a simple factor is not discrete (because it is strongly transitive), and we call *geometric completion* of positive (resp. negative) sign the closure $\overline{\Lambda}_+$ (resp. $\overline{\Lambda}_-$) of the image of Λ in the action on the positive (resp. negative) building. If we set $G := \overline{\Lambda}_- \times \overline{\Lambda}_+$, then Λ can be seen as a discrete subgroup of G via the diagonal embedding. If we denote by $W(t)$ the growth series of the common Weyl group W of X_- and X_+,

then the finiteness of $W(\frac{1}{q})$ implies that Λ is a lattice of G. The group Λ is generated by finitely many finite subgroups, which provides a length function ℓ_Λ. To any fundamental domain X for G/Λ is attached a cocycle $\alpha_X : G \times X \to \Lambda$ by: $\alpha_X(g,x) = \lambda \Leftrightarrow gx\lambda \in X$.

5.2. **Square-integrability.** The main difficulty with Kac-Moody lattices is that they are never cocompact; so we cannot apply to them 1.C. Still, in Y. Shalom's work on property (T) for quotients [8], a square-integrability criterion is proposed as a measure-theoretic substitute for the cocompactness of irreducible lattices. That this criterion is fulfilled by Kac-Moody lattices follows from the following theorem, the main result of [7].

Theorem 24. *Let Λ, G and W be as above. Then, there is a fundamental domain D for G/Λ, which is a countable union of compact open subsets $\{D_w\}_{w\in W}$ and such that for any $p \in [1;+\infty)$ and any $g \in G$, we have:*

$$\int_D \ell_\Lambda\big(\alpha_D(g,d)\big)^p \mathrm{d}\mu(d) < +\infty$$

whenever the minimal order q of the root groups satisfies $W(\frac{1}{q}) < +\infty$.

This, combined with Bader-Shalom's theorem, proves:

Corollary. *Kac-Moody lattices with irreducible Weyl group have the normal subgroup property.*

As mentioned in [5], the above square-integrability is also a useful hypothesis to prove super-rigidity results for some actions on non-positively curved metric spaces by irreducible lattices.

References

[1] U. Bader and Y. Shalom, *Factor and normal subgroup theorems for lattices in products of groups*, preprint, 2003.

[2] N. Bourbaki, Groupes et algèbres de Lie IV-VI (2nd edition), 1981, Masson.

[3] P. de la Harpe, A. Valette, La propriété (T) de Kazhdan pour les groupes localement compacts, Astérisque 175, 1989, Société mathématique de France.

[4] G.A. Margulis, Discrete subgroups of semisimple Lie groups, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 17, 1991, Springer.

[5] N. Monod, *Superrigidity for irreducible lattices and geometric splitting*, preprint, 2004.

[6] B. Rémy, *Sur les propriétés algébriques et géométriques des groupes de Kac-Moody*, Habilitation, 2003, Université de Grenoble 1.

[7] B. Rémy, *Integrability of induction cocycles for Kac-Moody groups*, Institut Fourier preprint 637, 2004.

[8] Y. Shalom, *Rigidity of commensurators and irreducible lattices*, Inventiones Mathematicæ 141 (2000) 1-54.

[9] R.J. Zimmer, Ergodic theory and semisimple groups, Monographs in Mathematics 81, Birkhäuser, 1984.

Combinatorial structure of some hyperbolic and Euclidean buildings

Alina Vdovina

We will call a *polyhedron* a two-dimensional complex which is obtained from several oriented p-gons by identification of corresponding sides. Consider a point of the polyhedron and take a sphere of a small radius at this point. The intersection of the sphere with the polyhedron is a graph, which is called the *link* at this point.

We construct several families of finite polyhedra with a given number p of sides of every face, such that the link of every vertex is a generalized 3-gon. Those polyhedra are interesting because of their universal coverings, which are hyperbolic buildings in the case $mp > 2m + p$ see, [4], and Euclidean buildings in the case $p = 3$, $m = 3$, see [1], [2].

We recall the definition of the polygonal presentation, introduced in [5].

Definition. Suppose we have n disjoint connected bipartite graphs $G_1, G_2, \dots G_n$. Let P_i and L_i be the sets of black and white vertices respectively in G_i, $i = 1, ..., n$; let $P = \cup P_i, L = \cup L_i$, $P_i \cap P_j = \emptyset$ $L_i \cap L_j = \emptyset$ for $i \neq j$ and let λ be a bijection $\lambda : P \to L$.

A set $\mathcal{K}$ of k-tuples $(x_1, x_2, \dots, x_k)$, $x_i \in P$, will be called a *polygonal presentation* over P compatible with λ if

(1) $(x_1, x_2, x_3, \dots, x_k) \in \mathcal{K}$ implies that $(x_2, x_3, \dots, x_k, x_1) \in \mathcal{K}$;
(2) given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \dots, x_k) \in \mathcal{K}$ for some $x_3, \dots, x_k$ if and only if x_2 and $\lambda(x_1)$ are incident in some G_i;
(3) given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \dots, x_k) \in \mathcal{K}$ for at most one $x_3 \in P$.

If there exists such $\mathcal{K}$, we will call λ a *basic bijection.*

Polygonal presentations for $n = 1$, $k = 3$ were listed in [3] with the incidence graph of the finite projective plane of order two or three as the graph G_1.

We can associate a polyhedron K on n vertices with each polygonal presentation $\mathcal{K}$ as follows: for every cyclic k-tuple $(x_1, x_2, x_3, \dots, x_k)$ from the definition we take an oriented k-gon on the boundary of which the word $x_1x_2x_3 \dots x_k$ is written. To obtain the polyhedron we identify the sides with the same label of our polygons, respecting orientation. We will say that the polyhedron K *corresponds* to the polygonal presentation $\mathcal{K}$.

Lemma 25. *([5]) A polyhedron K which corresponds to a polygonal presentation $\mathcal{K}$ has graphs $G_1, G_2, \dots, G_n$ as the links.*

Remark. Consider a polygonal presentation $\mathcal{K}$. Let s_i be the number of vertices of the graph G_i and t_i be the number of edges of G_i, $i = 1, ..., n$. If the polyhedron K corresponds to the polygonal presentation $\mathcal{K}$, then K has n vertices (the number of vertices of K is equal to the number of graphs), $k \sum_{i=1}^{n} s_i$ edges and $\sum_{i=1}^{n} t_i$ faces, all faces are polygons with k sides.

Let G be an incidence graph of a finite projective plane $\mathcal{P}^2(\mathbb{F}_q)$. Its black and white vertices correspond to points and lines of $\mathcal{P}^2(\mathbb{F}_q)$ respectively.

We mark black points of G with the different letters $x_1, ..., x_{q^2+q+1}$ of some group alphabet $\mathcal{A}$ and white points by the letters of another group alphabet $\mathcal{B} = \{y_1, \ldots, y_{q^2+q+1}\}$.

Let $\mathcal{T}_0$ be the triangle presentation with , described in 4 of [3] with $k = 3$, $n = 1$ and let G be the unique graph of this presentation, λ_0 be its basic bijection.

We consider three graphs G_1, G_2, G_3 such that G_1,G_2 are isomorphic to G and G_3 is isomorphic to G'. The black vertices of G_t are marked with letters of an alphabet $\mathcal{A}_t$, isomorphic to $\mathcal{A}$, $\mathcal{A}_t = \{x_1^t, \ldots, x_{q^2+q+1}^t\}$, $t = 1, 2, 3$.

The white vertices of G_t are marked with letters of an alphabet $\mathcal{B}_t$, isomorphic to $\mathcal{B}$, $\mathcal{B}_t = \{y_1^t, \ldots, y_{q^2+q+1}^t\}$, $t = 1, 2, 3$.

Let $P = \cup\mathcal{A}_t$, $L = \cup\mathcal{B}_t$. The bijection $\lambda : P \to L$ is defined as $\lambda : x_i^t \to y_i^{t+1}$, $t = 1, 2, 3; i = 1, ..., q^2 + q + 1$ ($t + 1$ and $t + 2$ are taken modulo 3). We construct a set of triples $\mathcal{T}$ and show later, that $\mathcal{T}$ is a polygonal presentation with $k = 3$, $n = 3$ and basic bijection λ.

We construct the set $\mathcal{T}$ as following:

Let x_i be a point of $\mathcal{P}^2(\mathbb{F}_q)$, $y_i = T(x_i)$. Let $I(y_i)$ be the set of points incident to the line y_i.

Definition. For each cyclic triple (x_i, x_j, x_l) from $\mathcal{T}_0$ we take to $\mathcal{T}$ three cyclic triples (x_i^1, x_j^2, x_l^3),(x_j^1, x_l^2, x_i^3),(x_l^1, x_i^2, x_j^3) if i, j and l are not equal pairwise and one cyclic triple (x_i^1, x_j^2, x_l^3), if $i = j = l$.

The polyhedron which corresponds to $\mathcal{T}$ contains three vertices, 1,2,3. For each cyclic triple $(x_i^1, x_j^2, x_l^3) \in \mathcal{T}$ we take an oriented triangle with letters x_i^1, x_j^2, x_l^3 on its sides. Vertex 1 lies between x_i^1 and x_j^2, vertex 2 lies between x_j^2 and x_l^3, vertex 3 lies between x_l^3 and x_i^1. The polyhedron is obtained by identification sides with the same labels respecting orientation.

Now we explain, how to construct a polyhedron with $k > 3$ vertices and faces, which are polygons with k sides. Let $w = z_1 \ldots z_k$ be a reduced word of length k in three letters a, b, c which does not contain proper powers of the letters a, b, c (each one of $z_1 \ldots z_k$ is a, b or c. We take $k - 3$ alphabets $\mathcal{A}_i = \{x_1^i, \ldots, x_{q^2+q+1}^i\}$, $i = 4, ..., k$ isomorphic to $\mathcal{A}_i$, $i = 1, 2, 3$ and the isomorphism induced by indexes. Now, each triple (x_i^1, x_j^2, x_l^3) from the main construction $\mathcal{T}$ we replace by a k-tuple such that $z_s = a$ has to be replaced by x_i^s; $z_s = b$ has to be replaced by x_j^s and $z_s = c$ has to be replaced by x_l^s. Then, with each k-tuple we consider all its cyclic permutations. The construction $\mathcal{P}_{k,k}$ just described is a polygonal presentation.

References

[1] W. Ballmann, M. Brin, *Orbihedra of nonpositive curvature*, Publications Mathématiques IHES, 82 (1995), 169-209.

[2] S. Barre, *Polyèdres finis de dimension 2 à courbure ≤ 0 et de rang 2*, Ann.Inst.Fourier 45, 4(1995),1037-1059.

[3] D. Cartwright, A. Mantero, T. Steger, A. Zappa, *Groups acting simply transitively on vertices of a building of type $\tilde{A}_2$*, Geometriae Dedicata 47 (1993), 143-166.

[4] D. Gaboriau, F. Paulin, *Sur les immeubles hyperboliques*, Geometriae Dedicata, (2001).

[5] A. Vdovina, *Combinatorial structure of some hyperbolic buildings,* Math. Z. 241, No. 3, 471–478 (2002).

Isomorphisms of groups acting on buildings

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(joint work with Bernhard Mühlherr)

We are interested in the isomorphism problem for two classes of groups:

- **Coxeter groups**, which act on thin buildings,
- **Kac-Moody groups**, which act on thick twin buildings.

Coxeter groups

A Coxeter group W possesses a set of involutory generators S such that all relations satisfied by pairs of elements of S provide a presentation of W. The ordered pair (W, S) is called a **Coxeter system**.

The isomorphism problem for Coxeter groups can be stated as follows.

Problem. *Determine all pairs of Coxeter systems (W_1, S_1), (W_2, S_2) such that W_1 and W_2 are isomorphic. Equivalently, given a Coxeter group W, determine all subsets $S \subseteq W$ such that (W, S) is a Coxeter system.*

Although the complete answer to this question is still unknown, many results in this direction have been obtained over the past 5 years. Moreover, it is conjectured in [2] that all irreducible Coxeter groups of finite rank are strongly rigid up to diagram twisting. The following result gives an example of a very favorable situation.

Theorem 26. *Let W be an infinite, irreducible Coxeter group of finite rank and 2-spherical type. Then W is strongly rigid. In other words, all subsets $S \subseteq W$ such that (W, S) is a Coxeter system are conjugate in W.*

The same property was proved to hold for Coxeter groups acting effectively, properly and cocompactly on contractible manifolds by Charney-Davis [7].

The proof of the previous result has two main steps. The first one is the reflection independence (namely, the fact that any automorphism of W leaves the union of the conjugacy classes of elements of S invariant). It is due to Haglund-Mühlherr [8]. The second step was completed in [4]. It uses a version of Kac' conjugation theorem due to Howlett-Rowley-Taylor [9] and valid for arbitrary Coxeter groups of finite rank.

Kac-Moody groups

A Kac-Moody group G over a field $\mathbb{K}$ (see [11] and [15] for the definitions) possesses a system of subgroups $((U_\alpha)_{\alpha\in\Phi(W,S)}, H)$ such that G is generated by $(\bigcup_{\alpha\in\Phi(W,S)} U_\alpha) \cup H$ and has a presentation in terms of these generators which is analogous to Steinberg's presentation of Chevalley groups. Here, the symbol

$\Phi(W,S)$ denotes the set of roots of some Coxeter system (W,S) of finite rank; if (W,S) has spherical type, then G is a Chevalley group over $\mathbb{K}$. The triple $(G,(U_\alpha)_{\alpha\in\Phi(W,S)},H)$ is called a **twin root datum** (see [11] for the definition).

The isomorphism problem for Kac-Moody groups can be stated as follows.

Problem. *Determine all pairs of twin root data* $(G_1,(U_{1,\alpha})_{\alpha\in\Phi(W_1,S_1)},H_1)$, $(G_2,(U_{2,\alpha})_{\alpha\in\Phi(W_2,S_2)},H_2)$ *such that* G_i *is a Kac-Moody group over a field* $\mathbb{K}_i$ *(*$i\in\{1,2\}$*) and* G_1 *is isomorphic to* G_2*. Equivalently, given a Kac-Moody group* G *over a field* $\mathbb{K}$*, determine all systems of subgroups* $((U_\alpha)_{\alpha\in\Phi(W,S)},H)$ *of* G *such that such that* $(G,(U_\alpha)_{\alpha\in\Phi(W,S)},H)$ *is a twin root datum.*

In the special case of Chevalley groups, a complete answer is known, due to Steinberg [13] and Borel-Tits [1]. It says that a Chevalley group G over a field $\mathbb{K}$ is involved in an essentially unique twin root datum of spherical type up to conjugation, except if the field $\mathbb{K}$ is very small (of cardinality ≤ 7) in which case there are some well known exceptions.

In the general case, we have the following result.

Theorem 27. *(*[3], [5]*) Let* $\mathcal{D}_i=(G_i,(U_{i,\alpha})_{\alpha\in\Phi(W_i,S_i)},H_i)$ *be a twin root datum coming from a Kac-Moody* G_i *over* $\mathbb{K}_i$*, where* $\mathbb{K}_i$ *is a finite field of cardinality* ≥ 4 *or an algebraically closed field (*$i\in\{1,2\}$*). Let* $\varphi:G_1\to G_2$ *be an isomorphism. Then* $\mathbb{K}_1\simeq\mathbb{K}_2$ *and* φ *induces an isomorphism of* $\mathcal{D}_1$ *to* $\mathcal{D}_2$*, except if* G_1 *and* G_2 *are both finite and* φ *is one of the exceptional isomorphisms mentioned above.*

This result was conjectured in the case $\mathbb{K}_1=\mathbb{K}_2=\mathbb{C}$ by Kac-Peterson [10], and proved for complex Kac-Moody groups of affine type by Carter-Chen [6] and certain Kac-Moody groups of hyperbolic type over finite fields by Rémy [12].

Remark. It is a fact that a Kac-Moody group G over a field $\mathbb{K}$ is finitely generated if and only if $\mathbb{K}$ is finite. Thus we may assume that $\mathbb{K}_1$ and $\mathbb{K}_2$ are either both finite or both algebraically closed in the previous statement.

Our proof of Theorem 27 rests heavily on the properties of the action of a Kac-Moody group on the associated twin building. The main idea is to reduce the problem to an analogous problem within a fixed apartment. We can then apply the aforementioned version of Kac' conjugation theorem to finish the proof. This reduction is achived by analyzing certain finite subgroups of the Kac-Moody groups under consideration. A crucial tool is the fixed point theorem for finite groups acting on arbitrary buildings. Another relevant tool is a theorem of Tits [14] on fixed points of $SL_2(\mathbb{K})$ acting on a tree.

References

[1] A. Borel and J. Tits. Homomorphismes "abstraits" de groupes algébriques simples. *Ann. Math.* **97**, pages 499–571, 1973.

[2] N. Brady, J. McCammond, B. Mühlherr, and W. Neumann. Rigidity of Coxeter groups and Artin groups. In *Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part I (Haifa, 2000)*, pages 91–109. Geom. Dedicata **94**, 2002.

[3] P.-E. Caprace and B. Mühlherr. Isomorphisms of Kac-Moody groups. *Preprint*, 2003.

[4] P.-E. Caprace and B. Mühlherr. Reflection rigidity of 2-spherical Coxeter groups. *Preprint*, 2003.
[5] P.-E. Caprace and B. Mühlherr. On certain isomorphisms between Kac-Moody groups. *Preprint*, 2004.
[6] R. W. Carter and Y. Chen. Automorphisms of affine Kac-Moody groups and related Chevalley groups over rings. *J. Algebra* **155**, pages 44–94, 1993.
[7] R. Charney and M. Davis. When is a Coxeter system determined by its Coxeter group? *J. of the London Math. Soc.* **61**, pages 441–461, 2000.
[8] F. Haglund and B. Mühlherr. Reflection independence of 2-spherical Coxeter groups. In preparation.
[9] R. B. Howlett, P. J. Rowley, and D. E. Taylor. On outer automorphism groups of Coxeter groups. *Manuscripta Math.* **93**, pages 499–513, 1997.
[10] V. G. Kac and D. H. Peterson. On geometric invariant theory for infinite-dimensional groups. *Springer Lecture Notes in Mathematics* **1271**, pages 109–142, 1987.
[11] B. Rémy. Groupes de Kac-Moody déployés et presque déployés. *Astérisque* **277**, 2002.
[12] B. Rémy. Immeubles de Kac-Moody hyperboliques, groupes non isomorphes de meme immeuble.*Geom. Dedicata* **90**, pages 29–44, 2002.
[13] R. Steinberg. Lecture on Chevalley groups. Yale University, 1968.
[14] J. Tits. A "theorem of Lie-Kolchin" for trees. *Contributions to Algebra*, pages 377–388, 1977. A collection of papers dedicated to Ellis Kolchin.
[15] J. Tits. Uniqueness and presentation of Kac-Moody groups over fields. *J. Algebra* **105**, pages 542–573, 1987.

Automorphisms of Rank One Buildings

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(joint work with Hendrik Van Maldeghem)

The title is not a misprint. Although buildings of rank one are, by definition, nothing but sets, there is additional structure obtained by extending constructions from higher rank buildings to this boundary case.

Let X be a noncompact symmetric space of rank one, and let U denote its boundary. We may think of X as a hyperbolic space (over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$), embedded in a projective space in such a way that the traces of lines are subspaces of minimal constant curvature (Beltrami–Klein model). The boundary U is homeomorphic to a sphere (of dimension d, say), and the non-trivial traces of lines on U are spheres of some fixed dimension s. Our aim is to determine all the automorphisms of the structure (U, B), where B consists of the non-trivial traces (blocks).

The present notes give an overview of results that are obtained in [6].

6. Hermitian Forms, Semi-Similitudes, and Reflections.

Adopting a general point of view — and ignoring hyperbolic spaces over $\mathbb{R}$ (where the boundary should be treated as a Möbius space) and the hyperbolic plane over $\mathbb{O}$ which also requires a different approach — we consider a (not necessarily commutative) field $\mathbb{K}$ with $\operatorname{char}\mathbb{K} \neq 2$, an involution σ on $\mathbb{K}$, and a non-degenerate σ-hermitian form f of Witt index 1 on $\mathbb{K}^n$. We write $\bar{x} := \sigma(x)$. The unital corresponding to this form is $U := \{v\mathbb{K} \mid v \in \mathbb{K}^n \setminus \{0\}, f(v, v) = 0\}$.

A semi-linear bijection $\gamma : \mathbb{K}^n \to \mathbb{K}^n$ (with companion automorphism α_γ) is called a *semi-similitude* of f if there is a scalar r_γ such that for all $v, w \in \mathbb{K}^n$ we have $f(\gamma(v), \gamma(w)) = r_\gamma \, \alpha_\gamma(f(v, w))$. The unitary transformations are just the semi-similitudes γ with $r_\gamma = 1$ and $\alpha_\gamma = \mathrm{id}_\mathbb{K}$. Clearly, every semi-similitude induces a bijection of the unital that preserves the system of blocks.

A linear transformation is called a transvection if it fixes every vector in some hyperplane H and every coset in $\mathbb{K}^n/H$. For a unitary transvection, the hyperplane of fixed points has to be the orthogonal space $v^\perp$ of some vector with $f(v, v) = 0$. The automorphism τ of (U, B) induced by a unitary transvection different from the identity fixes exactly one point of U (called the center of τ) and every block through that point.

It suffices to treat the case where $n = 3$, because automorphisms of the higher dimensional unitals can be reduced to automorphisms of plane sections. Replacing f by a scalar multiple and choosing a suitable basis, we may assume that f has the form $f(v, w) = \overline{v_0}\, w_2 + \overline{v_1}\, w_1 + \overline{v_2}\, w_0$.

We generalize a result obtained by J. Tits (see [7] for a proof under the assumption that $\mathbb{K}$ is commutative):

Proposition 28. *The group* T *generated by all transvections of* U *is normal in* $\mathrm{Aut}\, U, B$.

The main step in the proof of this basic observation consists of a purely geometric characterization of the generators of T in terms of their action on (U, B).

Remark. Using a standard argument due to Iwasawa, one proves that T is a simple group; in fact, it acts two-transitively on U, and the stabilizer T_u of $u \in U$ contains the group $\Xi(u)$ induced by all unitary transvections with center u as a nilpotent normal subgroup. For each block b through u, the commutator group $\Xi(u)'$ of $\Xi(u)$ acts sharply transitively on $b \setminus \{u\}$. This provides one way of describing the blocks in a purely group-theoretical way.

The group $\mathrm{Aut}\, U, B$ acts faithfully on T by conjugation, inducing exactly those automorphisms of T that preserve the system $\{\Xi(u) \mid u \in U\}$ of subgroups.

Our aim is to reconstruct an ambient building (namely, the projective space $\Delta(\mathbb{K})$ consisting of all subspaces of $\mathbb{K}^n$) in an $\mathrm{Aut}\, U, B$-equivariant way. To this end, we study the *unitary reflections*: a unitary involution γ is called a reflection if it fixes some hyperplane H pointwise, the orthogonal space $H^\perp$ is then called the *center* of γ.

A unitary reflection is called *exterior* if it fixes some point on U, it is called *interior* otherwise. Describing the reflections as suitable products of elements of T, we obtain:

Proposition 29. (1) *Every exterior reflection belongs to* T*, and each of its conjugates under* $\mathrm{Aut}\, U, B$ *is a reflection, as well.*

(2) *If the center of a reflection is spanned by some* v *with*

$$f(v, v) \in \{z \bar{x} x \mid z \in \mathrm{Z}(\mathbb{K}), x \in \mathbb{K}\}$$

then this reflection belongs to T*, and so do all its conjugates under* $\mathrm{Aut}\, U, B$.

(3) *If the center of a reflection is spanned by a vector v such that there exists $p \in \mathbb{K} \setminus \{0\}$ with $\bar{p} = -p$ and $f(v,v)\,p = p\,f(v,v)$ then this reflection belongs to* T*, and so do all its conjugates under* $\operatorname{Aut} U, B$.

Each of the unitals over $\mathbb{C}$ or $\mathbb{H}$ is covered by at least one of the cases mentioned in the previous proposition.

7. Reconstruction of the Ambient Building.

The unitals that we consider are described by subsets of projective spaces. After our reduction to $\mathbb{K}^3$, we actually deal with a projective plane. In order to reconstruct this structure, we note that every point outside U is the center of a uniquely determined unitary reflection. Mapping each subspace to its orthogonal space gives a polarity interchanging the center of a reflection with the axis of the same reflection, and a point $u \in U$ with the tangent to U at u.

This makes it possible to reconstruct the projective plane, as follows: points are the points of U and the reflections (used as names for their centers), lines are the points of U (used as names for the tangents) and the reflections (used as names for their axes). It remains to describe incidence: this is possible because the center of a reflection α lies on the axis of a reflection β if, and only if, the product of these reflections has order 2.

Thus the action of $\operatorname{Aut} U, B$ by conjugation on the set of all involutions in $\operatorname{Aut} U, B$ induces an action on the projective plane if, and only if, the set of reflections remains invariant. In several cases, we obtain that reflections belong to T, see the previous proposition. This gives the following result:

Theorem 30. *Under each one of the following assumptions, we can show that the group* $\operatorname{Aut} U, B$ *normalizes the set of reflections, that its action on U and on B extends to an action on the ambient building, and that* $\operatorname{Aut} U, B$ *consists of the bijections induced by semi-similitudes:*

(1) $\operatorname{Fix}(\sigma) \subseteq \{z\bar{x}x \mid z \in \operatorname{Z}(\mathbb{K}), x \in \mathbb{K}\}$.
(2) *For each $s \in \operatorname{Fix}(\sigma)$, there exists $p \in \mathbb{K} \setminus \{0\}$ with $\bar{p} = -p$ and $sp = ps$.*
(3) *Every reflection is an exterior one.*

Corollary. *In particular, these results cover the cases where $\mathbb{K}$ is commutative (this case had already been settled by M.E. O'Nan* [2] *in the finite case, and by J. Tits* [7] *in the infinite case), the case where σ is an involution of the second kind (i.e., $\operatorname{Z}(\mathbb{K}) \not\subseteq \operatorname{Fix}(\sigma)$), and the cases where $\mathbb{K}$ is a field of quaternions over any pythagorean field (because the norm $x \mapsto \bar{x}x$ is a surjection onto* $\operatorname{Fix}(\sigma)$ *for each involution on such a field).*

Remark. Applications of our results include a partial solution for the problem of recognizing the unitals defined by hermitian forms among more general ones (see [5], cf. also [3] and [4]), and answers to questions regarding uniqueness of topological, differentiable or symmetric structures.

As a more ambitious prospect, we hope that it becomes possible to obtain a new, conceptual proof for Mostow's rigidity theorem [1] in the rank one case.

References

[1] Mostow, G.D.: *Strong rigidity of locally symmetric spaces*, Annals of Mathematics Studies, No. 78. Princeton, N. J.: Princeton University Press and University of Tokyo Press, 1973.
[2] O'Nan, M.E.: *Automorphisms of unitary block designs*, J. Algebra **20** (1972), 495–511.
[3] Stroppel, M.: *Polarities of compact eight-dimensional planes*, Mh. Math., to appear.
[4] Stroppel, M.: *Polar unitals in compact eight-dimensional planes*, Arch. Math., to appear.
[5] Stroppel, M.: *Affine parts of topological unitals*, Adv. Geom., to appear.
[6] Stroppel, Markus; Van Maldeghem, Hendrik: *Automorphisms of unitals*, Manuscript (2004), in preparation.
[7] Tits, Jacques: *Résumé de cours et travaux*, Annuaire du Collège de France 1996–1997, 97^e année, Paris, pp. 89–102.

Geometry of orbits and holonomy

Carlos Olmos

Holonomy groups play a central role in Riemannian geometry. The holonomy group, i,e. the orthogonal group obtained by parallel transporting along based loops, measures the deviation of a Riemannian manifold from being flat. In the non-generic case (i.e. when it is not the full orthogonal group) it encodes very useful information about the space. Namely, the parallel tensors of M are just the extension of those algebraic tensors at a given point p that are invariant under the holonomy group. The reducibility of the holonomy group representation implies, via the de Rham decomposition theorem, the local product decomposition of the space. One of the most important and beautiful results in Riemannian geometry is the so called **Berger Holonomy Theorem**: *if the holonomy group of an irreducible Riemannian manifold M is not transitive on the tangent sphere, then M must be locally symmetric.*

The above theorem follows from the classification given by Marcel Berger [1] in 1955 of the possible holonomy groups of non-locally symmetric spaces. He used the fact that the curvature tensor and its covariant derivative at one point, take values in the holonomy algebra. Some years later James Simons [5] gave a purely algebraic proof of this fact. But his proof is long and involved, using case by case arguments and double induction. The problem of given a conceptual proof of Berger Theorem remained. The goal of this lecture is to give the main ideas of a recent conceptual proof of the above result which is given in [3]. It is based on submanifold geometry and relates Riemannian holonomy with normal holonomy. The basic tools can be found in [2]. The proof follows the following lines: let Mbe an irreducible Riemannian manifold with holonomy group Φ at p. Then

a) The normal space at v to any holonomy orbit $\Phi.v$ is totally geodesic in M, locally, when exponentiated (we call such a submanifold $N(v)$). This is a consequence of the Ambrose-Singer holonomy theorem, the Bianchi identity and a result of Cartan on the existence of a totally geodesic submanifolds with a given tangent space. Moreover, $N(v)$ splits off the direction of v.

b) The normal holonomy group of $Phi.v$ at v acts by isometry on $N(v)$. This is by producing a perpendicular variation $N(v(t))$ of totally geodesic submanifolds of

M (using the normal parallel transport in $\Phi.v$, along an arbitrary closed curve $v(t)$ starting at v). A perpendicular variation of totally geodesic submanifolds must be by isometries. The normal holonomy group must always contain the isotropy group Φ_v of Φ at v (restricted to the normal space $\nu_v(\Phi.v)$. This is a general result for full orbits which can be found in [2]. But, since $N(v)$ splits off the direction of v, we have that the intrinsic holonomomy of $N(v)$ must lie in the isotropy group Φ_v (restricted to the normal space). Then the isotropy of $N(v)$ contains the (intrinsic) holonomy of $N(v)$ at v. From this fact it is standard to show that $N(v)$ is locally symmetric.

c) If Φ is not transitive on the sphere then, for any principal vector w, there exists a line in the normal space of $\Phi.w$, not going through the origin, such that the normal spaces to Φ-orbits, through points of such a line, generate the full tangent space. Moreover, w belongs to the intersection of this one-parameter family of normal spaces. This is in fact, a general lemma about orthogonal (non-transitive) group actions. Such a line is in fact generic and passes eventually through focal orbits.

d) Therefore, for almost any w there exists a family of totally geodesic locally symmetric submanifolds of M whose tangent spaces generate the full tangent space T_pM and such that w belongs to the intersection of this family. This implies the local symmetry of M.

Using similar arguments it is given in [4] a proof of the theorem of Simons on holonomy systems.

References

[1] Berger, M. Sur les groupes d'holonomie homogènes de variétés á conexion affine et des variétés riemanniennes. *Bull. Soc. Math. Fr.* **283** (1955) 279–330.

[2] Berndt, J., Console, S. and Olmos, C. *Submanifolds and Holonomy.* CRC/Chapman and Hall Research Notes Series in Mathematics **434**, 2003 Boca Raton.

[3] Olmos, C. A geometric proof of Berger Holonomy Theorem. Preprint, 2004.

[4] Olmos, C. On the geometry of holonomy systems. Preprint, 2004.

[5] Simons, J. On the transitivity of holonomy systems. *Ann. of Math.* **76** (1962), 213–234.

Isoperimetric inequalities for quotients of buildings

Enrico Leuzinger

Let $\mathcal{B}$ be a Euclidean or hyperbolic building and let $G \subset \mathrm{Aut}\ \mathcal{B}$ be a locally compact unimodular group, which acts strongly transitively on $\mathcal{B}$. We use graphs $\mathcal{G}$, quasi-isometric to $\mathcal{B}$, to study asymptotic properties of quotients $\Gamma\backslash\mathcal{B}$, where Γ is a discrete subgroup of G. If G has Kazhdan's property (T) we show that such quotients satisfy strong isoperimetric inequalities. This yields new examples of graphs with positive Cheeger constant. Such graphs cannot be bi-Lipschitz embedded into Hilbert space. Moreover, simple random walks on such quotients are shown to be recurrent if and only if Γ is a uniform lattice in G.

Combinatorial Building Theory

Peter Abramenko and Hendrik Van Maldeghem

8. Characterizations of apartments

In the following, (W, S) denotes a Coxeter system and Δ a thick building of type (W, S). The set of chambers of Δ will be denoted by $\mathcal{C}(\Delta)$ and the Weyl-distance on $\mathcal{C}(\Delta)$ with values in W by δ. We summarize some of our results dealing with the interrelations between the fundamental notions of combinatorial building theory such as chambers, adjacency, W-distance, opposition (in the case of spherical buildings), convexity and apartments.

The first theorem to be mentioned here gives a nice combinatorial characterization of apartments in spherical buildings by means of the opposition relation. As pointed out in [1], a similar result also holds true for twin buildings. Our investigations of twin buildings eventually led to a new definition of them in [2]. In order to avoid additional notation, we confine ourselves to spherical buildings here.

Theorem 31. *Let Δ be spherical and $\mathcal{M}$ a non-empty subset of $\mathcal{C}(\Delta)$. For any C in $\mathcal{C}(\Delta)$, we denote by $n_{\mathcal{M}}(C)$ the number of chambers X in $\mathcal{M}$ which are opposite C. Then $\mathcal{M}$ is the set of chambers of an apartment of Δ if and only if the following two conditions are satisfied.*

(*i*) *For each $C \in \mathcal{M}$, we have $n_{\mathcal{M}}(C) = 1$*
(*ii*) *For each $C \in \mathcal{C}(\Delta) \setminus \mathcal{M}$, we have $n_{\mathcal{M}}(C) \equiv 0$ mod 2.*

Let us mention in passing that replacing (*ii*) by the requirement $n_{\mathcal{M}}(C) \geq 2$ for $C \in \mathcal{C}(\Delta) \setminus \mathcal{M}$ does not yield a characterization of apartments. Now observing that two chambers C and X in Δ are opposite if and only if $\delta(C, X) = w_0$, where w_0 is the element of maximal length in W, one can ask whether similar characterizations of apartments in not necessarily spherical buildings are available by using other Weyl distances w. So let Δ be an arbitrary (thick) building, $w \in W$ and $\mathcal{M}$ a non-empty subset of $\mathcal{C}(\Delta)$. For any C in $\mathcal{C}(\Delta)$, we denote by $n_{\mathcal{M},w}(C)$ the number of chambers $X \in \mathcal{M}$ satisfying $\delta(C, X) = w$. We now introduce the following Condition $(\mathrm{P}_{\mathcal{M},w})$:

$(\mathrm{P}_{\mathcal{M},w})$ For each $C \in \mathcal{M}$, we have $n_{\mathcal{M},w}(C) = 1$;
for each $C \in \mathcal{C}(\Delta) \setminus \mathcal{M}$, we have $n_{\mathcal{M},w}(C) \equiv 0 \bmod 2$.

It is proved in [3] that apartments always satisfy this condition.

Theorem 32. *If Σ is an apartment of Δ and $\mathcal{M} = \mathcal{C}(\Sigma)$, then Condition $(\mathrm{P}_{\mathcal{M},w})$ is satisfied for all $w \in W$.*

However, counter-examples show that Condition $(\mathrm{P}_{\mathcal{M},w})$ only characterizes apartments if we require it for "sufficiently many" w or combine it with the property of (gallery) connectedness. So the best possible result to be obtained here is the following (see [4]).

Theorem 33. *Let ℓ_0 be a positive integer which is not greater than the diameter of Δ and $\mathcal{M}$ a non-empty subset of $\mathcal{C}(\Delta)$. Assume that one of the following two assumptions is satisfied.*

(a) *Condition $(\mathrm{P}_{\mathcal{M},w})$ holds true for all $w \in W$ with length $\ell(w) \geq \ell_0$.*
(b) *$\mathcal{M}$ is connected, ℓ_0 is greater than or equal to the diameter of any spherical rank 2 residue of Δ and $(\mathrm{P}_{\mathcal{M},w})$ holds for all $w \in W$ with $\ell(w) = \ell_0$.*

Then $\mathcal{M} = \mathcal{C}(\Sigma)$ for some apartment Σ of Δ.

In both cases, the main work consists in verifying that $\mathcal{M}$ is a convex set of chambers. In the second case, the following characterization of convexity (which is not difficult to show but seems to have gone unnoticed before) proves to be useful.

Theorem 34. *A non-empty subset $\mathcal{M}$ of $\mathcal{C}(\Delta)$ is convex if and only if it is connected and its intersection with any spherical rank 2 residue of Δ is either empty or convex.*

9. Opposition and W-valued distance

We shall now state a general, rather technical theorem, and afterwards mention informally some consequences.

Let (W, S) be a Coxeter system, and let $w \in W$ be arbitrary. Denote by $S(w)$ all elements of S that appear in a reduced expression of w in elements of S. It is well known that $S_1(w) := \{s \in S : \ell(sw) < \ell(w)\}$ generates a spherical Coxeter group W_1 and, denoting the longest element in that group by w_1^o, that w can be written as $w = w_1^o w_1$, with $\ell(w) = \ell(w_1^o) + \ell(w_1)$. But now $S_1(w_1) =: S_2(w)$ again generates a spherical Coxeter group W_2 with some unique longest element w_2^o, and hence we may write $w = w_1^o w_2^o w_2$, with $\ell(w) = \ell(w_1^o) + \ell(w_2^o) + \ell(w_2)$. Going on like that, we obtain a unique reduced decomposition of $w \in W_1 W_2 \dots W_k$ as $w = w_1^o w_2^o \dots w_k^o$ for some natural number k, where w_j^0 is the longest word of the spherical Coxeter subgroup $W_{S_j(w)} =: W_j$, $1 \leq j \leq k$. We now have $S_j(w) = S(w_j^o)$, and $S(w)$ is the union of all $S_j(w)$. A similar reduced decomposition $v = v_1^o \dots v_m^o$ can be defined for $v = w^{-1}$, but note that $m \neq k$ is possible!

For a subset $T \subseteq S$, we say that two chambers are T-adjacent if they are i-adjacent for some $i \in T$.

Theorem 35. *Let Δ and Δ' be two thick buildings of type (W, S) and let $w \in W$. Let $\varphi : \mathcal{C}(\Delta) \to \mathcal{C}(\Delta')$ be a surjective map such that $\delta(C, D) = w$ if and only if $\delta'(\varphi(C), \varphi(D)) = w$, for all $C, D \in \mathcal{C}(\Delta)$. Then φ is a bijection and both φ and its inverse preserve $S_i(w)$-adjacency, for all $i \in \{1, 2, \dots, k\}$. Similarly for w^{-1} and $S_j(w^{-1})$-adjacency, for all $j \in \{1, \dots, m\}$. Finally, $\delta(C, D) = u$ if and only if $\delta'(\varphi(C), \varphi(D)) = u$, for all $u \in \{w_1^0, \dots, w_k^0, v_1^0, \dots, v_m^0\}$, with the decompositions $w = w_1^0 \dots w_k^0$ and $w^{-1} = v_1^0 \dots v_m^0$ introduced above.*

A major consequence is that, given a Coxeter system (W, S) and an element $w \in W$, with $S(w) = S$, *every 2-spherical building Δ of type (W, S) is determined by its set of chambers and all (ordered) pairs (C, D) of chambers, with $\delta(C, D) = w$.*

A similar conclusion holds for arbitrary types of buildings if w has a unique reduced decomposition. In other cases there are counterexamples available, see [3].

In particular, a spherical building is completely determined by its set of chambers and all pairs of opposite chambers.

The proof Theorem 35 is rather involved, but the special case of opposition mentioned in the previous paragraph is illuminating for the general case. We outline the proof.

It suffices to recover the adjacency relation on the set of chambers from the set of opposite chambers. Therefore, we prove that two chambers C, D are adjacent if and only if there exists a third chamber E (note that we assume the building to be thick!) such that *no chamber is opposite exactly one of* $\{C, D, E\}$. That this condition is sufficient is proved as follows. Suppose C and D are not adjacent. Choose an apartment Σ containing C, D, and let Σ' be an apartment containing a minimal path from D to C, except for C, and intersecting Σ in a half apartment. One verifies that Σ' contains exactly two chambers C', C'' opposite C, and one chamber D' opposite D. Moreover, $D' \notin \{C', C''\}$ precisely because we assumed that C and D are not adjacent. Our condition now implies that C', C'', D' are the only chambers of Σ' opposite E, and this contradicts Theorem 31.

Other consequences of Theorem 35 can be found in [3].

With some additional work, one can also prove that every 2-spherical building is determined by its set of chambers and all pairs (C, D) of chambers with $\delta(C, D) = w$, where we do not presuppose the type (W, S) of Δ, nor the element w, but where we know that $S(w) = S$! The same conclusion holds for arbitrary buildings if w has a unique reduced decomposition.

References

[1] P. Abramenko and H. Van Maldeghem, On opposition in spherical buildings and twin buildings, *Annals of Combinatorics* **4** (2000), 125 – 137.

[2] P. Abramenko and H. Van Maldeghem, 1-Twinnings of buildings, *Math. Z.* **218** (2001) no.1, 187 – 203.

[3] P. Abramenko and H. Van Maldeghem, Maps between buildings that preserve a given Weyl distance, to appear in Indagationes Mathematicae.

[4] P. Abramenko and H. Van Maldeghem, Combinatorial characterizations of convexity and apartments in buildings, submitted.

[5] K. Brown, *Buildings*, Springer-Verlag, 1989.

[6] M. A. Ronan, *Lectures on Buildings*, Academic Press, San Diego, *Persp. Math* **7**, 1989.

[7] J. Tits, *Buildings of Spherical Type and Finite BN-pairs*, Lecture Notes in Math. **386**, Springer, Berlin–Heidelberg–New York, 1974.

[8] J. Tits, Immeubles de type affine, **in** Buildings and the Geometry of Diagrams, *Lect. Notes Math.* **1181**, Springer-Verlag, Berlin, Heidelberg (1986), 157 – 190.

[9] H. Van Maldeghem, *Generalized Polygons*, Monographs in Mathematics **93**, Birkhäuser Verlag, Basel, Boston, Berlin, 1998.

[10] R. Weiss, *The Structure of Spherical Buildings*, Princeton University Press, Princeton, NJ, 2003.

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