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Classical Algebraic Geometry

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Introduction by the Organisers

The worshop on Classical Algebraic Geometry, organized by David Eisenbud (Berkeley), Joe Harris (Harvard) and Frank-Olaf Schreyer (Saarbrücken) was well attended by important senior researchers and many gifted young mathematicians.

Classical Algebraic Geometry is characterized by having very basic and concrete problems. However development of the abstract, sophisticated tools of algebraic geometry has often lead to remarkable progress on these problems. It was our intention to make a conference emphasizing progress on the classical problems, and featuring the new tools and their applications. We emphasize a few results:

Geometry of the Moduli Spaces of Curves. Based partly on motivations from mathematical physics, the enumerative geometry of the moduli spaces of curves has been an extremely active area for some time. Several of the major conjectures in the area have seen major progress, and may be close to resolution. The extension of Brill-Noether theory for line bundles to the case of bundles of higher rank is a long-standing problem (Mukai's conjecture), and this plays now a novel role in the search for interesting divisors on the moduli space in the work of Gavril Farkas. In the talk of Brendon Hassett we learned how other birational models of the Moduli space can be interpreted in terms of curves with nodes, cusps and tacnodes.

Derived Categories. Originally introduced in the late fifties as a tool for generalizing duality theory, derived categories have arisen in a several striking new contexts, including the characterization of birational transformations in the minimal model program, the analysis of the Bernstein-Gel'fand-Gel'fand and Fourier-Mukai transforms. One of the speakers of the conference (Mihnea Popa) suggested an amazing series of parallels between Castelnuovo theory and the Schottky problem, based on these last two contexts.

Enumerative Geometry. The high–tech tools of modern geometry have led to a better understanding of degenerations, which has in turn led to the solution of many classical enumerative problems. We heard talks on a solution via specialization of the long–standing problem of intersection theory on flag manifolds, and on the maximal degneration of complex structures through amoebas and tropical geometry.

We were particularly pleased by the number of extremely strong young (i.e. untenured) participants, among them for example Gavril Farkas and Mihnea Popa.

Workshop: Classical Algebraic Geometry

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Abstracts

Vector bundles on curves and classical geometry

Arnaud Beauville

Let C be a smooth projective curve, of genus $g \geq 2$. The moduli space $\mathcal{SU}_C(r)$ of semi-stable vector bundles of rank r on C, with trivial determinant, is a normal projective variety, which can be considered as a non-abelian analogue of the Jacobian variety of C. It is actually related to the Jacobian by the following construction, which goes back (at least) to [5]. Let J^{g-1} be the translate of JC parameterizing line bundles of degree $g-1$ on C, and $\Theta \subset J^{g-1}$ the canonical theta divisor. For $E \in \mathcal{SU}_C(r)$, consider the locus

$$\Theta_E := \{L \in J^{g-1} \mid H^0(C, E \otimes L) \neq 0\} \ .$$

Then either $\Theta_E = J^{g-1}$, or Θ_E is in a natural way a divisor in J^{g-1}, belonging to the linear system $|r\Theta|$. In this way we get a rational map

$$\theta : \mathcal{SU}_C(r) \dashrightarrow |r\Theta|$$

which can be identified to the map $\varphi_{\mathcal{L}} : \mathcal{SU}_C(r) \dashrightarrow \mathbb{P}(H^0(\mathcal{SU}_C(r), \mathcal{L})^*)$ given by the global sections of the determinant bundle $\mathcal{L}$, the positive generator of the Picard group of $\mathcal{SU}_C(r)$ [2].

For $r = 2$ the map θ is an embedding if C is not hyperelliptic [3]. We consider in this talk the higher rank case, where very little is known. We first look at the case $g = 2$. There a curious numerical coincidence occurs, namely

$$\dim \mathcal{SU}_C(r) = \dim |r\Theta| = r^2 - 1 \ .$$

For $r = 2$ θ is an isomorphism [5]; for $r = 3$ it is a double covering, ramified along a sextic hypersurface which is the dual of the "Coble cubic" [6]. Our result is:

Theorem 1. *For a curve C of genus 2, the map $\theta : \mathcal{SU}_C(r) \dashrightarrow |r\Theta|$ is generically finite* (or, equivalently, dominant)*. It admits some fibers of dimension $\geq [\frac{r}{2}] - 1$.*

Our method is to consider the fibre of θ over a reducible element of $|r\Theta|$ of the form $\Theta + \Delta$, where Δ is general in $|(r-1)\Theta|$. The main point is to show that this fibre restricted to the stable locus of $\mathcal{SU}_C(r)$ is finite. The other elements of the fibre are the classes of the bundles $\mathcal{O}_C \oplus F$, with $\Theta_F = \Delta$; reasoning by induction on r we may assume that there are finitely many such F, and this gives the first assertion of the theorem. The second one follows from considering the restriction of θ to a particular class of vector bundles, namely the symplectic bundles.

The method is not, in principle, restricted to genus 2 curves – but the geometry in higher genus becomes much more intricate. In the second part of the talk we apply it to rank 3 bundles in genus 3. Our result is:

Theorem 2. *Let C be a curve of genus 3. The map $\theta : \mathcal{SU}_C(3) \to |3\Theta|$ is a finite morphism.*

This means that a semi-stable vector bundle of rank 3 on C has always a theta divisor; or alternatively, that the linear system $|\mathcal{L}|$ on $\mathcal{SU}_C(3)$ is base point free.

This is not a big surprise since the result is already known for a *generic* curve of genus 3 [7]. We believe, however, that the method is more interesting than the result itself. In fact we translate the problem into an elementary question of projective geometry: what are the continuous families of planes in $\mathbb{P}^5$ such that any two planes of the family intersect? It turns out that this question has been completely (and beautifully) solved by Morin [4]. Translating back his result into the language of vector bundles we get a complete list of the stable rank 3 bundles E of degree 0 such that $\Theta_E \supset \Theta$. Theorem 2 follows as a corollary.

Details can be found in [1].

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Castelnuovo theory and the geometric Schottky problem

Mihnea Popa
(joint work with Giuseppe Pareschi)

The aim of this work is to show that Castelnuovo theory in projective space – as explained for example in [2] Ch.4 §3 – has a precise analogue for abelian varieties. This can be quite surprisingly related in a very concrete way to the geometric Schottky problem, namely the problem of identifying Jacobians among all principally polarized abelian varieties (ppav's) via geometric conditions on the polarization. The main result is that a ppav satisfies a precise analogue of the Castelnuovo Lemma if and only if it is a Jacobian. We prove or conjecture other results which show an extremely close parallel between geometry in projective space and Schottky type projective geometry on abelian varieties.

On a ppav (A, Θ) of dimension g one can make sense of what it means for a set of at least $g+1$ points to be in general position: we simply require for any g of them the existence of a translate of Θ containing them and avoiding all the others (we call this *theta general position*). It turns out that general points on an

Abel-Jacobi embedded curve C are in theta general position on the corresponding Jacobian $J(C)$, and impose the minimal number of conditions, namely $g+1$, on the linear series $|2\Theta_a|$, for $a \in J(C)$ general. The main result we prove is the following:

Theorem 3. *Let (A, Θ) be an irreducible principally polarized abelian variety of dimension g, and let Γ be a set of $n \geq g+3$ points on A in theta general position, imposing only $g+1$ conditions on the linear series $|2\Theta_a|$ for $a \in A$ general. Then (A, Θ) is a polarized Jacobian of a curve C, and $\Gamma \subset C$ or $-C$ for some Abel-Jacobi embedding $C \subset J(C)$.*

Roughly speaking, the key points in the proof of the Theorem are the following. First, a set Γ of points in theta general position imposing the minimal number of conditions on general 2Θ-transates satisfies in fact a strong version of general position: for every subset $Y \subset \Gamma$ of cardinality g, there exists a *unique* theta translate containing Y and avoiding all the other points of Γ. (For general points on an Abel-Jacobi embedded curve, this is essentially the Jacobi inversion theorem.) Second, the existence of points in Castelnuovo position implies, via the fact above, the existence of trisecants to the Kummer variety associated to (A, Θ). The specific result is the following:

Theorem 4. *Under the assumptions of Theorem 3, for any distinct points $p, q, r, s \in \Gamma$ we have that*

$$\Theta \cap \Theta_{p-q} \subset \Theta_{p-s} \cup \Theta_{r-q}.$$

Equivalenty, for every ξ such that $2\xi = s - p - q - r$, the images of the points

$$p + \xi, q + \xi \text{ and } r + \xi$$

lie on a trisecant to the Kummer variety of A.

The Trisecant Conjecture of Welters [6], as yet unproved, implies then the Castelnuovo-Schottky Lemma. However, our hypotheses contain the extra information needed in order to obtain the existence of one-dimensional families of trisecants, and apply directly the Gunning-Welters criterion, [3] and [6], for detecting Jacobians.

In view of the Matsusaka-Ran criterion for detecting Jacobians, beyond the Schottky type implication the main conclusion of our work can be stated as an almost perfect similarity between one-dimensional subvarieties of minimal degree in projective space on one hand and abelian varieties on the other hand. A conjecture of Debarre [1] predicts what all subvarieties of "minimal degree" in ppav's (i.e. those representing the minimal cohomology classes $\theta^d/d!$) should be: specifically the W_d's in Jacobians and the Fano surface of lines in the intermediate Jacobian of a smooth cubic threefold. These facts, plus the results of [4] on an abelian version of Castelnuovo-Mumford regularity, quite surprisingly suggest that the similarity should extend to (smooth) subvarieties of minimal degree of arbitrary dimension, relating rational normal scrolls to W_d's and Veronese surfaces to Fano surfaces. In the spirit of our work, this could potentially be realized via regularity, the analysis of Castelnuovo type genus bounds, or higher Castelnuovo theory, all with interesting Schottky type implications. We state this here as a general problem.

Question. *Is there a geometric correspondence relating rational normal scrolls to W_d's and Veronese surfaces to Fano surfaces of lines on cubic threefolds? What are the similar properties, from a Castelnuovo theory point of view, shared by the two sets of varieties?*

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Towards a canonical model for the moduli space of curves

Brendan Hassett
(joint work with David Hyeon)

We work over an algebraically closed field of characteristic zero.

Classical birational models of moduli spaces of curves of small genus tend to be unirational. Every curve of genus two admits a natural representation as a double cover of $\mathbb{P}^1$ branched over six points, so we have

$$M_2 \sim \mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^1}(6)))//\mathrm{SL}_2 = \mathbb{P}^6//\mathrm{SL}_2.$$

Every non-hyperelliptic curve of genus three is a plane quartic, so

$$M_3 \sim \mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^2}(4)))//\mathrm{SL}_3 = \mathbb{P}^{14}//\mathrm{SL}_3.$$

M_g is known to be unirational for $g \leq 14$ by work of Severi, Arbarello, Sernesi, Chang-Ran, and Verra.

On the other hand, Eisenbud, Harris, and Mumford have shown that $\overline{M}_g$ is of general type for $g \geq 24$. Standard conjectures of birational geometry then predict the existence of a *canonical model* for the moduli space

$$\overline{M}_g(0) = \mathrm{Proj}(\oplus_{n\geq 0}\Gamma(\overline{M}_g, nK_{\overline{\mathcal{M}}_g})).$$

However, as the canonical ring of $\overline{M}_g$ is not known to be finitely generated for *any* g, this remains an elusive object.

Our approach is to consider a more general object, the *log canonical model*

$$\overline{M}_g(\alpha) = \mathrm{Proj}(\oplus_{n\geq 0}\Gamma(\overline{M}_g, n(K_{\overline{\mathcal{M}}_g} + \alpha\delta))),$$

where $\alpha \in \mathbb{Q} \cap [0,1]$ is chosen so that $K_{\overline{\mathcal{M}}_g} + \alpha\delta$ is effective. For instance,

(1) for $9/11 < \alpha \leq 1$ we have $\overline{M}_g(\alpha) = \overline{M}_g$ (Cornalba-Harris);

(2) for $7/10 < \alpha \leq 9/11$, $\overline{M}_g(\alpha)$ is the space of pseudo-stable curves of D. Schubert, with nodes and cusps as singularities;
(3) for $2/3 < \alpha < 7/10$, $\overline{M}_g(\alpha)$ parametrizes curves with nodes, cusps, and tacnodes.

As α gets smaller more complicated singularities arise, like ramphoid cusps and higher-order tacnodes.

One attractive aspect of this formalism is that it includes many classical projective models for curves of small genus. For example, we have

$$\mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^2}(6)))//\mathrm{SL}_2 = \overline{M}_2(\alpha), \quad 7/10 < \alpha < 9/11$$

and

$$\mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^2}(4)))//\mathrm{SL}_3 = \overline{M}_3(\alpha), \quad 5/9 < \alpha < 17/28.$$

In these cases, we are working with the canonical class of the moduli stack rather than the moduli scheme. This is important because the locus of curves with automorphisms has codimension ≤ 1.

For curves of arbitrary genus, these spaces are constructed using the techniques of Geometric Invariant Theory. Let $\mathrm{Chow}_{\nu,g}$ denote the Chow variety of ν-pluricanonically imbedded curves of genus g

$$C \hookrightarrow \mathbb{P}^{N-1} \quad N = \begin{cases} (2\nu-1)(g-1) \text{ if } \nu > 1 \\ g \text{ if } \nu = 1 \end{cases}.$$

This admits a natural imbedding into projective space. Mumford has shown that the moduli space of stable curves can be realized as the quotient $\mathrm{Chow}_{5,g}//\mathrm{SL}_{9(g-1)}$. Schubert obtains $\overline{M}_g(\alpha), 7/10 < \alpha < 9/11$, as the quotient $\mathrm{Chow}_{4,g}//\mathrm{SL}_{7(g-1)}$; it is also possible to use $\nu = 3$. We realize $\overline{M}_g(7/10)$ as the quotient $\mathrm{Chow}_{2,g}//\mathrm{SL}_{3(g-1)}$.

Now consider the Hilbert scheme of ν-pluricanonically imbedded curves $\mathrm{Hilb}_{\nu,g}$ and the cycle-class map

$$\gamma : \mathrm{Hilb}_{\nu,g} \to \mathrm{Chow}_{\nu,g}.$$

The Hilbert scheme admits a number of natural imbeddings: For each $d \gg 0$, a curve determines a point in the Grassmannian of quotients

$$\begin{array}{rcl} \mathrm{Hilb}_{\nu,g} & \hookrightarrow & \mathrm{Gr}((2d\nu-1)(g-1), \binom{d+N-1}{N-1}) \\ C & \mapsto & \{\Gamma(\mathcal{O}_{\mathbb{P}^{N-1}}(d)) \to \Gamma(\mathcal{O}_C(d))\}. \end{array}$$

Let $(\mathrm{Hilb}_{\nu,g}//\mathrm{SL}_N)_d$ denote the invariant theory quotient arising from this linearization. We show that for small $\epsilon > 0$ and large $d \gg 0$,

$$\overline{\mathcal{M}}_g(7/10 - \epsilon) \simeq (\mathrm{Hilb}_{2,g}//\mathrm{SL}_{3(g-1)})_d.$$

Furthermore, the desired flip is induced by the cycle-class map:

$$\begin{array}{ccc} \mathrm{Hilb}_{2,g} & \stackrel{\gamma}{\longrightarrow} & \mathrm{Chow}_{2,g} \\ \downarrow & & \downarrow \\ (\mathrm{Hilb}_{2,g}//\mathrm{SL}_{3(g-1)})_d & \longrightarrow & \mathrm{Chow}_{2,g}//SL_{3(g-1)} \\ \downarrow \simeq & & \simeq \downarrow \\ \overline{M}_g(7/10-\epsilon) & \stackrel{\phi_-}{\longrightarrow} & \overline{M}_g(7/10) \end{array}$$

Applying the Hilbert-Mumford criterion for stable points to the Hilbert scheme, we find $\overline{M}_g(7/10-\epsilon)$ parametrizes curves with nodes, cusps, and tacnodes.

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Linear codes and algebraic geometry in higher dimensions

Noam D. Elkies

Let V be a variety over a field k, and choose a line bundle $L \to V$ and a vector space C of sections of L. Assume that C has finite dimension N and no base points in $V(k)$, and thus yields a map $\phi : V(k) \to \mathbf{P}^{N-1}(k)$. Further assume that there is no nonzero $c \in C$ that vanishes on all of $V(k)$. We consider the classical questions:

1) Describe the small subsets $S \subset V(k)$ that "fail to impose linearly dependent conditions on C" — that is, for which $\dim\bigl(\{f \in C \mid \forall p \in S : f(p) = 0\}\bigr)$ exceeds the expected dimension $N - \#(S)$, or equivalently, for which $\phi(S)$ is linearly dependent.

2) When k is finite, what is the distribution, as c varies over C, of the number of points of $\{p \in V(k) \mid c(p) = 0\}$ (a hyperplane section of $\phi(V(k))$, unless $c \neq 0$)?

When k is finite, these questions are related by a duality. Specifically, consider C as a *linear code*: a linear subspace of k^N, defined up to scaling each coordinate.

We have $\#\big(\{p \in V(k) \mid c(p) = 0\}\big) = N - \mathrm{wt}(c)$, where the *weight* $\mathrm{wt}(c)$ of any $c \in C$ is its number of nonzero coordinates. The *dual* of C is the linear code

$$C^* = \{c^* \in k^N \mid \forall c \in C, \sum_{p \in V(k)} c(p)c^*(p) = 0\}.$$

Thus small linear dependencies in $\phi(V(k))$ are nonzero $c^* \in C^*$ of low weight. The weight distributions of C and C^* are related by the *MacWilliams identity* [2]:

$$W_{C^*}(X,Y) = \frac{1}{\#(C)} W_C(X + (q-1)Y,\ X - Y),$$

where $q = \#(k)$, and W_C, the *weight enumerator* of C, is a generating function that encodes its weight distribution:

$$W_C(X,Y) := \sum_{c \in C} X^{N-\mathrm{wt}(c)} Y^{\mathrm{wt}(c)}.$$

Already in simple examples like linear or quadratic forms on $\mathbf{P}^n(k)$, the connection between questions (1) and (2) leads us to use polynomial identities such as the "q-binomial expansion" (the formula for the expansion of $\prod_{r=0}^{n-1}(1 + q^r T)$ in powers of T; see [1, Ch.10], for instance (10.0.10) on p.484, and again Cor. 10.2.2(c) on p.490, where this is attributed to Rothe 1811). For instance, we use these methods to give a new proof that there is no $S \in \mathbf{P}^{n-1}(k)$ of size $< 2n$ that imposes dependent conditions on quadrics, has no subsets that impose dependent conditions on quadrics, and linearly spans $\mathbf{P}^{n-1}(k)$. Equivalently, if C is the code of quadrics on $\mathbf{P}^{n-1}(k)$ then there is no $c^* \in C^*$ with $\mathrm{wt}(c^*) < 2n$ such that the support of c^* spans $\mathbf{P}^{n-1}(k)$. We also count c^* with $\mathrm{wt}(c^*) = 2n$ and spanning support: there are exactly

$$\#(\mathrm{GL}_n(k)) \cdot \left[\frac{(-1/2)^n}{n!} + \sum_{H=1}^{n} \frac{(-1/2)^{n-H}}{(2H)!(n-H)!} \prod_{r=1}^{H-1} (1 + q^r) \right].$$

For cubics on $\mathbf{P}^2(k)$, the weight enumerator of C was in effect determined by Schoof [3], using the arithmetic of elliptic curves over finite fields. In this case the $X^{N-w}Y^w$ coefficient of W_{C^*} is non-elementary once $w > 9$; for instance if q is prime then the $w = 10$ coefficient is given by a formula that involves the value $\tau(q)$ of the Ramanujan tau function.

The results we have obtained by exploiting the MacWilliams identity to relate questions (1) and (2) include the complete determination of W_C when $V = \mathbf{P}^3(k)$ and $C = \Gamma(\mathcal{O}(3))$. The following table lists, for each $T \in [-3, 6]$, the number of irreducible cubics f of weight $q^3 - Tq$ for which the surface $\{f = 0\}$ is not a cone.

(It is known that those 10 values of T are the only possible ones; the cubic surface then has $q^2 + (T+1)q + 1$ points rational over k.)

T	$51840/\#(\mathrm{GL}_4(k))$ times the number of f of weight $q^3 - Tq$
-3	$80(q+1)^2(q^2+q-3)$
-2	$\frac{45}{q+1}(77q^5+34q^4+90q^3+152q^2+281q-26)$
-1	$\frac{72}{q^3-q}(162q^7+325q^6-249q^5+205q^4+177q^3+670q^2+30q-360)$
0	$\frac{12}{q^2(q^2-1)(q^3-1)}(1735q^{11}+1329q^{10}+3314q^9-225q^8+6846q^7$ $-3993q^6+2546q^5+4785q^4+4999q^3+264q^2-12960q-4320)$
1	$\frac{72}{\#(\mathrm{GL}_2(k))}(182q^8-57q^7+90q^6+840q^5$ $-1262q^4+1907q^3+1350q^2-2690q+360)$
2	$\frac{90}{q-1}(27q^5+20q^4+136q^3-374q^2+1229q-990)$
3	$120(2q^4+9q^3-27q^2+182q-270)$
4	$36(q^4-5q^3+59q^2-235q+260)$
5	$72(q-4)(q-3)(q-2)$
6	$(q-5)^2(q-3)(q-2)$

Note that for each T, the number is given as a multiple of

$$\frac{\#(\mathrm{GL}_4(k))}{\#W(E_6)} = \frac{(q^4-1)(q^4-q)(q^4-q^2)(q^4-q^3)}{51840}.$$

Further work in this direction may include study of other (V, L, C), and attempted generalizations to complete intersections of codimension 2 and higher (via MacWilliams identities for higher weight enumerators) and perhaps to linear dependencies on non-reduced schemes of dimension zero.

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Valuative criteria for stable complexes

Dan Abramovich
(joint work with Alexander Polishchuk)

Motivated by work of Douglas and Aspinwal–Douglas in theoretical physics on so called Π-stability, Tom Bridgeland introduced a notion of *stability condition* in a triangulated category (see also work by Gorodentscev–Kuleshov–Rudakov). Considering the bounded derived category $D(X)$ of coherent sheaves on a smooth projective variety X, Bridgeland showed that the collection of all locally finite numerical stability conditions on $D(X)$ forms a complex manifold. This manifold is expected to have an important place in mirror symmetry.

Let X be a smooth projective variety. A stability condition on $D(X)$ consists of a pair

$$\sigma = (\mathcal{Z}, \mathcal{P})$$

where $\mathcal{Z} : K_0(X) \to \mathbb{C}$ is a group homomorphism and $\mathcal{P} = \{P(t)\}_{t\in\mathbb{R}}$ is a collection of full subcategories of $D(X)$, objects of which are called *semistable of phase* t. This pair is required to satisfy the following conditions:

(1) $P(t+1) = P(t)[1]$.
(2) If $t_1 > t_2$ and $E_i \in P(t_i)$ then $Hom(E_1, E_2) = 0$.
(3) For any $0 \neq E \in D(X)$ there exists a *Harder-Narasimhan diagram*

$$0 = E_0 \to E_1 \to \cdots \to E_n = E,$$

where $A_i := Cone(E_{i-1} \to E_i)$ is semistable of phase t_i, with

$$t_1 > t_2 \cdots > t_n.$$

(4) for any $0 \neq E \in P(t)$ we have $\mathcal{Z}([E]) = m(E)e^{i\pi\ t}$ with $m(E) > 0$.

The objects A_i are called the *Harder-Narasimhan constituents* of E.

In addition, we always make two more assumptions: first σ is *numerical*, namely $\mathcal{Z}$ factors through the Chern character $K_0(X) \to H^*(X,\mathbb{Q})$. Second, for a sufficiently small $\eta > 0$ and any real a the category $P(a-\eta, a+\eta)$ of objects with Harder-Narasimhan of phases in the interval $(a-\eta, a+\eta)$ is a quasi-abelian category of finite length.

We say that σ is *noetherian* if the similarly defined abelian category $P((0,1])$ is noetherian.

In the talk I first explain (following Bridgeland) in which way this notion of stability generalizes slope stability for coherent sheaves on a curve. Then I report on a result joint with A. Polishchuk, in the direction of finding a proper moduli space for stable objects:

We consider a fixed noetherian stability condition σ on $D(X)$. By a *family of objects of* $P(1)$ we mean an object of $F \in D(X \times S)$ such that the derived restriction $F_s := Li_s^*F$ to the fiber $X \times \{s\}$ is in $P(1)$ for all $s \in S$.

Theorem 5. *Let S be a smooth curve and $U \subset S$ a dense open subset.*

(1) *Every family F_U of objects in $P(1)$ over U extends to a family F of objects in $P(1)$ over S.*
(2) *Let F_1 and F_2 be families of objects in $P(1)$ over S, and let $\phi_U : (F_1)_U \to (F_2)_U$ be an isomorphism. Then Li^*F_1 and Li^*F_2 are S-equivalent.*
(3) *Every family F_U of objects in $P(1)$ over U extends, after a suitable finite surjective base change $g : S' \to S$, to a family F of objects in $P(1)$ over S' with polystable fibers in $S' - g^{-1}U$.*

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Rational points of varieties defined over a function field

Jason Starr
(joint work with A. J. de Jong)

This is a report of an ongoing project with A. J. de Jong extending the Tsen-Lang theorem regarding rational points of hypersurfaces over the function field of a surface, [6], [4], in the spirit of the generalization of Tsen's theorem proved in [2]. We propose a notion, *rational* 1*-connectedness* closely related to rational connectedness, outline a strategy for proving a rationally 1-connected variety with trivial Brauer obstruction defined over the function field of a surface has a rational point, state a theorem that is a special case of this strategy, and as a corollary give a new proof of de Jong's *Period-Index Theorem*, [1].

Let X be a smooth, geometrically connected, projective variety defined over the function field K of a variety over an algebraically closed field k. There is an exact sequence,

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}_{X/K}(K) \xrightarrow{\delta} \mathrm{Br}(K),$$

where $\mathrm{Pic}_{X/K}$ is the relative Picard scheme and $\mathrm{Br}(K)$ is the Brauer group of K. The map δ is a *Brauer obstruction* to existence of a K-point of X; if X has a K-point, $\delta = 0$.

Question. *What hypotheses on the geometric generic fiber $X \otimes_K \overline{K}$ guarantee the only obstruction to existence of a K-point is the Brauer obstruction?*

For tr.deg.$(K/k) = 1$, Tsen proved that a hypersurface of degree d in $\mathbb{P}^n$ has a K-point if $d \leq n$, [5]. Later Tsen and Lang proved for tr.deg.$(K/k) = b$, a hypersurface of degree d in $\mathbb{P}^n$ has a K-point if $d^b \leq n$, [6], [4].

Every smooth hypersurface in $\mathbb{P}^n$ of degree $d \leq n$ is Fano and thus *rationally connected*, i.e., every pair of $\overline{K}$-points are in the image of a morphism from $\mathbb{P}^1$. Graber, Harris and I proved *every* rationally connected variety over the function field of a curve in characteristic 0 has a K-point, [2].

Mazur asked for a notion of *rational n-connectedness* that is to rational connectedness what n-connectedness is to connectedness in topology. For rational 1-connectedness a working definition is rational connectedness of both X and the geometric generic fiber of the evaluation morphism $\overline{M}_{0,2}(X,\beta) \to X \times X$ for β "sufficiently positive". Here $\overline{M}_{0,2}(X,\beta)$ is any compactification of the space of 2-pointed genus 0 curves in X with curve class β, e.g. the Kontsevich compactification. One connection with the Tsen-Lang theorem is a theorem by Harris and myself (essentially) proving a general hypersurface of degree d in $\mathbf{P}^n$ is rationally 1-connected if $d^2 \leq n$, [3]. The main connection is the following theorem.

Theorem 6. *If char$(k) = 0$, tr.deg.$(K/k) = 2$, $\delta = 0$, Pic$_{X/K}(K) = \mathbb{Z}$, X is rationally 1-connected, there is an integral model $(\mathcal{X}, B)$ of (X, K) where B is a smooth projective surface and $\mathcal{X} \to B$ is smooth away from codimension 2 points in B, and if the evaluation morphism $\overline{M}_{0,1}(X,1) \to X$ is smooth with rationally connected fibers, then X has a K-point.*

Because $\mathrm{Pic}_{X/K}(K) = \mathbb{Z}$ and $\delta = 0$, there exists an ample invertible sheaf $\mathcal{O}_X(1)$ generating $\mathrm{Pic}_{X/K}(K)$, and $\overline{M}_{0,1}(X,1)$ is the space of pointed rational curves in X having $\mathcal{O}_X(1)$-degree 1. The hypothesis that there is an integral model $(\mathcal{X}, B)$ smooth away from codimension 2 and the hypothesis that the evaluation morphism is smooth with rationally connected fibers are technically useful, but hopefully can be removed from the final theorem.

The strategy is to reduce to [2]. Blowing up the base locus of a Lefschetz pencil on B gives a proper, flat morphism $\pi : B \to S$ of relative dimension 1 with geometrically connected fibers. There is a relative Hilbert scheme over S, Sec, parametrizing sections of the base-change of $\mathcal{X}$ over the base-change of B. There is a bijection between K-points of X and $k(S)$-points of $\mathrm{Sec} \otimes_{\mathcal{O}_S} k(S)$. By [2], there exists a $k(S)$-point if there is a subvariety $Z \subset \mathrm{Sec}$ such that $Z \otimes_{\mathcal{O}_S} \overline{k(S)}$ is rationally connected. A pair of sections $\sigma', \sigma'' \in \mathrm{Sec} \otimes_{\mathcal{O}_S} \overline{k(S)}$ define a point of $(\mathcal{X} \times_B \mathcal{X}) \otimes_{\mathcal{O}_S} \overline{k(S)}$. Because X is rationally 1-connected, there is a datum $(\Sigma, \tau', \tau'', f)$ of a surface Σ ruled by genus 0 curves over $B \otimes_{\mathcal{O}_S} \overline{k(S)}$, sections τ' and τ'' of this ruling, and a $B \otimes_{\mathcal{O}_S} \overline{k(S)}$-morphism $f : \Sigma \to \mathcal{X}$ such that $\sigma' = f \circ \tau'$ and $\sigma'' = f \circ \tau''$. If Image$(\tau')$ and Image(τ'') are linearly equivalent on Σ, then σ' and σ'' are contained in the image of a morphism from $\mathbb{P}^1$. Unfortunately this is rarely the case, but a version of this establishes *asymptotic rational connectedness* of the irreducible components of Sec (under the additional hypotheses).

If $X \otimes_K \overline{K}$ is a Grassmannian and the Brauer obstruction vanishes, the theorem implies X has a K-point. Let D be a division algebra with center K and

$\dim_K(D) = n^2$. Let m be the order of $[D] \in \mathrm{Br}(K)$. There is a *generalized Severi-Brauer variety* over K, X, whose L-points parametrize right ideals of $D \otimes_K L$ of dimension mn for every L/K. Because $m[D] = 0$, the Brauer obstruction of X vanishes. By the theorem, X has a K-point, i.e., D has a right ideal of dimension mn. The only right ideals of D are 0 and all of D. This gives a new proof of de Jong's Period-Index Theorem.

Theorem 7. *(de Jong [1]) The period equals the index for every element of the Brauer group of a function field of transcendence degree 2 over an algebraically closed field. In other words, if D is a division algebra with center K and $dim_K(D) = n^2$, the order of $[D] \in Br(K)$ is n.*

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$K3$ sections and the effective cone of $\overline{\mathcal{M}}_g$

Gavril Farkas

(joint work with Mihnea Popa)

This is an extended abstract of my talk given at Oberwolfach in June 2004 on joint work with Mihnea Popa partially contained in the papers [1] and [2].

We denote by $\overline{\mathcal{M}}_g$ the Deligne-Mumford moduli space of stable curves of genus g. A fundamental problem going back to Mumford is to determine the cone $\mathrm{Eff}(\overline{\mathcal{M}}_g)$ of effective divisors on $\overline{\mathcal{M}}_g$.

We recall a few basic facts about $\overline{\mathcal{M}}_g$. The boundary $\overline{\mathcal{M}}_g - \mathcal{M}_g$ corresponding to singular stable curves is a union of irreducible divisors $\Delta_0 \cup \ldots \cup \Delta_{[g/2]}$. We denote by $\delta_i := [\Delta_i] \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ the associated class in the Picard group of the moduli stack, and by λ the class of the Hodge bundle. We define the slope function $s : \mathrm{Eff}(\overline{\mathcal{M}}_g) \to \mathbb{R} \cup \{\infty\}$ by the formula

$$s(D) := \inf \{ \frac{a}{b} : a, b > 0 \text{ such that } a\lambda - b\delta - D \equiv \sum_{i=0}^{[g/2]} c_i \delta_i, \text{ where } c_i \geq 0 \}.$$

It is well-known that $s(D) < \infty$ for any D which is the closure of an effective divisor on $\mathcal{M}_g$ and in this case one has that $s(D) = a/\min_{i=0}^{[g/2]} b_i$.

The Slope Conjecture of Harris and Morrison predicts that

$$s_g := \inf_{D\in \mathrm{Eff}(\overline{\mathcal{M}}_g)} s(D) \geq 6 + 12/(g+1)$$

(cf. [3], Conjecture 0.1). This is known to hold for $g \leq 12, g \neq 10$. The conjecture has a number of interesting consequences, e.g. a positive answer would imply that the Kodaira dimension of $\mathcal{M}_g$ is $-\infty$ if and only if $g \leq 22$.

For g such that $g+1$ is composite, we can fix integers $r, d \geq 1$ such that $\rho(g,r,d) = g-(r+1)(g-d+r) = -1$ and denote by $\mathcal{M}^r_{g,d}$ the locus of curves carrying a $\mathfrak{g}^r_d$. It is known that $\mathcal{M}^r_{g,d}$ is an irreducible divisors and the class of its compactification has been computed by Eisenbud and Harris:

$$\overline{\mathcal{M}}^r_{g,d} \equiv c\big((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{[g/2}i(g-i)\delta_i\big).$$

Thus the Slope Conjecture singles out the Brill-Noether divisors as those having minimal slope and one can ask whether combinations of Brill-Noether divisors are the only effective divisors on $\overline{\mathcal{M}}_g$ of slope $6 + 12/(g+1)$.

The Slope Conjecture is closely related to the locus of curves $\mathcal{K}_g \subset \mathcal{M}_g$ lying on a $K3$ surface. We show that if $D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ is such that $s(D) < 6+12/(g+1)$ then $\mathcal{K}_g \subset D$, hence the locus $\mathcal{K}_g$ appears as a natural obstruction for an effective divisor to have small slope. Due to work of Mukai it is known that $\dim(\mathcal{K}_g) = 19+g$ for $g \geq 13$. This dimension count breaks down for $g = 10$, when because of the existence of a rational homogeneous variety associated to the Lie group G_2 we obtain that $\mathcal{K} = \mathcal{K}_{10}$ is a divisor on $\mathcal{M}_{10}$. We prove the following result:

Theorem 8. *The class of the compactification $\overline{\mathcal{K}}$ of the $K3$ divisor on $\mathcal{M}_{10}$ is*

$$\overline{\mathcal{K}} \equiv 7\lambda - \delta_0 - 5\delta_1 - 9\delta_2 - 14\delta_4 - b_5\delta_5,$$

where $b_5 \geq 6$. It follows that $s(\overline{K}) = 7 < 6 + 12/11$, hence $\overline{\mathcal{K}}$ is a counterexample to the Slope Conjecture.

To compute the class of $\overline{\mathcal{K}}$ we reinterpret curves on $K3$ surfaces in a way that makes no reference to $K3$ surfaces.

Theorem 9. *The divisor $\mathcal{K}$ has three incarnations as a subvariety of $\mathcal{M}_{10}$:*

(1) *(By definition) The locus of curves sitting on a $K3$ surface.*
(2) *The locus of curves C carrying a stable rank 2 vector bundle E with $\wedge^2(E) = K_C$ and $h^0(E) \geq 7$.*
(3) *The locus of curves C of genus 10 sitting on a quadric in an embedding $C \subset \boldsymbol{P}^4$ with $\deg(\mathrm{C}) = 12$.*

Descriptions (2) and (3) can be generalized to other genera to provide new divisors that violate the Slope Conjecture. For instance we can show that on $\mathcal{M}_{13}$ the locus of curves sitting on a quadric in an embedding $C \subset \mathbf{P}^5$ with $\deg(C) = 16$, is a divisor that has slope $< 6 + 12/14$. Similarly, the locus of curves C of genus 16 having an embedding $C \subset \mathbf{P}^7$ given by a $\mathfrak{g}^7_{21}$ such that the ideal of C is not generated by quadrics is a divisor on $\mathcal{M}_{16}$ and its compactification D on $\overline{\mathcal{M}}_{16}$

has slope $s(D) = 407/61 < 6 + 12/17$ hence it gives another counterexample to the Slope Conjecture. Finally, the locus of curves C of genus 22 carrying a linear system $\mathfrak{g}^{11}_{30}$ giving an embedding $C \subset \mathbf{P}^{11}$ which fails to satisfy the Green-Lazarsfeld condition N_2, is a divisor on $\mathcal{M}_{22}$ and its compactification has slope equal to $1655/256 = 6.503... < 6 + 12/23$.

We also studied divisors on $\overline{\mathcal{M}}_g$ having slope $6 + 12/(g+1)$ (for those g for which the Slope Conjecture holds on $\overline{\mathcal{M}}_g$). We proved the following:

Theorem 10. *The Iitaka dimension of the linear system of Brill-Noether divisors on $\overline{\mathcal{M}}_{11}$ is equal to* 19.

This is in stark contrast with the hypothesis formulated in [3] (and proven to be true for $g \leq 9$) that the Brill-Noether divisors are the only effective divisors on $\overline{\mathcal{M}}_g$ of slope $6 + 12/(g+1)$.

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Derived Equivalence of Stratified Mukai Flop

Yujiro Kawamata

Derived equivalence of different algebraic varieties provides deeper understanding of geometric phenomena on the varieties. There is a conjecture ([2], [7]):

Conjecture. *Let $f : Z \to X$ and $g : Z \to Y$ be birational morphisms between smooth projective varieties such that the pull-backs of canonical divisors are equal: $f^*K_X = g^*K_Y$. Then the bounded derived categories of coherent sheaves are equivalent: $D^b(Coh(X)) \cong D^b(Coh(Y))$.*

I considered the above conjecture in the case of stratified Mukai flops which are defined by Markman [10]. A stratified Mukai flop can be regarded as a degenerating family of usual Mukai flops.

I reviewed general known strategies toward the conjecture. The first one due to Bondal and Orlov [2] uses the semi-orthogonal decomposition. The second approach in [3] and [4] is based on the construction of the moduli space of perverse point sheaves. The third uses a locally free tilting sheaf as in [12] and [1].

From the Grassmannian variety $G(r, n)$, we can construct a stratified Mukai flop $X - \to Y$ in dimension $2r(n-r)+1$. In the case for $G(2, 4)$, Namikawa [11] proved that the natural choice $\Phi' = Rg_*Lf^*$ does not give an equivalence. I modified the functor to $\Phi(a) = Rg_*(Lf^*a \otimes \mathcal{O}_Y(E'_1)$ so that the conjecture is still confirmed in this case. A locally free generator of the derived category is given by Kapranov [5] in this case, but it is not tilting.

Question. *(1) It is expected that the conjecture should hold for a wider class of varieties.*

Let (X, B) *and* (Y, C) *be pairs of projective varieties with* $\mathbb{Q}$*-divisors which are KLT. Assume that there are birational morphisms* $f : Z \to X$ *and* $g : Z \to Y$ *from a smooth projective variety such that the pull-backs of log canonical divisors are equal:* $f^*(K_X + B) = g^*(K_Y + C)$*. Then we expect an equivalence of derived categories* $D(X, B) \cong D(Y, C)$*.*

The first question is to give a correct definition the category $D(X, B)$*. For example, if* X *has only quotient singularities, then we should have* $D(X) = D^b(Coh(\mathcal{X}))$*, where* $\mathcal{X}$ *is the smooth Deligne-Mumford stack associated to* X *([6], [9]).*

(2) If there is an equivalence $D(X) \cong D(Y)$*, then a posteriori one can regard* Y *as a moduli space of perverse point sheaves on* X *by using the kernel object of the equivalence functor ([12], [8]). The question is to give a priori definition of the perverse point sheaves.*

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Murphy's Law for the Hilbert scheme (and the Chow variety, and moduli spaces of surfaces of general type, and stable maps, and nodal and cuspidal plane curves, and . . .)

Ravi Vakil

Define an equivalence relation on singularities generated by: If $(X,p) \to (Y,q)$ is a smooth morphism, then $(X,p) \sim (Y,q)$. We say that *Murphy's Law* holds for a moduli space if every singularity type of finite type over $\mathbb{Z}$ appears on that moduli space.

Theorem 11. *The following moduli spaces satisfy Murphy's Law.*

- **1a.** *the Hilbert scheme of nonsingular curves in projective space*
- **1b.** *the moduli space of maps of smooth curves to projective space* $\mathcal{M}_g(\mathbb{P})$
- **1c.** *the Chow variety of nonsingular curves in projective space (where only seminormal singularities are allowed)*
- **2a.** *the (coarse or fine) moduli space of smooth surfaces (with ample canonical bundle)*
- **2b.** *the Hilbert scheme of nonsingular surfaces in* $\mathbb{P}^5$*, and the Hilbert scheme of surfaces in* $\mathbb{P}^4$
- **3.** *more generally, the moduli space of smooth* n*-folds* ($n > 1$) *(with ample canonical bundle)*
- **4a.** *branched covers of* $\mathbb{P}^2$ *with only simple branching (nodes and cusps), in characteristic not* 2 *or* 3
- **4b.** *the "Severi variety" of plane curves with a fixed numbers of nodes and cusps, in characteristic not* 2 *or* 3

In the lecture, more spaces were shown to satisfy Murphy's Law; for the sake of brevity we have kept the list short. A weaker equivalence relation may also be used.

We sketch some philosophy and history, and then outline the proof. I am grateful to the organizers and participants in the Oberwolfach workshop on Classical Algebraic Geometry for many comments, in particular for pointing out that Theorem 12 was first proved by Mnëv. I thank F. Catanese for sharing his expertise, and D. Abramovich for sharing his book [5]. The results stated here will appear in [10].

The moral of Theorem 11 is as follows: in algebraic geometry, we know that some moduli spaces of interest are "well-behaved" (e.g. equidimensional, having at worst finite quotient singularities, etc.), often because they are constructed as Geometric Invariant Theory quotients of smooth spaces: e.g. the moduli space of curves, the moduli space of vector bundles on a curve, the moduli space of branched covers of $\mathbb{P}^1$ (the Hurwitz scheme, or space of admissible or twisted covers), the Hilbert scheme of divisors on projective space. In other cases, there has been some effort to try to bound how "bad" the singularities can get. Theorem 11 in essence states that these spaces can be arbitrarily singular, and gives a means of constructing an example where any given behavior happens. To make this quite explicit, one can construct a smooth curve in projective space whose deformation space has any given number of components, each with a given singularity type, with any given non-reduced behavior along various associated subschemes. Similarly, one can give a smooth surface of general type in characteristic 17 that lifts to 17^7 but not to 17^8.

There is a folklore belief that the Hilbert scheme satisfies Murphy's Law, which was first explicitly stated in [4] p. 18. (The MathReview for this book MR1631825 shows the mathematical community's discomfort with the informal nature of the traditional statement of Murphy's Law.) I am not sure of the origin of Murphy's Law, but it seems reasonable to ascribe it to Mumford (see his famous "pathologies" paper [9]) and Hartshorne.

On the other hand, other moduli spaces were believed (or hoped) to be better-behaved. For example, Severi stated that the space of plane curves with given numbers of nodes and cusps was unobstructed (see the MathReview MR0897672 to [6]). J. Wahl gave the first counterexample in [12], and [6] gives another. Theorem 11 **4b** shows that Severi was "maximally wrong".

The proof is by drawing connections among various moduli spaces. We begin with a remarkable result of Mnëv. Define the *incidence scheme of points and lines in* $\mathbb{P}^2$, a locally closed subscheme of $(\mathbb{P}^2)^m \times (\mathbb{P}^{2*})^n = \{p_1, \dots, p_m, l_1, \dots, l_n\}$.

- We are given some specified incidences: For each pair (p_j, l_i), either p_j is required to lie on l_i, or p_j is required not to lie on l_i.
- The points are required to be distinct, and the lines are required to be distinct.
- Given any two lines, there a point required to be on both of them.
- Each line contains at least three points.

Theorem 12. *(Special case of Mnëv's Universality Theorem) Every singularity type appears on some incidence scheme.*

The original statement was in [7], [8]; for a later exposition see [5] p. 13.

Outline of Theorem 11. Given any singularity type, we begin with an incidence scheme, and a point on it with that singularity type. Then consider the surface S that is the blow-up of $\mathbb{P}^2$ at the points p_j, along with the marked divisor D that is the proper transform of the union of the l_i. Deformations of (S, D) correspond to deformations of the l_i and p_j preserving the incidences, so the deformation space of (S, D) has the same singularity type. Choose two other divisor classes on S that are equivalent to D modulo 2, and sufficiently ample, and choose effective divisors D', D'' in these classes. (This is a smooth choice; the resulting deformation space hence has the same singularity type.) Then use Catanese's construction [1] to produce a $(\mathbb{Z}/2)^3$ cover with branch divisor given by these three divisors, whose deformations are precisely those of (S, D, D', D''); this surface has ample canonical bundle. Then [2] Proposition 4.3 ensures that this surface has no extra automorphisms (other than $(\mathbb{Z}/2)^3$), ensuring that the coarse moduli space has the same singularity type as well. (In characteristic 2, we take $(\mathbb{Z}/2)^3$ covers instead, using Pardini's work on abelian covers.)

We have thus shown **2a**. By taking four or five sections of a sufficiently ample bundle, we obtain **2b**. By taking three sections, and using Wahl's results (showing that deformations of a generic branched cover of $\mathbb{P}^2$ are the same as deformations of the branch divisor preserving the nodes and cusps), we obtain **4**. By taking the product of the surface with high-genus curves, we obtain **3** (as deformations of the product are products of the deformations, by van Opstall's thesis [11]). By embedding the surface by a complete linear system corresponding to a sufficiently ample divisor, and slicing the surface with a sufficiently high degree hypersurface, we obtain **1**, using Fantechi and Pardini's key result of [3].

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Bounded complements: conjecture, examples and applications

Vyacheslav Shokurov

In early 90's complements were used in construction of 3-fold log flips [6]. In recent investigations they appear in a more general form as an important ingredient to establish

1. Borisovs-Alexeev's conjecture on the boundedness of ε-log canonical Fano varieties [1];

2. which in its turn is related to the log canonical thresholds [5], the canonical and anticanonical thresholds; and

3. ascending chain condition (acc) of the minimal log discrepancies (mld's).

Usual n-complements and their generalizations were discussed and, in particular, proposed

Conjecture (on bounded complements)**.** *Let $d > 0$ be a positive integral number and $\varepsilon \geq 0$ be a real number. Then there exists a finite set $\mathcal{N}_d(\varepsilon)$ of positive integral numbers such that, for any contraction of normal algebraic varieties $X \to Z$ and some (complementary index) $n \in \mathcal{N}_d(\varepsilon)$, if we take the general element D of the linear system $|-nK|$, where K denotes the canonical divisor, then the log pair $(X, D/n)$ has only ε-log canonical singularities in the mobil sense and locally over Z under the following conditions:*
a) $\dim X \leq d$;
b) X has only ε-log canonical singularities; and
c) $-K$ is nef and big over Z.

Moreover, it is expected that the last two conditions can be weakened:
b′) for $\varepsilon > 0$, X has only ε'-log canonical singularities for some $0 \leq \varepsilon' < \varepsilon$ depending on d, ε and $\mathcal{N}_d(\varepsilon)$; and
c′) there exists a positive integral number m and a divisor $D \in |-mK|$ such that $(X, D/m)$ has only ε-log canonical singularities.

Certain generalization on log pairs and other types of singularity restrictions are expected, too.

The following examples and applications illustrates the conjecture:

1. the surface case; 3-fold case for canonical singularities in the list of Hayakawa and Takeuchi [4];
2. the toric case: A. Borisov's classification [2] and closed subgroups of a real torus according to J. Lawrence [4];
3. the acc of mld's and of thresholds.

Sketch of the proof for 3. for the acc of mld's. The statement is conditional, that is, follows from the conjecture. Let $a_i = a(X_i, x_i)$ be a monotonic increasing sequence of mld's where $\dim X_i \leq d$ and X_i is considered over itself. Thus a) and c) are satisfied. Take $\varepsilon = \lim_{i\to\infty} a_i$. We need to verify a stabilization: $a_i = \varepsilon$ for all $i \gg 0$. The case $\varepsilon = 0$ is obvious since all $a_i \geq 0$ (or $= -\infty$). In the case $\varepsilon > 0$ we can apply b′) for all $i \gg 0$. In addition to the stabilization we get the boundedness of canonical indexes of points $x_i \in X_i$ since the complementary divisor $D = 0$. □

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Arcs and valuations

Shihoko Ishii

In [13], Nash introduces the Nash map which associates a family of arcs through the singularities on a variety (this family is called a Nash component in this talk) to an essential divisor over the variety. In other word, Nash map is a correspondence between the set of certain families of arcs and the set of certain divisorial valuations.

On the other hand, L. Ein, R. Lazarsfeld and M. Mustaţă ([4]) introduce a map from the set of irreducible cylinders for a non-singular variety to the set of divisorial valuations.

In this talk, we introduce a map from the set of fat arcs to the set of valuations. Here, a fat arc is an arc which does not factor through any proper closed subvarieties. This map is a generalization of Nash map and the map by Ein, Lazarsfeld and Mustaţă. We can see that some fat arcs correspond to divisorial valuations and the others to non-divisorial valuations. Here, we determine the fat arcs which correspond to divisorial valuations. By this characterization we obtain many examples corresponding to divisorial valuations including Nash components and cylinders in the arc space of a non-singular variety. As a cylinder and a Nash component are of infinite dimension, one may have an impression that an arc corresponding to a divisorial valuation should be of infinite dimension. But our characterization gives many finite dimensional families of arcs which correspond to divisorial valuations. Another example is the arc determined by a conjugacy class of a finite group G which gives the quotient variety $X = \mathbb{C}^n/G$ ([3]). The restriction of our map onto a subset of these arcs coincides with the "McKay correspondence" constructed in [9]. Therefore, one can think that the Nash map and the "Mckay correspondence" are brothers.

This talk also gives a partial answer to the Nash problem which asks if the Nash map is bijective. This problem was posed in Nash's preprint in 1968 (This preprint was published later as [13]). Inspired by this preprint, many people studied the arc spaces of singularities and divisors over the singular varieties (see, Bouvier [1], Gonzalez-Sprinberg [5], Hickel [6], Lejeune-Jalabert [10], [11], [12], Nobile [14], Reguera-Lopez [15]) Then, affirmative answer for the Nash problem is obtained for a minimal 2-dimensional singularity by Reguera-Lopez [15]. For non-minimal 2-dimensional singularities, we do not know the answer of the Nash problem even for a rational double point (Recently the author was informed that a French mathematician proved the affirmative answer for a rational double point). Last year, the Nash problem was answered affirmatively for a normal toric variety of arbitrary dimension, but negatively in general in [8]. Though there is a counter example for the Nash problem, it is still an interesting problem to clarify in which category the Nash problem is affirmatively answered. For example, this problem is still open for 2 and 3 dimensional singularities as the counter examples in [8] are normal singularity of dimension greater than or equal to 4. For non-normal singularities, nothing is known about the Nash problem. In this talk, we give the affirmative answer to the Nash problem for a pretoric variety. As a corollary, we obtain that for a non-normal toric variety the Nash problem is affirmative.

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Mondrian tableaux and the cohomology of flag varieties

Izzet Coskun

The k-step flag variety $F(a_1,\dots,a_k;n)$ parametrizes k-tuples of vector spaces $V_1 \subset \cdots \subset V_k \subset V$ of a fixed n-dimensional vector space V, where V_i has dimension a_i. These varieties are fundamental to geometry, representation theory and the theory of symmetric functions. Consequently, it is important to 'know' their cohomology rings. We now explain what we mean by 'knowing the cohomology ring' in the special case of Grassmannians.

Schubert cycles generate the cohomology of the Grassmannian $G(a,n)$. They are indexed by partitions $\lambda = (n-a \geq \lambda_1 \geq \cdots \geq \lambda_a \geq 0)$. The Schubert cycle σ_λ is the class of a variety defined by the following rank conditions with respect to a fixed complete flag $F_\bullet$

$$\Sigma_\lambda(F_\bullet) = \{W \in G(a,n) | \dim(W \cap F_{n-a+i-\lambda_i}) \geq i\}.$$

Given two Schubert cycles σ_λ and σ_μ their product $\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c^\nu_{\lambda,\mu} \sigma_\nu$ is expressed as a sum of Schubert cycles. The structure coefficients $c^\nu_{\lambda,\mu}$ are known as Littlewood-Richardson (LR for short) coefficients. We would like to have an effective rule (usually refered to as an LR rule) for computing these coefficients.

I would like to reiterate that the crucial requirement is that the rule be effective. For example, using the Giambelli formula one can express every Schubert cycle in terms of Pieri cycles (those for which $\lambda_i = 0$ for $i > 1$). Then the Pieri rule gives a way of multiplying Pieri cycles. This strategy does not qualify because the Giambelli formula is a determinantal formula and expresses the Schubert cycles as sums and *differences* of products of Pieri classes.

In the case of Grassmannians there are many LR rules in terms of Young tableaux [3], puzzles [4,5], and checkers [6]. For the two-step flag variety A. Knutson conjectured a rule in terms of puzzles (see [1]). However, for arbitrary flag varieties, except for multiplying very special classes (e.g. Monk's formula [3]), there was not even a conjectural rule. The purpose of this talk is to describe (and sketch the proof of) such a rule.

Our rule will be in terms of combinatorial objects called Mondrian tableaux. The Mondrian tableau associated to the Schubert cycle σ_λ in $G(a, n)$ is a collection of a nested squares of side lengths $n - a + i - \lambda_i$. In a Mondrian tableau a box of size s denotes a vector space of dimension s. If a box b_1 is contained in another box b_2, then the vector space represented by b_1 is a subspace of the vector space represented by b_2. A complete flag in V can be represented by a nested sequence of n boxes. The Mondrian tableau records those flag elements which have an additional dimension of intersection with the a-plane.

To multiply two Schubert cycles in $G(a, n)$ we place their corresponding Mondrian tableaux at the opposite corners of an $n \times n$ square. After some preliminary manipulations, we move the boxes at the lower left hand corner anti-diagonally up by one. These moves correspond to making the flags with respect to which the Schubert cycles are defined less transverse. At each step the intersection breaks into two pieces (except when the dimension of the next box at the left hand corner is one bigger than the one we are moving or the corresponding statement for the flag in the upper right hand corner holds). We continue moving the boxes and tracing each possibility until all the boxes are nested again. Once the boxes are nested the tableau again corresponds to a Schubert cycle. The figure shows the sample calculation $\sigma_1^2 = \sigma_2 + \sigma_{1,1}$ in $G(2, 4)$. The main result is as follows.

Theorem 13. *The LR coefficient $c^\nu_{\lambda,\mu}$ of $G(a, n)$ is equal to the number of times the Mondrian tableau associated to σ_ν occurs in the game starting with the Mondrian tableaux σ_λ and σ_μ in an $n \times n$ square.*

The rule for arbitrary flag varieties is similar. A Schubert cycle in $F(a_1, \ldots, a_k; n)$ is represented by a Mondrian tableau which has a_k boxes in k colors $C_1, \ldots, C_k$. Among the boxes exactly $a_i - a_{i-1}$ of them have color C_i. The flag $F_\bullet$ induces a complete flag on the vector space V_k. At the j-th element of this complete flag there exists a smallest index i for which the dimension of intersection of this flag

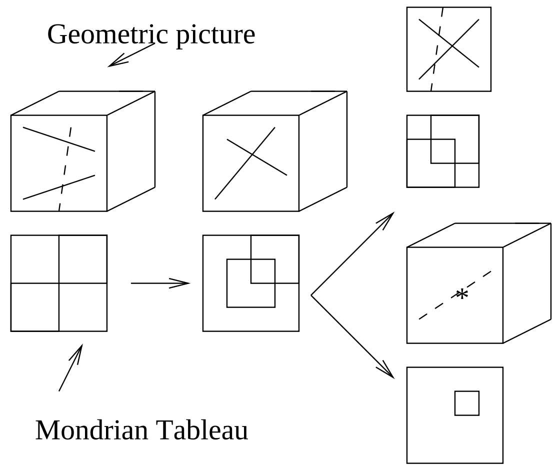

FIGURE 1. The product σ_1^2 in $G(2,4)$: Mondrian tableaux and the geometry corresponding to them.

element with V_i is one more than that of the previous flag element. We draw the corresponding box in the color C_i.

To multiply two Schubert cycles in the flag variety we place the corresponding Mondrian tableaux in opposite corners of an $n \times n$ square. We play a game similar to the one in the case of Grassmannians. Now there are more possibilities depending on which linear space among the V_i can meet the intersection when we specialize the flags. The rules list all the possibilities. The main theorem is the following.

Theorem 14. *The product of two Schubert cycles in $F(a_1, \ldots, a_k; n)$ equals the sum of all the Schubert cycles that result in the game of Mondrian tableaux starting from the given Schubert cycles in an $n \times n$ square.*

LR rules have many applications in geometry and representation theory. Due to lack of space here we mention only two immediate applications. A Schubert problem of expected dimension zero is called *enumerative over a field K* if all of the solutions can be realized over K for an appropriate choice of flags. A slight modification of the main theorem in [7] yields the following corollary.

Corollary 1. All Schubert problems for all flag varieties are enumerative over the real numbers.

The three-pointed Gromov-Witten invariants of the Grassmannian are the structure constants of the small quantum cohomology ring. Consequently, these coefficients are often refered to as the quantum LR coefficients. A theorem of Buch, Kresch and Tamvakis (in [1]) relates these coefficients to ordinary LR coefficients of two step flag varieties. Using this dictionary we obtain the following corollary.

Corollary 2. The Mondrian tableaux rules provide a quantum LR rule.

We conclude with a series of open problems. The techniques described here should yield at least some partial answers to these problems.

Problem. *Provide an LR rule for other types of Grassmannians/ flag varieties (i.e. of types B,C,D,E,F and G).*

For example, for the orthogonal Grassmannian the same method carried out on a quadric should lead to a rule. The new feature in this case is that there can be multiplicities of 2. This is currently work in progress. It would be interesting to see whether one can also provide rules for the exceptional groups by carrying out the degenerations on their associated cubic forms. Finally we can ask for a quantum LR rule for flag varieties.

Problem. *Provide a quantum LR rule for flag varieties.*

Combining the results discussed here with those of [2], one should be able to make progress on this problem.

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Hilbert's Theorem on non-negative ternary quartic forms

Claus Scheiderer
(joint work with V. Powers, B. Reznick and F. Sottile)

Let $f = f(x, y, z)$ be a ternary form of degree four with real coefficients. Assume that f is positive semidefinite (psd), i.e., has non-negative values on $\mathbb{R}^3$. In 1888, Hilbert [2] proved that f can be written as a sum of three squares of quadratic forms with real coefficients. His proof used arguments of classical algebraic geometry and is not always easy to understand. There exist modern accounts of, and variations on, Hilbert's proof, e.g. [3] or [4].

We are pursuing a new approach to Hilbert's theorem, which yields additional and new information. Call two quadratic representations

$$f = p_0^2 + p_1^2 + p_2^2 = q_0^2 + q_1^2 + q_2^2 \qquad (*)$$

equivalent if they can be obtained from each other by an orthogonal linear change. We are counting the number of inequivalent such representations. In particular, showing that this number is always at least one will re-prove Hilbert's theorem.

We are following an approach of C. T. C. Wall [5], which in turn is based on ideas of A. B. Coble [1]. Their work considers ternary quartic forms f over $\mathbb{C}$. Wall shows that, apart from one exceptional case, such f can always be written as a sum of three squares of (complex) quadratic forms. These representations are linked to non-trivial 2-torsion points on the Jacobian of the plane quartic curve $f = 0$, at least in the case when this curve is non-singular. Since the quartic has genus 3, there are 63 such points on the Jacobian, and accordingly, there are exactly 63 inequivalent quadratic representations of f (over $\mathbb{C}$).

For the situation of Hilbert's theorem, we are working over the reals. Assume that the real quartic curve $f = 0$ is non-singular. Since f is psd, the Jacobian has exactly 15 non-trivial real 2-torsion points. Using methods from Galois cohomology, we show that exactly 8 of these 15 points correspond to real sums of squares representations $(*)$ of f, while the remaining 7 correspond to signed representations ($f = p_0^2 + p_1^2 - p_2^2$ or $f = p_0^2 - p_1^2 - p_2^2$). Therefore, in the non-singular case, there are exactly 8 inequivalent ways to write f as a sum of three squares.

The analysis of the case where the quartic $f = 0$ is singular (irreducible) proceeds roughly along similar lines. Quadratic representations of f may have nonempty base loci contained in the singular locus of the curve. Case distinctions are necessary with respect to the possible base loci of such representations, depending on the singularities of the curve.

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Special linear systems in $\mathbb{P}^2$

Stephanie Yang

The purpose of this talk is to introduce a combinatorial technique to determine when general points in $\mathbb{P}^2$ fail to impose independent linear conditions on plane curves of a given degree. A well-known conjecture, formulated independently by

B. Segre, A. Gimigliano, B. Harbourne, and A. Hirschowitz (though most commonly known as the Harbourne-Hirschowitz conjecture) gives geometric meaning to when this is the case.

Let $m_1, \ldots, m_r$ be a sequence of positive integers which correspond to general points $p_1, \ldots, p_r \in \mathbb{P}^2$. Denote by $\mathcal{L} = \mathcal{L}_d(m_1, \ldots, m_k)$ the linear system of degree d curves with multiplicity m_i at p_i. The expected dimension of $\mathcal{L}$ is:

$$e(\mathcal{L}) = \max \left\{ \binom{d+2}{2} - \sum_{i=1}^{k} \binom{m_i+1}{2} - 1, -1 \right\}. \tag{1}$$

This is a sharp lower bound for the actual dimension of $\mathcal{L}$; when equality holds, we say that $\mathcal{L}$ is *non-special*, and otherwise, we say that $\mathcal{L}$ is *special*.

Let $\pi : V \to \mathbb{P}^2$ be the blow-up of the projective plane at the points $p_1, \ldots, p_r$. A curve $C \subseteq \mathbb{P}^2$ is called a (-1)-*curve* if it is rational and its proper transform $\widetilde{C} \subseteq V$ has self-intersection equal to -1. With this in our vocabulary, it is easy to state the Harbourne-Hirschowitz conjecture:

Conjecture (Harbourne-Hirschowitz)**.** *$\mathcal{L}$ is special if and only if it contains a multiple (-1)-curve in its base locus.*

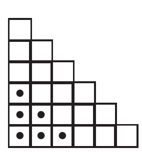
While one direction (the "if" part) of this equivalence is elementary, the other direction remains open except for a few special cases.

The Harbourne-Hirschowitz conjecture has a variety (no pun intended) of algebro-geometric consequences. First, a proof of the conjecture would settle the long-standing Nagata conjecture, posed in 1959 by Nagata after constructing a counterexample to Hilbert's 14th problem. (In short, Nagata conjectured if $n \geq 10$, then any degree d curve with n points of multiplicity m must satisfy $d > m\sqrt{n}$). The Harbourne-Hirschowitz conjecture also implies that a curve with negative self-intersection in the blow-up of $\mathbb{P}^2$ at (any number of) general points must have self-intersection -1, thus giving a complete description of the Mori cone of such surfaces.

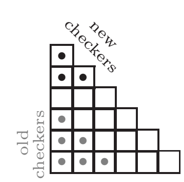
Checkers on a triangle

We now describe the combinatorial game. Given $\mathcal{L}_d(m_1, \ldots, m_k)$, form a $(d+1) \times (d+1)$ triangle of boxes. Our goal is to place checkers on the board using two types of moves:

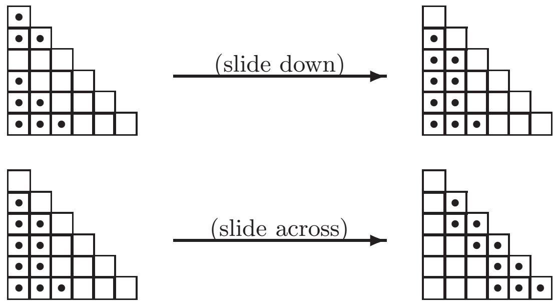
Type A: For each multiplicity m_i, we place $\binom{m_i+1}{2}$ checkers in one of the three corners of the box, forming an $m_i \times m_i$ triangle. If no corner of the box has enough empty squares available, then our only options are to quit the game, or perform moves of the other type in order to create empty squares. Two examples of valid moves are:



or



Type B: We may perform one of six “slides” which move all of the checkers towards one corner along rows, columns, or diagonals parallel to one side. Two examples of valid moves of this type are:



In [8], we prove the following fact: if all of the checkers can be placed on the board using moves only of the two types listed above, then $\mathcal{L}_d(m_1, \ldots, m_i)$ is non-special.

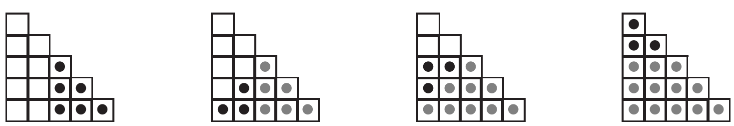
Examples

As a first example, consider linear system $\mathcal{L}_5(3, 2, 2, 2)$ of quintics with one triple point and three double points. When we perform the triangle algorithm as follows:

Step 1: Place six checkers (for the triple point) on one corner (bottom right) of the triangle

Steps 2–4: Place three checkers (for a double point) on the top corner of the triangle, and perform two slides: first slide down in columns, and then slide to the right along the rows.

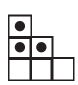
The successive steps look like:



Step 1 Step 2 Step 3 Step 4

and thus $\mathcal{L}_5(3, 2, 2, 2)$ is non-special (and empty).

Now consider the special linear system $\mathcal{L} = \mathcal{L}_2(2, 2)$ of conics through two double points. After placing first three checkers in any corner of the triangle, we cannot place another triangle of three checkers onto the board, even after any sequence of slides. This of course is due to the fact that $\mathcal{L}_2(2, 2)$ is special.



Application

In [8], we modify a well-known degeneration of $\mathbb{P}^2$ first introduced by Ciliberto and Miranda in [1]. Let Δ be a one-parameter family, and denote by X the blow-up of the three-fold $\mathbb{P}^2 \times \Delta$ at a point. The fibers of X over Δ can be viewed as a degeneration of $\mathbb{P}^2$ to a reducible surface with two rational components. On X we can create a family of linear systems of plane curves with multiple points on each fiber, and use this degeneration to "break" the family of linear systems into systems defined on each of the two rational components of the special fiber of X. This gives us a recursive bound for the dimension of such plane curves. A consequence of this degeneration is the following fact:

Theorem 1. *For any positive integer M, there exists a bound $D = D(M)$ such that:*

$$\left\{\begin{matrix}\text{The Harbourne-Hirschowitz conjecture is true for all linear systems}\\ \mathcal{L}_d(m_1,\dots,m_k) \text{ with } d < D(M) \text{ and } m_i \le M.\end{matrix}\right\}$$
$$\Longrightarrow \left\{\begin{matrix}\text{The Harbourne-Hirschowitz conjecture is true for all linear systems}\\ \mathcal{L}_d(m_1,\dots,m_k) \text{ with } m_i \le M.\end{matrix}\right\}$$

The base points above are allowed to have mixed multiplicity. (Most recent results have applied only to collections of base points with all or all but one points assigned equal multiplicity.) Also note that the list of possible linear systems on the left is finite while those on the right are infinite. The exact formula for $D(M)$ is given in [8], and selected values are given below:

M	5	6	7	8	9	10	11	12	$\cdots$	$M >> 0$
$D(M)$	21	25	29	34	42	51	61	71	$\cdots$	$O(M^2)$

In particular, $D(7) = 29$ and the number of possible linear systems 28 or less, with multiple points of order 7 or less, is approximately 10^8. One hundred million cases sounds daunting to all but the computer-minded. We wrote a program (in `C++`) to enumerate this long list of cases and play the combinatorial game (of checkers on a triangle) on each linear system. Remarkably, the game worked to prove the Harbourne-Hirschowitz conjecture in all but a few dozen of linear systems, which are then handled with ad hoc methods in the last section of this paper, to prove:

Theorem 2. *The Harbourne-Hirschowitz conjecture is true for all linear systems of plane curves with base points having multiplicity up to* 7.

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Pfaffians of extrasymmetric antisymmetric matrices, and moduli spaces

Fabrizio Catanese

(joint work with Ingrid Bauer and Roberto Pignatelli)

The motivation for our work stems from the questions posed by F. Enriques in Chapter VIII of his book "Le superficie algebriche" ([4]) about surfaces of general type with $p_g = 4$.

For these, $K^2 \geq 4$, and the cases $K^2 = 4, 5$ were completely classified by Enriques. Enriques also discussed at length the case $K^2 = 6$, which was later completely classified by Horikawa in [7].

The existence question posed by Enriques for $K^2 \geq 7$ was later solved by virtue of the contributions of several authors, and we now know that such surfaces exist, even with a birational canonical map, for $7 \leq K^2 \leq 32$, cf. e.g. [3].

The classification project progresses more slowly: the case $K^2 = 7$ was finally completely classified in the monograph by the first coauthor ([1]).

The challenging open problem here is to understand the structure of the moduli space, i.e., to determine the incidence correspondence of the several locally closed strata which are described in the classification.

Horikawa in [5] showed that the moduli space for $K^2 = 5$ is connected, with two irreducible components meeting along a divisor, and he showed ([7]) that there are at most three connected components for $K^2 = 6$.

We were able to prove:

Theorem 3. *Consider the moduli space of surfaces with* $p_g = 4, K^2 = 6$*. Then it has at most two connected components.*

In particular, there is a deformation of surfaces of type (IIIb) to surfaces of type (II).

Question. *It the above moduli space (for* $p_g = 4, K^2 = 6$*) disconnected?*

A possible reason for this could be that the surfaces of both components degenerate to surfaces with a genus two pencil, but in one case the braid monodromy is transitive, in the other case it is not.

Sketch of the proof of Theorem 3. We use the following fact: the canonical model X of surfaces of Type (II) are hypersurfaces of degree 9 in the weighted projective space $\mathbb{P}(1,1,2,3)$. As such, these surfaces are always singular (this explains the result of Horikawa that the moduli space is non reduced on this open set), and the canonical divisor is 2-divisible as a Weil divisor on X.

Similarly occurs for type (IIIb), so for both type of surfaces we have a semicanonical ring $\mathcal{B}$, and we would like to find a flat family of deformations of the semicanonical ring.

The ring $\mathcal{B}$ is a Gorenstein ring, of codimension 1 in case (II), of codimension 4 in case (IIIb), where X is embedded in $\mathbb{P}(1,1,2,3,4,5,6)$.

In order to describe the semicanonical ring and its deformations in case (IIIb), we use, as in [2], the format of extrasymmetric antisymmetric 6×6 matrices.

This format applies because surfaces of type (IIIb) have a pencil of hyperelliptic curves of genus 3, and one can lift this graded ring of dimension 1 to the semicanonical ring of the surface.

The deformation trick is similar to the one used in [2] for the canonical ring: filling entries of homogeneous degree 0 in the matrix with parameters. When these parameters are non zero, three of the given Pfaffians allow to eliminate the 3 variables of respective weights $4,5,6$. We obatin then a semicanonical ring of type (II). □

We want now to briefly discuss the cited method of extrasymmetric antisymmetric 6×6 matrices.

The main point here is the lack of a structure theorem for Gorenstein subvarieties of codimension 4 (for codimension 3 we have the celebrated theorem of Buchsbaum and Eisenbud!).

Several explicit formats were proposed by Dicks, Reid and Papadakis.

The geometric roots for the above one lie in the fact that the Segre product $\mathbb{P}^2\times\mathbb{P}^2$ is embedded in $\mathbb{P}^8$ as the variety of 3×3 matrices A of rank 1, hence defined there by 9 quadratic equations, admitting 16 relations.

If however one writes $A = B + C$, with C symmetric , B antisymmetric, then one can form the antisymmetric 6×6-matrix D:

$$\begin{matrix} B & C \\ -C & B \end{matrix}$$

The matrix D has an extrasymmetry from which follows indeed that the 15 4×4 Pfaffians of D are not linearly independent, but exactly reduce to the 9 above quadratic equations.

Using a flat family of deformations of the above subvariety, and interpreting the entries of the matrix as indeterminates to be specialized, one obtains an easy construction of Gorenstein subvarieties of codimension 4, which seems to be rather ubiquitous.

We refer to [11] for a thorough discussion of the problem of understanding Gorenstein rings in codimension 4. Our result shows that this moduli space could be rather complicated, since we obtain a deformation from codimension 4 to codimension 1, but we observe that one cannot pass through the lower codimensions.

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Finite and infinite generation of Nagata invariant ring

Shigeru Mukai

An m-dimensional linear representation of an (algebraic) group G induces an action on the polynomial ring $\mathbf{C}[z_1, \dots, z_m]$ of m variables. This is called a *linear action* on the polynomial ring. In 1890, Hilbert showed that the invariant ring was finitely generated for classical representations of the general and special linear groups. The following is known as his (original) fourteenth problem ([1]):

Question. *Is the invariant ring $\mathbf{C}[z_1, \dots, z_m]^G$ of a linear action of an algebraic group finitely generated?*

The answer is affirmative for the (1-dimensional) additive algebraic group $\mathbf{G}_a$ ([3]). In 1958, Nagata considered the standard unipotent linear action

$$(t_1, \dots, t_n) \in \mathbf{C}^n \ \downarrow \ \mathbf{C}[x_1, \dots, x_n, y_1, \dots, y_n] =: S \qquad (1)$$

$$\begin{cases} x_i \mapsto x_i \\ y_i \mapsto y_i + t_i x_i \end{cases}, \quad 1 \le i \le n,$$

of $\mathbf{C}^n$ on the polynomial ring S of $2n$ variables and showed that the invariant ring S^G with respect to a general linear subspace $G \subset \mathbf{C}^n$ of codimension 3 was not finitely generated for $n = 16$. I studied this example systematically and obtained the following:

Theorem 4. *The invariant ring S^G of (1) with respect to a general linear subspace $G \subset \mathbf{C}^n$ of codimension r is finitely generated if and only if*

$$\frac{1}{2} + \frac{1}{r} + \frac{1}{n-r} > 1.$$

This inequality is equivalent to the finiteness of the Weyl group $W(T_{2,n-r,r})$ of the Dynkin diagram $T_{2,n-r,r}$ with three legs of length 2, $n-r$ and r. There are four infinite series [I]–[IV] and five exceptonal cases [V]–[IX] where this holds:

	[I]	[II]	[III]	[IV]	[V]	[VI]	[VII]	[VIII]	[IV]
r	1		2		3	3	4	3	5
$n-r$		1		2	3	4	3	5	3
diagram	A_n	A_n	D_n	D_n	E_6	E_7	E_7	E_8	E_8

The 'if' part of the theorem is proved case by case. In the cases [I] and [III], the invariant ring is very explicit and the proof is immediate. The case [II] is classical and the invariant ring S^G is the homogeneous coordinate ring of a Grassmannian variety. In the case [IV], that is, $\dim G = 2$, the invariant ring is the total coordinate ring, or the Cox ring, of the moduli space of parabolic 2-bundles on an n-pointed projective line. Note that the following part of the 14th problem seems still open:

Question. *Is the invariant ring $\mathbf{C}[z_1, \ldots, z_m]^G$ of a linear action of the 2-dimensional additive group $G = \mathbf{G}_a \times \mathbf{G}_a$ finitely generated?*

See [2] for the 'only if' part.

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Geometry of tropical curves

Grigory Mikhalkin

The talk gives a description of *tropical curves*, their inner structure, and outlines some of their applications in complex and real algebraic geometry.

Tropical curves arise as 1-dimensional varieties over the so-called *tropical semifield* $\mathbb{R}_{\text{trop}}$, the semifield of real numbers equipped with two operations: taking the maximum (treated as the semifield addition) and addition (treated as the semifield multiplication).[1] The counterpart of the complex torus $(\mathbb{C}^*)^n = (\mathbb{C} \setminus 0)^n$

[1] It is convenient to include $-\infty$ (the additive zero) to $\mathbb{R}_{\text{trop}}$ (among other things such an inclusion would make projective tropical curves compact), however for the sake of simplicity we exclude $-\infty$ from $\mathbb{R}_{\text{trop}}$ in this talk.

is $\mathbb{R}^n_{\text{trop}} \approx \mathbb{R}^n$ (since exclusion of $-\infty$ from $\mathbb{R}_{\text{trop}}$ resulted in absence of tropical additive zero).

Consider a tropical polynomial

$$f(y) = \text{“}\sum_j a_j y^j\text{”} = \max_j (jy + a_j).$$

Here $y \in \mathbb{R}^n$, $a_j \in \mathbb{R}_{\text{trop}}$, $j \in A$, where $A \subset \mathbb{Z}^n$ is finite, jy is the scalar product in $\mathbb{R}^n$. The operations in quotation marks refer to the tropical semifield operations. Note that f is a convex piecewise-linear function $f : \mathbb{R}^n \to \mathbb{R}$, it is continuous and smooth everywhere except for an $(n-1)$-dimensional locus. This locus in $\mathbb{R}^n$ is called the tropical hypersurface defined by f.

Tropical curves appear as 1-dimensional objects (graphs) by tropical hypersurfaces. Such curves are piecewise-linear graphs in $\mathbb{R}^n$. They inherit a certain structure from $\mathbb{R}^n \approx \mathbb{R}^n_{\text{trop}}$. Let us take a somewhat different point of view and start by defining *abstract tropical curves* as graphs equipped with certain structure (in a fashion similar to the definition of Riemann surfaces as abstract smooth surfaces equipped with a conformal structure).

Let Γ be a finitely-valent graph (i.e. a topological space homeomorphic to a locally finite 1-dimensional CW-complex, we do not assume that Γ is compact). Every point $a \in \Gamma$ is l_a-valent, $l_a \in \mathbb{N}$, (a point inside an edge is 2-valent). There exists a small open neighborhood $U_a \ni a$, $U_a \subset \Gamma$ such that $U_a \setminus \{a\}$ consists of l_a components each homeomorphic to an open interval. Let $\phi_a : U_a \to \mathbb{R}^{l_a - 1}$ be an embedding such that every component of $\phi_a(U \setminus \{a\})$ is a straight open interval stretching from $\phi_a(a)$ to a point $p_j \in \mathbb{R}^{l_a-1}$, $j = 1, \dots, l_a$. Suppose that this interval has a rational slope, i.e. $p_j - \phi(a)$ is a positive multiple of a primitive (non divisible in $\mathbb{Z}^{l_a-1}$) vector $v_j \in \mathbb{Z}^{l_a-1}$. The map ϕ_a is called *a $\mathbb{Z}$-affine chart* if any $l_a - 1$ out of vectors $v_1, \dots, v_{l_a}$ form a basis of the lattice $\mathbb{Z}^{l_a-1}$ and

$$\sum_{j=1}^{l_a} v_j = 0.$$

Note that according to this definition a 1-valent vertex cannot have a $\mathbb{Z}$-affine chart. A map $\Phi : \mathbb{R}^m \to \mathbb{R}^n$ is called *a $\mathbb{Z}$-affine linear map* if it is an affine-linear map whose derivative is linear over $\mathbb{Z}$. In other words Φ is defined as a composition of a $\mathbb{Z}$-linear map and a translation by an arbitrary real vector.

A $\mathbb{Z}$-affine structure on Γ is a collection of $\mathbb{Z}$-affine charts for every point $a \in \Gamma$ such that for any two charts

$$\phi_a : U_a \to \mathbb{R}^{l_a-1} \text{ and } \phi_b : U_b \to \mathbb{R}^{l_b-1}$$

there exists a $\mathbb{Z}$-affine linear map $\Phi_{ab} : \mathbb{R}^{l_a-1} \to \mathbb{R}^{l_b-1}$ such that

$$\Phi_{ab} \circ \phi_a|_{U_a \cap U_b} = \phi_b|_{U_a \cap U_b}.$$

A $\mathbb{Z}$-affine structure is called *proper* if for any chart $\phi_U : U \to \mathbb{R}^n$ and any compact $K \subset \mathbb{R}^n$ the closure of $\phi_U^{-1}(K) \subset \Gamma$ is compact.

Definition. *A tropical curve* is a locally compact graph equipped with a proper $\mathbb{Z}$-affine structure.

A tropical curve of genus g with k punctures is a tropical curve such that its underlying graph is connected, has k ends at infinity and has $\dim(H_1(\Gamma)) = g$.

A primitive tangent vector v_a to a point a of tropical curve is a primitive $\mathbb{Z}^{l_a-1}$-vector tangent to $\phi_a(U_a)$. It is easy to see that Φ_{ab} maps v_a to a primitive $\mathbb{Z}^{l_b-1}$-vector tangent to $\phi_b(U_b)$ (since Φ_{ba} exists). Thus a tropical curve Γ determines a natural metric on Γ once we set the length of a primitive tangent vector to be equal to 1. Since the $\mathbb{Z}$-affine structure is proper, the length of any unbounded edge of Γ is infinite and the metric is complete. Vice versa, it is possible to reconstruct a tropical curve from a locally compact graph without 1-valent vertices equipped with a complete inner metric.

Thus we have a natural 1-1 correspondence between tropical curves and complete metric graphs without 1-valent vertices. The metric description of a tropical structure is specific for dimension 1, in higher dimension we have to stick to the $\mathbb{Z}$-affine structure approach. We do not discuss the definition of such structure in higher dimension here, but let us remark that $\mathbb{R}^n$ has a tautological $\mathbb{Z}$-affine structure.

We say that the ends of Γ are *marked* if they are numbered by numbers 1 through k.

Definition. A map

$$h : \Gamma \to \mathbb{R}^n,$$

where Γ is a tropical curve with k marked ends, is called *tropical* if for every point $a \in \Gamma$ there exists a $\mathbb{Z}$-affine linear map

$$\psi_a : \mathbb{R}^{l_a-1} \to \mathbb{R}^n$$

such that $h|_{U_a} = \psi_a \circ \phi_a$.

The *degree* β of h is the k-tuple $(v_1, \ldots, v_k)$, where each $v_k \in \mathbb{Z}^n$ is the image under h of the primitive tangent vector to the corresponding end in the outbound direction. (Note that $\sum\limits_{j=1}^{k} v_j = 0$.)

One can form *the moduli space* $\mathcal{M}_{g,k}^{\text{trop}}$ of all tropical curves of genus g with k marked punctures. It turns out that in the case $g = 0$, $k \geq 3$ this moduli space itself has a proper $\mathbb{Z}$-affine structure of dimension $k-3$, i.e. is a tropical manifold, which is proper (complete) but non-compact (unless $k = 3$). If $g > 0$ then it admits a completion $\overline{\mathcal{M}}_{g,k}^{\text{trop}} \supset \mathcal{M}_{g,k}^{\text{trop}}$ that has a structure of a tropical orbifold of dimension $3g - 3 + k$ (unless $g = 1$ and $k = 0$). Similarly, one can define *the moduli space* $\mathcal{M}_{g,k}^{\text{trop},\beta}(\mathbb{R}^n)$ of tropical maps of degree β from genus g curves with k punctures to $\mathbb{R}^n$ and its completion $\overline{\mathcal{M}}_{g,k}^{\text{trop},\beta}(\mathbb{R}^n)$.

As in the classical case we have *forgetting maps*

$$\text{ft}_j : \overline{\mathcal{M}}_{g,k}^{\text{trop}} \to \overline{\mathcal{M}}_{g,k-1}^{\text{trop}},$$

$j = 1, \dots, k$, which associate to a tropical curve Γ another tropical curve obtained by forgetting the jth end (i.e. obtained by removing the corresponding unbounded edge). Also we have evaluation maps

$$\mathrm{ev}_j : \overline{\mathcal{M}}_{g,k}^{\mathrm{trop},\beta}(\mathbb{R}^n) \to \mathbb{R}^n,$$

defined for j such that $v_j = 0$ by associating to a tropical map $h : \Gamma \to \mathbb{R}^n$ the point in $\mathbb{R}^n$ that is the image of the jth end of Γ.

It turns out that the intersection theory in $\overline{\mathcal{M}}_{g,k}^{\mathrm{trop},\beta}(\mathbb{R}^n)$ coincides with that of its classical counterpart $\overline{\mathcal{M}}_{g,k}^{\beta}((\mathbb{C}^*)^n)$ for $g = 0$ or for arbitrary g in the case of $n = 2$. This can be used for computation of the Gromov-Witten invariants of toric varieties as well as for computation of their real counterparts and this makes one of the main application areas of tropical curves.

Consider a tropical map $h : \Gamma \to \mathbb{R}$, where Γ is a tropical curve of positive genus. Its degree is a k-tuple of integer numbers. Let $\bar{\Gamma}$ be the compact tropical curve obtained from Γ by forgetting all its punctures. The degree of h yields a divisor on $\bar{\Gamma}$ (supported on at most k points). This divisor has degree zero and is called *the divisor of* h. As in Classical Geometry we form *the tropical Picard group* $\mathrm{Pic}(\bar{\Gamma})$ by taking the group of all finitely supported divisors and setting all divisors of meromorphic functions (i.e. of tropical maps $h : \Gamma \to \mathbb{R}$) equivalent to zero.

Let $\omega_1, \dots, \omega_g \in H^1(\bar{\Gamma}; \mathbb{Z})$ be a basis. (Here we use cellular cocycle representatives for the classes ω_j.) Each of ω_j can be integrated along a path $\gamma : [0, l] \to \bar{\Gamma}$. The easiest is to parameterize γ by its arclength so that $\gamma'(t)$ is a primitive tangent vector for any $t \in [0, l]$. The integral $\int\limits_{\gamma} \omega_j$ is the sum of the (oriented) lengths of γ on every edge of $\bar{\Gamma}$ multiplied by the value of ω_j on that edge. With the help of such integration the basis $\omega_1, \dots, \omega_g$ defines the embedding $H_1(\bar{\Gamma}; \mathbb{Z}) \to \mathbb{R}^g$. We form *the Jacobian variety*

$$\mathrm{Jac}(\bar{\Gamma}) = \mathbb{R}^g / H_1(\bar{\Gamma}; \mathbb{Z}).$$

It is a torus $(S^1)^g$ equipped with a natural $\mathbb{Z}$-affine structure which does not depend on the choice of the basis $\omega_1, \dots, \omega_g$.

Suppose that D is a divisor on $\bar{\Gamma}$ of degree zero. Then there is a collection of paths in $\bar{\Gamma}$ that has D as its boundary. *The tropical Mittag-Leffler problem* is to tell whether D comes from a meromorphic function. As in the classical case the answer is given by the *tropical Abel-Jacobi theorem*: D is the divisor of a meromorphic function if and only if integration against γ is a $H_1(\bar{\Gamma}; \mathbb{Z})$-point in $\mathbb{R}^g$, i.e. if it is zero in $\mathrm{Jac}(\bar{\Gamma})$. Furthermore, the tropical Abel-Jacobi theorem states that such integration yields a bijection

$$A : \mathrm{Pic}_0(\bar{\Gamma}) \to \mathrm{Jac}(\bar{\Gamma}),$$

where $\mathrm{Pic}_0(\bar{\Gamma}) \subset \mathrm{Pic}(\bar{\Gamma})$ is a subgroup corresponding to the zero-degree divisors.

Finally I'd like to mention an application of tropical curve to construction of real algebraic curves with a controlled topology. (This was the original motivation for the patchworking construction for real algebraic curves in the plane introduced by

Viro in 1979 and patchworking is one of the main ingredients of tropical geometry.) A tropical curve $h : \Gamma \to \mathbb{R}^n$, where Γ is a tropical curve of genus g with k punctures, is called *regular* if it varies in a $(k+(n-3)(1-g))$-dimensional family (it can be shown that it always varies in *at least* $(k+(n-3)(1-g))$-dimensional family). Here we allow to vary both the map h and the $\mathbb{Z}$-affine structure on the curve Γ. It turns out that a regular tropical curve can be approximated by *amoebas* $\mathrm{Log}_t(C)$, where $C \subset (\mathbb{C}^*)^n$ is a holomorphic curve of the corresponding genus and degree, $\mathrm{Log}_t(z_1, \ldots, z_n) = (\log_t |z_1|, \ldots, \log |z_n|)$ and $t > 1$ is a sufficiently large number. With the help of this observation (which is also used in tropical computation of the Gromov-Witten invariants of toric varieties discussed above) one can construct algebraic curves of a given degree in real toric varieties, in particular those that contain components with a given knot type in real toric 3-folds.

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