Mathematisches Forschungsinstitut Oberwolfach

Report No. 39/2004

Mini-Workshop: Ehrhart Quasipolynomials: Algebra, Combinatorics, and Geometry

Organised by
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August 15th–August 21st, 2004

Introduction by the Organisers

The mini-workshop Ehrhart Quasipolynomials: Algebra, Combinatorics, and Geometry, organised by Jesús De Loera (Davis) and Christian Haase (Durham), was held August 15th-21st, 2004. A small group of mathematicians and computer scientists discussed recent developments and open questions about *Ehrhart quasipolynomials*. These fascinating functions are defined in terms of the lattice points inside convex polyhedra. More precisely, given a rational convex polytope P for each positive integer n, the Ehrhart quasipolynomials are defined as $i_P(n) = \#\left(nP \cap \mathbb{Z}^d\right)$. This equals the number of integer points inside the dilated polytope $nP = \{nx : x \in P\}$. The functions $i_P(n)$ appear in a natural way in many areas of mathematics. The participants represented a broad range of topics where Ehrhart quasipolynomials are useful; e.g. combinatorics, representation theory, algebraic geometry, and software design, to name some of the areas represented.

Each working day had at least two different themes, for example the first day of presentations included talks on how lattice point counting is relevant in compiler optimization and software engineering as well as talks about tensor product multiplicities in representation theory of complex semisimple Lie Algebras. Some special activities included in the miniworkshop were (1) a problem session, a demonstration of the software packages for counting lattice points `Ehrhart` (by P. Clauss), `LattE` (by J. De Loera et al.), and `Barvinok` (by S. Verdoolaege), (2) a guest speaker from one of the research in pairs groups (by R. Vershynin),and (3) a nice expository event where each of the three mini-workshops sharing the Oberwolfach facilities had a chance to introduce the hot questions being pursued to the others. The atmosphere was always very pleasant and people worked very actively. For instance, two of the talks reported on new theorems obtained during the mini-workshop. The organizers and participants sincerely thank MFO for providing a wonderful working environment, perhaps unique around the world. We also thank Günter M. Ziegler for his support and encouragement. In what follows we present the abstracts of talks following the order in which talks were presented.

MSC Classification:

Mini-Workshop: Ehrhart Quasipolynomials

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Abstracts

Nested families for vector partition function

Charles Cochet
(joint work with V. Baldoni-Silva, M. Beck, M. Vergne)

The main goal of this talk is to describe a new method for computing vector partition functions and associated Ehrhart quasi-polynomials. As an application to representation theory, we will also discuss about the efficient computation of weight multiplicity and tensor product coefficients (or Littlewood-Richardson coefficients).

Let Φ be a $r \times N$ integral matrix, with columns ϕ_j. For any vector $a \in \mathbb{R}^r$ lying in the cone generated by columns of Φ, we define the convex polyhedron $P(\Phi, a)$ as the set of non-negative N-dimensional vectors x satisfying $\Phi x = a$. We assume that Φ has full rank. We also assume that $\ker(\Phi) \cap \mathbb{R}^r_+ = \{0\}$, so that the polyhedron is a polytope (bounded). Then the vector partition function $k(\Phi, a)$ counts the number of integral points in $P(\Phi, a)$.

As the motivation, we begin with recalling a previous algorithm for vector partition function by Baldoni-Vergne ([BSV01]), specialized by Baldoni-DeLoera-Vergne ([BSDLV03]) to the case of ths matrix Φ which vectors are positive roots for the Lie algebra A_r. Roughly speaking, this algorithm gives vector partition function as a sum of *iterated residues* of rational functions, over the set of *special permutations*. When compared to classical algorithms, this method can not only give associated Ehrhart polynomial, but is also unaffected by an increase of the size of weights. However, since the residue operation is expensive, this algorithm could not perform computation in spaces of as high dimension, like previous algorithms.

An increase of the time spent to compute the set which we sum over can be affordable, if this set is finally smaller (and consequently decreases the total number of residues to compute). This is the starting point of the new method that we describe in this talk.

This new algorithm still relies on sums of *iterated residues*. But sums are over *nested families*, an object arising from specific flags associated to the set of vectors which are columns of Φ. As expected, the number of residues to compute has much decreased. Although nested families are complicated to compute, the extra time needed is neglectible with regards to the time saved on residue computation.

This method was implemented in the case where columns of Φ are positive roots of the simple Lie algebra of type B. This led to a Maple procedure, named `KostantB`. We emphasize that this procedure is not sensitive to the size of coordinates of input vector.

Now a few words about an application in representation theory (by C). We are interested in the two following computational problems in the case of a classical Lie algebra: the multiplicity c^{μ}_{λ} of the weight μ in the representation $V(\lambda)$ of highest weight λ; the multiplicity $c^{\nu}_{\lambda\,\mu}$ of the representation $V(\nu)$ in the tensor product of representations of highest weights λ and μ (Littlewood-Richardson coefficient).

Softwares LiE (from van Leewen *et al.*, see [vL94]) and GAP (from Geck *et al.*, see [GAP]), and MAPLE packages `coxeter/weyl` (from Stembridge, see [Ste95]), use Freudenthal's and Klimyk's formulas, and work for any semi-simple Lie algebra $\mathfrak{g}$. Unfortunately, these formulas are very sensitive to the size of weights.

Our approach to these two problems is through Kostants' and Steinberg's formulas, giving weight multiplicities and tensor product coefficients as sums of vector partition function. Using the above new method for vector partition function, we get a fast algorithm for weight multiplicities and tensor product coefficients, that can handle associated Ehrhart quasi-polynomials, and specially designed for weights with huge coordinates.

For the moment, the computation of weight multiplicities and tensor product coefficients (*not yet Ehrhart quasi-polynomials*) for B_r has been implemented in a MAPLE program named `Multiplicities.B.mws` (C, see [Coc]). This program, efficient up to B_6, is complementary to previous softwares and packages. It follows previous work for A_r (C, see [Coc03]), using other results on vector partition function (Baldoni-DeLoera-Vergne, see [BSDLV03]).

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Two Conjectured Generalizations of the Saturation Theorem

Tyrrell B. McAllister

(joint work with J. De Loera)

We present two conjectures regarding polytopes arising in the representation theory of complex semisimple Lie algebras. Both of these conjectures imply the Saturation Theorem of Knutson and Tao as a special case.

Given highest weights λ, μ, and ν for a complex semisimple Lie algebra $\mathfrak{g}$, we denote by $C^{\nu}_{\lambda\mu}$ the multiplicity of the irreducible representation V_ν in the tensor product of V_λ and V_μ; that is,

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu} C^{\nu}_{\lambda\mu} V_\nu.$$

The values $C^{\nu}_{\lambda\mu}$ are known as *Clebsch-Gordan coefficients*. In the particular case in which $\mathfrak{g}$ is of type A_r, they are also called *Littlewood-Richardson coefficients*.

Given highest weights λ, μ, and ν for a Lie algebra of type A_r, there is a polytope $H^{\nu}_{\lambda\mu}$, called a *hive polytope*, with the property that $C^{\nu}_{\lambda\mu}$ equals the number of integer lattice points in $H^{\nu}_{\lambda\mu}$. Knutson and Tao introduced these polytopes and used them to prove the Saturation Theorem ([5]; see also a nice exposition of their proof in [2]).

Theorem. *(Saturation) Given highest weights λ, μ, and ν for a Lie algebra of type A_r, and given an integer $N > 0$, the Littlewood-Richardson coefficient $C^{\nu}_{\lambda\mu}$ satisfies*

$$C^{\nu}_{\lambda\mu} \neq 0 \quad \Longleftrightarrow \quad C^{N\nu}_{N\lambda,N\mu} \neq 0.$$

The hive polytope $H^{\nu}_{\lambda\mu}$ is defined to be the set of solutions to a particular system of linear equalities and inequalities. We put

$$H^{\nu}_{\lambda\mu} = \left\{ h \in \mathbb{R}^{(r+1)(r+2)/2} : \begin{array}{l} Bh = b(\lambda,\mu,\nu), \\ Rh \leq 0 \end{array} \right\}, \tag{1}$$

where B and R are integral matrices, and $b(\lambda,\mu,\nu)$ is a $3r$-dimensional integral vector depending on λ, μ, and ν. Knutson and Tao proved the Saturation Theorem by showing that every nonempty hive polytope contains a nonintegral vertex. We conjecture that this integrality result extends to the larger class of polytopes of the form

$$\left\{ h \in \mathbb{R}^{(r+1)(r+2)/2} : \begin{array}{l} Bh = b, \\ Rh \leq c \end{array} \right\},$$

where b and c may take on any integral values for which the resulting polytope is nonempty.

We show that, to prove this conjecture, and therefore to prove the Saturation Theorem, it suffices to prove following.

Conjecture 1. *Fix an integer $r > 0$ and let B and R be the matrices in (1). Then the convex hull of the points whose coordinates are the columns of the block matrix*

$$\begin{bmatrix} B & 0 \\ R & I \end{bmatrix}$$

has a unimodular triangulation by simplices whose vertices are among these points.

We have verified this conjecture up to $r = 6$ using placing triangulations.

Our second conjecture concerns a class of polytopes which perform for arbitrary semisimple Lie algebras the same role that Hive polytopes serve in type A_r. Let highest weights λ, μ, and ν for a semisimple Lie algebra $\mathfrak{g}$ be given. Berenstein and Zelevinsky define ([1]) a polytope $BZ^{\nu}_{\lambda\mu}$ which contains exactly $C^{\nu}_{\lambda\mu}$ integral lattice points, where $C^{\nu}_{\lambda\mu}$ is a Clebsch–Gordan coefficient for $\mathfrak{g}$.

Using Barvinok's lattice point enumeration algorithm, we can compute the Ehrhart quasi-polynomials $C^{N\nu}_{N\lambda,N\mu}$ of the BZ–polytopes. These computations motivate our second conjecture.

Conjecture 2. *Given highest weights λ, μ, and ν of a Lie algebra of type A_r, B_r, C_r, or D_r, put*

$$C^{N\nu}_{N\lambda,N\mu} = \begin{cases} f_1(N) & \text{if } N \equiv 1 \mod M, \\ \vdots & \\ f_M(N) & \text{if } N \equiv M \mod M. \end{cases}$$

Then the coefficients of each f_i are all nonnegative.

The type A_r case of this conjecture was conjectured by King, Tollu, and Toumazet in [4]. That Conjecture 2 implies the Saturation Theorem follows from a result of Derksen and Weyman ([3]) showing that the Ehrhart quasi-polynomials of Hive polytopes are in fact just polynomials.

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Enumerating Integer Projections of Parametric Polytopes

Sven Verdoolaege

(joint work with M. Bruynooghe)

Many compiler techniques depend on the ability to count the number of integer points that satisfy a given set of linear inequalities. Typically, a subset of the variables are identified as the *parameters* $\vec{p}$ and the number of possible values for the other, "counted" variables $\vec{y}$ needs to be counted as a function of those parameters. If each variable is either a parameter or counted, then the problem is equivalent to the enumeration of parametric polytopes, which can be computed efficiently using a technique based on Barvinok's decomposition of unimodular cones [1, 3, 5].

In general, the linear inequalities may also involve additional, existentially quantified variables $\vec{\epsilon}$ and then the object is to count the number of possible integer values for $\vec{y}$ in function of the parameters $\vec{p}$ This problem is equivalent to enumerating an integer projection of a parametric polytope. Pugh [4] addresses the closely related problem of counting the number of solutions to Presburger formulas, but his technique seems underspecified and has apparently never been implemented. One of the substeps is also clearly exponential in the input size. More recently, a technique was outlined for integer projections of non-parametric polytopes [2], but it appears not to have been implemented yet and it is not immediately clear how easily it can be extended to the parametric case.

Consider the polyhedron P defined by the given set of linear inequalities in $(d + d' + n)$-dimensional space, where d, d' and n are the number of counted variables, existential variables and parameters. Our technique manipulates this polyhedron directly through a set of simplification rules, which either reduce the number of existential variables, shrink the polyhedron or split the problem into several smaller subproblems.

Pick an existential variable ϵ_i and take a pair of inequalities such that the coefficients of ϵ_i have opposite signs. The two inequalities determine a range for ϵ_i as a function of the other variables. If no such pair exists or if this range admits at least one integer solution over the whole of P, then ϵ_i can be eliminated. If a pair exists such that at least one of the inequalities is independent of the other existential variables and such that the range admits at most one solution, then ϵ_i can be treated as a counted variable.

In general, P will need to be split such that each part is closer to removal of an existential variable. If the splitting constraint is independent of the existential variables, the enumerations of both parts can simply be summed. Otherwise, we need to take the disjunction of both enumerations E_1 and E_2. If $d = 0$, this can be computed as $E_1 + E_2 - E_1E_2$. If $d \neq 0$, we first treat all counted variables as parameters and then sum the resulting enumeration over the counted variables. Finding good splitting constraints remains a challenge, though.

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Ehrhart Polynomials for Program Analysis, Optimization and Compiling

Philippe Clauss

Ehrhart polynomials have many applications concerning compiler design. Their use in this field was initiated as i re-discovered Eugène Ehrhart's results and met him in Strasbourg in 1994. I proposed an extension to any number of parameters [1, 2] and linked my work to Loechner and Wilde's work on the parametric vertices of a parameterized polyhedron [3], resulting in the first ever made Ehrhart polynomials computation program in 1996. The implemented algorithm consists in the following steps :

- given a system of parameterized linear rational inequations,
- compute the validity domains (domains where the vertices have constant definitions) and the parametric coordinates of the vertices,
- for each validity domains,
  - since the general form of the associated Ehrhart polynomial is known (degree, period),
  - count the number of points for some initial values of the parameters (using a loop scanning the polytope),
  - solve a system of linear equations whose solutions are the Ehrhart polynomial coefficients.

One well-known model in computer science used to analyze programs is the so-called *polytope model* [4]: all iterations of a nested loop are represented as integer points whose coordinates are the indices values. Since these values span a convex domain, the integer points define a lattice polytope. Moreover, the considered nested loops are often parameterized by some size parameters. Hence we must consider parameterized polytopes.

Between the many applications in program analysis and optimization, we present data layout transformations defined by Ehrhart polynomials for data spatial locality optimization. This technique, detailed in [5, 6], consists in reorganizing in memory array elements referenced in a nested loop in the same order as they are accessed during the execution. This improves the cache behavior of the nested loop since all data of any loaded cache block will be entirely and successively accessed. The number of cache misses are therefore significantly reduced. This is done by computing the number of iterations executed before the one referencing a given array element. We show an example where the execution time becomes 16 times faster.

Despite the several software implementations for computing Ehrhart polynomials presented in this mini-workshop, it is unfortunately not yet possible to consider such computations into a compiler like *gcc*. However, embedded systems designers can already be Ehrhart polynomials users since more time is given to conceive a system which will be produced and distributed in a huge number of units. Anyway, some more efforts are still needed to accelerate Ehrhart polynomials computation.

Moreover, many other extensions should be considered since applications do exists in compiler design. Some kind of non-linear equations arise in program analysis that are not yet handled by nowadays Ehrhart polynomials computation methods, although Ehrhart has shown in [7] that problems of the form $P+nP' > 0$, where P and P' are polynomials in 2 variables and n is a parameter define a set of integer solutions whose count is an Ehrhart polynomial.

In many cases, it would be sufficient to approximate an Ehrhart polynomial by only computing its higher degree coefficient. A cheaper and faster algorithm for this purpose would have many applications as well. But since it is known that this coefficient is the volume of the considered polytope, the cost is related to the computation cost of the volume.

Due to the growing complexity of computer systems, dynamic analysis of programs is also considered. It consists in collecting traces during the run of a program and then model them in order to understand the program behavior. Since computer programs generally have a relatively repetitive and periodic behavior, it could be interesting to detect a kind of "Ehrhart polynomial behavior". Such a representation model would then provide an interesting way to reuse static analysis techniques of the polytope model for dynamic analysis.

Finally, another motivating objective is to be able, starting from a given Ehrhart polynomial, to find a polytope whose number of lattice points is equal to this Ehrhart polynomial. This would allow to generate program codes (loops scanning the so-found polytopes) having some interesting properties.

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On the unimodality of h^*-vectors

Christos A. Athanasiadis

Let P be a d-dimensional integral polytope in $\mathbb{R}^N$ with Ehrhart polynomial $i(P,r) = \#(rP \cap \mathbb{Z}^N)$ and h^*-vector $h^*(P) = (h_0^*, h_1^*, \ldots, h_d^*)$ defined by

$$\sum_{r \geq 0} i(P,r)t^r = \frac{h_0^* + h_1^* t + \cdots + h_d^* t^d}{(1-t)^{d+1}}. \tag{1}$$

It is well known that the h_i^* are nonnegative integers. Our main concern will be to describe sufficient conditions on P for $h^*(P)$ to be unimodal, meaning that $h_0^* \leq h_1^* \leq \cdots \leq h_j^* \geq h_{j+1}^* \geq h_d^*$ for some $0 \leq j \leq d$. More specifically we draw our attention to the following two conjectures. Let R_P denote the semigroup ring of P over a field $\mathbb{K}$, graded so that (1) is the Hilbert series of R_P.

Conjecture 1. If R_P is standard and Gorenstein then $h^*(P)$ is unimodal.

Conjecture 2. If $h_i^* = h_{d-i}^*$ for all $0 \leq i \leq d$ then $h^*(P)$ is unimodal.

Conjecture 1 was stated more generally for standard graded Gorenstein domains by Hibi [2] and Stanley [5]. Conjecture 2 is due to Hibi [3].

We will discuss the proof of a recent partial result [1] towards Conjecture 1, namely an affirmative answer under the additional assumption that all pulling triangulations of P are unimodular, as well as applications to the motivating case of order polytopes of graded partially ordered sets, originally discovered by Reiner and Welker [4], and to the case of the Birkhoff polytope of doubly stochastic $n \times n$ matrices. We also discuss some features that possible counterexamples to these conjectures should have.

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Quasi-polynomials Arising from Group Actions

Petr Lisoněk

We adhere to the standard definition [4] according to which a sequence is *quasi-polynomial* if its ordinary generating function (o.g.f.) is a univariate rational function with rational coefficients such that each root of the denominator is some root of unity. We will describe two general paradigms in which quasi-polynomials arise in enumeration of orbits of group actions. For the second paradigm we demonstrate a connection with Ehrhart quasi-polynomials.

Let $\mathbf{n} := \{1, 2, \dots, n\}$. For $f \in \mathbb{N}^{\mathbf{n}}$ let $c_f := \sum_{x \in \mathbf{n}} f(x)$. Let G be a subgroup of $S_{\mathbf{n}}$ and consider the natural action of G on $\mathbb{N}^{\mathbf{n}}$. For any $f \in \mathbb{N}^{\mathbf{n}}$, the orbit of f under G is called a *G-partition of number c_f*. Thus, for example, $S_{\mathbf{n}}$-partitions correspond to unordered partitions into at most n parts, and $1_{\mathbf{n}}$-partitions (where $1_{\mathbf{n}}$ is the trivial subgroup of $S_{\mathbf{n}}$) correspond to compositions with n non-negative parts. For any $c \in \mathbb{N}$, the number of G-partitions of c will be denoted by $P_G(c)$. The o.g.f. $\sum_{c \geq 0} P_G(c) t^c$ is obtained by Pólya-substitution of the formal power series $1/(1-t) \;=\; 1 + t + t^2 + \cdots$ in the cycle index of G's action on $\mathbf{n}$, that is, for each $1 \leq i \leq n$, the variable z_i of the cycle index is substituted by $1/(1-t^i)$. Because the cycle index is a multivariate polynomial, we get:

Proposition 1. Let n be a positive integer. For each $G \leq S_{\mathbf{n}}$, the number of G-partitions of c is quasi-polynomial in c.

As an illustration let us present a very short proof of the following identity which holds for each positive integer n:

$$\prod_{k=1}^{n} \frac{1}{1-x^k} = \sum_{a \vdash n} \prod_k \frac{1}{a_k!} \left(\frac{1}{k(1-x^k)} \right)^{a_k}$$

where the sum extends over all $a = (a_1, a_2, ...)$ such that $a_1 \cdot 1 + a_2 \cdot 2 + \ldots = n$ and $a_i \geq 0$ for $1 \leq i \leq n$. On the left-hand side we have the o.g.f. for the number of $S_{\mathbf{n}}$-partitions while on the right-hand side we have the Pólya-substitution of $1/(1-t)$ in the cycle index of $S_{\mathbf{n}}$'s natural action on $\mathbf{n}$ [2]. This identity is proved in several classical texts (e.g. MacMahon, Riordan) but the connection with group actions seems to have been overlooked.

By using appropriate subgroups of $S_{\mathbf{n}}$, Proposition 1 furnishes many examples of quasi-polynomial combinatorial enumeration sequences, such as for example the number of symmetry classes of 0,1-matrices with a fixed number of columns (where two matrices belong to the same class if they differ only by a row and column permutation), the number of unlabelled multigraphs on a fixed number of vertices, and many more.

Another family of quasi-polynomial combinatorial enumeration sequences arises when instead of considering just one group action, we study a sequence of actions—one action per each "size" of the objects that we wish to enumerate. In these situations we can often observe a direct connection with *Ehrhart quasi-polynomials.*

Let us demonstrate this paradigm on a concrete example. By a *polygon dissection* we mean each subdivision of the interior of a convex s-gon into smaller polygons by means of non-intersecting, but possibly touching diagonals. If the s-gon is *regular,* we can count symmetry classes of dissections under the cyclic or dihedral symmetry. Thus we have a sequence of actions, one action per each s. Denote by r the number of cells arising in the dissection, i.e. we use $r-1$ diagonals to create the dissection. Let $H_{r,s}$ denote the number of dissection classes under the cyclic symmetry, and let $h_{r,s}$ denote the number of dissection classes under the dihedral symmetry. Thus, for example, $H_{3,6} = 4$ and $h_{3,6} = 3$. In [3] we used Pólya theory to prove:

Proposition 2. For any fixed r, sequences $(H_{r,s})$ and $(h_{r,s})$ are quasi-polynomial in s.

A connection with *Ehrhart quasi-polynomials* can now be observed as follows. (We thank to F. Santos for an interesting discussion of this topic during the workshop.) Consider the tree structure (in the graph theory sense) that represents the adjacencies between r cells of the dissection. The tree has r vertices, and edges of the tree correspond to diagonals of the dissection, and a plane embedding of the tree is fixed. For each fixed r, there is a finite number of such trees. Now a symmetry class of dissections is described by an assignment of a sequence of integers to each vertex of the tree (the length of the sequence at vertex v equals the degree of v), subject to certain linear constraints that these integers must satisfy. (These linear constraints represent the geometry of the s-gon and they also ensure that each dissection class is counted exactly once in cases when the tree possesses non-trivial automorphisms.) The integers add up to s. As a consequence, for each tree the number of symmetry classes of dissections of s-gons characterized by this tree is equal to the number of lattice points in the projection on $2(r-1)$ coordinates of one or more rational polytope(s) parameterized by the value of s, and as such this number is quasi-polynomial in s. (The corresponding o.g.f. can be computed using the algorithm in [1].) The total number of dissections, being a finite sum of quasi-polynomials, is also quasi-polynomial.

It appears that there are other interesting types of combinatorial structures (with two parameters and a natural group action depending on the second parameter) for which the enumeration sequence of symmetry classes is quasi-polynomial once the value of the first parameter is fixed, and where the proof strategy outlined above may be applicable. We are currently pursuing the case of symmetry (isometry) classes of non-linear binary codes, where the first parameter is the size of the code (the number of codewords) and the second parameter is the block length of the code.

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Computational Complexity and Periods of Ehrhart Quasipolynomials

Kevin Woods

This talk has three primary goals: to examine some results and open problems about the period of the Ehrhart quasipolynomial of a polytope, to provide a toolbox of rational generating function algorithms which can be used in a wide variety of problems, and to apply these tools to the question of computing the period.

If $P \subset \mathbb{R}^d$ is a rational polytope, let $i_P(t) = \left|tP \cap \mathbb{Z}^d\right|$. Ehrhart proved (see [4]) that if $\mathcal{D} = \mathcal{D}(P)$ is the least common denominator of all of the coordinates of all of the vertices of P, then $i_P(t)$ is a quasi-polynomial of period $\mathcal{D}$. Sometimes, however, $i_P(t)$ may have a smaller period, and we provide several examples of this phenomenon (see a large class of examples in [6], Gelfand-Tsetlin polytopes in [2], and hive polytopes in [5] and Corolary 3 of [3]). An interesting set of open problems is to develop necessary and/or sufficient conditions for the period to be other than $\mathcal{D}$.

Next we present some tools for using rational generating functions of the form

$$\sum_{i \in I} \frac{\mathbf{x}^{a_i}}{(1-\mathbf{x}^{b_{i1}})\cdots(1-\mathbf{x}^{b_{ik}})}$$

to solve this and other problems in polynomial time (for fixed dimension d). Most of these tools are presented in [1]. We also talk about one new tool: if $f(\mathbf{x})$ is a rational generating function, then we may decide whether f is identically zero in polynomial time (for fixed d and fixed k).

Finally, we use these tools to prove that, given a polytope $P \subset \mathbb{R}^d$ and a number n, we may decide in polynomial time (for fixed d) whether n is a period of $i_P(t)$. In particular, we may decide whether $i_P(n)$ is of period 1, that is, whether it is a polynomial. This yields an algorithm to compute the minimum period of $i_P(t)$, but we must factor $\mathcal{D}(P)$ to do this. We conjecture that we can, in fact, determine the period of $i_P(t)$ in polynomial time.

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Universality of Table Polytopes

Shmuel Onn
(joint work with J. De Loera)

We show that any rational convex polytope P is polynomial time representable as a polytope T of all nonnegative $n \times n \times 3$ arrays with fixed line sums, for suitable n and line sums
$u_{i,j}, v_{i,k}, w_{j,k}$,

$$T = \{x \in \mathbb{R}_+^{n\times n\times 3} \ : \ \sum_k x_{i,j,k} = u_{i,j}, \sum_j x_{i,j,k} = v_{i,k}, \sum_i x_{i,j,k} = w_{j,k}\}.$$

The representation provides a bijection between P and T and between the set of integer points in P and the set of integer points ("tables") in T. In particular, this implies that any Ehrhart quasipolynomial is the Ehrhart quasipolynomial of some such table polytope T.

Further, it shows that computational problems over P such as linear programming, integer programming, and counting integer points, can be reduced to the analogous problems over T.

One interesting remaining open problem is whether any *real polytope* is also representable as some table polytope. Another problem concerns the possibility of the existence of a *strongly* polynomial time algorithm for linear programming, since our universality shows that any rational polytope is representable as one described by a simple $(0, 1)$ system of equations depending on the single parameter n only.

The talk was based on the following two papers, available on my home page http://ie.technion.ac.il/ onn :

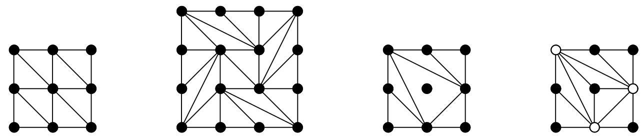
(1) The complexity of three-way statistical tables SIAM Journal on Computing, 33:819–836, 2004.
(2) All rational polytopes are transportation polytopes and all polytopal integer sets are contingency tables Proceedings of the 10th IPCO (Annual Mathematical Programming Society Symposium on Integer Programming and Combinatorial Optimization), Lecture Notes in Computer Science, 3064:338–351, 2004.

Quadratic triangulations

Christian Haase
(joint work with L. Piechnik)

A triangulation of a lattice polytope into lattice simplices is *quadratic* if

- it is regular,
- it is unimodular, and
- its minimal non-faces have two elements.



Quadratic triangulations are rare. They are cool because they provide a squarefree quadratic Gröbner basis for the ideal defining the projective toric variety associated with the given polytope. (There is a whole hierarchy of interesting covering properties; see Francisco Santos' talk.)

The goal of this talk was to exhibit some examples of polytopes which admit nice triangulations.

Paco's Lemma [5, 4] *Suppose all lattice points in P are vertices. Then P is compressed (all pulling triangulations are unimodular) if and only if P has facet width one.*

Corollary. *The following polytopes have regular unimodular triangulations.*

- *kP if P has one* [5]
- *polytopes with totally unimodular collection of facet normals (e.g., transportation/flow polytopes)* [3]
- *order polytopes* [4]
- *stable polytopes of perfect graphs* [4]

Examples of polytopes with quadratic triangulation include polygons with ≥ 4 boundary lattice points [1], products of polytopes with quadratic triangulation [3], smooth polytopes with all lattice points vertices [2], smooth reflexive 4-polytopes [2], and certain 3×3 transportation polytopes [2].

For the last two example classes, one uses a projection lemma that in special situations allows to lift quadratic triangulations from a d-polytope to a $(d+1)$-polytope.

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A geometric view of Ehrhart coefficients

Achill Schürmann

Due to the work of McMullen [3] on lattice invariant valuations we have a "local formula"

$$e_i(P) = \sum_{F \leq P, \dim F = i} \mu(N(F,P)) \cdot \mathrm{vol}(F) \tag{1}$$

for the i–th Ehrhart coefficient of a lattice polytope P. Here, μ is a real valued function on rational cones and $N(F,P)$ denotes the outer normal cone of F with respect to P. The existence of a rational valued μ satisfying (1) can be attained by application of the Hirzebruch–Riemann–Roch theorem to the local formula

$$\mathrm{Td}(X_\Sigma) = \sum_{\sigma \in \Sigma} \mu(\sigma)[V(\sigma)]$$

for the Todd class of a toric variety X_Σ with associated lattice fan Σ (see [1], [4]). The advantage of a local formula is obvious: It is possible to attain information on the Ehrhart coefficients for large classes of polytopes with the same normal fans, e.g., all transportation polytopes of a fixed dimension.

So how can we construct such μ? Recently, Pommersheim and Thomas [5] gave constructions of rational valued μ (see abstract of Pommersheim), answering a question of Danilov [1]. We give an independent, elementary construction, which allows to compute values of μ as differences of summed volumes. We hereby give an elementary "geometric interpretation" of Ehrhart coefficients.

Let Λ denote an arbitrary lattice in $\mathbb{R}^d$ with $\det(\Lambda) = 1$, e.g. $\Lambda = \mathbb{Z}^d$. For each sublattice L of Λ we have the freedom to choose a lattice tile T_L in the linear hull $\mathrm{lin}(L)$ of L, that is, a compact subset with $\mathrm{vol}(T_L) = \det(L)$ and $L + T_L = \mathrm{lin}(L)$. A canonical choice is for example the Dirichlet–Voronoi–cell of L (see [2]). Additionally we choose a lattice tile T of Λ which may differ from T_Λ. The freedom of these choices allows us to construct many different μ. In order to describe the construction in greater detail, let $\langle \cdot, \cdot \rangle$ denote the usual dot product on $\mathbb{R}^d$ and $N^* = \{x \in \mathbb{R}^d : \langle x, y \rangle \leq 0, \forall y \in N\}$ the dual (rational) cone of a rational cone N.

The idea is that a possible choice for $\mu(N)$ is the average volume of $(N^* \cap \Lambda) + T$ in $N + \mathrm{lin}(N)^\perp$, up to a possibly large error term. This error term depends on $\mu(\bar{N})$, $\bar{N} < N$, that is, on all true subfaces $\bar{N}$ of N. It is computed from volumes that we derive from "pieces" R_N. These pieces allow different dissections of $\mathbb{R}^d$ and in a sense give a "geometric meaning" to μ.

If $\dim(N) = 0$ we set $R_N = T_\Lambda$ and $\mu(N) = 1$. Otherwise, we first compute $R_{\bar{N}}$ and $\mu(\bar{N})$ for all $\bar{N} < N$. Then, for all $\bar{N} < N$, we consider lattice points $x \in \bar{L} = \Lambda \cap \mathrm{lin}(\bar{N})^\perp$ with the property that the bounded "parts" $P_{\bar{N}}(x) = (x + R_{\bar{N}}) \cap \bar{N}^*$ arecontained in N^* and do not overlap other $P_{\bar{N}'}(x')$, with $\bar{N}' < N$, $\dim(\bar{N}') = \dim(\bar{N})$ and $x' \in \bar{L}'$. Let $X_{\bar{N},N}$ denote the maximal subset of $\bar{L}$ with these properties and set

$$R_N = \left(\mathbb{R}^d \setminus \bigcup_{\bar{N} < N} (X_{\bar{N},N} + R_{\bar{N}}) \right) \cap (T_L + \mathrm{lin}(N)) \,.$$

The part in big parentheses is invariant with respect to lattice invariant translations of $L = \Lambda \cap \mathrm{lin}(N)^{\perp}$. So we have a dissection

$$\mathbb{R}^d = (L + R_N) \cap \bigcup_{\bar{N} < N} (X_{\bar{N},N} + R_{\bar{N}}).$$

If we think of larger and larger polytopes with the same outer normal cones, we can dissect $\mathbb{R}^d$ by covering corresponding faces with translates of R_N, up to some part along their boundary. By our construction, this remaining part can be covered by pieces $R_{\bar{N}}$, $\bar{N} > N$.

With R_N we define $\mu(N)$: Let

$$v_N = \mathrm{vol}\,(R_N \cap ((N^* \cap \Lambda) + T))$$

and for all $\bar{N} < N$ let

$$v_{\bar{N},N} = \mathrm{vol}\left(R_N \cap N^* \cap \mathrm{lin}(\bar{N})\right).$$

Then

$$\mu(N) = \left(v_N - \sum_{\bar{N} < N} v_{\bar{N},N} \cdot \mu(\bar{N}) \right) / \ \mathrm{vol}(T_L).$$

Although the computation of μ via volumes is messy, the construction might be of use for estimations of appropriate μ. Note also that it can be applied directly to any rational cone and not only to unimodular cones. Simple examples indicate that there is a choice of lattice tiles giving the same μ as given by the construction of Pommersheim and Thomas. If at all possible, it is is not clear so far how this choice might look like in general. By allowing the lattice tiles to be non–convex and/or non–symmetric with respect to the origin we attain certainly many μ which are not covered by the construction of Pommersheim and Thomas.

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Crystals, polytopes, and representation theory

Peter Littelmann

The aim of the talk was to present an overview on some recent developments in the use of polyhedral combinatorial methods in classical representation theory of semisimple Lie algebras. The main emphasis was put on the connections between character formulas / tensor product formulas, crystal graphs, integral points in convex polytopes and flat deformations of flag varieties.

For the general linear group $GL_n(\mathbb{C})$, Gelfand and Cetlin [5] associated to a partition $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ of length at most n (i.e., an irreducible polynomial representation of $GL_n(\mathbb{C})$) a real polytope in $\mathbb{R}^{\frac{1}{2}n(n-1)}$, the so called *Gelfand–Tsetlin pattern* of shape λ. The number of these patterns is the dimension of the corresponding representation. It is also easy then to deduce similar kind of formulas for the dimension of weights spaces. Extensions of this notion to other classical groups have been given Gelfand and Cetlin [6] and Zhelobenko [12, 13].

A new way to associate convex polytopes to representations came together with the introduction of the notion of the crystal graph of a representation. Note that Luzstig [11] showed that the theory of finite dimensional representations of a semisimple complex Lie algebra $\mathfrak{g}$ and its quantum group $U_q(\mathfrak{g})$ over $\mathbb{C}(q)$ (q a variable) are essentially the same, both are classified by integral dominant weights. Further, Lusztig introduced integral forms $U_{\text{int}}(\mathfrak{g})$, $V_{\text{int}}(\lambda)$ of the quantum group and the representations, defined over the ring of Laurent polynomials, and showed that these specialize (essentially) for $q = 1$ to the classical enveloping algebra of $\mathfrak{g}$ and the corresponding representation. Kashiwara [7] analyzed in a similar way the situation at $q = 0$ and found that in this case the representation in the limit has a very nice canonical basis, the crystal basis. Hence one can associate to a finite dimensional representation $V(\lambda)$ of $\mathfrak{g}$ a *crystal graph* $G(\lambda)$ having as vertices the elements of the crystal basis, and two elements b, b' are joined by an arrow, colored with the simple root α, if the Kashiwara operator f_α maps b onto b'.

One way to get a combinatorial model for the crystal graph is the path model [8]. These piecewise linear paths have been encoded into integral points in a union of polytopes by Dehy [4]. A nice algebraic geometric connection of these polytopes with toric varieties has been found by Chirivi [3]. He generalized the notion of a *Hodge algebra* or *Algebra with Straightening Law* and, using the description of the coordinate ring of flag varieties in [10], he showed that they admit a flat defomation into a union of toric varieties such that the associated polytopes correspond exactly to those in [4].

Another approach was used in [9]. The idea of the construction is rather simple: let b be an element of the crystal basis. If b is not the highest weight element, then there is at least one incoming arrow with color a simple root α_1. Denote by n_1 the maximal integer one can move up incoming arrows with this labe. By repeating the procedure, one can associate to every element a sequence of integers $(n_1, \ldots, n_r)$. To make the construction uniform for all elements of the crystal base, one fixes a reduced decomposition of the longest word in the Weyl group W and applies the root operators according to the appearance in this decomposition.

It has been shown in [9] that this set of sequences is the set of integral points in a rational convex polytope C_λ. In fact, there exists a convex rational cone $C_{\mathfrak{g}}$ such that C_λ can be obtained from $C_{\mathfrak{g}}$ by cutting out certain halfspaces.

The defining equation for the cone $C_{\mathfrak{g}}$ depends on the choice of the reduced decomposition of the longest word in the Weyl group. Many simple examples for the defining equations can be found in [9], an algorithm for an arbitrary decomposition can be found in [1]. In this paper they go a step beyond the problem of finding character formulas, they solve the problem to find explicit polyhedral combinatorial expressions for multiplicities in the tensor product of any two simple modules. Again, Berenstein and Zelevinsky make use of a lot of technics related to the theory of quantum groups and canonical basis.

A beautiful connection with the geometry of toric varieties is given by Caldero in [2], he shows that for every reduced decomposition there exists a flat deformation of the flag variety into a toric variety such that the corresponding graded algebra corresponds to the semigroup of integral points in the cone $C_{\mathfrak{g}}$.

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Local cycle expressions for the Todd class of a toric variety

James Pommersheim
(joint work with H. Thomas)

This talk is based on a paper which is to appear in the (*J. Amer. Math. Soc.*, to appear). We show how to construct cycles which represent the Todd class of a toric variety. Given a lattice with an inner product, we assign a rational number $\mu(\sigma)$ to each rational polyhedral cone σ in the lattice in such a way that for any fan Σ, the Todd class of the corresponding toric variety X_Σ is given by

$$\text{Td } X_\Sigma = \sum_{\sigma \in \Sigma} \mu(\sigma)[V(\sigma)]. \tag{1}$$

Our Todd class construction yields a construction of a local formula for the number of lattice points in a lattice polytope. Formulas of this type were first obtained by P. McMullen in 1983, though non-constructively. In particular, given a lattice M with an inner product, we may associate a rational number $\mu(\sigma)$ to each rational polyhedral cone in the dual lattice, in such a way that for any polytope P in M, we have

$$\#(P \cap M) = \sum_{F \subset P} \mu(N(P,F))\text{Vol}(F), \tag{2}$$

where the sum is over all faces F of P, and $N(P,F)$ denotes the outer normal cone to P at F.

More generally, instead of choosing an inner product on the lattice, we may choose a *complement map*, a suitable assignment of linear subspaces to cones. We show that under certain conditions, each such choice of complement map yields a Todd class formula, and hence a local lattice point formula.

We also note that the functions μ constructed here are suitably additive under subdivision, and this implies their polynomial-time computability using Barvinok's algorithm for writing any cone as the sum of unimodular cones with coefficients 1 and -1. Indeed, existing software such as LattE or Barvinok could be modified to compute the function μ efficiently for any choice of complement map. One could thereby reduce the computation of the Ehrhart polynomial of a lattice polytope P to the volumes of the faces of P. One can also avoid the computation of these volumes, by directly intersecting the Todd class of the toric variety with the Chern character of a certain line bundle associated to the polytope, a standard method in this theory (see, for example, the survey article of Barvinok and Pommersheim, 1999). This yields an algorithm for computing the Ehrhart polynomial of a lattice polytope. It would be interesting to compare this algorithm with the usual Barvinok's algorithm, both in theory and practice, especially on some class of non-unimodular polytopes.

One question that arises naturally out of this work is to classify all functions μ which give a local formula for the Todd class, i.e. which satisfy Equation (1) for all fans Σ. Similarly, one might ask for a classification of all functions μ which satisfy the lattice point formula of Equation (2). Our work shows the any complement map can be used to construct such μ, but leaves open the possibility that there exist μ that do not come from this construction.

Minkowski's theorem: counting lattice points and cells

Roman Vershynin

Every convex body K in $\mathbb{R}^n$ admits a coordinate projection that contains at least $|\frac{1}{6}K|$ cells of the integer lattice, provided this volume is at least one. This main result is a cell-counting variant of the classical Minkowski's theorem, which states that K contains at least $|\frac{1}{2}K|$ points of the integer lattice (provided K is origin symmetric). The proof of the cell-counting theorem relies on an extension to Z^n the known Sauer-Shelah theorem in the extremal combinatorics.

Bounds on the lattice point enumerator of convex bodies

Martin Henk

For a convex body $K \subset \mathbb{R}^n$ we denote by $G(K) = \#(K \cap \mathbb{Z}^n)$ the lattice point enumerator of K. It is still an unsolved problem to find a *good* upper bound on $G(K)$ in terms of other functionals of the convex body. In analogy to the Ehrhart polynomial for lattice polytopes in would be desirable to have an upper bound of the type

$$G(\lambda K) \leq \sum_{i=0}^{n} \lambda^i \alpha_i(K), \quad \lambda \in \mathbb{R}_{\geq 0}, \tag{1}$$

for certain functionals $\alpha_i()$. In particular, it should hold $\alpha_n(K) = \mathrm{vol}(K)$, where $\mathrm{vol}(K)$ denotes the volume of K, and $\alpha_{n-1}(K)$ should be related to the surface area of K.

In 1973 Wills conjectured that (1) holds with $\alpha_i(K) = V_i(K)$, where $V_i()$ denotes the i-th intrinsic volume. Note that $V_0(K) = 1$, $V_n(K) = \mathrm{vol}(K)$ and $V_{n-1}(K)$ is half of the surface area K. Although this conjecture has been verified for $n \leq 3$ and for certain classes of convex bodies, in general it is false. The first counterexample is due to Hadwiger in dimensions ≥ 441. Later it was shown that Wills' conjecture is *even false* for $n \geq 207$ [1]. An open problem in this context is the question whether *at least* the first two coefficients in Wills' conjecture are correct, i.e., does there exist a constant $c(n)$ depending only on the dimension such that

$$G(K) \leq V_n(K) + V_{n-1}(K) + c(n) \sum_{i=0}^{n-2} V_i(K) \quad ?$$

A stronger conjecture of the same type is due to Ehrhart. He proved for dimensions ≤ 3 that $G(K) \leq V_n(K) + V_{n-1}(K) + \sum_{i=0}^{n-2} V_i(Q)$, where Q is the smallest lattice box containing K, and conjectured that this is true for all dimensions.

The oldest general upper bound on $G(K)$ is due to Blichfeldt. To this let K be a convex body containing $n+1$ affinely independent lattice points. Then $G(K) \leq n! \operatorname{vol}(K) + n$, which follows from the fact that such a body must contain at least $G(K) - n$ distinct lattice simplices. A classical lower bound on $G(K)$ is due to Bokowski, Hadwiger and Wills which can be expressed as

$$V_n(K) - V_{n-1}(K) < G(K).$$

Here it is an open problem how this bound can be generalised to arbitrary lattices.

Now in the following let K be a 0-symmetric convex body. The most prominent inequality relating lattice points and volume of a 0-symmetric convex body is due to Minkowski. He showed that

$$G(\operatorname{int}(K)) = 1 \Longrightarrow \operatorname{vol}(K) \leq 2^n. \tag{2}$$

There are several possibilities to look for generalisations of this result. For instance, let $C^n = [-1,1]^n$ be the unit cube and let $G_i(P)$, $0 \leq i \leq n$, be the coefficients of the Ehrhart polynomial of a 0-symmetric lattice polytope P. Then Wills asked

$$G(\operatorname{int}(P)) = 1 \Longrightarrow G_i(P) \leq G_i(C^n) = 2^i \binom{n}{i} \quad ?$$

Since $G_n(P) = \operatorname{vol}(P)$ and in view of (2) the answer is certainly yes for $i = n$. Furthermore, the case $i = n-1$ has been proven by Wills. Of course, one can also ask for an inequality of type (2) for arbitrary convex bodies. However, in order to bound the volume in this case additional assumptions are necessary. Let us assume that the centre of gravity of K is the origin. Ehrhart conjectured that

$$G(\operatorname{int}(K)) = 1 \Longrightarrow \operatorname{vol}(K) \leq \frac{(n+1)^n}{n!} \quad ?$$

So far this inequality has only been proven in the planar case. The bound would be best possible as the family of lattice simplices $\{x \in \mathbb{R}^n : x_i \geq -1, \sum x_i \leq 1\}$ show. Minkowski himself proved a very strong generalisation of (2). To this end let $\Lambda \subset \mathbb{R}^n$ be an arbitrary lattice and for $1 \leq i \leq n$ let $\lambda_i(K, \Lambda) = \min\{\lambda > 0 : \dim(\lambda K \cap \Lambda) \geq i\}$ be the i-th successive minimum of K (with respect to Λ). With this notation (2) can be reformulated as $\lambda_1(K,\Lambda)^n \operatorname{vol}(K) \leq 2^n \det \Lambda$ and Minkowski also proved the stronger inequality

$$\lambda_1(K,\Lambda) \cdot \lambda_2(K,\Lambda) \cdot \ldots \cdot \lambda_n(K,\Lambda) \operatorname{vol}(K) \leq 2^n \det \Lambda. \tag{3}$$

In [2] it is conjectured that a similar relation holds if the volume is replaced by the lattice point enumerator, namely

$$G_\Lambda(K) = \#(K \cap \Lambda) \leq \prod_{i=1}^{n} \left\lfloor \frac{2}{\lambda_i(K,\Lambda)} \right\rfloor \quad ? \tag{4}$$

It is easy to see that (4) implies (3). Since $\lambda_i(\mu K, \Lambda) = \frac{1}{\mu}\lambda_i(K,\Lambda)$ the upper bound in (4) may be regarded as a quasi-polynomial. In [2] the conjecture has been verified for $n = 2$ and also the weaker inequality $G_\Lambda(K) \le \lfloor 2/\lambda_1(K,\Lambda) \rfloor^n$ was shown. In [5] it was proven that

$$G_\Lambda(K) < 2^{n-1} \prod_{i=1}^{n} \left\lfloor \frac{2}{\lambda_i(K,\Lambda)} \right\rfloor .$$

For more information on bounds on the lattice point enumerator as well as for references of the presented inequalities we refer to the survey [4] and the book [3].

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Basis expansions and roots of Ehrhart polynomials

Julian Pfeifle

(joint work with M. Beck, J. De Loera, M. Develin, and R. P. Stanley)

The Ehrhart polynomial i_P of a d-dimensional lattice polytope $P \subset \mathbb{R}^d$ is usually written in the *power basis* of the vector space of polynomials of degree d:

$$i_P(n) \;=\; \sum_{i=0}^{d} c_i \, n^i \, .$$

In this talk, we argued that comparing this representation with the basis expansion

$$i_P(n) \;=\; \sum_{i=0}^{d} a_i \binom{n+d-i}{d}$$

yields useful information about i_P. Note that in the literature sometimes the notation h_i^* is used instead of a_i.

(1) The inequalities $a_i \ge 0$ (that follow from the fact that i_P is the Hilbert function of a semi-standard graded Cohen-Macaulay algebra) are used to derive all other known inequalities [1] [2] [3] [6] for the coefficients c_i, with the exception of the inequality $c_{d-1} \ge \frac{1}{2} \cdot$ (normalized surface area) that comes from geometry.

(2) Some of the coefficients in this representation have nice interpretations:

$$\begin{aligned} a_1 &= i_P(1) - (d+1)\,, \\ a_2 &= i_P(2) - (d+1)i_P(1)\,, \\ a_{d-1} &= (-1)^d\big(i_P(-2) - (d+1)\,i_P(-1)\big)\,, \\ a_d &= (-1)^d\,i_P(-1) \;=\; \#\{\text{inner points}\}\,. \end{aligned}$$

(3) Expressing the Ehrhart polynomial in this basis makes it easy to prove relations such as

$$\binom{d}{\ell}\,\Delta^k i_P(0) \;\le\; \binom{d}{k}\,\Delta^\ell i_P(0),$$

where $\Delta^k i_P$ is the k-th difference of i_P.

We also present new linear inequalities satisfied by the coefficients of Ehrhart polynomials and relate them to known inequalities.

Next, we investigated the roots of Ehrhart polynomials:

Theorem.

(a) *The complex roots of Ehrhart polynomials of lattice d-polytopes are bounded in norm for fixed d.*

(b) *All real roots of Ehrhart polynomials of d-dimensional lattice polytopes lie in the half-open interval $[-d, \lfloor d/2 \rfloor)$. For $d = 4$, the real roots lie in the interval $[-4, 1)$.*

(c) *For any positive real number t, there exists an Ehrhart polynomial of sufficiently large degree with a real root strictly larger than t. In fact, for every d there is a d-dimensional $0/1$-polytope whose Ehrhart polynomial has a real zero α_d such that $\lim_{d\to\infty} \alpha_d/d = 1/(2\pi e) = 0.0585\cdots$. In particular, the upper bound in* (b) is tight up to a constant.

An experimental investigation of the Ehrhart polynomials of cyclic polytopes leads to the following conjecture:

Conjecture. *Let $P = C_d(n)$ be any cyclic polytope realized with integer vertices on the standard moment curve $t \mapsto (t, t^2, \ldots, t^d)$ in $\mathbb{R}^d$. Then the Ehrhart polynomial of P reads*

$$i_P(n) \;=\; \sum_{k=0}^{d} \mathrm{vol}_k\big(\pi_k(P)\big)\, n^k,$$

where $\mathrm{vol}_d(\cdot)$ is the standard d-dimensional volume, $\mathrm{vol}_k(\cdot)$ for $k = 0, 1, \ldots, d-1$ is the normalized lattice volume, and $\pi_k : \mathbb{R}^d \to \mathbb{R}^k$ is the projection to the first k coordinates.

Problem. *Find an explicit expression for the Todd class of the toric variety associated to the outer normal fan of $P = C_d(n)$.*

This problem has been solved for $0 \le d \le 3$ by using the expressions for the codimension ≤ 3 parts of the Todd class from [4] and the techniques of [5]. In particular, the conjecture has been proven for $0 \le d \le 3$.

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On normal polytopes without regular unimodular triangulations

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A lattice polytope $P \subset \mathbb{R}^d$ is *normal* if $nP \cap \mathbb{Z}^d = n(P \cap \mathbb{Z}^d)$ for every $n \in \mathbb{N}$. Normal polytopes arise naturally in algebraic geometry and in combinatorial optimization [8]. Starting with [6], it has been repeatedly observed that normality of a polytope is closely related to its being covered by unimodular simplices. More precisely, from [6, 3, 5] one can extract the following sequence of properties, each of which implies the next one. In all of them, $S = P \cap \mathbb{Z}$. A simplex is unimodular if its vertices are a basis for the affine lattice $\mathbb{Z}^d$. A triangulation is unimodular if all its simplices are.

(1) All simplices with vertices in S are unimodular. (P is *totally unimodular*).
(2) P is *compressed*. (All its *pulling* triangulations are unimodular).
(3) P has a unimodular *regular triangulation*.
(4) P has a unimodular triangulation.
(5) P has a unimodular *binary cover*. This is a property introduced by Firla and Ziegler [3], whose significance comes from the fact that it is much easier to check algorithmically than any other of the properties (3) to (8).
(6) P has a *unimodular cover*. (Every $x \in P$ lies in some unimodular simplex).
(7) For every n, every integer point in nP is an integer positive combination of an affinely independent subset of points of S. (This is called the *Free Hilbert Cover* property in [1])
(8) For every n, every integer point in nP is an integer positive combination of at most $d + 1$ points of S. (The *Integral Carathéodory Property* of [3]).
(9) For every n, every integer point in nP is an integer positive combination of an affinely independent subset of points of S. (P is normal).

It is very easy to find examples that prove $3 \not\Rightarrow 2$ and $2 \not\Rightarrow 1$, but not so easy for any of the other implications. Ohsugi and Hibi [5] found the first normal polytope without regular unimodular triangulations, which turned out to give $4 \not\Rightarrow 3$. Then Bruns and Gubeladze [1] proved $8 \Leftrightarrow 7$ and found an example for $9 \not\Rightarrow 8$ [2]. The implications from 7 to 4 remain open. (There is an example of a *cone*, not a polytope, disproving $5 \Rightarrow 4$ in [3]).

In this talk, after reviewing the above concepts, we intend to give a new look at the Hibi-Ohsugi and Bruns-Gubeladze examples. On the one hand, using polyhedral combinatorics tools we give a new simple proof of the following theorem, found independently by Hibi-Ohsugi [4] and Simis-Vasconcelos-Villarreal [7]. Our proof adds the word "binary", not present in the original statements, to part (c).

We recall that the *edge polytope* P_G of a connected graph G with d vertices and n edges is the convex hull of the n points $\{e_i + e_j : ij \in G\} \subset \mathbb{R}^d$. It lies in an affine hyperplane and in a lattice of index 2 (which means all the properties above need to be modified accordingly). It has dimension $d-1$, unless G is bipartite in which case has dimension $d-2$ and is totally unimodular.

Theorem. *The following properties are equivalent for a graph G:*

(a) *Every two vertex-disjoint cycles in G are joined by an edge.*
(b) *P_G is normal.*
(c) *P_G has a unimodular binary cover.*

We also provide a simplified proof of the fact that Ohsugi and Hibi's normal polytope has no regular unimodular triangulation.

We next observe that by a unimodular change of coordinates the normal polytope of Bruns and Gubeladze becomes a 0/1 5-dimensional lift of the 4-dimensional polytope $\Delta_{2,5} := \mathrm{conv}(\{e_i + e_j : i, j = 1, \ldots, 5\}) \subset \mathbb{R}^5$. This is known as the second 4-dimensional hypersimplex [8] and is the edge polytope of the complete graph K_5. In particular, similar methods to those applied in the first example can be used here (with some care) to prove the following:

Theorem. *Let P be the normal polytope without the integral Carathéodory's property appearing in* [1, 2].

(1) *P is a projection of the Hibi-Ohsugi polytope in* [5]. *The projection π is unimodular and $\pi(P \cap \mathbb{Z}^9) = \pi(P) \cap \mathbb{Z}^4$.*
(2) *For every n, every integer point in nP can be written as an integer positive combination of at most* **7** *vertices of P.*

Part (2) allows us to conclude that the *Carathéodory rank* of P is 7 (we prove the upper bound and the lower bound is the main result in [2]). Part (1) suggests one should study the intermediate projections between P and $\pi(P)$, and probably will get new examples of normal polytopes without regular unimodular triangulations, hopefully disproving some of the remaining implications in our initial list. Another interesting approach is to try to find an extension of Theorem 1 to the class of "0/1 liftings of edge polytopes".

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