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Geometrie

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Introduction by the Organisers

The workshop was organized by V. Bangert (Freiburg), Yu. D. Burago (St. Petersburg) and U. Pinkall (Berlin). Out of the 47 participants 22 came from Germany, 8 from the United States, 7 from Switzerland, 4 from Russia, 4 from England and 2 from France.

The official program consisted of 20 lectures and included four lectures by B. Kleiner (Ann Arbor) on "Perelman's work on Ricci flow". The program covered a wide range of new developments in geometry. To name some of them, we mention the topics "Metric space geometry in the style of Alexandrov/Gromov", "Finsler geometry", "Constant mean curvature surfaces in Thurston geometries".

Workshop: Geometrie

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Abstracts

Proof of projective Lichnerowicz-Obata conjecture

Vladimir S. Matveev

We prove the following classical conjecture: Let a connected Lie group G act on a complete connected Riemannian manifold (M^n, g) of dimension $n \geq 2$ by projective transformations. Then, it acts by affine transformations, or g has constant positive sectional curvature.

Definition: *Let (M^n, g) be smooth Riemannian manifold. A diffeomorphism $F : M^n \to M^n$ is called a* **projective transformation***, if it takes unparameterized geodesic to geodesics. A diffeomorphism $F : M^n \to M^n$ is called an* **affine transformation***, if it preserves the Levi-Civita connection of g.*

Theorem 1 (Projective Lichnerowicz Conjecture): *Let a connected Lie group G act on a complete connected Riemannian manifold (M^n, g) of dimension $n \geq 2$ by projective transformations. Then, it acts by affine transformations, or g has constant positive sectional curvature.*

Corollary (Projective Obata Conjecture): *Let a connected Lie group G act on a closed connected Riemannian manifold (M^n, g) of dimension $n \geq 2$ by projective transformations. Then, it acts by isometries, or g has constant positive sectional curvature.*

For dimension 2, projective Obata Conjecture was proved in [12, 14]. All assumptions in Theorem are important: we can construct counterexamples, if one of the assumptions is omitted.

Any connected simply-connected Riemannian manifold of constant positive curvature is a round sphere. All projective transformations of the round sphere are known (essentially, since Beltrami 1865); so that Theorem and Corollary close the theory of nonisometric infinitesimal projective transformations of complete manifolds.

Bruce Kleiner suggested the following (alternative) way to formulate the main result. Denote by $\mathrm{Proj}(M^n, g)$ and $\mathrm{Aff}(M^n, g)$ the groups of projective and affine transformations of (M^n, g). Clearly, $\mathrm{Aff}(M^n, g)$ is a subgroup of $\mathrm{Proj}(M^n, g)$.

Theorem 2 (Alternative way to formulate projective Lichnerowicz Conjecture): *Let (M^n, g) be a complete connected Riemannian manifold of dimension $n \geq 2$. If the index of $\mathrm{Aff}(M^n, g)$ in $\mathrm{Proj}(M^n, g)$ is infinite, then g has has constant positive sectional curvature.*

History: First nontrivial examples of projective transformations are due to Beltrami [2]. The problem of finding metrics (on surfaces) whose groups of projective transformations are bigger than the groups of isometries was stated by Lie [5]. For complete manifolds, this problem was formulated by Schouten [17].

The local theory of projective transformations was well understood thanks to efforts of several mathematicians, among them Dini, Levi-Civita, Fubini, Eisenhart, Weyl and Solodovnikov.

Projective transformations were extremely popular objects of investigation in 50th–80th. One of the reasons for it is their possible applications in physics, see the surveys [1, 16] for details.

Most results on projective transformations require additional geometric assumptions written as a tensor equation. For example, Corollary was proved under the assumption that the metric is Einstein or Kähler (Couty [4]), or that the scalar curvature is constant negative (Yamauchi [21]).

The only result which does not require additional tensor assumptions is due to Solodovnikov [18, 19]. He proved the Lichnerowicz conjecture under the assumptions that the dimension of the manifold is greater than two and that all objects (the metric, the manifold, the projective transformations) are real-analytic.

Methods: Our proof of Lichnerowicz-Obata conjecture is quite long (∼40 p.). It can be found in [15]. Roughly speaking, we use the following methods:

The classical methods came from the local theory of projectively equivalent metrics (Beltrami, Dini, Levi-Civita, Fubini, Eisenhart, Cartan, Weyl, Solodovnikov (see, for example, [18]).)

The newer methods came from theory of integrable systems: the main observation (see any of the papers [6, 7, 8, 9, 10, 11, 3, 13, 20]) is that, for a given Riemannian metric g, the existence of a projectively equivalent metric allows one to construct commuting integrals for the geodesic flow of g.

And the general idea came from the singularity theory. The role of singularities play the points where the eigenvalues of the Lie derivative of the metric bifurcate. We describe behavior of the metric near the simplest singular points, show that the simplest singular points always exist, and explain how the structure near singular points can be extended to the whole manifold.

Acknowledgements: I am grateful to Prof. Kleiner for his comments during my talk that lead to Theorem 2. I thank DFG-programm 1154 (Global Differential Geometry) and Ministerium für Wissenschaft, Forschung und Kunst Baden-Württemberg (Eliteförderprogramm Postdocs 2003) for partial financial support.

References

[1] A. V. Aminova, *Projective transformations of pseudo-Riemannian manifolds. Geometry, 9.* J. Math. Sci. (N. Y.) **113**(2003), no. 3, 367–470.

[2] E. Beltrami, *Resoluzione del problema: riportari i punti di una superficie sopra un piano in modo che le linee geodetische vengano rappresentante da linee rette*, Ann. Mat., **1**(1865), no. 7, 185–204.

[3] Alexei V. Bolsinov, Vladimir S. Matveev, *Geometrical interpretation of Benenti's systems*, J. of Geometry and Physics, **44**(2003), 489–506.

[4] R. Couty, *Sur les transformations des variétés riemanniennes et kählériennes,* Ann. Inst. Fourier. Grenoble **9**(1959), 147–248. eometry and Physics, **44**(2003), 489–506.
[5] S. Lie, *Untersuchungen über geodätische Kurven,* Math. Ann. **20**(1882); Sophus Lie Gesammelte Abhandlungen, Band 2, erster Teil, 267–374. Teubner, Leipzig, 1935.
[6] V. S. Matveev, P. J. Topalov, *Trajectory equivalence and corresponding integrals*, Regular and Chaotic Dynamics, **3**(1998), no. 2, 30–45.
[7] V. S. Matveev and P. J. Topalov, *Integrability in theory of geodesically equivalent metrics*, J. Phys. A., **34**(2001), 2415–2433.
[8] V. S. Matveev, P. J. Topalov, *Quantum integrability for the Beltrami-Laplace operator as geodesic equivalence,* Math. Z. **238**(2001), 833–866.
[9] Vladimir S. Matveev, *Geschlossene hyperbolische 3-Mannigfaltigkeiten sind geodätisch starr,* Manuscripta Math. **105**(2001), no. 3, 343–352.
[10] Vladimir S. Matveev, *Three-dimensional manifolds having metrics with the same geodesics,* Topology **42**(2003) no. 6, 1371-1395.
[11] Vladimir S. Matveev, *Hyperbolic manifolds are geodesically rigid,* Invent. math. **151**(2003), 579–609.
[12] Vladimir S. Matveev, *Solodovnikov's theorem in dimension two*, Dokl. Math. **69**(2004), no. 3, 338–341.
[13] Vladimir S. Matveev, *Projectively equivalent metrics on the torus,* Diff. Geom. Appl. **20**(2004), 251-265.
[14] Vladimir S. Matveev, *Die Vermutung von Obata für Dimension 2,* Arch. Math. **82**(2004), 273–281.
[15] Vladimir S. Matveev, *Projective Lichnerowicz-Obata conjecture*, `ArXiv.org/math.DG/0407337.`
[16] J. Mikes, *Geodesic mappings of affine-connected and Riemannian spaces. Geometry, 2.,* J. Math. Sci. **78**(1996), no. 3, 311–333.
[17] J. A. Schouten, *Erlanger Programm und Übertragungslehre. Neue Gesichtspunkte zur Grundlegung der Geometrie,* Rendiconti Palermo **50**(1926), 142-169.
[18] A. S. Solodovnikov, *Projective transformations of Riemannian spaces,* Uspehi Mat. Nauk (N.S.) **11**(1956), no. 4(70), 45–116.
[19] A. S. Solodovnikov, *The group of projective transformations in a complete analytic Riemannian space,* Dokl. Akad. Nauk SSSR **186**(1969), 1262–1265.
[20] P. J. Topalov and V. S. Matveev, *Geodesic equivalence via integrability,* Geometriae Dedicata **96**(2003), 91–115.
[21] K. Yamauchi, *On infinitesimal projective transformations,* Hokkaido Math. J. **3**(1974), 262–270.

On the structure of hyperbolic space forms

Ruth Kellerhals

A hyperbolic n-space form Q is a quotient of the standard hyperbolic space H^n by a discrete group Γ of hyperbolic isometries. If Γ acts without fixed points, then the quotient is a hyperbolic n-manifold denoted by M. In the other case, Q is a hyperbolic n-orbifold.

In the first part of the talk, we discuss some global geometrical and topological properties of hyperbolic n-manifolds M for arbitrary $n \geq 2$ (cf. [K4]). For $\varepsilon > 0$, consider the thick and thin decomposition $M = M_{>\varepsilon} \cup M_{\leq\varepsilon}$ into the part $M_{>\varepsilon}$ where the injectivity radius is everywhere larger than ε and into the closure of the thin part $M_{<\varepsilon}$. By the Margulis lemma, there is a universal constant

$\varepsilon(n) > 0$ with the following properties. For all $\varepsilon \leq \varepsilon(n)$, the thick part $M_{>\varepsilon}$ is compact, and the components of the thin part $M_{\leq\varepsilon}$ are either neighborhoods of simple closed geodesics of length $\leq \varepsilon$ homeomorphic to ball bundles over the circle or cusp neighborhoods homeomorphic to products of compact flat manifolds with a real half line. Explicit estimates for the Margulis constant $\varepsilon(n)$ have immediate limiting consequences for characteristic numbers such as volume. For low dimensions, there are the estimates $\varepsilon(2) \geq \operatorname{arsinh}(1) \simeq 0.8813$ due to Buser [Bu, §4], $\varepsilon(3) \geq 0.104$ due to Meyerhoff [M], and, for $n = 4, 5$, $\varepsilon(n) \geq \sqrt{3}/9\pi \simeq 0.0612$ (cf. [K2], [K3]).

The main result which we discuss is an estimate of the Margulis constant $\varepsilon(n)$ for arbitrary $n \geq 2$. The proof has different ingredients. First, we interpret hyperbolic isometries by means of Clifford matrices according to Vahlen, Maass and Ahlfors. Next, define $\nu = \left[\frac{n-1}{2}\right]$, and

$$c_\nu = \frac{2^{\nu+1}}{\pi^\nu} \cdot \frac{\Gamma(\frac{\nu+2}{2})^2}{\Gamma(\nu+2)} = \frac{2}{\pi^\nu} \int_0^{\pi/2} \sin^{\nu+1} t \, dt \quad .$$

In this setting, Cao and Waterman [CW] derived an important n-dimensional collar theorem. They showed that each simple closed geodesic g in M of length $l(g) \leq l_0 = c_\nu/3^{\nu+1}$ admits a tube $T_g(r)$ embedded in M of radius r satisfying

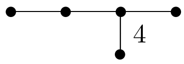
$$\cosh(2r) = \frac{1-3\kappa}{\kappa} \quad , \quad \text{where} \quad \kappa = 2\,(l(g)/c_\nu)^{\frac{2}{\nu+1}} \quad .$$

Another ingredient is the existence of certain extremal cusp neighborhoods, called canonical cusps, in non-compact hyperbolic manifolds of finite volume (cf. [H], [K2], [S]). Moreover, a formula for the displacement rate of a loxodromic Möbius transformation in n variables as well as results and techniques developped in [K2], [K3] play an important role. These methods allow to show that, for $\varepsilon \leq \frac{c_\nu}{3^{\nu+1}}$, each of the finitely many connected components of the thin part $M_{<\varepsilon}$ of M is either contained in a tube as described above or in a canonical cusp.

We discuss some implications about the coarse geometry of hyperbolic n-manifolds. For example, there is a point $p \in M$ with injectivity radius $i_p(M) > 1/(n+3)\pi^{n-1}$ implying universal lower bounds for volume. By combining this result with those of Przeworski [P], it follows that in each manifold M with a simple closed geodesic of length $\leq l_0$, there is a point $p \in M$ with injectivity radius $i_p(M) > 0.2217$.

These results improve already existing estimates as described in [BK, Proposition 2.5.3], [CW, Theorem 9.8], and they extend our earlier contributions for cusped hyperbolic manifolds [K1]. We also provide a geometrical inequality relating injectivity radius $i(M)$ and diameter $\operatorname{diam}(M)$ of a compact manifold M by using a result of Heintze and Karcher [HK]. Finally, we consider the counting function $\rho_n(V)$ on the set of non-isometric hyperbolic manifolds of dimension $n \geq 4$ with volume bound V and estimate the constant $\alpha(n)$ in $\rho_n(V) \leq V^{\alpha(n)V}$. This latter inequality was discovered by Burger, Gelander, Lubotzky and Mozes [BGLM] by exploiting the thick and thin decomposition for $\varepsilon \leq \varepsilon(n)$.

In the last part of the talk, we present a brief overview over small volume hyperbolic orbifolds in low dimensions. In particular, we mention the recent result of [HiK] where it is shown that the smallest volume hyperbolic 4-orbifold is the quotient space $Q_* = H^4/\Gamma_*$ of H^4 by the Coxeter group Γ_* with diagram



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As such, the orbifold Q_* is unique, arithmetic and of volume $\pi^2/1440$.

References

[BGLM] M. Burger, T. Gelander, A. Lubotzky, S. Mozes, Counting hyperbolic manifolds, Geom. Funct. Anal. **12** (2002), 1161−1173.

[Bu] P. Buser, Geometry and spectra of compact Riemann surfaces, Birkhäuser, 1992.

[BK] P. Buser, H. Karcher, Gromov's almost flat manifolds, Astérisque **81**, Soc. Math. France, Paris 1981.

[CW] C. Cao, P. L. Waterman, Conjugacy invariants of Möbius groups, in: Quasiconformal Mappings and Analysis, A Collection of Papers Honoring F. W. Gehring, P. L. Duren et al, Editors, Springer-Verlag, 1998.

[HK] E. Heintze, H. Karcher, A general comparison theorem with applications to volume estimates for submanifolds, Ann. scient. Éc. Norm. Sup. **11** (1978), 451−470.

[H] S. Hersonsky, Covolume estimates for discrete groups of hyperbolic isometries having parabolic elements, Michigan Math. J. **40** (1993), 467−475.

[HiK] T. Hild, R. Kellerhals, The fcc lattice and the smallest volume cusped hyperbolic 4-orbifold, Preprint 2004.

[K1] R. Kellerhals, Volumes of cusped hyperbolic manifolds, Topology **37** (1998), 719−734.

[K2] R. Kellerhals, Collars in $PSL(2, \mathbb{H})$, Ann. Acad. Sci. Fenn. Math. **26** (2001), 51−72.

[K3] R. Kellerhals, Quaternions and some global properties of hyperbolic 5–manifolds, Can. J. Math. **55** (2003), 1080–1099.

[K4] R. Kellerhals, On the structure of hyperbolic manifolds, to appear in Isr. J. Math.

[M] R. Meyerhoff, A lower bound for the volume of hyperbolic 3-manifolds, Can. J. Math. **39** (1987), 1038−1056.

[P] A. Przeworski, Cones embedded in hyperbolic manifolds, J. Diff. Geom. **58** (2001), 219−232.

[S] H. Shimizu, On discontinuous groups operating on the product of the upper half planes, Ann. Math. **77** (1963), 33−71.

Skew and totally skew embeddings and immersions

Serge Tabachnikov

(joint work with M. Ghomi)

In 1960s, H. Steinhaus asked whether there existed smooth closed space curves without parallel tangents (he conjectured that such curves did not exist). Call a closed space curve without parallel tangents a skew loop. More generally, a skew brane is a smooth closed manifold immersed into an affine space in codimension 2 and free from pairs of parallel tangent spaces.

B. Segre [3, 4] was the first to answer the Steinhaus question by constructing examples of skew loop and obtaining many results on their geometry. More recent results in this direction is obtained by Ghomi and Solomon [1]. Let M be a smooth surface in 3-space with positive curvature. If M is quadratic then every closed immersed curve on M has a pair of tangent lines, parallel in the ambient space, and if M is not quadratic then there is a skew loop on M. Another version of this question was considered by White [6] who proved the next theorem: let M^n be a smooth closed manifold immersed into the unit sphere $S^{n+1} \subset \mathbf{R}^{n+2}$; then M has a pair of points at which the tangent n-spaces are parallel in the ambient space. In other words, M is not a skew brane. Another recent result [7] is that every knot type can be realized by a skew loop.

In [5] I extended the results of Ghomi-Solomon and White to quadrics of all dimensions and signatures. The main result is an analog of White's theorem in which the unit sphere is replaced by any quadratic hypersurface $S = \{Q(x,x) = 1\}$ where Q is a quadratic form. My approach is as follows. Consider the "squared distance" function $f : S \times S \to \mathbf{R}$ given by $f(x,y) = Q(x,y)$. If a pair (x,y) is a critical point of f and $y \neq \pm x$ then the tangent spaces T_xS a nd T_yS are parallel. Then one uses Morse theory to estimate below the number of critical points of the function f and to show that the desired critical pairs exist.

An interesting open problem is to extend the "cylinder lemma" from [1] to higher dimensions. This lemma asserts that if γ is closed non-centrally symmetric curve in the horizontal plane then there is a section of the vertical cylinder over γ which is a skew brane. A generalization leads to the following conjecture, of interest on its own right: let M be a closed manifold with zero Euler characteristic and α is a differential 1-form on M such that $d\alpha \neq 0$. Then there exists a function f on M such that the 1-form $\alpha + df$ has no zeros. In the application to skew branes, M is an odd-dimensional real projective space.

Another result from [5] is that a ruled developable disc in space cannot support a closed immersed skew loop; if the requirement to be ruled is relaxed then there exists examples of skew loops on a developable disc. The proof follows from a lemma whose proof is elementary but not easy: if γ is a closed immersed curve in a plane domain foliated by straight lines then there exists points $x, y \in \gamma$ that belong to the same leave of the foliation and such that $T_x\gamma$ and $T_y\gamma$ are parallel. It would be interesting to find a variational proof of this claim (known in the cases when the lines are parallel or concurrent).

A different class of submanifolds was considered in [2]. A submanifold $M^n \subset \mathbf{R}^N$ is called totally skew if, for all pairs of distinct points $x, y \in M$, the tangent spaces T_xM and T_yM do not contain intersecting or parallel lines. The notion of a totally skew submanifold is the result of extending the notion of a skew manifold from affine to projective geometry. The general problem is, given M, to find the least N such that M admits a totally skew embedding to $\mathbf{R}^N$. Denote this number by $N(M^n)$ and shorthand it to $N(n)$ if M^n is a disc.

It is easy to prove that $2n + 1 \leq N(M^n) \leq 4n + 1$. In [2], we studied the number $N(n)$. We found a close relation of this problem with some fundamental and well-studied problems in topology: the generalized vector field problem, non-singular bilinear maps and the immersion problem for real projective spaces. One of our results is as follows. Denote by ξ_p the canonical linear vector bundle over

$\mathbf{RP}^p$; the Whitney sum of r copies of ξ_p is denoted by $r\xi_p$. Then if there exists a totally skew n-dimensional disc in $\mathbf{R}^{2n+q}$ then the vector bundle $(n+q)\xi_{n-1}$ admits $n+1$ linearly independent sections. As a consequence, one has the following lower bounds on $N(n)$ for small values of n:

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$N(n) \geq$	3	6	7	12	13	14	15	24	25	27	28	31	36	37	38	48	49

Among many corollaries of this theorem, if $N(n) = 2n+1$, the lowest possible value, then $n = 1, 3$ or 7. Another corollary: if n is a power of 2 then $N(n) \geq 3n$.

An interesting question is whether these lower bounds are sharp. This is so for $n = 1$ and $n = 2$: the map $x \mapsto (x, x^2, x^3)$ over the reals or complex numbers provides an example. Does there exist a 3-dimensional totally skew disc in 7-dimensional space?

Another result in [2] provides constructions of totally skew spheres: if there exists a bilinear symmetric non-singular map $g : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^{n+q}$ then there exists a totally skew sphere $S^n \subset \mathbf{R}^{2n+q+1}$. Multiplication of polynomials, an example due to Hopf, provides such a bilinear map and, as a consequence, there exists a totally skew sphere $S^n \subset \mathbf{R}^{3n+2}$. Using multiplication of complex polynomials, one also constructs a totally skew sphere $S^n \subset \mathbf{R}^{3n+1}$ for even n. I do not know of other examples of non-singular symmetric bilinear maps.

References

[1] M. Ghomi, B. Solomon. *Skew loops and quadratic surfaces* Comm. Math. Helv. **77** (2002), 767–782.

[2] M. Ghomi, S. Tabachnikov. *Totally skew embeddings of manifolds*, Preprint `math.DG/0307044`.

[3] B. Segre. *Global differential properties of closed twisted curves*, Rend. Sem. Mat. Fis., Milano **38** (1968), 256–263.

[4] B. Segre. *Sulle coppie di tangenti fra loro parallele relative ad una curva chiusa sghemba*, Hommage au Professeur Lucien Godeaux, pp. 141–167, Louvain 1968.

[5] S. Tabachnikov. *On skew loops, skew branes and quadratic hypersurfaces*, Moscow Math. J. **3** (2003), 681–690.

[6] J. White. *Global properties of immersions into Euclidean spheres*, Indiana Univ. Math. J., **20** (1971), 1187–1194.

[7] Y.-Q. Wu. *Knots and links without parallel tangents*, Bull. London Math. Soc., **34** (2002), 681–690.

Conformal geometry of gravitational plane waves

Wolfgang Kühnel

(joint work with H.-B. Rademacher)

One basic question in the early days of relativity was the following: *Which Einstein spaces can be mapped conformally into another Einstein space or into itself?* A classical result states the following:

Theorem A. (Brinkmann, implicitly in [1]) *A 4-dimensional Riemannian or Lorentzian Einstein space admitting a non-homothetic conformal vector field is either locally conformally flat or is locally a vacuum pp-wave (plane–fronted wave).*

The class of *pp*-waves in general is given by all Lorentzian metrics on open parts of $\mathbb{R}^4 = \{(u, v, x, y)\}$ which are of the form

$$ds^2 = -2H(u, x, y)du^2 - 2dudv + dx^2 + dy^2$$

with an arbitrary function H, the *potential*, which does not depend on v. For the history see [7].

Brinkmann's theorem was better understood and also independently rediscovered in various papers, see [3, Thm.3]. A specialization to the Ricci-flat case reads as follows:

Theorem B. *A vacuum spacetime admitting a non-homothetic conformal vector field is either locally flat or is locally a pp-wave (plane–fronted wave).*

However, this result does not immediately lead to a classification of those metrics and those conformal vector fields which actually can occur. In [6] conformal vector fields on *pp*-waves are studied, and a few examples are given. The converse to Theorem B is not true: Not every vacuum *pp*-wave admits a non-homothetic conformal vector field. By [4] the conformal group is at most 7-dimensional. Our main result is the following [5]:

Theorem C. *All vacuum spacetimes admitting a 7-dimensional conformal group (together with the vector fields themselves) can be explicitly determined in terms of elementary functions. More precisely H is a linear combination of the real and imaginary parts of one of the following three cases:*

<u>Case 1:</u> $2H(u, x, y) = c(x + iy)^2 \exp(2\kappa ui)$ where c and κ are constants,

<u>Case 2:</u> $2H(u, x, y) = c\dfrac{(x + iy)^2}{u^2} \exp(2\kappa ui)$ where c and κ are constants,

<u>Case 3:</u> $2H(u, x, y) = c \cdot \dfrac{(x + iy)^2}{(u^2 + \alpha u + \beta)^2} \exp\left(2\gamma i \displaystyle\int \frac{du}{u^2 + \alpha u + \beta}\right)$

where c, α, β, γ are constants and where a non-homothetic conformal vector field can be chosen as

$$V = Z_1 + \alpha(u\partial_u - v\partial_v) + \beta\partial_u + \gamma(y\partial_x - x\partial_y)$$

where Z_1 is the standard conformal field $Z_1 = u^2\partial_u + \frac{1}{2}(x^2 + y^2)\partial_v + ux\partial_x + uy\partial_y$ which we know from flat Minkowski space (the case $H = 0$).

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Theorem D. *The class of all vacuum spacetimes admitting a 6-dimensional conformal group cannot be described in terms of a finite dimensional space of parameters. More precisely, there are solutions which are C^∞ but not real analytic, depending on the arbitrary choice of a real C^∞-function.*

Corollary. *A Lorentzian Einstein manifold is not necessarily real analytic.*

This is in sharp contrast with the Riemannian case where Einstein spaces are always real analytic in appropriate coordinates, see [2].

Example:

$$H(u,x,y) = \begin{cases} \exp\left(\frac{1}{u^2-1}\right)(x^2-y^2) & \text{if } u^2 < 1 \\ 0 & \text{if } u^2 \geq 1. \end{cases}$$

Here the Ricci tensor vanishes identically. The curvature tensor vanishes for $u^2 > 1$ but does not vanish for $u^2 < 1$. Therefore, the metric is not analytic, independently of the choice of coordinates. The isometry group is 5-dimensional, together with the homothetic vector field $Y_2 = 2v\partial_v + x\partial_x + y\partial_y$ we obtain a 6-dimensional conformal group.

References

[1] H. W. Brinkmann, *Einstein spaces which are mapped conformally on each other*, Math. Ann. **94** (1925), 119–145

[2] D.M. De Turck and J. Kazdan, *Some regularity theorems in Riemannian geometry*, Ann. Sci. Ecole Norm. Sup. (4) **14** (1981), 249-250

[3] D. Eardley, J. Isenberg, J. Marsden and V. Moncrief: *Homothetic and conformal symmetries of solutions to Einstein's equations*, Commun. Math. Phys. **106** (1986), 137–158

[4] G. S. Hall and J. D. Steele, *Conformal vector fields in general relativity*, J. Math. Phys. **32**, 1847–1853 (1991)

[5] W. Kühnel and H.-B. Rademacher, *Conformal geometry of gravitational plane waves*, Geom. Dedicata (to appear), for the preprint see www.igt.uni-stuttgart.de/LstDiffgeo/Kuehnel

[6] R. Maartens and S.D. Maharaj, *Conformal symmetries of pp-waves*, Class. Quantum Grav. **8** (1991), 503–514

[7] R. Schimming, *Riemannsche Räume mit ebenfrontiger und mit ebener Symmetrie*, Math. Nachr. **59**, 129–162 (1975)

Embedding and nonembedding results in asymptotic geometry

Sergei V. Buyalo

(joint work with Victor Schroeder)

In asymptotic geometry one studies metric spaces from a large scale point of view, when the local geometry plays a minor role or even does not exists, e.g. the space might be discrete. The following notion is an important example of such approach. A map $f : X \to Y$ between metric spaces is called quasi-isometric, if for some $a \geq 1$, $b \geq 0$ we have

$$\frac{1}{a}|xx'| - b \leq |f(x)f(x')| \leq a|xx'| + b$$

for all x, $x' \in X$. In my talk the following results obtained jointly with V. Schroeder (see [BS1], [BS2]) are discussed.

Theorem 1. *For any $n \geq 2$ there exists a quasi-isometric embedding $f : \mathrm{H}^n \to T_1 \times \cdots \times T_n$ of the hyperbolic space H^n into n-fold product of metric trees $T_1, \ldots, T_n$.*

Theorem 2. *For any $n \geq 2$ there is no quasi-isometric embedding* $\mathrm{H}^n \to T_1 \times \cdots \times T_{n-1} \times \mathbb{R}^m$ *for any $m \geq 0$ and any metric trees $T_1, \dots, T_{n+1}$.*

The stabilizing factor $\mathbb{R}^m$ is introduced here to make the situation nontrivial.

Embedding. The embedding from Theorem 1 is given by an explicit construction in a horospherical coordinates, see [BS1]. Applying a similar construction, my student A. Egorov has embedded the Hadamard manifold X with metric given in horospherical coordinates by

$$ds^2 = dt^2 + e^{2t}(dx^2 + dy^2) + e^{4t}dz^2$$

into the product of four trees T^4. It is proved by P. Pansu (see [Pa]) that X is not quasi-isometric to H^n. However, for the complex hyperbolic plane $\mathrm{H}^2_{\mathbb{C}}$ the question is open whether it admits a quasi-isometric embedding in the 4-fold product of metric trees. Recall that the metric of $\mathrm{H}^2_{\mathbb{C}}$ is given by

$$ds^2 = dt^2 + e^{2t}(dx^2 + dy^2) + e^{4t}(dz - xdy)^2.$$

It is known that $\mathrm{H}^2_{\mathbb{C}}$ admits a uniform embedding into the metric product of five (locally finite) trees (this follows from a result of A. Dranishnikov, [Dr]), and that there is no quasi-isometric embedding into the metric product of three trees stabilized by any Euclidean factor, see Theorem 3 below.

Nonembedding. To prove Theorem 2, we introduce a quasi-isometry invariant called the *hyperbolic dimension.* Its definition is similar to the definition of Gromov's asymptotic dimension. Recall that the asymptotic dimension of a metric space X is defined as $\operatorname{asdim} X = \min n$ such that for every $d > 0$ there is a uniformly bounded covering $\mathcal{U}$ of X with Lebesgue number $L(\mathcal{U}) \geq d$ and multiplicity $\leq n + 1$. Recall also that the Lebesgue number $L(\mathcal{U})$ is the maximal radius such that any (open) ball in X of that radius is contained in some element of the covering, and that the multiplicity of a covering is the maximal number of its elements having a common point.

The hyperbolic dimension is defined in the same way with only difference that we allow unbounded elements of the covering which are however in a sence small. Roughly speaking, we consider coverings whose elements are asymptotically doubling with some uniformity condition.

The hyperbolic dimension has the following properties

• monotonicity: if $X \to X'$ q.i, then $\operatorname{hypdim}(X) \leq \operatorname{hypdim}(X')$;

• product thm: $\operatorname{hypdim}(X \times X') \leq \operatorname{hypdim}(X) + \operatorname{hypdim}(X')$;

• $\operatorname{hypdim}(\mathbb{R}^m) = 0$ for every $m \geq 0$ (this property distinguishes the hyperbolic dimension from the asymptotic dimension since $\operatorname{asdim}(\mathbb{R}^m) = m$;

• $\operatorname{hypdim}(T) \leq 1$ for any metric tree T.

Our main result about hypdim is this.

Theorem 3. $\operatorname{hypdim} \mathrm{H}^n \geq n$ *for every $n \geq 2$ (actually,* $\operatorname{hypdim} X \geq \dim \partial_\infty X + 1$ *for any Gromov hyperbolic space X whose boundary at infinity $\partial_\infty X$ is doubling w.r.t. any visual metric).*

Using these properties, we have $\operatorname{hypdim}(T_1\times\cdots\times T_k\times\mathbb{R}^m)\le k$. Hence, if there is a quasi-isometric $X\to T_1\times\cdots\times T_k\times\mathbb{R}^m$, then by monotonicity, $\operatorname{hypdim}X\le k$. Together with Theorem 3, this shows that $\mathrm{H}^n\not\to T_1\times\cdots\times T_{n-1}\times\mathbb{R}^m$.

Questions. A problem related to obstacles to quasi-isometric embeddings is the Gromov conjecture that $\operatorname{asdim}\Gamma=\dim\partial_\infty\Gamma+1$ for any Gromov hyperbolic group Γ (see [Gr, 1.E_1']). The estimate $\operatorname{asdim}\Gamma\ge\dim\partial_\infty\Gamma+1$ is easy, and the question is whether $\operatorname{asdim}\Gamma\le\dim\partial_\infty\Gamma+1$. The positive answer would imply a number of new nonembedding results.

• In this respect one can show that $\operatorname{asdim}\Gamma\le\operatorname{mdim}\partial_\infty\Gamma+1$, where the metric dimension mdim is defined as follows. Let $\mathcal{U}$ be a covering of a metric space Z, $\operatorname{mesh}(\mathcal{U})=\sup\{\operatorname{diam}U:\,U\in\mathcal{U}\}$, $L(\mathcal{U})$ be the Lebegues number. We define *the capacity* of $\mathcal{U}$ by

$$c(\mathcal{U})=\frac{L(\mathcal{U})}{\operatorname{mesh}(\mathcal{U})}\in[0,1].$$

For $\tau>0$, $\delta\in(0,1)$ and an integer $m\ge0$ we put $c_\tau(Z,m,\delta)=\sup_{\mathcal{U}}c(\mathcal{U})$, where the supremum is taken over all open coverings $\mathcal{U}$ of Z with multiplicity $\le m+1$ and $\delta\tau\le\operatorname{mesh}(\mathcal{U})\le\tau$. Now, we put

$$c(Z,m,\delta)=\liminf_{\tau\to0}c_\tau(Z,m,\delta),$$

$c(Z,m)=\lim_{\delta\to0}c(Z,m,\delta)$ and finally

$$\operatorname{mdim}(Z)=\inf\{m:\,c(Z,m)>0\}.$$

One always has $\dim Z\le\operatorname{mdim}Z$. Therefore, the Gromov conjecture is reduced to the question whether $\dim\partial_\infty\Gamma=\operatorname{mdim}\partial_\infty\Gamma$ (for some visual metric on $\partial_\infty\Gamma$).

• Another open question is what is the hyperbolic dimension of $\mathrm{H}^2\times\mathrm{H}^2$? One easily sees that it must be 3 or 4. We strongly believe that it is 4. This would imply an interesting nonembedding result that $\mathrm{H}^2\times\mathrm{H}^2\not\to T_1\times T_2\times T_3\times\mathbb{R}^m$ for any $m\ge0$. More generally, we conjecture that the hyperbolic dimension of any symmetric space of noncompact type without Euclidean factors is at least the dimension of the space. This also would imply a number of new nonembedding results.

References

[BS1] S. Buyalo and V. Schroeder, *Embedding of hyperbolic spaces in the product of trees*, `arXive:math.GT/0311524`.

[BS2] S. Buyalo and V. Schroeder, *Hyperbolic dimension of metric spaces,* `arXive:math.GT/0404525`.

[Dr] A. Dranishnikov, *On hypersphericity of manifolds with finite asymptotic dimension*, Trans. Amer. Math. Soc. 355 (2003), no. 1, 155–167 (electronic).

[Gr] M. Gromov, *Asymptotic invariants of infinite groups*, in “Geometric Group Theory”, (G.A. Niblo, M.A. Roller, eds.), London Math. Soc. Lecture Notes Series 182 Vol. 2, Cambridge University Press, 1993.

[Pa] P. Pansu, L^p-cohomology and pinching, in M. Burger, A. Iozzi, Edt., Rigidity in Dynamics and Geometry, Springer, 2002, 379–389.

Generalized Hopf differentials

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(joint work with Harold Rosenberg)

The basic global results in the theory of constant mean curvature (cmc) surfaces in space forms are the theorems of A. D. Alexandrov and H. Hopf from the 1950ies [4, 8]. Alexandrov's theorem states that a closed, embedded cmc surface in $\mathbb{S}^3_+$, $\mathbb{R}^3$, or $\mathbb{H}^3$ space form is necessarily a standard distance sphere. Its proof is based on a moving planes argument that is amazingly flexible and has been applied in many other contexts since. It has even turned out to be fruitful for the theory of nonlinear elliptic equations [7].

Hopf's theorem on the other hand states that an immersed cmc sphere in a space form M^3_κ is necessarily a standard distance sphere. The basic idea in Hopf's argument is to observe that the $(2,0)$–part of the second fundamental form $h_\Sigma = \langle\,.\,, A\,.\,\rangle$ of such a cmc surface Σ^2 is a *holomorphic* quadratic differential, a fact that is also one of the foundations of the theory of cmc tori in space forms [1, 5, 6].

1. New Results for Cmc Surfaces in Product Spaces

It is straightforward to extend Alexandrov's result and prove that a closed, embedded cmc surface Σ^2 in either one of the product spaces $\mathbb{S}^2_+ \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ is an embedded, rotationally-invariant sphere S^2_H and is therefore uniquely determined up to congruence by the value of its mean curvature H. In the first case the restriction to hemispheres is again crucial and, in fact, much more serious than in the case of the 3–sphere.

Hopf's result on the other hand is not that easy to generalize; the $(2,0)$–part $\pi_{2,0}(h_\Sigma)$ of the second fundamental form is in general *not holomorphic*, since the ambient space curvature term in the Codazzi equations does not vanish anymore. Our basic new result is that for cmc surfaces in the product spaces $M^2_\kappa \times \mathbb{R}$ holomorphicity can be restored with the help of an explicit, geometrically defined correction term [3]:

Theorem 1. *Let $(\kappa, H) \neq 0$, and let L be the symmetric bilinear form corresponding to the field of projectors onto the vertical lines in the product space $M^2_\kappa \times \mathbb{R}$. Then the expression*

$$Q := 2H \cdot \pi_{2,0}(h_\Sigma) - \kappa \cdot \pi_{2,0}(\iota^\star L)\ .$$

defines a natural holomorphic quadratic differential on any immersed cmc surface $\iota\colon \Sigma^2 \looparrowright M^2_\kappa \times \mathbb{R}$ with mean curvature H.

The remaining two theorems in this section are also established in [3, see]. First of all, applying ODE techniques to the fundamental equations of surface theory, we can classify the cmc surfaces with $Q \equiv 0$.

Theorem 2. *Let $(\kappa, H) \neq 0$, and let $\iota\colon \Sigma^2 \looparrowright M^2_\kappa \times \mathbb{R}$ be a complete surface with constant mean curvature H and vanishing holomorphic quadratic differential Q.*

Furthermore, let $\theta := \arcsin(d\xi\cdot\nu)$ *denote the angle between the unit normal field* ν *and the vertical lines. Then the following holds*:

- *if* $\kappa+4H^2 > 0$*, then* Σ^2 *is congruent to one of the embedded, rotationally-invariant cmc spheres* S^2_H.
- *if* $\kappa + 4H^2 \leq 0$*, then* Σ^2 *is a complete* open *surface. Depending on the sign of the function* $4H^2 + \kappa\cos^2(\theta)$*, it is either congruent to a disk-like surface* D^2_H *or a particular parabolic surface* P^2_H *or a surface* C^2_H *of catenoidal type.*

Since the space of holomorphic quadratic differentials on the sphere $\mathbb{S}^2 = \mathbb{CP}^1$ is trivial, Theorems 1 and 2 yield the following analogue of Hopf's result:

Theorem 3. *Any immersed cmc sphere* S^2 *in a product space* $M^2_\kappa\times\mathbb{R}$ *is congruent to one of the embedded, rotationally-invariant cmc spheres* S^2_H.

2. Further Generalizations

In this section we investigate the *scope* where our generalized Hopf differentials can be defined. In particular, we ask for which (orientable) Riemannian 3–manifold (M^3, g) does there exist a correction field L that induces a holomorphic quadratic differential on any immersed cmc surface $\iota\colon \Sigma^2 \looparrowright (M^3, g)$. In this generality, it is of course no longer possible to write down an explicit expression for the correction L. However, the following holds:

Theorem 4. *Fix some constant* $H \in \mathbb{R}$*. Let* (M^3, g) *be an oriented Riemannian manifold, and let* L_0 *be a* $\mathbb{C}$*–valued, traceless, symmetric bilinear form on* M^3*. Then the expression*

$$Q := \pi_{2,0}(h_\Sigma + \iota^\star L_0)$$

defines a holomorphic *quadratic differential on any surface* $\iota\colon \Sigma^2 \looparrowright (M^3, g)$ *with constant mean curvature* H*, if and only if* L_0 *solves the differential equation*

$$(*)\qquad \mathrm{D}_X\, L_0 \;=\; \tfrac{1}{2}\, i\cdot\left[\star X, G - 2H\, L_0\right]\,.$$

Here the square brackets denote the commutator, and $\star X$ stands for the skew-symmetric endomorphism $Y \mapsto X\times Y$ induced by the cross-product. Restricting to traceless fields L_0 is in fact a mere normalization. The ODE-system $(*)$ is strongly *overdetermined*, and thus one should expect that the corresponding integrability conditions impose serious *restrictions on the geometry* of the underlying Riemannian 3–manifold:

Theorem 5. *Let* $(\tilde{M}^3, g)$ *be a simply-connected, oriented Riemannian manifold, and let* $H \in \mathbb{R}$ *be some real constant. Then equation* $(*)$ *is solvable if and only if* $(\tilde{M}^3, g)$ *is a* homogeneous space *with an at least* 4*-dimensional isometry group.*

Recall that homogeneous Riemannian 3–manifolds $(\tilde{M}^3, g)$ come with 6–, 4–, or 3–dimensional isometry groups. Those with 6–dimensional isometry groups are the space forms, whereas those with 4*–dimensional isometry groups* admit natural equivariant Riemannian submersions with 1–dimensional, totally-geodesic fibers [10, 11]. Up to isometry they are classified by the curvature κ of the quotient surface and the bundle curvature τ. In this class of homogeneous 3–manifolds, one distinguishes six different *homogeneous structures*:

	$\kappa > 0$	$\kappa = 0$	$\kappa < 0$
$\tau = 0$	$\mathbb{S}^2 \times \mathbb{R}$	$\mathbb{R}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$\tau \neq 0$	$\mathbb{S}^3_{\text{Berger}}$	$\mathsf{Nil}(3)$	$\widetilde{\mathsf{Sl}}(2, \mathbb{R})$

At this point we have constructed holomorphic quadratic differentials on cmc surfaces in homogeneous 3–manifolds corresponding to 7 of the eight maximal homogeneous structures that appear in Thurston theory; only the geometries corresponding to $\mathsf{Solv}(3)$ are missing.

Holomorphic quadratic differentials on cmc surfaces in the target spaces listed in the second row of the table were not known beforehand. Inspecting the proof of Theorem 5, one finds that equation $(*)$ always admits a homogeneous solution L_0. Following the argument leading to Theorem 2, it is possible to classify the cmc surfaces where the corresponding holomorphic quadratic differential Q vanishes identically, and thus we can generalize Hopf's result even further:

Theorem 6. *Any immersed cmc sphere S^2 in a simply-connected homogeneous space $(\tilde{M}^3, g)$ with an at least 4–dimensional isometry group is in fact an embedded, rotationally-invariant cmc sphere.*

It seems natural to think of the holomorphic quadratic differential Q constructed in Theorems 4 and 5 as a *family of first integrals for the cmc equation* that is due to the 1–dimensional isotropy groups of the bundle geometries and the 3–dimensional isotropy groups of the space forms, respectively. For the proofs of all 3 theorems presented in this section we refer the reader to the forthcoming paper [2].

3. Conclusions

It is a common feature of the bundle geometries that the isotropy group of any point p contains the 180°–rotations around all horizontal geodesics through p. This property makes it feasible to construct global minimal surfaces from Plateau solutions with suitable boundary polygons, using the Schwarz reflection principle. Together with the principal results of the preceding section, this observation provides a lot of evidence for the *thesis* that homogeneous 3–manifolds with at least 4–dimensional isometry groups are the proper setting for studying global properties of minimal surfaces and cmc surfaces.

The talk ended discussing this thesis in the context of the Heisenberg group. After describing the equivariant minimal surfaces in $\mathsf{Nil}(3)$ as classified by Mercuri and Pedrosa [9], we presented some local and global analogues of the doubly-periodic Scherk surface. With this background, we discussed the possibility of half-space theorems and Bernstein theorems for minimal surfaces in $\mathsf{Nil}(3)$. These results will be the subject of a forthcoming joint paper.

References

[1] U. Abresch, *Constant mean curvature tori in terms of elliptic functions*, J. Reine Angew. Math. **374** (1987), 169–192.
[2] U. Abresch, *A Hopf differential for constant mean curvature surfaces in Riemannian* 3*–manifolds*, in preparation.
[3] U. Abresch, H. Rosenberg, *A Hopf differential for constant mean curvature surfaces in* $\mathbb{S}^2 \times \mathbb{R}$ *and* $\mathbb{H}^2 \times \mathbb{R}$, Preprint, Bochum und Paris, December 2003.
[4] A. D. Alexandrov, *A characteristic property of spheres*, Ann. Mat. Pura Appl. **58** (1962), 303–315.
[5] A. I. Bobenko, *All constant mean curvature tori in* $\mathbb{R}^3$*,* $\mathbb{S}^3$*,* $\mathbb{H}^3$ *in terms of theta-functions*, Math. Ann. **290** (1991), 209–245.
[6] F. E. Burstall, D. Ferus, F. Pedit, U. Pinkall, *Harmonic tori in symmetric spaces and commuting Hamiltonian systems on loop algebras*, Ann. of Math. (2) **138** (1993), 173–212.
[7] B. Gidas, W. M. Ni, L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.
[8] H. Hopf, *Differential geometry in the large*, Lecture Notes in Mathematics, **1000**, Springer-Verlag, Berlin 1983.
[9] Ch. B. Figueroa, F. Mercuri, R. Pedrosa, *Invariant surfaces of the Heisenberg groups*, Ann. Mat. Pura Appl. (4) **177** (1999), 173–194.
[10] P. A. Scott, *The geometries of* 3*–manifolds*, Bull. London Math. Soc. **15** (1983), 401–487.
[11] W. P. Thurston, *Three-dimensional geometry and topology*, Princeton Math. Series **35**, Princeton Univ. Press, Princeton, New Jersey 1997.

Biquotients, curvature and homotopy type

Wilderich Tuschmann
(joint work with Vitali Kapovitch and Anton Petrunin)

This work is motivated by the finiteness theorems in Riemannian geometry and a question of S.-T. Yau which asks whether there always exists only a finite number of diffeomorphism types of closed smooth manifolds of positive sectional curvature that are homotopy equivalent to a given positively curved manifold ([Yau93], Problem 11).

If one relaxes the condition $\sec > 0$ to $\sec \geq 0$ then the answer to Yau's question is known to be false in all dimensions ≥ 7, even in the category of simply connected manifolds. This follows from combining a result of Grove and Ziller [GZ00] that the total space of any linear sphere bundle over S^4 admits a Riemannian metric with nonnegative curvature with the fact that, for each $k \geq 3$, the total spaces of linear S^k-bundles over S^4 fall into infinitely many homeomorphism, but only finitely many homotopy types. However, the examples of Grove and Ziller do not satisfy a uniform upper curvature bound when rescaled to have uniformly bounded diameter, and, more generally, it is thus natural to look at the following question.

Question 1. *Given fixed $n \in \mathbb{N}, D > 0$ and $c, C \in \mathbb{R}$, are there at most finitely many diffeomorphism classes of pairwise homotopy equivalent closed Riemannian n-dimensional manifolds with sectional curvature $c \leq \sec \leq C$ and diameter $\leq D$?*

The answer is known to be positive in some special situations. For example, it is true if M is 2-connected by [PT99], if $D = D(C, c, n)$ is sufficiently small by Gromov's theorem on almost flat manifolds [BK81] and the rigidity of infranilmanifolds [Aus60], or if $C \leq 0$ and $n \geq 5$ by results of Farrell and Jones [FJ90, FJ93]. Remarkably enough, in the latter case one actually does not even need the lower curvature and upper diameter bound. In other words, for $n \geq 5$ the answer to the analogue of Yau's original question for nonpositive curvature (which in this case is a special case of the Borel conjecture) is yes.

Here we are mostly concerned with the following special case of Question 1:

Question 2. *Given fixed $n \in \mathbb{N}$ and $D, C > 0$, are there at most finitely many diffeomorphism types of pairwise homotopy equivalent closed Riemannian n-manifolds with sectional curvature $0 \leq \sec \leq C$ and diameter $\leq D$?*

This question is also of interest in the following respects.

First of all, in general dimensions the diffeomorphism finiteness theorems in Riemannian geometry (see, e.g., [AC91, Che70, Pet84, FR00, FR02, KGW91, PT99, Tus02]) leave it completely open.

On the other hand, sequences of lens spaces $M_k = S^{2m+1}/\mathbb{Z}_k$ already show that as soon as one drops the condition that the homotopy type be fixed, this question must definitely be answered in the negative. Moreover, starting from dimension $n = 6$, from [GZ00] one may infer the existence of infinite sequences of closed simply connected nonnegatively curved n-manifolds of mutually distinct homotopy type, and in dimensions $n > 8$, $n \neq 10$ by [Tot03] (see also [FR01]) there even exist infinite sequences of closed simply connected nonnegatively pinched Riemannian n-manifolds with pairwise non-isomorphic rational cohomology rings that also satisfy uniform upper diameter bounds.

Our first main result shows that if $n \geq 10$, the answer to Question 2 is in general negative, even under the extra assumption of positive Ricci curvature.

Theorem A. *In each dimension $n \geq 10$ there exist infinite sequences $(M^n_k)_{k \in \mathbb{N}}$ of pairwise homotopy equivalent but mutually non-homeomorphic closed simply connected Riemannian n-manifolds satisfying*

$$0 \leq \sec(M^n_k) \leq 1, \ \mathrm{Ric}(M^n_k) > 0 \ \textit{and} \ \mathrm{diam}(M^n_k) \leq D = D(n).$$

In view of the fact that for closed simply connected manifolds of nonnegative sectional curvature till now no obstructions to the existence of a Riemannian metric with positive sectional curvature are known, this result provides a geometric answer to the bounded curvature case of Yau's question as close as possible.

It is quite likely that the dimensional restriction $n \geq 10$ in Theorem A is not optimal, and it is an interesting question to find the minimal dimension where examples satisfying the conclusion of Theorem A can occur. By [FR00, Tus02], this dimension must be at least 7. If one does not require the sectional curvature to be nonnegative, we show that the answer to Question 1 is indeed already negative in all dimensions ≥ 7:

Theorem B. *For any $n \geq 7$ there exist infinite sequences of homotopy equivalent but mutually non-homeomorphic closed Riemannian n-manifolds M_k^n with*

$$|\sec(M_k^n)| \leq 1 \text{ and } \operatorname{diam}(M_k^n) \leq D = D(n).$$

If $n \neq 8$, all these manifolds can in addition be chosen to be simply-connected.

Notice that for simply connected manifolds, by [Tus02, FR00], $n = 7$ is indeed the smallest dimension where such examples can occur.

In [Bel03] Belegradek constructed examples of manifolds admitting infinitely many nonnegatively curved metrics with mutually non-diffeomorphic souls. We sharpen this result by constructing such examples which in addition have uniform bounds on the curvature of the manifolds and the diameters of the souls:

Theorem C. *For any $k > 10$ the manifold $S^2 \times S^2 \times S^3 \times S^3 \times \mathbb{R}^k$ admits an infinite sequence of complete nonnegatively curved metrics g_i with pairwise non-homeomorphic souls S_i such that*

$$0 \leq \sec(M, g_i) \leq 1 \text{ and } \operatorname{diam}(S_i) \leq D$$

where D is a positive constant independent of k.

The manifolds in Theorems A and souls in Theorem C can be written as biquotient manifolds which, if simply connected, are known to carry metrics of positive Ricci curvature. The ideas of the proofs of Theorems A,B and C can be described as follows.

To prove Theorem A we fix a rank 2 bundle ξ over $S^2 \times S^2 \times S^2$ and look at the sphere bundle P of $\xi \oplus \epsilon^{k-1}$ with $k \geq 3$. We then look at various circle bundles $S^1 \to M_i \to P$. A topological argument shows that with an appropriate choice of ξ, infinitely many of such bundles have total spaces homotopy equivalent to $S^2 \times S^2 \times S^3 \times S^k$ but distinct first Pontrjagin classes and thus are mutually non-homeomorphic.

We can represent all M_is as $S^3 \times S^3 \times S^3 \times S^k / T_i^2$ where $T_i^2 \subset T^3$ which acts freely and isometrically on $S^3 \times S^3 \times S^3 \times S^k$. This easily implies that all M_i satisfy all the geometric constraints in Theorem A.

To prove Theorem C we put $k = 3$ and fix a rank 2 bundle ζ over P and look at the pullbacks of $\zeta \oplus \epsilon^{l-2}$ to M_i. By the same reasons as before, the total spaces of these pullbacks have metrics satisfying the geometric restrictions of Theorem C with souls isometric to M_i. A topological argument shows that with an appropriate choice of ζ the total spaces of the pullbacks are diffeomorphic to $S^2 \times S^2 \times S^3 \times S^3 \times \mathbb{R}^l$ if $l > 10$.

To prove Theorem B we look at a 6-manifold X^6 which is homotopy equivalent to $S^2 \times S^2 \times S^2$ but has nontrivial first Pontrjagin class. By an easy topological argument, among S^1-bundles over X^6 there are infinitely many spaces homotopy equivalent to $S^2 \times S^2 \times S^3$ but having distinct Pontrjagin classes. They all admit metrics of bounded curvature and diameter by the same argument as in the proof of Theorem A.

References

[AC91] Michael T. Anderson and Jeff Cheeger. Diffeomorphism finiteness for manifolds with Ricci curvature and $L^{n/2}$-norm of curvature bounded. *Geom. Funct. Anal.*, 1(3):231–252, 1991.

[Aus60] Louis Auslander. Bieberbach's theorems on space groups and discrete uniform subgroups of Lie groups. *Ann. of Math. (2)*, 71:579–590, 1960.

[Bel03] Igor Belegradek. Vector bundles with infinitely many souls. *Proc. Amer. Math. Soc.*, 131(7):2217–2221 (electronic), 2003.

[BK81] P. Buser and H. Karcher. *Gromov's almost flat manifolds*, volume 81 of *Astérisque*. Société Mathématique de France, 1981.

[Che70] J. Cheeger. Finiteness theorems for Riemannian manifolds. *Am. J. Math.*, 92:61–74, 1970.

[FJ90] F.T. Farrell and L.E. Jones. Rigidity and other topological aspects of compact non-positively curved manifolds. *Bull. Am. Math. Soc., New Ser.*, 22(1):59–64, 1990.

[FJ93] F. T. Farrell and L. E. Jones. Topological rigidity for compact non-positively curved manifolds. In *Differential geometry: Riemannian geometry (Los Angeles, CA, 1990)*, volume 54 of *Proc. Sympos. Pure Math.*, pages 229–274. Amer. Math. Soc., Providence, RI, 1993.

[FR00] Fuquan Fang and Xiaochun Rong. Fixed point free circle actions and finiteness theorems. *Commun. Contemp. Math.*, 2(1):75–86, 2000.

[FR01] Fuquan Fang and Xiaochun Rong. Curvature, diameter, homotopy groups, and cohomology rings. *Duke Math. J.*, 107(1):135–158, 2001.

[FR02] Fuquan Fang and Xiaochun Rong. The second twisted Betti number and the convergence of collapsing Riemannian manifolds. *Invent. Math.*, 150(1):61–109, 2002.

[GZ00] K. Grove and W. Ziller. Curvature and symmetry of Milnor spheres. *Ann. of Math. (2)*, 152(1):331–367, 2000.

[KGW91] P. Peterson K. Grove and J.-Y. Wu. Geometric finiteness theorems via controlled topology. *Invent. Math.*, 99:205–213, 1991.

[Pet84] S. Peters. Cheeger's finiteness theorem for diffeomorphism classes of Riemannian manifolds. *J. Reine Angew. Math.*, 349:77–82, 1984.

[PT99] A. Petrunin and W. Tuschmann. Diffeomorphism finiteness, positive pinching, and second homotopy. *Geom. Funct. Anal.*, 9(4):736–774, 1999.

[Tot03] Burt Totaro. Curvature, diameter, and quotient manifolds. *Math. Res. Lett.*, 10(2-3):191–203, 2003.

[Tus02] Wilderich Tuschmann. Geometric diffeomorphism finiteness in low dimensions and homotopy group finiteness. *Math. Ann.*, 322(2):413–420, 2002.

[Yau93] Shing-Tung Yau. Open problems in geometry. In *Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990)*, volume 54 of *Proc. Sympos. Pure Math.*, pages 1–28. Amer. Math. Soc., Providence, RI, 1993.

Tame and wild integrability

Iskander A. Taimanov

Let M^{2n} be a symplectic manifold with the Poisson bracket $\{\cdot,\cdot\}$ induced by the symplectic form. Any smooth function H on M^{2n} generates a Hamiltonian flow on M^{2n} such that the evolution of an arbitrary smooth function $f : M^{2n} \to \mathbb{R}$ along the trajectories is given by the equation $\frac{df}{dt} = \{f, H\}$. A function which is preserved by the flow is called a first integral or an integral of the motion.

This is, for instance, the so-called Hamiltonian function H (that follows form the skew-symmetry of the Poisson brackets).

A Hamiltonian flow is called *completely integrable* if it admits a family of first integrals $I_1 = H, \dots, I_n$ such that these integrals are in involution: $\{I_j, I_k\} = 0, \;\; 1 \leq j, k \leq n$, and they are functionally independent almost everywhere, i.e. outside some nowhere dense set Σ which is called the singular locus.

More generally it is said that the flow is integrable if it admits $n + k$ first integrals $I_1 = H, \dots, I_{n+k}$ such that they generate a Lie algebra L with respect to the Poisson brackets such that L has a commutative subalgebra of dimension $n - k$ and these first integrals are functionally independent almost everywhere.

The condition that the functions f and g are in involution means that the Hamiltonian flows generated by f and g commute. Therefore the integrability means that a generic level surface $V_c = \{I_1 = c_1, \dots, I_{n+k} = c_{n+k}\}$ admits an action of $\mathbb{R}^{n-k}$ by translations along trajectories of commuting flows. We conclude that 1) this surface is a quotient of $\mathbb{R}^{n-k}$ and such a compact surface is diffeomorphic to a torus, and 2) the commuting flows are linearized on V_c.

In analogous manner the integrability of the level surface $\{H = h\}$, $h = \text{const}$, is defined.

There is some freedom in the definitions when we say about functional independence of first integrals almost everywhere. It could be that

- they are functionally independent on an open dense set;
- given a smooth measure on M^{2n} such that the measure of M^{2n} is finite, the first integrals are functionally independent on the full measure subset.
- first integrals $I_1, \dots, I_n$ are analytic (i.e. the flow is analytically integrable).[1]
- there is a finite smooth (or even analytic) simplicial decomposition of the phase space M^{2n} such that a singular locus Σ forms a subcomplex of this decomposition and the complement to it is cutted by another subcomplex of positive codimension to a union of finitely many sets U_α which are foliated by invariant tori over their images under the momentum map.[2]

It appears that there is an important difference between this notions which is similar to the difference between wild and tame sets in geometric topology.

The analytic integrability implies the geometric simplicity [6] and these cases has to be treated as a tame integrability since a tame behavior of the flows. Moreover it is possible to apply methods of topology to establishing topological obstructions to such an integrability.

The first obstruction was found by Kozlov in 1989 when he proved that if the geodesic flow on a compact oriented analytic Riemannian two-manifold is analytically integrable then the manifold is diffeomorphic to a sphere or to a torus [5].

In 1984 the we found obstructions in the high-dimensional situation [6] by proving the following statement:

[1] This situation was mostly studied in the classical analytic mechanics of the 19th century

[2] This is the so-called *geometric simplicity* introduced in [6].

If the geodesic flow on a compact analytic manifold M^n is geometrically simple then there is an invariant torus $T^n \subset SM^n$ such that the natural projection $\pi : SM^n \to M^n$ induced a homomorphism $\pi_* : \pi_1(T^n) \to \pi_1(M^n)$ those image is a subgroup of finite index in $\pi_1(M^n)$

and this implies that

1) $\pi_1(M^n)$ of M^n contains a commutative subgroup of finite index;

2) $H^*(M;\mathbb{R})$ of M^n contains a subring A which is isomorphic to $H^*(T^k;\mathbb{R})$ where T^k is a k-torus and k is the first Betti number of M^n;

3) moreover $b_1 = \dim M^n$ then $H^*(M^n;\mathbb{R}) = H^*(T^n;\mathbb{R})$.

In 1999 Butler [2] established an integrability of the geodesic flow on a three-dimensional nilmanifold in terms of C^∞ first integrals and in 2000 in our joint paper with Bolsinov we proved an analogous result for a three-dimensional solvmanifold [1]. This flow on the solvmanifold

1) has positive topological entropy although its entropy with respect to any smooth invariant metric vanishes;

2) contains an Anosov subsystem being analytic and integrable in terms of C^∞ functions.

Recently we proved with Knauf that another example of such flow which is integrable only in terms of C^∞ functions is the classical n-centre problem at high energy levels [4].

All these flows are very complicated being completely integrable. That explains by the complicated geometry of their singular locii which can be treated as wild subsets in the phase spaces.

We argue that a good definition of a complete integrability has to include a condition of tame integrability. In this case the flow can be described in a simple way. A mathematical background for such a definition can be given by theory of o-minimal structures [3], i.e. we have to assume that the first integrals are definable functions in some such a structure.

The topological obstructions results proved by Kozlov and the author are straightforwardly extended for such a notion of tame integrability.

The following problems are very important for understanding topological obstructions to integrability:

- can we generalize the Kozlov theorem for the C^∞ case (i.e. drop the assumption of analyticity)?
- does the analytic (or more generally tame) integrability of the flow implies that its topological entropy vanishes?

In both cases the expected answers are unclear to us.

References

[1] Bolsinov, A.V., Taimanov, I.A. Integrable geodesic flows with positive topological entropy. Inventiones Math. **140** (2000), 639–650.

[2] Butler, L.T. New examples of integrable geodesic flows. Asian J. Math. **4** (2000), 515–526.

[3] van den Dries, L. Tame topology and o-minimal structures. London Math. Society Lecture Notes **248**, Cambridge University Press, Cambridge, 1998.

[4] Knauf, A., Taimanov, I.A. On the integrability of the n-centre problem (to appear in Mathematische Annalen).
[5] Kozlov, V.V. Topological obstacles to the integrability of natural mechanical systems. Soviet Math. Dokl. **20** (1979), 1413–1415.
[6] Taimanov, I.A. Topological obstructions to the integrability of geodesic flows on non-simply connected manifolds. Math. USSR-Izv. **30** (1988), 403–409.

Distortion of Knotted Curves

John M. Sullivan
(joint work with Elizabeth Denne)

Gromov introduced the notion of distortion for curves as the maximum ratio of arclength to chord length, and showed that any closed curve has distortion at least $\pi/2$, that of a round circle. He then asked [5] whether every knot type can be built with distortion less than, say, 100.

When studying quadrisecants of knots—lines in space that intersect the knot four times—Kuperberg [6] introduced a way to say which secants of the knot are topologically nontrivial or *essential*. Denne [2] has further developed these ideas to show that knotted curves have essential alternating quadrisecants, and with Diao [3], we used such quadrisecants to get a good lower bound on the ropelength of nontrivial knots. Here, we deduce $\delta \geq \pi$ for any knotted curve, merely using the existence of an essential secant along with results from [3] that characterize how a family of secants can become essential. In [4], we carry these ideas further to show $\delta > 3.99$ for any knot. For comparison, trefoil knots with distortion less than 8.2 can be exhibited.

For us, a *knot* will mean a closed, oriented, rectifiable curve K embedded in $\mathbb{R}^3$ with finite length $\ell(K)$. Such a knot has a Lipschitz parameterization by arclength.

Two points p, q along a knot K separate K into two complementary arcs, γ_{pq} and γ_{qp}. (Here γ_{pq} is the arc from p to q following the orientation of K.) We let ℓ_{pq} denote the length of γ_{pq}. We are mainly interested in the shorter arclength distance $d(p,q) := \min(\ell_{pq}, \ell_{qp}) \leq \ell(K)/2$. We contrast this with the straight-line (chord) distance $|p-q|$, the length of the segment $\overline{pq} \subset \mathbb{R}^3$.

The arclength parametrization of K has Lipschitz constant 1 by definition. The distortion of K is a Lipschitz constant for the inverse map:

Definition. The *distortion* between distinct points p and q on the knot K is

$$\delta(p,q) := \frac{d(p,q)}{|p-q|} \geq 1.$$

The distortion of K is the supremal distortion between any distinct points.

For any knot K, we have $\delta \geq \pi/2$, with equality only for a round circle; a proof following Gromov can be found in [7, Prop. 2.1]. Our main result in [4] says that a nontrivial knot (of finite total curvature) must have distortion at least 3.99.

To get this lower bound we use the notion of essential arcs, which we introduced in [3] as an extension of ideas of Kuperberg [6]. Note that generically a knot K

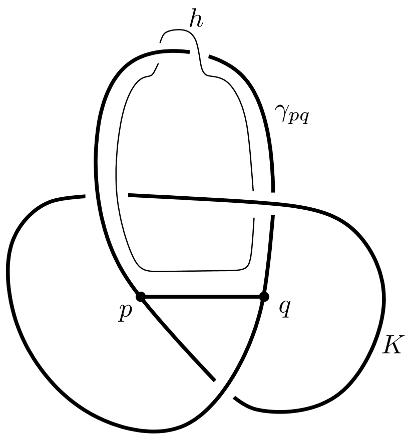

FIGURE 1. This arc γ_{pq} is essential in the knot K because the parallel h, whose linking number with K is zero, is homotopically nontrivial. In this example, γ_{qp} is also essential, so pq is essential.

together with a chord $\overline{pq}$ forms a Θ-graph in space; being essential is a topological feature of this knotted graph, as shown in Figure 1.

Definition. Suppose K is a knot and $p, q \in K$. Assuming $\overline{pq}$ has no interior intersections with K, we define a loop $h = h(\gamma_{pq})$ in the free homotopy of the knot complement $X := \mathbb{R}^3 \smallsetminus K$. Namely, h is represented by a parallel curve to $\gamma_{pq} \cup \overline{pq}$, chosen to have linking number zero with K. (If $\overline{pq}$ does intersect K, we perturb it first, as in [3].) Then we say γ_{pq} is an *essential arc* of K if h is a nontrivial free homotopy class (or equivalently, if $\gamma_{pq} \cup \overline{pq}$ bounds no disk whose interior is disjoint from K). We say the secant pq is *essential* if both arcs γ_{pq} and γ_{qp} are essential.

If K is unknotted then any subarc is inessential. Conversely, Dehn's lemma can be used [3, Thm. 5.2] to show that, if for some $p, q \in K$ both γ_{pq} and γ_{qp} are inessential, then K is unknotted. What will be most important for us is the following theorem which describes borderline-essential arcs.

Proposition ([3], Thm. 7.1). *Suppose γ_{pr} is in the boundary of the set of essential arcs for a knot K. (That is, γ_{pr} is essential, but there are inessential arcs of K with endpoints arbitrarily close to p and r.) Then K must intersect the interior of segment $\overline{pr}$ at some point $q \in \gamma_{rp}$ for which the secants pq and qr are both essential.* □

The proof follows simply by tracking the change in the homotopy class $h(\gamma_{pr})$ as the secant $\overline{pr}$ passes from one side of q to the other. This change equals the commutator of the meridian loop of K with any one of the elements $h(\gamma_{pq})$, $h(\gamma_{qp})$, $h(\gamma_{qr})$, or $h(\gamma_{rq})$.

The approximation results of [3] show that any knot K of finite total curvature has a shortest essential arc. Our main theorem is now based on analysis of this arc and the two related essential secants guaranteed by the Proposition.

Theorem. *Every nontrivial knot of finite total curvature has distortion $\delta \geq \pi$.*

Proof. Let γ_{pr} be a shortest essential arc for the knot K, and let δ be the distortion of K. For convenience, rescale the knot so that $\ell_{pr} = \delta$. Then any essential secant ab has $|a-b| \geq 1$, for otherwise the shorter of the essential arcs γ_{ab} and γ_{ba} would have length at most $|a-b|\delta < \delta$, contradicting the definition of γ_{pr}.

Since some nearby arcs are shorter and thus inessential, the Proposition is applicable to γ_{pr}, giving us $q \in \overline{pr} \cap \gamma_{rp}$ with pq and qr essential. Now let m be the midpoint of γ_{pr}, so that $\ell_{pm} = \delta/2 = \ell_{mr}$. If follows that for any $x \in \gamma_{pm}$ we have $d(q,x) \geq \delta + \ell_{px}$, while for $y \in \gamma_{mr}$ we have $d(q,y) \geq \delta + \ell_{yr}$. By the definition of distortion, if follows that $|q-x| \geq 1 + \ell_{px}/\delta \geq 1$. In particular, the whole arc γ_{pr} stays outside $B_1(q)$. It follows immediately that $\delta = \ell_{pr} \geq \pi$. □

We note that this bound, which is twice the minimum distortion possible for a closed unknotted curve, is not sharp. In particular, the midpoint m of γ_{pr} must be further away: $|q-m| \geq 3/2$. A more detailed analysis in [4] shows that $\delta = \ell_{pr} \geq 3.9945$, where the minimum length involves a logarithmic spiral at constant distortion from q.

References

[1] J. Cantarella, G. Kuperberg, R.B. Kusner, J.M. Sullivan, *The Second Hull of a Knotted Curve*, Amer. J. Math, **125**:6 (2003), 1335–1348. `arXiv:math.GT/0204106`

[2] E. Denne, *Alternating Quadrisecants of Knots*, Ph.D. thesis, Univ. Illinois, Urbana (2004).

[3] E. Denne, Y. Diao, J.M. Sullivan, *Quadrisecants Give New Lower Bounds for the Ropelength of a Knot*. `arXiv:math.GT/0408026`

[4] E. Denne, J.M. Sullivan, *The Distortion of a Knotted Curve*. `arXiv:math.GT/0409438`

[5] M. Gromov, *Homotopical Effects of Dilatation*, J. Diff. Geom. **13** (1978), 303–310.

[6] G. Kuperberg, *Quadrisecants of knots and links*, J. Knot Thy. Ramif., **3**:1 (1994), 41–50. `arXiv:math.GT/9712205`

[7] R.B. Kusner and J.M. Sullivan, *On Distortion and Thickness of Knots*, in IMA Vol. **103** (Springer, 1997), 67–78. `arXiv:dg-ga/9702001`

Perelman's work on Ricci flow

Bruce Kleiner

G. Perelman posted two preprints [1,2] on the ArXiv, which introduced new techniques for studying Hamilton's Ricci flow, and presented an argument for Thurston's Geometrization Conjecture. The four lectures discussed these papers. The first two lectures gave an overview of Perelman's approach. The third and fourth discussed his noncollapsing estimate and some of his results on κ-solutions.

[1] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, http://front.math.ucdavis.edu/math.DG/0211159

[2] G. Perelman, *Ricci flow with surgery on three-manifolds*, http://front.math.ucdavis.edu/math.DG/0303109

Volumes on normed and Finsler spaces: Introduction and update

Juan Carlos Álvarez Paiva[1]

This note is a short introduction to the theory of volumes on normed and Finsler spaces. It is also a quick update to the survey of Álvarez and Thompson [3], where the reader will find proofs, references, and some of the history of the subject.

The theory starts with a seemingly simple question: *how to measure volumes on normed and Finsler spaces?* In other words, we would like to assign a volume density to each finite-dimensional normed and Finsler space in a way that naturally extends the definition of volume of Euclidean and Riemannian spaces. I follow, with slight modifications, the axiomatic approach of [6].

1. Basic axioms

After a moment's reflection, most geometers would agree that a reasonable notion of volume on finite-dimensional normed spaces should satisfy the following three properties:

Axiom 1. Compact subsets of a normed space have finite volume and open subsets of a normed space have positive volume.

Axiom 2. The volume of an open set in a Euclidean space is its Euclidean volume.

Axiom 3. An affine map between normed space of the same dimension that does not increase distances does not increase volumes.

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As a consequence of Axiom 3, isometries between normed spaces are volume-preserving. In particular, translations are volume-preserving. This, together with Axiom 1 and Haar's theorem, implies that the volume density in a finite-dimensional normed space X is a constant multiple of the Lebesgue measure on X. The constant may be determined by prescribing the volume of the unit ball B_X of X. This volume cannot be prescribed arbitrarily: once we have a definition of volume on normed spaces satisfying Axioms 1–3, the assignment $B_X \mapsto \mathrm{vol}_X(B_X)$ is a linear invariant of centered convex bodies. Moreover, the value of this invariant on any n-dimensional ellipsoid (the unit ball of some n-dimensional Euclidean space) must be equal to ϵ_n—the volume of the unit ball in $\mathbb{R}^n$.

Still, this restriction on $B_X \mapsto \mathrm{vol}_X(B_X)$ does not uniquely determine a canonical choice of volume on normed spaces. Some possible choices are:

The Busemann definition. Assign to every n-dimensional normed space the multiple of the Lebesgue measure for which the volume of its unit ball is ϵ_n.

The Holmes-Thompson definition. Assign to every n-dimensional normed space X the multiple of the Lebesgue measure for which the volume of its unit ball B_X is the symplectic volume of $B_X \times B_X^* \subset X \times X^*$ divided by ϵ_n.

Gromov's mass* or Benson definition. Assign to every n-dimensional normed space the multiple of the Lebesgue measure for which the minimal volume of a parallelotope circumscribing its unit ball is 2^n.

Infinitely many other choices are possible. For example, we may assign to every n-dimensional normed space the multiple of the Lebesgue measure for which the minimal volume of an ellipsoid circumscribing its unit ball is ϵ_n.

This wealth of definitions suggests that we may need to impose yet another axiom on our notion of volume. Such an axiom readily suggests itself if we notice that that having defined volumes we have defined areas.

2. Areas in normed spaces

If we have a notion of volume on normed spaces, we can define the k-dimensional area of a measurable subset S of a k-dimensional affine subspace in a normed space X: translate the subspace so that it passes through the origin, induce a norm on the resulting vector subspace Y from its embedding in X, and measure the volume of S in the k-dimensional normed space Y.

This remark allows us to measure the area of any polyhedron in a normed space and, more generally, to define the area integrands in all dimensions. The fourth axiom is a weak ellipticity condition on the area integrands.

Axiom 4. If T is a simplex in a normed space, then the area of any face of T does not exceed the sum of the areas of the remaining faces.

Some equivalent formulations of this axiom are:

(1) The hypersurface area functional in a normed space is lower semi-continuous.
(2) A domain contained in a hyperplane of a normed space is area-minimizing.

(3) If X is an n-dimensional normed space, the function whose value at an $(n-1)$-vector $v_1 \wedge \cdots \wedge v_{n-1}$ is the area of the parallelotope formed by $v_1, \ldots, v_{n-1}$ is a norm on the space $\Lambda^{n-1}X$ of $(n-1)$-vectors of X.

It is known that the Busemann, Holmes-Thompson, and Benson (mass*) definitions of volume satisfy Axiom 4. Moreover, they and their convex combinations are the only **known** definitions that do so. However, even if it could be proved that these are the only definitions to satisfy Axiom 4, we still have infinitely possible definitions. Differential geometry may help us to pick the right one(s).

3. Volumes and areas in Finsler spaces

In Finsler geometry, we only deal with *Minkowski norms* on vector spaces: norms that are smooth away from the origin and have positive-definite Hessians at every nonzero point. A *Finsler metric* on a manifold is simply a continuous function on its tangent bundle that is smooth away from the zero section and such that its restriction to each tangent space is a Minkowski norm.

Given a definition of volume on normed spaces, we have a definition of volume on Finsler manifolds: the volume density on an n-dimensional Finsler manifold M assigns to each parallelotope formed by tangent vectors $v_1, \ldots, v_n \in T_xM$ its volume in the normed space T_xM. The condition that the volume density be smooth is satisfied by both the Busemann and Holmes-Thompson definition, but not by mass*.

The Busemann and Holmes-Thompson definition are both very natural in the Finsler context: the Busemann volume of a Finsler manifold is its Hausdorff measure ([5]) and The Holmes-Thompson volume of an n-dimensional Finsler manifold equals the symplectic volume of its unit codisc bundle divided by the volume of the unit ball in n-dimensional Euclidean space ([2]).

If we define the area of a submanifold of a Finsler space as the volume of the submanifold with its induced Finsler metric, a natural condition to impose on a definition of volume on normed and Finsler spaces is that totally geodesic submanifold be extremal for the area functional. This condition is satisfied by the Holmes-Thompson definition:

Theorem (Berck, [4]). *Totally geodesic submanifolds of Finsler spaces are extremal for the Holmes-Thompson area functional.*

However, as the following theorem shows, the Hausdorff measure fails this test in a very strong way.

Theorem (Álvarez and Berck, [1]). *For any real number λ, all geodesics of the Finsler metric*

$$\varphi_\lambda(\mathbf{x}, \mathbf{v}) = \frac{(1 + \lambda^2\|\mathbf{x}\|^2)\|\mathbf{v}\|^2 + \lambda^2\langle \mathbf{x}, \mathbf{v}\rangle^2}{\|\mathbf{v}\|}$$

are straight lines. However, the only value of λ for which all (totally geodesic) planes are extremals of the Hausdorff area functional of φ_λ is $\lambda = 0$.

References

[1] J.C. Álvarez-Paiva and G. Berck, *What is wrong with the Hausdorff measure in Finsler spaces*, arXiv:math.DG/0408413, 2004.
[2] J.C. Álvarez-Paiva and E. Fernandes, *Fourier transforms and the Holmes-Thompson volume of Finsler manifolds*, Int. Math. Res. Notices **19** (1999), 1031–1042.
[3] J.C. Álvarez-Paiva and A.C. Thompson, *Volumes in normed and Finsler spaces*, A Sampler of Riemann-Finsler geometry (Cambridge) (D. Bao, R. Bryant, S.S. Chern, and Z. Shen, eds.), Cambridge University Press, 2004, pp. 1–49.
[4] G. Berck, *Minimalité des sous-variétés totalement géodésiques en géométrie finslerienne*, arXiv:math.DG/0409320, 2004.
[5] H. Busemann, *Intrinsic area*, Ann. of Math. **48** (1947), 234–267.
[6] A.C. Thompson, *Minkowski Gometry*, Encyclopedia of Math. and its Applications, vol. 63, Cambridge University Press, Cambridge, 1996.

Almost hermitian 6-manifolds and a generalization of Kirichenko's theorem

Thomas Friedrich

Fix a subgroup $\mathrm{G} \subset \mathrm{SO}(n)$ of the special orthogonal group and decompose the Lie algebra $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$ into the Lie algebra $\mathfrak{g}$ of G and its orthogonal complement $\mathfrak{m}$. The different geometric types of G-structures on a Riemannian manifold correspond to the irreducible G-components of the representation $\mathbb{R}^n \otimes \mathfrak{m}$. Indeed, consider a Riemannian manifold (M^n, g) and denote its Riemannian frame bundle by $\mathcal{F}(M^n)$. It is a principal $\mathrm{SO}(n)$-bundle over M^n. A G-structure is a reduction $\mathcal{R} \subset \mathcal{F}(M^n)$ of the frame bundle to the subgroup G. The Levi-Civita connection is a 1-form Z on $\mathcal{F}(M^n)$ with values in the Lie algebra $\mathfrak{so}(n)$. We restrict the Levi-Civita connection to $\mathcal{R}$ and decompose it with respect to the decomposition of the Lie algebra $\mathfrak{so}(n)$,

$$Z\big|_{T(\mathcal{R})} \;:=\; Z^* \,\oplus\, \Gamma\,.$$

Then, Z^* is a connection in the principal G-bundle $\mathcal{R}$ and Γ is a 1-form on M^n with values in the associated bundle $\mathcal{R} \times_{\mathrm{G}} \mathfrak{m}$. If $\Gamma = 0$, then the Levi-Civita connection preserves the G-strucuture (integrable geometries). Some authors call Γ the *intrinsic torsion* of the G-structure. There is a second notion, namely the *characteristic connection* and the *characteristic torsion* of a G-structure. It is a G-connection ∇^{c} with totally skew symmetric torsion tensor. Not any type of geometric G-structures admits a characteristic connection. In order to formulate the condition, we embed the space of all 3-forms into $\mathbb{R}^n \otimes \mathfrak{m}$ using the morphism

$$\Theta \,:\, \Lambda^3(\mathbb{R}^n) \longrightarrow \mathbb{R}^n \otimes \mathfrak{m}\,, \quad \Theta(\mathrm{T}) \;:=\; \sum_{i=1}^{n} e_i \otimes \mathrm{pr}_{\mathfrak{m}}(e_i \,\lrcorner\, \mathrm{T})\,.$$

A G-structure admits a characteristic connection ∇^{c} if and only if the intrinsic torsion Γ belongs to the image of the Θ. In this case, the intrinsic torsion is given by the equation $2\,\Gamma = -\,\Theta(\mathrm{T}^{\mathrm{c}})$. For interesting geometric structures Θ is injective, i. e. the condition that the torsion is totally skew symmetric singles out a unique characteristic connection substituting the Levi-Civita connection. This characteristic torsion form has been computed explicitely in terms of the underlying geometric data. Formulas of that type are known for almost hermitian structures, almost metric contact structures, G_2-structures in dimension 7 and Spin(7)-structures in dimension 8. If $M^n = G_1/G$ is naturally reductive, the characteristic connection coincides with the *canonical* connection of the reductive space. In this sense, we can understand the characteristic connection of a Riemannian G-structure as a generalization of the canonical connection of a Riemannian naturally reductive space. The canonical connection of a naturally reductive space has parallel torsion form and parallel curvature tensor. For arbitrary G-structures and their characteristic connections, these properties do not hold anymore. Corresponding examples will be discussed. Non-integrable geometric structures and their characteristic connections are important in type II string theory. Indeed, their torsion forms serve as candidates for a NS-3-form involved in the so called Strominger model. This is a 6-tuple $(M^n, g, \nabla, \mathrm{T}, \Phi, \Psi)$ consisting of a Riemannian manifold (M^n, g), a metric connection ∇ with totally skew symmetric torsion form T, a dilation function Φ and a spinor field Ψ. The string equations can be written in the following way:

$$\begin{aligned} \mathrm{Ric}^{\nabla} + \frac{1}{2}\,\delta(\mathrm{T}) + 2\,\nabla^g(d\Phi) &= 0\,, \quad \delta(\mathrm{T}) \;=\; 2\,(\mathrm{grad}(\Phi) \,\lrcorner\, \mathrm{T})\,, \\ \nabla\Psi &= 0\,, \quad (2\cdot d\Phi - \mathrm{T})\cdot\Psi \;=\; 0\,. \end{aligned}$$

The first fermionic equation $\nabla\Psi = 0$ means that the spin holonomy of the connection preserves a spinor. We study the holonomy group of metric connections with totally skew symmetric torsion. For examples, in case of the flat euclidian space this group is always semisimple and does not preserve any non-degenerate 2-form or any spinor. On compact Riemannian manifolds we prove similar results using suitable integral formulas. Generalizations involving the torsion form T of the Schrödinger-Lichnerowicz formula and the Parthasarathy formula for the square of the Dirac operator will be discussed. In particular, these formulas yield an operator Ω acting on spinor fields and defined for any triple (M^n, g, ∇) with totally skew symmetric torsion T,

$$\begin{aligned} \Omega \;&:=\; (D^{1/3})^2 + \frac{1}{8}\,(d\mathrm{T} - 2\,\sigma_{\mathrm{T}}) + \frac{1}{4}\,\delta(\mathrm{T}) - \frac{1}{8}\,\mathrm{Scal}^g - \frac{1}{16}\,||\mathrm{T}||^2 \\ &=\; \Delta_{\mathrm{T}} + \frac{1}{8}\,(3\,d\mathrm{T} - 2\,\sigma_{\mathrm{T}} + 2\,\delta(\mathrm{T}) + \mathrm{Scal})\,. \end{aligned}$$

We call Ω the *Casimir operator* of the triple (M^n, g, ∇), since in case of a symmetric space it coincides with the group theoretical Casimir operator. Ω has some remarkable properties. For example, its kernel contains all ∇-parallel spinors, the 3-form T acts in its kernel etc. We investigate the integrability condition for parallel spinors as well as the Casimir operator for all the characteristic connections T^c of non integrable structures in dimension $n = 5, 6, 7$. Moreover, in these dimensions we will construct explicit solutions of the spinor Killing equation on naturally reductive spaces, for examples on Aloff-Wallach spaces. On the other side, any 7-dimensional 3-Sasakian manifold admits a two-parameter family of metric connections with totally skew symmetric torsion and parallel spinors. The parallelism $\nabla^c T^c = 0$ of the torsion form of a characteristic connection is an important property. The first reason is that $\nabla^c T^c = 0$ implies the conservation law $\delta(T^c) = 0$. Moreover, if the torsion is parallel, several formulas for differential operators acting on spinors simplify and it is possible to investigate the space of parallel or harmonic spinors in more detail. Sasakian structures or nearly Kähler structures (a Theorem of Kirichenko) have a parallel characteristic torsion form, even if they are not reductive. This motivates the investigation of Riemannian G-structures with a parallel characteristic torsion form in general. In dimension $n = 6$ we study the non integrable geometries with this property generalizing, in this sense, Kirichenko's result. Any almost hermitian manifold of type G_1 admits a unique characteristic connection. The U(3)-orbit type of the characteristic torsion is constant. It turns out that there exist only two orbits with a non abelian isotropy (holonomy) group in dimension six. The manifolds under consideration are torus fibrations over some special 4-manifold, twistor spaces or a non Kählerian hermitian structure on the Lie group $SL(2, \mathbb{C})$. Finally we classify all naturally reductive hermitian $\mathcal{W}_3$-manifolds with small (abelian) isotropy group of the characteristic torsion.

References

[1] Th. Friedrich and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian Journ. Math. 6 (2002), 303-336.

[2] Th. Friedrich and S. Ivanov, *Almost contact manifolds, connections with torsion and parallel spinors*, Journ. Reine u. Angew. Math. 559 (2003), 217-236.

[3] Th. Friedrich and S. Ivanov, *Killing spinor equations in dimension 7 and geometry of integrable G_2-manifolds*, Journ. Geom. Phys. 48 (2003), 1-11.

[4] Th. Friedrich, *On types of non-integrable geometries*, Suppl. Rend. Circ. Mat. di Palermo Ser. II, 71 (2003), 99-113.

[5] Th. Friedrich, Spin(9)*-structures and connections with totally skew-symmetric torsion*, Journ. Geom. Phys. 47 (2003), 197-206.

[6] I. Agricola, *Connections on naturally reductive spaces, their Dirac operator and homogeneous models in string theory*, Comm. Math. Phys. 232 (2003), 535-563.

[7] I. Agricola and Th. Friedrich, *On the holonomy of connections with skew-symmetric torsion*, Math. Ann. 328 (2004), 711-748.

[8] I. Agricola and Th. Friedrich, *On the Casimir operator of a connection with skew-symmetric torsion*, Journ. Geom. Phys. 50 (2004), 188-204.

[9] I. Agricola and Th. Friedrich, *Killing spinors in supergravity with 4-fluxes*, Class. Quant. Grav. 20 (2003), 4707-4717.

[10] B. Alexandrov, Th. Friedrich and N. Schoemann, *Almost hermitian 6-manifolds revisited*, to appear in Journ. Geom. Phys.

Curvature tensors of singular spaces

Andreas Bernig

We refer to [2] for a survey concerning generalizations of curvature notions to singular spaces and to [1] for the material presented here.

Using Geometric Measure Theory, we provide the space of compact submanifolds of Euclidean space $\mathbb{R}^N$ with a natural topology, called *tame topology.* To each compact submanifold X, one associates the (Federer-Fleming-) $N-1$-current $\tilde{X}$, given by integration over the unit normal bundle of X. This current has the following properties:

a) $\partial \tilde{X} = 0$, i.e. $\tilde{X}$ is a cycle.
b) $\tilde{X}$ is integral.
c) $\tilde{X} \llcorner \alpha = 0$, where α is the canonical 1-form on $S\mathbb{R}^N$, i.e. $\tilde{X}$ is Legendrian.
d) The support of $\tilde{X}$ is compact.

Let $\mathcal{LC}(S\mathbb{R}^N)$ denote the space of $N-1$-currents with these properties, endowed with the weak topology. Besides compact submanifolds, some singular spaces admit such a *normal cycle*: piecewise linear spaces, convex bodies, sets with positive reach, compact subanalytic sets, Lipschitz manifolds with bounded curvature.

To each compact set $X \subset \mathbb{R}^N$ admitting a normal cycle $\tilde{X}$, we introduce a sequence of tensor-valued Borel measures $\Lambda_{k,d}(X,-), 0 \leq d \leq k \leq N$ such that, B being a Borel set, the following properties hold.

a) $\Lambda_{k,d}(X,B) \in \mathrm{Sym}^2 \Lambda^d \mathbb{R}^d$.
b) Valuation property:

$$\Lambda_{k,d}(X_1,B) + \Lambda_{k,d}(X_2,B) = \Lambda_{k,d}(X_1 \cap X_2, B) + \Lambda_{k,d}(X_1 \cup X_2, B).$$

c) Translation invariance: $\Lambda_{k,d}(X+t, B+t) = \Lambda_{k,d}(X,B)$ for all $t \in \mathbb{R}^N$.
d) Rotation covariance:

$$\Lambda_{k,d}(\rho X, \rho B) = \rho \Lambda_{k,d}(X,B)$$

for all $\rho \in SO(N)$.
e) Continuity: If $X_i \to X$ in the flat topology, then $\Lambda_{k,d}(X_i,-)$ converges weakly to $\Lambda_{k,d}(X,-)$.
f) Homogeneity: Let $\lambda > 0$. Then $\Lambda_{k,d}(\lambda X, \lambda B) = \lambda^k \Lambda_{k,d}(X,B)$.
g) Let $\mathrm{tr}_{d,2d} : \otimes^{2d}\mathbb{R}^N \to \otimes^{2d-2}\mathbb{R}^N, d \geq 1$ denote contraction of the d-th and the $2d$-th coordinate. Then

$$\mathrm{tr}_{d,2d} \Lambda_{k,d}(X,B) = \frac{k-d+1}{d} \Lambda_{k,d-1}(X,B).$$

Let s, ric, R denote scalar curvature, Ricci tensor and Riemann curvature tensor of a compact submanifold X of dimension $n < N$. We set $E := \frac{s}{2}g - \mathrm{ric}$ the Einstein tensor and $\hat{R} := R - \mathrm{ric} \cdot g + \frac{s}{4} g \cdot g$ a modification of the Riemann tensor and let $E^\#, \hat{R}^\#$ denote the corresponding $(2,0)$ and $(4,0)$-tensor fields dual to E and $\hat{R}$. Then

a) $\Lambda_{n-2,0}(X,B) = \frac{1}{4\pi} \int_{X \cap B} s \mu_g$ if $n \geq 2$;
b) $\Lambda_{n-2,1}(X,B) = \frac{1}{2\pi} \int_{X \cap B} E^\# \mu_g$ if $n \geq 3$;
c) $\Lambda_{n-2,2}(X,B) = \frac{1}{4\pi} \int_{X \cap B} \hat{R}^\# \mu_g$ if $n \geq 4$.

Given an arbitrary $X \subset \mathbb{R}^N$ admitting a normal cycle, we *define* scalar curvature measure, Einstein measure and modified Riemann measure to be the corresponding tensor-valued measures $4\pi\Lambda_{n-2,0}(X,-)$, $2\pi\Lambda_{n-2,1}(X,-)$ and $4\pi\Lambda_{n-2,2}(X,-)$. This extends some of the classical curvature formalism to singular spaces.

The same construction can be applied with the ambient space $\mathbb{R}^N$ replaced by any Riemannian manifold (M,g) of dimension N. The $\Lambda_{k,d}(X,-)$ are no longer tensor-valued measures (which would not make any sense), but tensor-valued distributions. Given a section $T \in \Gamma(\otimes^{2d}T^*M)$, $\Lambda_{k,d}(X,T)$ is a real number and the map $T \mapsto \Lambda_{k,d}(X,T)$ is continuous in an appropriate topology.

One of the main results is that these tensor-valued distributions are intrinsic in the following sense:

Let $\tau : (M,g) \to (M',g')$ be an isometric embedding of oriented Riemannian manifolds. Suppose that $X \subset M$ is a compact subset admitting a normal cycle. Let $T' \in \Gamma(\otimes^{2d}T^*M')$ and $T := \tau^*T'$ its restriction to M. Then for $0 \le d \le k \le N$

$$\Lambda_{k,d}(X,T) = \Lambda_{k,d}(\tau(X),T').$$

A surprising fact is that there is no continuous, intrinsic tensor-valued distribution corresponding to the Ricci tensor. Since scalar curvature and metric tensor are only defined as measures, there is no way to compute ric from E, s, g as in the smooth case.

As an application, the above construction permits to find approximations of the Einstein tensor of a compact smooth submanifold by looking at sufficiently good piecewise polyhedral approximations. This was already used (in the 3-dimensional case) in Computational Geometry.

References

[1] BERNIG, A.: Curvature tensors of singular spaces. Preprint 2003.
[2] BERNIG, A.: On some aspects of curvature. Preprint 2004.

Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions

Urs Lang
(joint work with Thilo Schlichenmaier)

The lecture reported on our recent work [7] on a variation of Gromov's notion of asymptotic dimension [4], with a view towards applications in analysis on metric spaces. The invariant considered was introduced and named Nagata dimension by Assouad [1]; indeed it is closely related to a theorem of Nagata characterizing the topological dimension of metrizable spaces.

Let $X = (X,d)$ be a metric space, and let $\mathcal{B} = (B_i)_{i \in I}$ be a family of subsets of X. The family $\mathcal{B}$ is called *D-bounded* for some constant $D \ge 0$ if $\operatorname{diam} B_i := \sup\{d(x,x') \colon x,x' \in B_i\} \le D$ for all $i \in I$. For $s > 0$, the *s-multiplicity* of $\mathcal{B}$ is the infimum of all integers $n \ge 0$ such that every subset of X with diameter $\le s$ meets no more than n members of the family. The *asymptotic dimension* $\operatorname{asdim} X$

of X is defined as the infimum of all integers n such that for all $s > 0$, X possesses a D-bounded covering with s-multiplicity at most $n+1$ for some $D = D(s) < \infty$. This imposes no condition on small scales as it is not required that $D(s) \to 0$ for $s \to 0$. The *Nagata dimension* (or *Assouad–Nagata dimension*) $\dim_{\mathrm{N}} X$ of X is the infimum of all integers n with the following property: There exists a constant $c > 0$ such that for all $s > 0$, X has a cs-bounded covering with s-multiplicity at most $n+1$. Note that this notion takes into account *all* scales of the metric space in an equal manner. Clearly $\dim_{\mathrm{N}} X \geq \operatorname{asdim} X$. The number $\dim_{\mathrm{N}} X$ is unaffected if the covering sets are required to be open, or closed, or if the 'test set' with diameter $\leq s$ is replaced by an open or closed ball of radius s (the minimal constant c may change, however).

It is easily seen that $\dim_{\mathrm{N}} X \leq \dim_{\mathrm{N}} Y$ whenever $f\colon X \to Y$ is a map between metric spaces satisfying, for instance, $a\, d(x,x')^p \leq d(f(x),f(x')) \leq b\, d(x,x')^p$ for all $x,x' \in X$ and for some constants $a,b,p > 0$. For every metric space X, the topological dimension $\dim X$ never exceeds $\dim_{\mathrm{N}} X$. The product theorem $\dim_{\mathrm{N}}(X \times Y) \leq \dim_{\mathrm{N}} X + \dim_{\mathrm{N}} Y$ holds; the inequality may be strict. Each subset X of $\mathbb{R}^n$ containing interior points satisfies $\dim_{\mathrm{N}} X = n$. Every doubling metric space has finite Nagata dimension. For $X = Y \cup Z$, $\dim_{\mathrm{N}} X = \sup\{\dim_{\mathrm{N}} Y, \dim_{\mathrm{N}} Z\}$. Hence, every compact n-dimensional riemannian manifold X satisfies $\dim_{\mathrm{N}} X = n$. Every product of n non-trivial metric trees and every euclidean building of rank n has Nagata dimension n. By a *metric tree* we mean a geodesic metric space all of whose geodesic triangles are degenerate, i.e. isometric to tripods; no local finiteness or compactness assumption is made. A geodesic metric space X that is hyperbolic in the sense of Gromov has finite Nagata dimension if it satisfies a respective condition on small scales. Every homogeneous Hadamard manifold has finite Nagata dimension.

A map f from a metric space X into another metric space Y is called *quasisymmetric* if it is injective and there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that $d(x,z) \leq t\, d(x',z)$ implies $d(f(x),f(z)) \leq \eta(t)\, d(f(x'),f(z))$ for all $x,x',z \in X$ and $t \geq 0$. Then $f^{-1}\colon f(X) \to X$ is also quasisymmetric, and f is a homeomorphism onto its image.

Theorem 1. Let X, Y be two metric spaces, and let $f\colon X \to Y$ be a quasisymmetric homeomorphism. Then $\dim_{\mathrm{N}} X = \dim_{\mathrm{N}} Y$.

Assouad's theorem [2] asserts that for every doubling metric space (X,d) and every exponent $p \in (0,1)$, there is an N such that the metric space (X,d^p) admits a bi-Lipschitz embedding into $\mathbb{R}^N$. Dranishnikov [3] showed that every geodesic metric space with bounded geometry and asymptotic dimension at most n admits a large-scale uniform embedding into the product of $n+1$ locally finite metric trees.

Theorem 2. Let (X,d) be a metric space with $\dim_{\mathrm{N}} X \leq n < \infty$. Then for all sufficiently small exponents $p \in (0,1)$, there exists a bi-Lipschitz embedding of (X,d^p) into the product of $n+1$ metric trees.

In particular, (X,d) admits a quasisymmetric embedding into the product of $n+1$ metric trees. It follows that a metric space X satisfies $\dim_{\mathrm{N}} X < \infty$ if and only if it admits a quasisymmetric embedding into the product of finitely many metric trees.

We say that a pair of metric spaces (X,Y) possesses the *Lipschitz extension property* if there is a constant C such that for every subset $Z \subset X$ and for every Lipschitz map $f\colon Z \to Y$, there is a Lipschitz extension $\bar{f}\colon X \to Y$ of f with constant $\mathrm{Lip}(\bar{f}) \le C \,\mathrm{Lip}(f)$. A comprehensive characterization of such pairs is still missing. However, we obtain complete results if one of the two spaces has finite Nagata dimension. We call a metric space Y *Lipschitz m-connected* for some integer $m \ge 0$ if there is a constant c_m such that every Lipschitz map $f\colon S^m \to Y$ has a Lipschitz extension $\bar{f}\colon B^{m+1} \to Y$ with constant $\mathrm{Lip}(\bar{f}) \le c_m \,\mathrm{Lip}(f)$; here S^m and B^{m+1} denote the unit sphere and closed ball in $\mathbb{R}^{m+1}$ equipped with the induced metric. This condition is easily verified in the presence of an appropriate weak convexity property of the metric. In particular, every Banach space and every geodesic metric space with convex metric is Lipschitz m-connected for all $m \ge 0$.

Theorem 3. Suppose that X,Y are metric spaces, $\dim_{\mathrm{N}} X \le n < \infty$, and Y is complete. If Y is Lipschitz m-connected for $m = 0,1,\dots,n-1$, then the pair (X,Y) has the Lipschitz extension property.

As a corollary we obtain the known fact that for a complete metric space Y, the pair $(\mathbb{R}^n, Y)$ has the Lipschitz extension property if and only if Y is Lipschitz m-connected for $m = 0,\dots,n-1$.

Theorem 4. Suppose that Y is a complete metric space with $\dim_{\mathrm{N}} Y \le n < \infty$, and Y is Lipschitz m-connected for $m = 0,1,\dots,n$. Then Y is an absolute Lipschitz retract; equivalently, the pair (X,Y) has the Lipschitz extension property for every metric space X. In particular, Y is Lipschitz m-connected for all $m \ge 0$.

This result is obtained as a corollary of a more general theorem which provides Lipschitz extensions for maps $f\colon Z \to Y$ defined on a set $Z \subset X$ with $\dim_{\mathrm{N}} Z \le n$. When combined with the estimates for hyperbolic and nonpositively curved spaces mentioned earlier, Theorem 4 unifies and generalizes the results obtained in [5, 1.2] and [6, 4.6 and 6.5].

[1] P. Assouad, Sur la distance de Nagata, C. R. Acad. Sci. Paris Sér. I Math. 294 (1982), no. 1, 31–34.

[2] P. Assouad, Plongements lipschitziens dans $\mathbb{R}^n$, Bull. Soc. Math. France 111 (1983), 429–448.

[3] A. Dranishnikov, On hypersphericity of manifolds with finite asymptotic dimension, Trans. Amer. Math. Soc. 355 (2003), 155–167.

[4] M. Gromov, Asymptotic Invariants of Infinite Groups, pp. 1–295 in: G. A. Niblo, M. A. Roller (eds.), Geometric Group Theory, Vol. 2, London Math. Soc. Lecture Note Series, no. 182, Cambridge Univ. Press 1993.

[5] U. Lang, B. Pavlović, V. Schroeder, Extensions of Lipschitz maps into Hadamard spaces, Geom. Funct. Anal. (GAFA) 10 (2000), 1527–1553.

[6] U. Lang, C. Plaut, Bilipschitz embeddings of metric spaces into space forms, Geom. Dedicata 87 (2001), 285–307.
[7] U. Lang, T. Schlichenmaier, Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions, preprint 2004, arXiv:math.MG/0410048.

On the nondegeneracy of constant mean curvature surfaces

[1]

(joint work with Nick Korevaar and Jesse Ratzkin[2])

We prove that many complete, noncompact, constant mean curvature (CMC) surfaces $f : \Sigma \to \mathbb{R}^3$ are nondegenerate; that is, the Jacobi operator $\Delta_f + |A_f|^2$ has no L^2 kernel. In fact, if Σ has genus zero with k ends, and if $f(\Sigma)$ is embedded (or Alexandrov immersed) in a half-space, then the dimension of the L^2 kernel is at most $k - c - l$, where c the number of cylindrical ends of $f(\Sigma)$, and $l = 2$ or 3 is the dimension of the subspace of $\mathbb{R}^3$ spanned by the vertices of the spherical classifiying polygon for $f(\Sigma)$. Our main tool is a conjugation operation on Jacobi fields which linearizes the conjugate cousin construction. Consequences include partial regularity for CMC moduli space, a larger class of CMC surfaces to use in gluing constructions, and a surprising characterization of CMC surfaces via spinning spheres.

Constant mean curvature surfaces in $\mathbb{R}^3$ are equilibria for the area functional, subject to an enclosed-volume constraint. The mean curvature is nonzero when the constraint is in effect, so we can scale and orient the surfaces to make their mean curvature 1, a condition we abbreviate by CMC. Over the past two decades a great deal of progress has been made on understanding complete CMC surfaces and their moduli spaces, however many interesting open problems remain. One of the most important questions concerns the possibility of decaying Jacobi fields on complete CMC surfaces, that is, the Morse-theoretic degeneracy of these equilibria. Our main result is to rule out such Jacobi fields on a large class of complete CMC surfaces.

For a given immersed surface $f : \Sigma \to \mathbb{R}^3$, its mean curvature H_f is determined by the quasilinear elliptic equation

$$\Delta_f f = 2H_f \nu_f,$$

where $\nu = \nu_f$ is the (mean curvature, or inner) unit normal to f and Δ_f is the Laplace-Beltrami operator. The surface $f(\Sigma)$ is CMC if $H_f \equiv 1$. The oldest examples of CMC surfaces are the sphere of radius 1 and cylinder of radius 1/2. Interpolating between these two examples are the Delaunay unduloids, which are rotationally symmetric and periodic. A Delaunay unduloid is determined (up to rigid motion) by its necksize n, which is the length of the smallest closed geodesic on the surface. A necksize of $n = \pi$ corresponds to a cylinder of radius 1/2, and as

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$n \to 0$ one obtains the singular limit of a chain of mutually tangent unit spheres. The ODE determining the Delaunay surfaces still has global solutions when the necksize parameter is any negative number; in this case the resulting Delaunay nodoids are not embedded.

Here we will consider the slightly larger class of CMC surfaces in $\mathbb{R}^3$ which are *Alexandrov-immersed*. A proper immersion $f : \Sigma \to \mathbb{R}^3$ is an Alexandrov immersion if one can write $\Sigma = \partial M$, where M is a three-manifold into which the mean curvature normal ν points, and f extends to a proper immersion of M into $\mathbb{R}^3$. In the finite topology CMC setting, M is necessarily a handlebody with a solid cylinder attached for each end. For example, the Delaunay unduloids are Alexandrov-immersed (in fact, embedded), but the Delaunay nodoids are not.

It is a theorem of Alexandrov [A] that the only compact, connected, Alexandrov-immersed, CMC surfaces are unit spheres. For noncompact surfaces, Korevaar, Kusner and Solomon [KKS] proved that each end of a finite-topology CMC surface is exponentially asymptotic to a Delaunay unduloid, that two-ended CMC surfaces are unduloids, and that three-ended CMC surfaces have a plane of reflection symmetry. In fact, all *triunduloids* (three-ended, genus zero CMC surfaces) were constructed and classified by Große-Brauckmann, Kusner and Sullivan [GKS], as were all *coplanar k–unduloids* (k–ended, genus zero CMC surfaces whose asymptotic axes all lie in a plane [GKS2]). These authors define a classifying map assigning each coplanar k–unduloid an immersed polygonal disc with k geodesic edges in S^2, whose edge-lengths are the asymptotic necksizes of the corresponding k–unduloid.

The classifying map of [GKS, GKS2] is a homeomorphism, and gives information about the topological structure of moduli space of coplanar k–unduloids. To obtain information about the smooth structure of moduli space, one needs to understand the linearization of the mean curvature operator, which is the Jacobi operator

$$\mathcal{L}_f = \Delta_f + |A_f|^2,$$

where $|A_f|$ is the length of the second fundamental form of f. Solutions to the Jacobi equation $\mathcal{L}_f u = 0$ are called *Jacobi fields*, and correspond to normal variations of the CMC surface $f(\Sigma)$ which preserve the mean curvature to first order. More precisely, if u is a Jacobi field, then the one-parameter family of immersions $f(t) = f + tu\nu$ has mean curvature $H(t) = 1 + O(t^2)$. Thus one can think of Jacobi fields as tangent vectors to the moduli space of constant mean curvature surfaces.

Definition. A CMC surface $f : \Sigma \to \mathbb{R}^3$ is nondegenerate if the only solution $u \in L^2$ to $\mathcal{L}_f u = 0$ is the zero function.

Near a nondegenerate CMC surface $f(\Sigma)$, a theorem of Kusner, Mazzeo and Pollack [KMP] shows that the moduli space of CMC surfaces is a real-analytic manifold with coordinates derived from the asymptotic data of [KKS] (that is, the axes, necksizes, and neckphases of the unduloid asymptotes). In general the CMC moduli space is a real-analytic variety. Indeed, on a degenerate CMC surface, there would be a nonzero L^2 Jacobi field u, which (by [KMP]) decays exponentially on all ends. The presence of such a Jacobi field means there exists a one-parameter family of surfaces $f(t)$ with the same asymptotic data and with mean curvature $1+O(t^2)$, indicating a possible singularity in the CMC moduli space. Thus, proving nondegeneracy eliminates the potential for such singularities.

Our main theorem bounds the dimension of the space of L^2 Jacobi fields on a large class of CMC surfaces:

Theorem. Let $f : \Sigma \to \mathbb{R}^3$ be a coplanar k–unduloid. Then the space of L^2 Jacobi fields on $f(\Sigma)$ is at most $(k-2)$–dimensional. Moreover, if the span of the vertices of the classifying geodesic polygon in S^2 is a subspace of $\mathbb{R}^3$ with dimension l (nesessarily 2 or 3), and if c is the number of cylindrical ends of $f(\Sigma)$, then the space of L^2 Jacobi fields on $f(\Sigma)$ is at most $(k-c-l)$–dimensional.

As a corollary, we deduce that almost all triunduloids are nondegenerate. Recall ([GKS] and our earlier discussion) that a triunduloid uniquely determines a spherical triangle whose edge-lengths are the asymptotic necksizes n_1, n_2, n_3. The spherical triangle inequalities imply $n_1 + n_2 + n_3 \leq 2\pi$ and $n_i + n_j \geq n_k$. When these inequalities are strict, the vertices of the classifying triangle span $\mathbb{R}^3$, and so our main theorem asserts that the space of L^2 Jacobi fields vanishes:

Corollary. Let $f : \Sigma \to \mathbb{R}^3$ be a triunduloid. Then f is nondegenerate if its necksizes satisfy the strict spherical triangle inequalities or if there is a cylindircal end (with necksize $n = \pi$).

The main tool we develop is a conjugate Jacobi field construction, which converts Neumann fields to Dirichlet fields. This conjugate variation field arises from linearizing the conjugate cousin construction of [GKS]. Our construction is motivated by the analogous nondegeneracy result of Cosín and Ros [CR] for coplanar minimal surfaces. However, the geometry in the present case, and thus the proof, is quite different, with interesting consequences. For example, we obtain new insight into the classifying map for triunduloids and, more generally, for coplanar k–unduloids (see [GKS, GKS2]). The conjugate Jacobi field construction also yields a simple, synthetic characterization of constant mean curvature in terms of spinning a sphere at double-speed along the surface.

We conclude by mentioning some naturally related open problems concerning Jacobi fields on CMC surfaces and the moduli space theory of CMC surfaces:

Our main theorem gives upper bounds for the dimension of the space of L^2 Jacobi fields on coplanar k–unduloids. Is this bound sharp? In particular, up to scaling, there is at most one nonzero L^2 Jacobi field on any triunduloid satisfying $n_1 + n_2 + n_3 = 2\pi$ or $n_i + n_j = n_k$. Does this Jacobi field ever exist?

Is it possible to extend our technique to a wider class of CMC surfaces? For instance, there are many CMC surfaces which are not Alexandrov symmetric but do have some symmetry (*e.g.* tetrahedral symmetry). Can one use our methods to bound either the necksizes or the dimension of the space $\mathcal{V}$ of L^2 Jacobi fields on such surfaces?

The question of integrability of a Jacobi field is also open. According to [KMP], any tempered (sub-exponential growth) Jacobi field on a nondegenerate CMC surface is integrable, in the sense that it is the velocity vector field of a one-parameter family of CMC surfaces. It would be useful to decide whether tempered Jacobi fields are always integrable in this sense.

References

[A] A. D. Alexandrov. *A characteristic property of spheres.* Ann. Mat. Pura Appl. 58:303–315, 1962.

[CR] C. Cosín and A. Ros. *A Plateau problem at infinity for properly immersed minimal surfaces with finite total curvature.* Indiana Univ. Math. J. 50:847–879, 2001.

[GKS] K. Große-Braukmann, R. Kusner and J. Sullivan *Triunduloids: Embedded constant mean curvature surfaces with three ends and genus zero.* J. Reine Angew. Math. 564:35–61, 2003.

[GKS2] K. Große-Brauckmann, R. Kusner and J. Sullivan. *Coplanar constant mean curvature surfaces.* In preparation.

[KKR] N. Korevaar, R. Kusner and J. Ratzkin. *On the Nondegeneracy of Constant Mean Curvature Surfaces.* Preprint: arXiv.math.DG/0407153.

[KKS] N. Korevaar, R. Kusner and B. Solomon. *The Structure of Complete Embedded Surfaces with Constant Mean Curvature.* J. Differential Geom. 30:465–503, 1989.

[KMP] R. Kusner, R. Mazzeo and D. Pollack. *The moduli space of complete embedded constant mean curvature surfaces.* Geom. Funct. Anal. 6:120–137, 1996.

A geometric approach to exotic involutions in dimensions 5,6,13, and 14

Thomas Püttmann

(joint work with U. Abresch, C. Duran, and A. Rigas)

A differentiable involution σ of the sphere $\mathbb{S}^n$ is called exotic if it is fixed point free and not conjugate by diffeomorphisms to the antipodal map. The quotient $\mathbb{S}^n/\{\mathrm{id}, \sigma\}$ is homotopy equivalent but not diffeomorphic to the standard real projective space $\mathbb{RP}^n$.

Starting from a Wiedersehen metric on the Gromoll-Meyer sphere we obtain simple formulas for exotic involutions of $\mathbb{S}^5$, $\mathbb{S}^6$, $\mathbb{S}^{13}$, and $\mathbb{S}^{14}$. In order to give the formula for the involution of $\mathbb{S}^6$, for example, let w be a quaternion and p be an imaginary quaternion such that $|w|^2 + |p|^2 = 1$. Moreover, let $b : \mathbb{S}^6 \to \mathbb{S}^3$ be the map $b(p, w) = \frac{w}{|w|} e^{\pi p} \frac{\bar{w}}{|w|}$. Here, e^p denotes the exponential map of the unit sphere in the quaternions.

Theorem. *The map $\mathbb{S}^6 \to \mathbb{S}^6$,*

$$(p, w) \mapsto \big(-b(p, w)p\,\overline{b(p, w)}, -b(p, w)p\,\overline{b(p, w)}\big)$$

is an exotic involution.

The exotic involution of $\mathbb{S}^5$ is obtained by restricting the involution of $\mathbb{S}^6$ to purely imaginary w. The exotic involutions of $\mathbb{S}^{14}$ and $\mathbb{S}^{13}$ are given by substituting quaternions by octonions. The exotic involution of $\mathbb{S}^5$ can be described in terms of 3-dimensional Euclidean geometry only (see [2]). It would be very interesting to see directly from this visualizable construction that the involution is exotic.

Applications of our geometric approach to these involutions are discussed. These include a non-cancellation phenomenon in group actions and explicit cohomogeneity one actions on the standard $\mathbb{S}^5$ that are not conjugate to a linear action.

References

[1] U. Abresch, C. E. Duran, T. Püttmann, A. Rigas, *Wiedersehen metrics, exotic involutions, and non-cancellation phenomena*, Preprint, October 2004.

[2] U. Abresch, C. E. Duran, T. Püttmann, A. Rigas, *An exotic involution of the* 5*-sphere*, QuickTime movie available at `http://www.ruhr-uni-bochum.de/mathematik8/puttmann`.

Generalized Cylinders and Applications

Christian Bär
(joint work with P. Gauduchon and A. Moroianu)

I will report on joint work with P. Gauduchon and A. Moroianu that will appear in [2]. We give various applications of a construction which we call *generalized cylinders*. Let M be a manifold and let g_t be a smooth 1-parameter family of semi-Riemannian metrics on M, $t \in I \subset \mathbb{R}$. Then we call the manifold $\mathcal{Z} = I \times M$ with the metric $g^{\mathcal{Z}} := dt^2 + g_t$ a *generalized cylinder* over M. The generalized cylinder is an $(n+1)$-dimensional semi-Riemannian manifold (with boundary if I has boundary) of signature $(r+1, s)$ if the signature of g_t is (r, s). The vector field $\nu := \frac{\partial}{\partial t}$ is spacelike of unit length and orthogonal to the hypersurfaces $M_t := \{t\} \times M$. The t-lines are geodesics. Let W denote the Weingarten map of M_t with respect to ν and let H be the mean curvature.

On the one hand, this ansatz is very flexible. Locally, near a semi-Riemannian hypersurface with spacelike normal bundle every semi-Riemannian manifold is of this form. The restriction to spacelike normal bundle, i. e. to the positive sign in front of dt^2 in the metric of $\mathcal{Z}$ is made for convenience only. Changing the signs of the metrics on M as well as on $\mathcal{Z}$ reduces the case of a timelike normal bundle to that of a spacelike normal bundle.

On the other hand, this ansatz still allows to closely relate the geometries of M and $\mathcal{Z}$. The relevant formulas relating the curvatures of (M, g_t) and of $(\mathcal{Z}, g^{\mathcal{Z}})$ are the following:

$$
\begin{array}{llcl}
(1) & \langle W(X), Y\rangle & = & -\frac{1}{2}\dot{g}_t(X,Y), \\
(2) & \left\langle R^{\mathcal{Z}}(U,V)X, Y\right\rangle & = & \left\langle R^{M_t}(U,V)X, Y\right\rangle \\
& & & +\frac{1}{4}\left(\dot{g}_t(U,X)\dot{g}_t(V,Y) - \dot{g}_t(U,Y)\dot{g}_t(V,X)\right), \\
(3) & \left\langle R^{\mathcal{Z}}(X,Y)U, \nu\right\rangle & = & \frac{1}{2}\Big((\nabla^{M_t}_Y \dot{g}_t)(X,U) - (\nabla^{M_t}_X \dot{g}_t)(Y,U)\Big), \\
(4) & \left\langle R^{\mathcal{Z}}(X,\nu)\nu, Y\right\rangle & = & -\frac{1}{2}\left(\ddot{g}_t(X,Y) + \dot{g}_t(W(X),Y)\right), \\
(5) & \mathrm{ric}^{\mathcal{Z}}(\nu,\nu) & = & \mathrm{tr}(W^2) - \frac{1}{2}\mathrm{tr}_{g_t}(\ddot{g}_t), \\
(6) & \mathrm{ric}^{\mathcal{Z}}(X,\nu) & = & d_X\,\mathrm{tr}(W) - \Big\langle \mathrm{div}^M(W), X\Big\rangle, \\
(7) & \mathrm{ric}^{\mathcal{Z}}(X,Y) & = & \mathrm{ric}^{M_t}(X,Y) + 2\langle W(X), W(Y)\rangle \\
& & & -\mathrm{tr}(W)\langle W(X), Y\rangle - \frac{1}{2}\ddot{g}_t(X,Y), \\
(8) & \mathrm{Scal}^{\mathcal{Z}} & = & \mathrm{Scal}^{M_t} + 3\,\mathrm{tr}(W^2) - \mathrm{tr}(W)^2 - \mathrm{tr}_{g_t}(\ddot{g}_t),
\end{array}
$$

where $X, Y, U, V \in T_pM$, $p \in M$.

The first application of this construction concerns the fundamental theorem for hypersurfaces. If M is a hypersurface of a semi-Riemannian model space $\mathbb{M}^{r+1,s}_\kappa$ of signature $(r+1,s)$ and constant sectional curvature $K \equiv \kappa$ with Weingarten map A, then the Codazzi-Mainardi and Gauss equations must hold:

$$
\begin{array}{llcl}
(9) & (\nabla^M_X A)Y & = & (\nabla^M_Y A)X, \\
& R^M(X,Y)Z & = & \langle A(Y), Z\rangle A(X) - \langle A(X), Z\rangle A(Y) \\
(10) & & & +\kappa(\langle Y,Z\rangle X - \langle X,Z\rangle Y)
\end{array}
$$

for all $X, Y, Z \in T_pM$, $p \in M$.

The fundamental theorem asserts that these conditions are also sufficient in the simply connected case.

Theorem 1. Let (M^n, g) be a simply connected semi-Riemannian manifold of signature (r,s), let $\kappa \in \mathbb{R}$ and let A be a field of symmetric endomorphisms of TM satisfying the two equations (9) and (10) above.

Then M can be isometrically immersed as a semi-Riemannian hypersurface into the model space $\mathbb{M}^{r+1,s}_\kappa$ with Weingarten map A. Any two such immersions differ by an isometry of $\mathbb{M}^{r+1,s}_\kappa$.

For the proof one defines a family of metrics on M by

$$g_t(X,Y) := g((\mathfrak{c}_\kappa(t)\,\mathrm{id} - \mathfrak{s}_\kappa(t)A)^2X, Y).$$

Here $\mathfrak{s}_\kappa, \mathfrak{c}_\kappa : \mathbb{R} \to \mathbb{R}$ denote the generalized sine and cosine functions satisfying $\mathfrak{s}_\kappa(0) = 0$, $\mathfrak{c}_\kappa(0) = 1$, $\kappa\mathfrak{s}_\kappa^2 + \mathfrak{c}_\kappa^2 = 1$, $\mathfrak{s}'_\kappa = \mathfrak{c}_\kappa$, and $\mathfrak{c}'_\kappa = -\kappa\mathfrak{s}_\kappa$. Then one checks that $M_0 \subset \mathcal{Z}$ has Weingarten map $W = A$ and that the generalized cylinder $\mathcal{Z}$ has constant curvature κ. Hence $\mathcal{Z}$ is locally isometric to $\mathbb{M}_\kappa^{r+1,s}$ and we have constructed local embeddings. The global statement follows using a standard continuation procedure.

The second application concerns the identification of the spinor bundles with respect to different metrics and the variation formula for the Dirac operator. Let g_t be a smooth 1-parameter family of semi-Riemannian metrics, $t \in [0, 1]$. We want to relate spinors w. r. t. g_0 to those w. r. t. g_1. We show that the identification given in [3] coincides with parallel transport on the generalized cylinders along the t-lines. The variational formula in [3] then follows directly from a simple commutator formula.

The third application helps to better understand generalized Killing spinors.

Theorem 2. Let (M^n, g) be a semi-Riemannian spin manifold and let A be a field of symmetric endomorphisms of TM satisfying equation (9) on M. Let ψ be a spinor on (M^n, g) satisfying for all $X \in TM$

$$\nabla_X^{\Sigma M}\psi = \frac{1}{2}A(X)\cdot\psi. \tag{11}$$

Then the generalized cylinder $\mathcal{Z} = I\times M$ with the metric dt^2+g_t, where $g_t(X, Y) = g((\mathrm{id} - tA)^2X, Y)$, and with the spin structure inducing the given one on $\{0\} \times M$ by restriction has a parallel spinor, whose restriction to the leaf $\{0\} \times M$ is just ψ.

In [1] this was used for $A = \lambda\cdot\mathrm{id}$ to classify the geometries admitting (classical) Killing spinors. The case that A is parallel was treated in [5]. Generalized Killing spinors are also closely related to T-Killing spinors studied in [4].

References

[1] C. Bär, *Real Killing spinors and holonomy*, Commun. Math. Phys. **154** (1993), 509–521.

[2] C. Bär, Paul Gauduchon and A. Moroianu, *Generalized Cylinders in semi-Riemannian and spin geometry*, to appear in Mathem. Zeitschrift

[3] J.-P. Bourguignon and P. Gauduchon, *Spineurs, opérateurs de Dirac et variations de métriques*, Commun. Math. Phys. **144** (1992), 581–599.

[4] T. Friedrich and E. C. Kim, *Some remarks on the Hijazi inequality and generalizations of the Killing equation for spinors*, J. Geom. Phys. **37** (2001), 1–14.

[5] B. Morel, *The energy-momentum tensor as a second fundamental form*, Preprint, 2003, `math.DG/0302205`.

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