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Arbeitsgemeinschaft mit aktuellem Thema: Polylogarithms

Organised by
Spencer Bloch (Chicago)
Guido Kings (Regensburg)
Jörg Wildeshaus (Villetaneuse)

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Introduction by the Organisers

The k-th polylogarithm function is defined on $|z| < 1$ by

$$Li_k(z) = \sum_{n \geq 1} \frac{z^n}{n^k}.$$

In the past 25 years or so, polylogarithms have appeared in many different areas of Mathematics. The following list is taken, for the most part, from [Oe]: volumes of polytopes in spherical and hyperbolic geometry, volumes of hyperbolic manifolds of dimension 3, combinatorial description of characteristic classes, special values of zeta functions, geometry of configurations of points in $\mathbb{P}^1$, cohomology of $GL_n(\mathbb{C})$, calculation of Green's functions associated to perturbation expansions in quantum field theory, Chen iterated integrals, regulators in algebraic K-theory, differential equations with nilpotent monodromy, Hilbert's problem on cutting and pasting, nilpotent completion of $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$, Bethe's Ansatz in thermodynamics, and combinatorial problems in quantum field theory.

Of course, these problems are not all unrelated. One common thread is that values of polylogarithms appear naturally as periods of certain mixed Hodge structures associated to mixed Tate motives over cyclotomic fields. How these periods are related to special values of L-functions is a part of the Beilinson conjectures, which were discussed in a previous Arbeitsgemeinschaft. Since that time, the general picture has clarified. A number of talks are devoted to aspects of this more general philosophy (talk 2-5, 9, 10 13-17). The p-adic aspects of the theory have been studied and will be explained in the eleventh talk. In addition, a vast generalization, multiple polylogarithms of the form

$$Li_{s_1,\ldots,s_k} := \sum_{n_1 > \ldots > n_k \geq 1} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1^{s_1} \cdots n_k^{s_k}},$$

have come to play a role as presented in the fourth and twelfth talk.

The connection between volumes of hyperbolic manifolds and polylogarithms is described in the sixth talk. The seventh talk introduces higher torsion and explains why polylogarithms occur in that setting.

Polylogarithms play a role in physics. The eighth talk explains how zeta values, polylogarithms, and multiple polylogarithms appear in calculations of perturbative expansions in quantum field theory.

The final talks 13-17 return to the relationship with special values of L-functions. The thirteenth and fourteenth talk are devoted to the Zagier conjectures for number fields. The fifteenth talk concerns the elliptic polylogarithm sheaves and Zagier's conjecture for elliptic motives. Finally, talk 16 and 17 describe how polylogarithms relate to Euler systems and to the Bloch-Kato conjectures on special values of L-functions.

The *Arbeitsgemeinschaft* was organized by Spencer Bloch (University of Chicago), Guido Kings (Universität Regensburg) and Jörg Wildeshaus (Université Paris 13). It was held October 3rd – October 9th, 2004 with 46 participants.

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Abstracts

Function theory of higher logarithms

Elisenda Feliu

In this first talk we define the higher logarithms Li_k and prove some of their analytic properties. All the results are taken from [Oe].

The functions $Li_k(z)$. For every integer $k \geq 1$, we consider the following power series with disc of convergence $|z| < 1$:

$$Li_k(z) = \sum_{n \geq 1} \frac{z^n}{n^k}.$$

Since for $k = 1$ and $|z| < 1$, the series $Li_1(z)$ is just the development in a neighbourhood of 0 of the function $-\log(1-z)$, $Li_1(z)$ admits an analytic continuation on $\mathbb{C} \setminus [1, +\infty)$. We call this function *the principal branch of Li_1.*

Observe that for every $k \geq 2$ and $|z| < 1$, the following *derivative relation* is satisfied:

$$Li'_k(z) = \frac{1}{z} Li_{k-1}(z).$$

We deduce by recurrence, and using the fact that $\mathbb{C} \setminus [1, +\infty)$ is a simply connected domain, that all $Li_k(z)$ can be analytically continued to a holomorphic function on $\mathbb{C} \setminus [1, +\infty)$. For every k, this holomorphic function is called *the principal branch of the k-logarithm.*

Monodromy. These functions are in fact *multi-valued functions* on $\mathbb{C} \setminus \{0, 1\}$: consider (X, x_0) the universal cover of $(\mathbb{C} \setminus \{0, 1\}, \frac{1}{2})$. As a set, this is:

$$X = \left\{ \text{homotopy classes of paths in } \mathbb{C} \setminus \{0, 1\} \text{ starting at } \frac{1}{2} \right\}.$$

There is an action of $G := \pi_1(\mathbb{C} \setminus \{0, 1\}, \frac{1}{2})$ on X given by $g \cdot [c] = [\alpha \cdot c]$, for $g = [\alpha] \in G,\ [c] \in X$.

By a multi-valued function on $\mathbb{C} \setminus \{0, 1\}$ one means a holomorphic function on X. Since $-\log(1-z) = \int_\gamma \frac{du}{1-u}$ for γ any path starting at 0 and with end point z, $Li_1(z)$ is a multi-valued function on $\mathbb{C} \setminus \{0, 1\}$. Again by the derivative relation, for all k, $Li_k(z)$ is a multi-valued function on the same domain.

For $[c] \in X$ with c a path ending at z, one writes $Li_k^{[c]}(z) := Li_k([c])$.

To study the action of G on the values $Li_k^{[c]}(z)$, it suffices to know the action of the two generators of G, $c_0(t) := \frac{1}{2} e^{2\pi i t}$ and $c_1(t) := 1 + \frac{1}{2} e^{2\pi i t}$.

Theorem 1 (Monodromy relations)**.** *Let $z \in \mathbb{C} \setminus \{0, 1\}$ and c any path from $\frac{1}{2}$ to z in $\mathbb{C} \setminus \{0, 1\}$. Then, for all $k \geq 1$,*

(a) $Li_k^{[c_0 c]}(z) = Li_k^{[c]}(z), \quad Li_k^{[c_1 c]}(z) = Li_k^{[c]}(z) - 2\pi i \frac{(\log^{[c]}(z))^{k-1}}{(k-1)!}.$

(b) $\log^{[c_0 c]}(z) = \log^{[c]}(z) + 2\pi i, \quad \log^{[c_1 c]}(z) = \log^{[c]}(z).$

One can express these monodromy relations using some matrices. We introduce, for every integer $n \geq 0$, the $(n+1) \times (n+1)$ matrix (indexed from 0 to n),

$$L_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ -Li_1 & 1 & 0 & \cdots & \cdots & 0 \\ -Li_2 & \log & 1 & \cdots & \cdots & 0 \\ -Li_3 & \frac{\log^2}{2} & \log & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ -Li_n & \frac{\log^{n-1}}{(n-1)!} & \cdots & \frac{\log^2}{2} & \log & 1 \end{pmatrix},$$

and we define $A_n = L_n \cdot \tau(2\pi i)$, where $\tau(\lambda) = diag(1, \lambda, \ldots, \lambda^n)$.

We write $L_n^{[c]}(z)$ and $A_n^{[c]}(z)$ for the value of these matrices at z relatively to a path c with $c(1) = z$.

Consider the morphism ρ_n from G to $GL_{n+1}(\mathbb{Q})$ given by

$$\begin{aligned} \rho_n([c_0]) &= \exp(e_0), \\ \rho_n([c_1]) &= \exp(e_1), \end{aligned}$$

$$e_0 = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Then, the monodromy relation is expressed by a matrix equality as:

$$A_n^{g[c]}(z) = A_n^{[c]}(z)\rho_n(g), \quad g \in G.$$

The functions P_k. We define some real analytic functions

$$P_k : \mathbb{C} \setminus \{0, 1\} \to i^{k-1}\mathbb{R},$$

associated to the multi-valued functions $Li_k(z)$. They are introduced here as entries of some matrices $\log T_n(z)$.

Consider for any $z \in \mathbb{C} \setminus \{0, 1\}$, the matrix

$$T_n(z) = A_n^{[c]}(z)\overline{A_n^{[c]}(z)^{-1}}\tau(-1),$$

where c is any path ending at z (since the image of ρ is real, T_n does not depend on this choice). By the definitions above, $T_n(z)$ is a unipotent lower triangular matrix and hence, it is the exponential of a nilpotent lower triangular matrix $\log T_n(z)$. Moreover, since $\overline{T_n(z)} = \tau(-1)T_n(z)^{-1}\tau(-1)$, $\overline{\log T_n(z)} = -\tau(-1) \log T_n(z)\tau(-1)$ and therefore the entry (i, j) of the matrix $\log T_n(z)$ is real if $i - j$ is odd and purely imaginary if $i - j$ is even.

After some computations, we obtain that

$$\log T_n(z) = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 \\ -2P_1(z) & 0 & \cdots & \cdots & \cdots & 0 \\ -2P_2(z) & \log z\bar{z} & \ddots & & & \vdots \\ -2P_3(z) & 0 & \log z\bar{z} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ -2P_n(z) & 0 & \cdots & 0 & \log z\bar{z} & 0 \end{pmatrix}$$

with

$$P_k(z) = \begin{cases} \sum_{0\leq l<k} \frac{b_l}{l!} \log^l(z\bar{z}) \operatorname{Re}(Li^{[c]}_{k-l}(z)) & k \text{ odd}, \\ \sum_{0\leq l<k} \frac{b_l}{l!} \log^l(z\bar{z}) \operatorname{Im}(Li^{[c]}_{k-l}(z)) & k \text{ even}, \end{cases}$$

for c any path from $\frac{1}{2}$ to z in $\mathbb{C} \setminus \{0, 1\}$.

Observe that $P_2(z) = iD(z)$, where $D(z)$ is the *Bloch-Wigner dilogarithm.*

If we define $P_k(0) = P_k(\infty) = 0$ and $P_k(1) = \zeta(k)$, for k odd, and $P_k(1) = 0$ otherwise, each P_k is a continuous function on $\mathbb{P}_1(\mathbb{C})$.

Functional equations. Restricting to the case $k = 2$, for every $x, y \in \mathbb{P}_1(\mathbb{C})$, $(x, y) \neq (0, 0), (1, 1), (\infty, \infty)$, the following equation is satisfied:

$$P_2\left(\frac{1-y^{-1}}{1-x^{-1}}\right) - P_2\left(\frac{1-y}{1-x}\right) + P_2\left(\frac{y}{x}\right) - P_2(y) + P_2(x) = 0.$$

If we define, for any four different points a, b, c, d in $\mathbb{P}_1(\mathbb{C})$,

$$\tilde{P}_2(a, b, c, d) := P_2(r(a, b, c, d)),$$

($r(\cdot)$ being the cross-ratio), the last equation can be rewritten as

$$\sum_{i=1}^{5} (-1)^i \tilde{P}_2(a_1, \ldots, \hat{a}_i, \ldots, a_5) = 0,$$

for any $a_1, \ldots, a_5$ distinct points in $\mathbb{P}_1(\mathbb{C})$. Moreover this relation characterizes $D(z)$ (up to a scalar):

Theorem 2 (Bloch)**.** *The set of measurable functions* $f : \mathbb{P}_1(\mathbb{C}) \times \overset{4}{\cdots} \times \mathbb{P}_1(\mathbb{C}) \to \mathbb{R}$ *which are invariant under the diagonal action of* $PGL_2(\mathbb{C})$ *and satisfy the cocycle condition*

$$\sum_{i=1}^{5} (-1)^i f(a_1, \ldots, \hat{a}_i, \ldots, a_5) = 0,$$

forms a one dimensional $\mathbb{R}$*-vector space generated by* $D(r(a_1, \ldots, a_4))$*.*

(See [Bl], theorem 7.4.4).

The Bloch-Wigner Dilogarithm

Stefan Kühnlein

0. History and statement of the desired result

For an imaginary quadratic number field F with ring of integers $\mathcal{O}_F$, the group $\mathrm{SL}_2(\mathcal{O}_F)$ acts on hyperbolic 3-space properly discontinuously, and the covolume of this action is known to be

$$\mathrm{cov}(\mathrm{SL}_2(\mathcal{O}_F)) = \frac{|d_F|^{3/2}}{(2\pi)^2} \cdot \zeta_F(2),$$

where ζ_F is the Dedekind zeta-function of F.
There are several approaches to this formula (cf. [EGM]):
• compute the covolume after some change of variables by counting binary hermitean forms over $\mathcal{O}_F$ of bounded discriminant and using Dirichlet's expanding domains principle.
• use Eisenstein series for $\mathrm{SL}_2(\mathcal{O}_F)$.
• use the theory of Tamagawa numbers.

Hyperbolic volumes are related to higher polylogarithms (cf. talk number 6); in our case the values of the dilogarithm should play a rôle. We will use the Bloch-Wigner-dilogarithm D which is a real analytic function on $\mathbb{C} \setminus \{0,1\}$, defined on a neighbourhood of $\frac{1}{2}$ by

$$D(z) = \Im(\mathrm{Li}_2(z)) + \log|z| \cdot \arg(1-z).$$

It satisfies the two equations

$$D(\overline{z}) = -D(z), \quad D(x) - D(y) + D(\frac{y}{x}) - D(\frac{1-y}{1-x}) + D(\frac{1-y^{-1}}{1-x^{-1}}) = 0.$$

The second (so-called 5-term-) relation can be motivated geometrically. The first implies that D vanishes on the real line (where it can be prolonged to be a continuous function).
Now let F be any number field with ring of integers $\mathcal{O}_F$, discriminant d_F, r_1 real and r_2 complex places. Choose non-real embeddings $\sigma_1, \ldots, \sigma_{r_2}$ from F to $\mathbb{C}$ (one from every pair of complex conjugated embeddings).
The following is a special case of a conjecture of Don Zagier, proven by Sasha Goncharov:
Theorem 0: *The value $\zeta_F(2)$ of the Dedekind zeta-function can be expressed as*

$$\zeta_F(2) = \pi^{2(r_1+r_2)}|d_F|^{-1/2} \cdot \det(A),$$

where A is an $r_2 \times r_2$-matrix with entries

$$a_{ij} = \sum_k q_{k,j} D(\sigma_i(f_{k,j}))$$

for some rational numbers $q_{k,j}$ and elements $f_{k,j} \in F$.

1. Preparation: the Borel-regulator
Let $\Gamma_N := \mathrm{SL}_N(\mathcal{O}_F) \subseteq G_N := \mathrm{SL}_N(F \otimes_{\mathbb{Q}} \mathbb{R}) = \mathrm{SL}_N(\mathbb{R})^{r_1} \times \mathrm{SL}_N(\mathbb{C})^{r_2}$.
We define the primitive cohomology $PH^j(\Gamma_N, \mathbb{C})$ to be the quotient of $H^j(\Gamma_N, \mathbb{C})$ by the subspace generated by products of elements of lower degree. This is dual to the primitive homology $PH_j(\Gamma_N, \mathbb{C})$, the orthogonal complement to the space generated by products of cohomology classes of lower degree. (Similar definition for other groups and other coefficient rings...)
Borel has shown that for fixed $j \in \mathbb{N}$ and large enough N we have

$$PH^j(\Gamma_N, \mathbb{R}) \cong PH^j(G_N, \mathbb{R}),$$

and this has (by explicit calculation) dimension 0, if j is even, and dimension d_m if $j = 2m-1$, where d_m is the order of vanishing of $\zeta_F(s)$ at $s = 1-m$.
Note, that $d_m = r_1 + r_2$, if $m \geq 3$ is odd and $d_m = r_2$, if $m \geq 2$ is even. This comes from the functional equation of the Dedekind zeta-function.
$PH^{2m-1}(\mathrm{SU}_N, \mathbb{Z} \cdot (2\pi\mathrm{i})^m)$ is free of rank one. Choose a generator e_m of it. Using the fact that the Cartan decompositions of $\mathfrak{sl}_N$ and $\mathfrak{su}_N$ are closely related by

$$\mathfrak{sl}_N = \mathfrak{so}_N \oplus \mathfrak{p}_N, \quad \mathfrak{su}_N = \mathfrak{so}_N \oplus \mathrm{i}\mathfrak{p}_N,$$

we find (using van Est's theorem) a canonical isomorphism

$$H^{2m-1}(\mathrm{SU}_N, \mathbb{R} \cdot (2\pi\mathrm{i})^m) \cong H^{2m-1}_{\mathrm{ct}}(\mathrm{SL}_N(\mathbb{C}), \mathbb{R} \cdot (2\pi\mathrm{i})^{m-1}).$$

On the right hand side we use continuous group cohomology; note the different power of $2\pi\mathrm{i}$ occuring there. This isomorphism sends e_m to an element

$$b_m \in H^{2m-1}_{\mathrm{ct}}(\mathrm{SL}_N(\mathbb{C}), \mathbb{R} \cdot (2\pi\mathrm{i})^{m-1}).$$

Let Σ be the set of all embeddings $F \longrightarrow \mathbb{C}$. Then we get a map

$$J_m : PH_{2m-1}(\Gamma_N, \mathbb{R}) \longrightarrow \mathrm{maps}(\Sigma, \mathbb{R} \cdot (2\pi\mathrm{i})^{m-1})$$

by sending $\kappa \in PH_{2m-1}(\Gamma_N, \mathbb{R})$ to the element

$$J_m(\kappa) = [\Sigma \ni \sigma \mapsto \langle b_m, \sigma_*(\kappa) \rangle],$$

where σ_* is the map which is induced on homology by the embedding of Γ_N into $\mathrm{SL}_N(\mathbb{C})$ via σ. J_m is injective with image the set of invariants under complex conjugation which acts on $\mathrm{maps}(\Sigma, \mathbb{R} \cdot (2\pi\mathrm{i})^{m-1})$ by $(\overline{f})(\sigma) := \overline{(f(\overline{\sigma}))}$ (this is the statement proving the dimension of the primitive homology space). Therefore we view J_m as an isomorphism

$$J_m : PH_{2m-1}(\Gamma_N, \mathbb{R}) \longrightarrow \mathrm{maps}(\Sigma, \mathbb{R} \cdot (2\pi\mathrm{i})^{m-1})_+ \cong (\mathbb{R} \cdot (2\pi\mathrm{i})^{m-1})^{d_m}.$$

Both sides have a natural $\mathbb{Q}$-structure, and one considers the regulator of J_m, which by definition is the determinant of the image of a $\mathbb{Q}$-basis of $PH_{2m-1}(\Gamma_N, \mathbb{Q})$ in terms of a $\mathbb{Q}$-basis of $(\mathbb{Q} \cdot (2\pi\mathrm{i})^{m-1})^{d_m}$.
The following theorem is due to Borel (cf. [Bo1] and [Bo2] – this is worth to be read anyway). Its proof relies heavily on the theory of Tamagawa numbers.

Theorem 1: *Up to a non-zero rational factor, the regulator of J_m is*

$$d_F^{1/2}(\pi \mathrm{i})^{-(r_1+2r_2)\cdot m}\zeta_F(m).$$

2. Towards Theorem 0

Using Theorem 1 for $m = 2$, one can deduce Theorem 0.

2a First approach

Hidden in our remarks above there is an isomorphism (for $N \geq 2$)

$$H^3_{\mathrm{ct}}(\mathrm{SL}_N(\mathbb{C}), \mathbb{R}) \cong H^3_{\mathrm{ct}}(\mathrm{SL}_2(\mathbb{C}), \mathbb{R}).$$

We now use the Bloch-Wigner dilogarithm in order to construct a cocycle in the class of b_2. To that end, let $a \in \mathbb{P}^1(\mathbb{Q})$. For most choices of 4 elements $g_0, \dots, g_3 \in \mathrm{SL}_2(\mathbb{C})$ the value

$$\kappa(g_0, \dots, g_3) := D(\text{cross ratio}(g_0a, g_1a, g_2a, g_3a))$$

is defined. When everything is defined, the (homogeneous) 3-cocycle condition

$$\sum_{j=0}^{4}(-1)^j\kappa(g_0, \dots, \hat{g}_j, \dots, g_4) = 0$$

holds. This comes from the 5-term-relation satisfied by D: use the GL_2-invariance of the cross ratio in order to replace $g_0a, \dots, g_4a$ by $0, 1, \infty, x, y$ respectively.
κ is measurable and bounded, and therefore (using a result of P. Blanc) defines a continuous cohomology class in $H^3_{\mathrm{ct}}(\mathrm{SL}_2(\mathbb{C}), \mathbb{R})$. One can check that $\mathrm{i}\kappa$ represents b_2, and on inserting it in the definition of J_2 and evaluating it on a $\mathbb{Q}$-basis of $PH_3(\Gamma_N, \mathbb{Q})$ gets Theorem 0 by comparison with Theorem 1.

2b Second approach

For every $\sigma \in \Sigma$, the map $\mathrm{i} \cdot (D \circ \sigma)$ defines a homomorphism on the free abelian group $\mathbb{Z}[F \setminus \{0, 1\}]$ with values in $\mathbb{R} \cdot (2\pi \mathrm{i})$. Inside the kernel of this map (due to the 5-term relation) sits the subgroup R generated by the expressions

$$[x] - [y] + [\frac{y}{x}] - [\frac{1-y}{1-x}] + [\frac{1-y^{-1}}{1-x^{-1}}], \quad x, y \in F \setminus \{0, 1\}.$$

Let $B_2(F)$ be the kernel of the (well defined!) map

$$\mathbb{Z}[F \setminus \{0, 1\}]/R \longrightarrow \bigwedge^2 F^\times, \quad [x] \mapsto x \wedge (1-x).$$

The image of $(\mathrm{i} \cdot (D \circ \sigma))_{\sigma \in \Sigma} : B_2(F) \longrightarrow \mathrm{maps}(\Sigma, \mathbb{R} \cdot (2\pi \mathrm{i}))$ sits inside the invariants under the action of complex conjugation mentioned above (due to the first relation satisfied by D). Bloch and Suslin construct an isomorphism $B_2(F) \cong K_3(F)$. Moreover, following Milnor, Moore and the localisation exact sequence for K-theory,

$$K_3(F) \otimes \mathbb{Q} \cong K_3(\mathcal{O}_F) \otimes \mathbb{Q} \cong PH_3(\Gamma_N, \mathbb{Q}).$$

It can be shown that this last isomorphism composed with J_2 coincides with the Bloch-Suslin map composed with $(\mathrm{i} \cdot (D \circ \sigma))_{\sigma \in \Sigma}$. Therefore the image of J_2 is spanned by elements of the form needed for a proof of Theorem 0.

Interpretation in terms of mixed variations of Hodge structure

Manuel Breuning

In this talk we first recall the definition and some properties of a (variation of) mixed Hodge structure. We then define the variation of mixed Hodge structure $pol^{(N)}$ on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ which encodes the function theory of the functions Li_k. The main reference for this talk is [BD2, §1].

Mixed Hodge structures. Let $\mathbb{F} = \mathbb{Q}$ or $\mathbb{R}$. Recall that for an integer k one defines an $\mathbb{F}$-Hodge structure of weight k to be a finite-dimensional $\mathbb{F}$-vector space with a finite decreasing filtration $F^\bullet$ of $V_\mathbb{C} = V \otimes_\mathbb{F} \mathbb{C}$ (the Hodge filtration) which satisfies $V_\mathbb{C} = F^p V_\mathbb{C} \oplus \overline{F^{k-p+1} V_\mathbb{C}}$ for all p. A mixed Hodge structure defined over $\mathbb{F}$ (abbreviated $\mathbb{F}$-MHS) consists of a finite-dimensional $\mathbb{F}$-vector space V together with a finite increasing filtration $W_\bullet$ of V (the weight filtration) and a finite decreasing filtration $F^\bullet$ of $V_\mathbb{C}$ (the Hodge filtration), such that for every $k \in \mathbb{Z}$ the weight graded quotient $\mathrm{Gr}_k^W V$ is an $\mathbb{F}$-Hodge structure of weight k with respect to the filtration induced by $F^\bullet$. A morphism of $\mathbb{F}$-MHSs $(V, W_\bullet, F^\bullet) \to (V', W'_\bullet, F'^\bullet)$ is an $\mathbb{F}$-linear map $V \to V'$ which is compatible with weight and Hodge filtrations. The category of $\mathbb{F}$-MHSs is abelian.

Tate objects. For $k \in \mathbb{Z}$ one defines the Tate object $\mathbb{F}(k)$ to be the $\mathbb{F}$-MHS given by $V = \mathbb{F} \cdot (2\pi i)^k \subset \mathbb{C} = V_\mathbb{C}$ with weight filtration $W_l V = 0$ for $l < -2k$ and $W_l V = V$ for $l \geq -2k$, and Hodge filtration $F^p V_\mathbb{C} = V_\mathbb{C}$ for $p \leq -k$ and $F^p V_\mathbb{C} = 0$ for $p > -k$. There exists a natural notion of tensor product of $\mathbb{F}$-MHSs, and if V is any $\mathbb{F}$-MHS we write $V(k)$ for $V \otimes \mathbb{F}(k)$. We also note that for $k \geq 1$ there exists an isomorphism

$$\mathbb{C}/(2\pi i)^k \mathbb{Q} \cong \mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(k))$$

where Ext^1 denotes the group of equivalence classes of 1-extensions in the category of $\mathbb{Q}$-MHSs.

Variations of mixed Hodge structure. Let X be a complex manifold. A variation of MHS defined over $\mathbb{F}$ (abbreviated $\mathbb{F}$-VMHS) on X consists of a local system V of $\mathbb{F}$-vector spaces on X, an increasing filtration $W_\bullet$ of V by local subsystems, and a decreasing filtration $F^\bullet$ of $V_\mathcal{O} = V \otimes_\mathbb{F} \mathcal{O}_X$ by holomorphic subbundles, such that under the natural connection $\nabla : V_\mathcal{O} \to \Omega_X^1 \otimes_{\mathcal{O}_X} V_\mathcal{O}$ one has $\nabla F^p V_\mathcal{O} \subseteq \Omega_X^1 \otimes_{\mathcal{O}_X} F^{p-1} V_\mathcal{O}$ for all p (Griffiths' transversality), and at each point $x \in X$ the fibre V_x is an $\mathbb{F}$-MHS with respect to the induced filtrations. A VMHS is called graded-polarizable if for each $k \in \mathbb{Z}$ there exists a 'nice' bilinear form on $\mathrm{Gr}_k^W V$, and admissible if it satisfies certain conditions 'on the boundary of X'. See [Ha3, §7] for more details in the case of curves.

Let S be a smooth variety over $\mathbb{C}$. For every integer k we consider $\mathbb{Q}(k)$ as a (constant) $\mathbb{Q}$-VMHS on the complex manifold $S(\mathbb{C})$. There is a canonical map

$$\mathcal{O}^*(S) \to \mathrm{Ext}^1_{S(\mathbb{C})}(\mathbb{Q}(0), \mathbb{Q}(1)), \quad f \mapsto [f],$$

where $\mathrm{Ext}^1_{S(\mathbb{C})}$ denotes equivalence classes of 1-extensions in the category of $\mathbb{Q}$-VMHSs on $S(\mathbb{C})$. The $\mathbb{Q}$-VMHS $[f]$ is graded-polarizable and admissible.

The variation $pol^{(N)}$. Let N be a positive integer. Recall that (using the functions Li_k) a matrix A_N of multi-valued functions on $\mathbb{C}\setminus\{0,1\}$ was defined in the first talk. The rows and columns of this matrix are indexed by the set $[0,N]=\{0,1,\dots,N\}$. Using the matrix A_N we now define a $\mathbb{Q}$-VMHS $pol^{(N)}$ on $\mathbb{P}^1\setminus\{0,1,\infty\}$ as follows. The holomorphic vector bundle is $V_{\mathcal{O}}=\mathcal{O}^{[0,N]}$ and the local system is $V=A_N(z)\mathbb{Q}^{[0,N]}\subset V_{\mathcal{O}}$ (this is well-defined by the monodromy properties of $A_N(z)$). The weight filtration is given by $W_lV=0$ for $l<-2N$, $W_{-2l}V=W_{-2l+1}V=A_N(z)\mathbb{Q}^{[l,N]}$ for $0<l\le N$, and $W_lV=V$ for $l\ge 0$. The Hodge filtration is $F^pV_{\mathcal{O}}=V_{\mathcal{O}}$ for $p\le -N$, $F^{-p}V_{\mathcal{O}}=\mathcal{O}^{[0,p]}$ for $0\le p<N$, and $F^pV_{\mathcal{O}}=0$ for $p>0$. It is easy to check that for each $x\in\mathbb{P}^1\setminus\{0,1,\infty\}$ the fibre V_x is a $\mathbb{Q}$-MHS. From the equation $\frac{d}{dz}A_N(z)=(\frac{e_0}{z}+\frac{e_1}{z-1})A_N(z)$ where e_0 and e_1 are the matrices from the first talk one deduces that the natural connection $\nabla: V_{\mathcal{O}}\to\Omega^1\otimes V_{\mathcal{O}}$ is given by $\nabla f=df-(\frac{dz}{z}e_0+\frac{dz}{z-1}e_1)f$. This implies $\nabla F^pV_{\mathcal{O}}\subseteq\Omega^1\otimes F^{p-1}V_{\mathcal{O}}$. The $\mathbb{Q}$-VMHS $pol^{(N)}$ is graded-polarizable and admissible (see [Ha3, Theorem 7.1]).

Properties of $pol^{(N)}$. If $N\le M$ then the projection $\mathcal{O}^{[0,M]}\to\mathcal{O}^{[0,N]}$ induces a morphism $pol^{(M)}\to pol^{(N)}$. The projective system $pol=(pol^{(N)})_N$ is called the 'classical polylogarithm'. We note the following properties of $pol^{(N)}$.

(1) $pol^{(N)}=pol^{(M)}/W_{-2N-2}pol^{(M)}$ for $N\le M$.
(2) $\mathrm{Gr}^W_{-2k}pol^{(N)}=\mathbb{Q}(k)$ for $0\le k\le N$.
(3) $pol^{(1)}=[1-z]$ as extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$.
(4) There is a canonical isomorphism $W_{-2}pol^{(N)}\cong\mathrm{Sym}^{N-1}([z])(1)$. The map $\mathrm{Gr}^WW_{-2}pol^{(N)}\xrightarrow{\cong}\bigoplus_{k=1}^N\mathbb{Q}(k)$ induced by (2) corresponds to

$$\begin{aligned}&\mathrm{Gr}^W\mathrm{Sym}^{N-1}([z])(1)=\mathrm{Sym}^{N-1}\mathrm{Gr}^W([z])(1)\\&\quad=\mathrm{Sym}^{N-1}\big(\mathbb{Q}(0)\oplus\mathbb{Q}(1)\big)(1)=\bigoplus_{k=1}^N\mathbb{Q}(k)\xrightarrow{(N-k)!\text{ on }\mathbb{Q}(k)}\bigoplus_{k=1}^N\mathbb{Q}(k).\end{aligned}$$

The transition maps of the projective system $W_{-2}pol^{(N)}$ correspond to the derivation of degree -1 of $\mathrm{Sym}^*([z])$ which on $\mathrm{Sym}^1([z])$ is the projection $\mathrm{Sym}^1([z])=[z]\to\mathbb{Q}(0)=\mathrm{Sym}^0([z])$.

The functions P_k. The single-valued functions P_k introduced in talk 1 appear naturally in this context. Denote the diagonal matrix $\mathrm{diag}(1,\lambda,\dots,\lambda^N)$ by $\tau(\lambda)$, and the Lie algebra generated by the matrices e_0 and e_1 by $\langle e_0,e_1\rangle$. Passing from the $\mathbb{Q}$-VMHS $pol^{(N)}$ to an $\mathbb{R}$-VMHS gives the $\mathbb{R}$-local system $A_N(z)\mathbb{R}^{[0,N]}\subset\mathcal{O}^{[0,N]}$ which clearly depends only on the image of $A_N(z)$ in $\mathrm{GL}_{N+1}(\mathbb{C})/\mathrm{GL}_{N+1}(\mathbb{R})$. For this image there exists a canonical representative. Indeed, for every $z\in\mathbb{C}\setminus\{0,1\}$ there exists a unique matrix $M(z)\in\exp(\langle e_0,e_1\rangle)$ satisfying $M(z)=\tau(-1)\overline{M(z)}^{-1}\tau(-1)$ and $A_N(z)=M(z)\tau(2\pi i)$ in $\mathrm{GL}_{N+1}(\mathbb{C})/\mathrm{GL}_{N+1}(\mathbb{R})$. The first column of the matrix $-\log(M(z))$ is $(0,P_1(z),\dots,P_N(z))$.

Mixed Hodge structure on π_1

Gabriel Herz and Gereon Quick

We report on Hain's work [Ha1] in which he gives an elementary construction of a mixed Hodge structure on the completed fundamental group of a smooth quasi-projective algebraic $\mathbb{C}$-Variety using iterated integrals. A detailed explanation of iterated integrals and mixed Hodge structure is also given in [SP].
Given a smooth manifold M an iterated integral (of length s) is a linear combination of integrals of the form $\int \omega_1\omega_2\cdots\omega_r$ (with $r \leq s$) viewed as an $\mathbb{R}$-valued function on the set of piecewise smooth paths on M where each $\omega_j \in \mathcal{A}^1(M)$ is a smooth 1-form on M. We extend the function determined by $\mathcal{I}$ by linearity to 1-chains on M and denote the value of an iterated integral $\mathcal{I}$ evaluated at a 1-chain α on M by $< \mathcal{I}, \alpha > \in \mathbb{R}$. The starting point for the construction of the mixed Hodge structure is a theorem of Chen and the observation that there are obvious filtrations on the vector space of iterated integrals given by the length of integrals. Let J be the augmentation ideal of the group ring $\mathbb{Z}G$ with $G := \pi_1(M, x)$. In the proof Hain assumes G to be finitely generated. To relate iterated integrals to the completed fundamental group-ring $\widehat{\mathbb{Z}G}$ Hain gives an elementary proof of a special case of a theorem by Chen (the original statement and the proof are in [Ch1, thm. 2.1.1], but also compare to [Ch2, 2.6 cor. 1]). Hain proves the isomorphy

$$\phi : H^0(B_s(M), x) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G/J^{s+1}, \mathbb{R})$$

where the left hand side denotes the vector space of iterated integrals of length s which depend only on the homotopy class of a loop based at x. This theorem holds more generally if one replaces $B_s(M)$ by $B_s(\mathcal{A}')$, where $\mathcal{A}'$ is a quasi-isomorphic differential graded subalgebra of $\mathcal{A}$, and $B_s(\mathcal{A}')$ denotes the iterated integrals with differentials in $\mathcal{A}'$.
Hain gives a more elementary proof of this theorem using the relation of iterated integrals and smooth connections on a trivial bundle $V \times M$. Such a connection determines and is determined by a connection-1-form $\omega \in \mathcal{A}^1(M) \otimes \mathrm{End}(V)$. Via the well-known construction of parallel transport on a manifold along a connection one gets an $\mathbb{R}$-valued transport function T on piecewise smooth paths on M. Via basic facts on differential equations one can show that T is a converging series of iterated integrals

$$T = \mathrm{Id} + \int \omega + \int \omega\omega + \int \omega\omega\omega + \dots$$

If the connection 1-form ω lies in a nilpotent subalgebra of $\mathcal{A}^1(M) \otimes \mathrm{End}(V)$ this function is in fact an iterated integral, since the series stops. This construction is the crucial idea for Hain's proof of the surjectivity assertion in Chen's theorem.
One constructs the flat vector bundle $E := V \times_G \tilde{M}$, where $V := \mathbb{R}G/J^{s+1}$, $\tilde{M}$ is the universal covering space of M and G acts on V by right multiplication. E has a canonical flat connection and is trivial as a smooth bundle. The induced connection form ω on the trivial bundle satisfies $\omega^{s+1} = 0$. It follows that its associated transport functional T is an element in $V \otimes H^0(B_s(M), x)$ and for an

element $\gamma \in G$ the endomorphism $T(\gamma)$ agrees with right multiplication by γ on the algebra V. So one has the map

$$\hat{V} \xrightarrow{T} H^0(B_s(M), x) \xrightarrow{\phi} \hat{V}$$

whose dual is the identity. So ϕ is surjective. The injectivity is clear.
In order to put a mixed Hodge structure on the truncated fundamental group, Hain uses Chen's Theorem, puts filtrations on a certain space of iterated integrals and proves that these give rise to a mixed Hodge structure on the truncated π_1.
Given a smooth, quasi-projective algebraic $\mathbb{C}$-variety X, we write $G := \pi_1(X, x)$ for its fundamental group. Then Hain proves that

(1) $\mathbb{Z}G/J^{s+1}$ has a mixed Hodge structure for all $s \geq 1$,
(2) the mixed Hodge structure is natural with respect to pointed morphisms of varieties and
(3) the pointed map $\mathbb{Z}G/J^{s+1} \to \mathbb{Z}G/J^{t+1}, (t \leq s)$ is a morphism of mixed Hodge structures.

The statement is true without assuming X smooth or quasi-projective (cf. [Ha2, thm. 6.3.1]). The more elementary proof in [Ha1] proceeds as follows: Because of X being smooth and quasi-projective it exists a smooth and projective completion $\bar{X}$ of X. Let $D := \bar{X} - X$. D is a divisor with normal crossings. Let $\mathcal{A} < D >$ be the differential graded algebra of smooth forms with logarithmic singularities along D. It is quasi-isomorphic to $\mathcal{A}$.
The Hodge and weight filtrations on $\mathcal{A} < D >$ are:

$$F^p\mathcal{A} < D > = \{\text{forms with} \geq p \text{ many } dz\text{'s}\},$$

$$W_l\mathcal{A} < D > = \{\text{forms with} \leq l \text{ many } \tfrac{dz}{z}\text{'s}\}.$$

This gives filtrations on $B_s(\mathcal{A} < D >)$ as follows:

$$F^pB_s(\mathcal{A} < D >) := \left\langle \int \omega_1 \cdots \omega_r \in B_s(\mathcal{A} < D >) \right| \begin{array}{c} w_j \in F^{p_j}\mathcal{A} < D >, \\ \sum_{j=1}^s p_j \geq p \end{array} \left. \vphantom{\int} \right\rangle$$

and

$$W^lB_s(\mathcal{A} < D >) := \left\langle \int \omega_1 \cdots \omega_r \in B_s(\mathcal{A} < D >) \right| \begin{array}{c} w_j \in W_{l_j}\mathcal{A} < D >, \\ \sum_{j=1}^s l_j + r \leq l \end{array} \left. \vphantom{\int} \right\rangle.$$

This induces filtrations on $H^0(B_s(\mathcal{A} < D >), x)$ which we call HB_s for short. Now the aim is to prove that these filtrations define a mixed Hodge structure on HB_s. Hain proceeds by induction on s. For $s = 1$ exists an isomorphism $\mathbb{C} \oplus H^1(X, \mathbb{C}) \cong \mathrm{HB}_1$. So HB_1 carries a mixed Hodge structure. For the induction step one uses the short exact sequence

$$0 \longrightarrow \mathrm{HB}_{s-1} \longrightarrow \mathrm{HB}_s \xrightarrow{p} \mathrm{im}p \longrightarrow 0,$$

where $\mathrm{im}p \subset H^1(X, \mathbb{C})^{\otimes s}$, and

$$p : H^0(B_s(X), x) \longrightarrow H^1(X, \mathbb{C})^{\otimes s}$$

maps $\int \omega_1 \cdots \omega_s$ to the function

$$\otimes_{i=1}^s H_1(X,\mathbb{C}) \to \mathbb{C},\ \otimes_{i=1}^s a_i \mapsto \left\langle \int \omega_1 \cdots \omega_s, \prod_{j=1}^s (a_j - 1) \right\rangle.$$

By the aid of Eilenberg-Moore spectral sequence one can prove, that im$p$ is defined over $\mathbb{Q}$ and that it is a sub-mixed Hodge structure on $H^1(X,x)^{\otimes s}$ (for details see [Ha2]). Then the middle term HB_s carries a mixed Hodge structure by [GS, 1.16]. In the case X projective or $W_1 H^1(X) = 0$ (as in the case of $\mathbb{P}^1 - \{0,1,\infty\}$) this step can also be done elementary. Since in this case the image of p is the kernel of the map

$$\sum_{i=1}^{s-1} c_i : H^1(X,\mathbb{C})^{\otimes s} \longrightarrow \sum_{i=1}^{s-1} H^1(X,\mathbb{C})^{\otimes i} \otimes H^2(X,\mathbb{C}) \otimes H^1(X,\mathbb{C})^{\otimes^{s-1-i}}$$

where

$$c_i(\otimes_{i=1}^s z_i) = \otimes_{j=1}^i z_j \otimes (z_i \wedge z_{i+1}) \otimes_{j=i+2}^s z_j.$$

Mixed structure on the fundamental group of the projective line minus three points

Mathias Lederer

The talk is divided into two parts. In the first part, we give an explicit description of the machinery developed in the previous talk, for the special case where $X = \mathbb{P}^1 - \{0,1,\infty\}$. In the second part we give an introduction to the motivic fundamental group as defined by Deligne in [De], in particular the Hodge and weight filtration on the de Rham realisation.

For the first part, set $X = \mathbb{P}^1 - \{0,1,\infty\}$, $\overline{X} = \mathbb{P}^1$. Fix some $x \in X$. We want to study the group ring $\mathbb{C}\pi_1(X,x)$, more precisely its completion

$$\mathbb{C}\pi_1(X,x)\widehat{\ } = \varprojlim \mathbb{C}\pi_1(X,x)/J^N\,,$$

where J is the augmentation ideal of $\mathbb{C}\pi_1(X,x)$. We first define the Hodge filtration on $\mathbb{C}\pi_1(X,x)\widehat{\ }$ following [Ha3] and then see how this fits into the picture of the previous talk. Set $\Omega^1(X) = H^0(\overline{X}, \Omega^1_{\overline{X}}(\log D))$, i.e. $\Omega^1(X) = \langle \omega_0, \omega_1 \rangle$, where $\omega_0 = \frac{dz}{z}$ and $\omega_1 = \frac{dz}{1-z}$. Consider the algebra $A = \oplus_{n\geq 0} \Omega^1(X)^{\otimes(-n)}$ and its ideal $I = \oplus_{n\geq 1} \Omega^1(X)^{\otimes(-n)}$. Set

$$A\widehat{\ } = \varprojlim A/I^N\,.$$

We will construct an isomorphism

$$\theta_x : \mathbb{C}\pi_1(X,x)\widehat{\ } \xrightarrow{\sim} A\widehat{\ }.$$

Let $\omega \in \Omega^1(X) \otimes \Omega^1(X)^*$ be the element corresponding to the identity on $\Omega^1(X)$. Then ω is integrable, i.e. $d\omega + \omega \wedge \omega = 0$, thus the iterated integral $1 + \int \omega + \int \omega\omega + \ldots$ depends only on the homotopy class of the closed path involved. Thus since $\int_{\alpha\beta} w_1 \ldots w_r = \sum_{j=0}^r \int_\alpha w_1 \ldots w_j \int_\beta w_{j+1} \ldots w_r$, the iterated integral defines a homomorphism from $\pi_1(X,x)$ to the group of units in $A\hat{}$. This extends to an algebra homomorphism $\mathbb{C}\pi_1(X,x) \to A\hat{}$, under which J is mapped to I. Hence this extends to a continuous algebra homomorphism θ_x as above. It remains to show that θ_x is an isomorphism. Clearly θ_x induces a homomorphism on the graded rings of $\mathbb{C}\pi_1(X,x)\hat{}$ and $A\hat{}$. The 0-th graded parts of these are both $\mathbb{C}$. The 1-st graded parts are J/J^2 on the left and I/I^2 on the right. We have $J/J^2 \xrightarrow{\sim} H_1(X)$ (homology is the abelianised fundamental group) and $H_1(X) \xrightarrow{\sim} I/I^2$ (since the latter is isomorphic to $\Omega^1(X)^*$). The composition of these two isomorphisms is indeed induced by θ_x. Now the 1-st graded parts of the graded rings generate the graded rings as $\mathbb{C}$-algebras. Therefrom follows that θ_x induces a surjection on the graded rings, hence also θ_x is surjective. For the injectivity, we use the observation that the graded ring of $A\hat{}$ is a (noncommutative) polynomial ring in 2 variables Y_0, Y_1, say. Take X_0, X_1 in $\mathbb{C}\pi_1(X,x)\hat{}$ mapping to Y_0, Y_1 under θ_x. Therefrom one constructs a mapping $g : A\hat{} \to \mathbb{C}\pi_1(X,x)\hat{}$ (take Y_0, Y_1 back to X_0, X_1). Then $\theta_x \circ g$ is clearly the identity, hence g is injective. The surjectivity of g is proved as the surjectivity of θ_x, thus $\theta_x = g^{-1}$.

A is graded with $A_1 = \Omega^1(X)^*$ of Hodge type $(-1,-1)$. Thus it is reasonable to define the Hodge and weight filtrations $F^{-p}A = \oplus_{n\le p} A_n$ and $W_{-m} = \oplus_{n\ge m/2} A_n$ and transfer them to the truncated parts $\mathbb{C}\pi_1(X,x)/J^N$ of $\mathbb{C}\pi_1(X,x)\hat{}$ via the isomorphism $\theta_x : \mathbb{C}\pi_1(X,x)/J^N \xrightarrow{\sim} A/I^N$. It follows that $\mathbb{C}\pi_1(X,x)\hat{} = \mathbb{C}\langle\langle X_0, X_1\rangle\rangle$, the noncommutative power series in 2 variables. The weight graduation on the truncated parts yields

$$\mathrm{Gr}^W_m(\mathbb{C}\pi_1(X,x)/J^N) = \begin{cases} J^r/J^{r+1} & \text{if } m = -2r, 0 \le r \le N \\ 0 & \text{else,} \end{cases}$$

and the Hodge filtration on these is

$$F^{-r}(J^r/J^{r+1}) = J^r/J^{r+1},\ F^{-r+1}(J^r/J^{r+1}) = 0 \text{ for } 0 \le r < N\,.$$

Compare this to the filtrations of the previous talk, which were defined on the space $H^0(B_{N-1}(X),x)$, which was defined to be the $\mathbb{C}$-span of all $\int w_1 \dots w_r$, where $r \le N-1$, every w_j is a $\mathcal{C}^\infty$-form, and the integral is homotopy invariant. By Chen (and is in fact easy to see in our case here), this is isomorphic to $(\mathbb{C}\pi_1(X,x)/J^N)^* = \oplus_{n\le -1}\Omega^1(X)^{\otimes n}$.

In the second part of the talk, let us consider a more general setting (as Deligne did in [De]): Let k be a number field, U in $\mathrm{Spec}(\mathcal{O}_k)$ open, and étale over $\mathrm{Spec}(\mathbb{Z})$, $\overline{X}_U$ proper and smooth over U, D_U a divisor on $\overline{X}_U$ with normal crossings, $X_U = \overline{X}_U - D_U$. Further, let $\overline{X}$, D, X_U be the general fibres of $\overline{X}_U$, D_U, X_U, and fix some $x \in X_U(U)$, i.e. $x \in X_U(k)$ with good reduction in U. We will assume throughout that $H^0(\overline{X}, \mathcal{O}_{\overline{X}}) = k$ and $H^1(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$.

We use the following notation: If Γ is a group and Z^{N+1} the $(N+1)$-st group in its descending central series, define $\Gamma^{(N)} = \Gamma/Z^{N+1}$ and $\Gamma^{[N]}$ to be $\Gamma^{(N)}$ modulo torsion. (We use the same notation also for Lie algebras instead of groups.) The goal is to construct $\pi_1(X,x)^{(N)}_{\mathrm{mot}}$ for all N, then the motivic fundamental group is $\pi_1(X,x)_{\mathrm{mot}} = \varprojlim \pi_1(X,x)^{(N)}_{\mathrm{mot}}$. This is going to be a prounipotent group scheme in the tannakian category of realisations over U. Let us describe the various realisations (except for the crystalline realisation) of $\pi_1(X,x)^{(N)}_{\mathrm{mot}}$.

For the Betti realisation, fix an embedding $\sigma : k \hookrightarrow \mathbb{C}$. Then the Betti realisation of $\pi_1(X,x)^{(N)}_{\mathrm{mot}}$ is $\pi_1(X,x)^{(N)}_{\sigma} = \pi_1(X(\mathbb{C}),x)^{[N]\,\mathrm{alg\ un}}$. Here we use the *unipotent algebraic hull* of a group Γ, which is defined as follows: The functor "forget the Γ-action" from the category of nilpotent finite dimensional representations of Γ over $\mathbb{C}$ is a fibre functor, and its (tensor) automorphism group is $\Gamma^{\mathrm{alg\ un}}$. This is a prounipotent algebraic group.

For the ℓ adic realisation, let $\overline{k}$ be an algebraic closure of k. Compute the algebraic fundamental group $\hat{\pi}_1(X_{\overline{k}},x)$ (which is isomorphic to the profinite completion of $\pi_1(X(\mathbb{C}),x)$), consider $\hat{\pi}_1(X_{\overline{k}},x)^{[N]}$ (which is isomorphic to the profinite completion of $\pi_1(X(\mathbb{C}),x)^{[N]}$), and decompose this group into a product of ℓ-adic Lie groups, indexed by primes ℓ. The ℓ-adic realisation of $\pi_1(X,x)^{(N)}_{\mathrm{mot}}$ is $\hat{\pi}_1(X_{\overline{k}},x)^{[N]}_{\ell}$. The group $\mathrm{Gal}(\overline{k}|k)$ acts not only on the whole algebraic fundamental group but also on each $\hat{\pi}_1(X_{\overline{k}},x)^{[N]}_{\ell}$.

For the de Rham realisation, let $\mathcal{V}ec^{\mathrm{nil}}$ be the category of vector bundles with nilpotent connection on X, i.e. successive extensions of $(\mathcal{O},d)$. Every such bundle $\mathcal{V}$ has a canonical extension to $\overline{X}$, call it $\mathcal{V}_{\mathrm{can}}$. This process is compatible with $\otimes$, and by our hypothesis on H^1, $\mathcal{V}_{\mathrm{can}}$ is a trivial bundle. There is an equivalence of categories between the category of trivial vector bundles over $\overline{X}$ and the category Vec_k, given by $\mathcal{W} \mapsto H^0(\overline{X},\mathcal{W})$. Its inverse is $W \mapsto W \otimes \mathcal{O}_{\overline{X}}$. By composition with the above, we get a fibre functor

$$F_{DR} : \mathcal{V}ec^{\mathrm{nil}} \to Vec_k : \mathcal{V} \mapsto H^0(\overline{X},\mathcal{V}_{\mathrm{can}})\,.$$

Let $\pi_1(X)_{\mathrm{DR}}$ be its (tensor) automorphism group. The goal is to define filtrations thereon.

Let V be a k-vector space and $\nabla = d + \omega$ a connection on $V \otimes \mathcal{O}_{\overline{X}}$. Thus ω lives in $H^0(\overline{X},\Omega^1(\log D)) \otimes \mathrm{End}(V)$, and ∇ is integrable iff $d\omega + \frac{1}{2}[\omega,\omega] = 0$, thus iff $[\omega,\omega] = 0$, where $[\ ,\ ]$ is defined by $[\alpha\otimes u, \beta\otimes v] = \alpha\wedge\beta\otimes[u,v]$. The same formula defines $[\ ,\ ]^{\sim}$ on $\bigwedge^2 H^0(\overline{X},\Omega^1(\log D)) \otimes \mathrm{End}(V)$. Let K be the kernel of $\bigwedge^2 H^0(\overline{X},\Omega^1(\log D)) \to H^0(\overline{X},\Omega^2(\log D))$. Then ∇ is integrable iff $[\omega,\omega]^{\sim} \in K \otimes \mathrm{End}(V)$. Set $H = H^0(\overline{X},\Omega^1(\log D))^*$ (the dual), and let $\phi : H \to$ End(V) correspond to ω. Define $\mathrm{Lib}(H)$ to be the free Lie algebra on the vector space V. This is graded, with $\mathrm{Lib}(H)_1 = H$, $\mathrm{Lib}(H)_2 = \bigwedge^2 H$ etc. Let $K^{\perp} \subset \bigwedge^2 H$ be orthogonal to K with respect to $[\ ,\ ]^{\sim}$. ϕ corresponds to a Lie algebra homomorphism $\rho : \mathrm{Lib}(H) \to \mathrm{End}(V)$, and ∇ is integrable iff $\rho(K^{\perp}) = 0$. Hence the datum of ∇ is equivalent to the datum of an $L(H,K^{\perp}) = \mathrm{Lib}(H)/(K^{\perp})$-action on V. Applying this to $\mathcal{V} \in \mathcal{V}ec^{\mathrm{nil}}$ and the corresponding $V = H^0(\overline{X},\mathcal{V}_{\mathrm{can}})$, we conclude that F induces an equivalence of categories between $\mathcal{V}ec^{\mathrm{nil}}$ the category of nilpotent representations of $L(H,K^{\perp})$.

The algebras $\mathrm{Lib}(H)$ and $(K^{\perp})$ are graded, hence also $L(H, K^{\perp})$ is graded, and in the central series, we have $Z^{N+1} = \oplus_{n \geq N+1} L(H, K^{\perp})_n$. A nilpotent representation of $L(H, K^{\perp})$ factors through some $L(H, K^{\perp})^{(N)}$, which is a nilpotent Lie algebra. By integration, one gets a representation of the corresponding unipotent algebraic group. Therefrom follows: $\pi_1(X)_{\mathrm{DR}} = \varprojlim G^{(N)}$, where $G^{(N)}$ is the unique unipotent algebraic group with $\mathrm{Lie}(G^{(N)}) = L(H, K^{\perp})^{(N)}$. Let us define the Hodge and weight filtrations on the level of the Lie algebras: $F^{-n} L(H, K^{\perp})^{(N)}$ is the sum of all components of degree $\leq n$, and $W_{-2n} L(H, K^{\perp})^{(N)} = W_{-2n+1} L(H, K^{\perp})^{(N)}$ is the sum of all components of degree $\geq n$. This is the same as the $(n+1)$-st algebra in the central series of $L(H, K^{\perp})^{(N)}$. One defines the weight filtration in the same way also in the other realisations: Go over to the Lie algebra and take the $(n+1)$-st algebra in the central series.

Finally, we compare the connection between the first and the second part of the talk. $\mathbb{C}\pi_1(X, x)/J^N$ is a Hopf algebra with comultiplication $\Delta g = g \otimes g$ for all $g \in \pi_1(X, x)$. The Malcev Lie algebra $[\gamma, \delta] = \gamma\delta - \delta\gamma$ is the subset $\mathfrak{g}_N$ of $\mathbb{C}\pi_1(X, x)/J^N$ consisting of all elements such that $\Delta\gamma = \gamma \otimes 1 + 1 \otimes \gamma$. The multiplication on $\mathfrak{g}_N$ is given by $[\gamma, \delta] = \gamma\delta - \delta\gamma$; this makes $\mathfrak{g}_N$ indeed a nilpotent Lie algebra, hence we can integrate it and get a unipotent algebraic group G_N. It turns out that $\varprojlim G_N = \pi_1(X(\mathbb{C}), x)^{\text{alg un}}$. But mind that G_N and $G^{(N)}$ do not coincide, just the limits do.

Volumes of hyperbolic manifolds

Thilo Kuessner

The aim of this talk was to show that odd-dimensional hyperbolic manifolds M of finite volume give rise to nontrivial classes in the algebraic K-theory of number fields, by the following theorem of Goncharov, which is the main result of [Go5].

Theorem 1. *: To each compact hyperbolic manifold of odd dimension n, there exists an element*

$$\gamma(M) \in K_n\left(\overline{\mathbb{Q}}\right) \otimes \mathbb{Q}$$

such that the Borel regulator $r_n : K_n\left(\overline{\mathbb{Q}}\right) \otimes \mathbb{Q} \rightarrow \mathbb{R}$ maps

$$r_n(\gamma(M)) = vol(M).$$

There are two constructions of $\gamma(M)$, a homological and a Hodge-theoretic one.

We discussed in our talk the homological construction ([Go5], section 2), which uses the description of $K_n\left(\overline{\mathbb{Q}}\right) \otimes \mathbb{Q}$ as the subspace of indecomposable elements in $H_n\left(GL\left(\overline{\mathbb{Q}}\right); \mathbb{Q}\right)$ (see [Sr], p.29), and of the Borel regulator r_n as being defined by pairing with the Borel class $b_n \in H_c^n(GL(\mathbb{C}); \mathbb{Q})$. ($H_c^*$ denotes the continuous cohomology. It is known that $H_c^*(GL(N; \mathbb{C}); \mathbb{Q}) \cong H^*(U(N); \mathbb{Q})$ and b_{2m-1} is defined to be the pull-back of the fundamental class $\left[S^{2m-1}\right] \in H^{2m-1}\left(S^{2m-1}\right)$ under the canonical map $U(m) \rightarrow S^{2m-1}$.)

Each closed hyperbolic manifold $M = \Gamma \backslash \mathbb{H}^n$ gives rise, using an embedding $j : \Gamma \rightarrow SO\left(n, 1; \overline{\mathbb{Q}}\right)$, to a class $j_*[M] \in H_n\left(SO\left(n, 1; \overline{\mathbb{Q}}\right)\right)$. In the case $n = 3$

one can use the isomorphism $SO(n,1;\mathbb{R}) \cong PSL(2,\mathbb{C})$ to make $j_*[M]$ a class in the homology of GL and hence in K-theory. In general, however, the construction is more complicated: one uses the isomorphism $H_n\left(SO\left(n,1;\overline{\mathbb{Q}}\right)\right) \cong H_n\left(Spin\left(n,1;\overline{\mathbb{Q}}\right)\right)$ and constructs the homology class as $s_{+*}j_*[M]$, where $s_+ : Spin(n,1) \to GL(N,\mathbb{C})$ is the spin representation (for n odd and $N = 2^{\frac{n+1}{2}+1}$).

We now explain the proof of Theorem 1 in some detail.
Recall that a manifold M^n is called hyperbolic if it is diffeomorphic to $\Gamma\backslash\mathbb{H}^n$, for $\Gamma < SO(n,1)$ discrete and torsionfree. If $\Gamma\backslash\mathbb{H}^n$ has finite volume, then Γ can be conjugated to be a subgroup of $SO\left(n,1;\overline{\mathbb{Q}}\right)$ (Weil). There is a canonical identification $H_*(M) \cong H_*(\Gamma)$. In particular, if M^n is compact and orientable, then the fundamental class $[M] \in H_n(M)$ corresponds to a class in $H_n(\Gamma)$.

One can define a continuous n-cocycle on $Isom^+(\mathbb{H}^n) = SO(n,1)$ by sending each $(g_0,\ldots,g_n)$ to the volume of the geodesic simplex with vertices $g_0x,\ldots,g_nx$, for a fixed $x \in \mathbb{H}^n$. (The cocycle property follows from the additivity of volume.) Its cohomology class is called the Lobachevsky class $v_n \in H_c^n(SO(n,1))$. The proof of Theorem 1 will follow from Lemma 1 and Lemma 2.

Lemma 1. *: If M is a compact, orientable, hyperbolic n-manifold, then $< v_n, j_*[M] >= vol(M)$ for the inclusion $j : \Gamma \to SO(n,1)$.*

Proof: There exists a triangulation of M by geodesic simplices $\sigma_1,\ldots,\sigma_r$, such that all vertices of all σ_i are in $x_0 \in M$. We have that $[M]$ is represented by $\sigma_1+\ldots+\sigma_r$. Moreover, each σ_i has n+1 edges which are closed loops and thus can be considered as an n+1-tuple of elements of $\Gamma = \pi_1(M,x_0)$. Hence, we get an element $j_*\sigma_i$ in the bar resolutions of Γ resp. $SO(n,1)$. (According to Eilenberg-McLane, this map defines the isomorphism $H_n(M) \to H_n(\Gamma)$.) From the definition of the Lobachevsky class we have $< v_n, j_*(\sigma_i) >= vol(\sigma_i)$, since the geodesic simplex with vertices $g_0\tilde{x}_0, g_1\tilde{x}_0,\ldots,g_n\tilde{x}_0$, where $g_0,\ldots,g_n \in \pi_1(M,x_0)$ are represented by the edges of σ_i, is a lift of σ_i to $\widetilde{M} = \mathbb{H}^n$. Hence $< v_n, j_*([M]) >= vol(\sigma_1) + \ldots + vol(\sigma_r) = vol(M)$. □

Using the spin representation $s_+ : Spin(n,1) \to GL(N;\mathbb{C})$, one can relate the Lobachevsky class to the Borel class.

Lemma 2. *: $s_+^* b_n = c_n v_n \in H_c^n(SO(n,1))$ with $c_n \neq 0$.*

Now one defines $\gamma(M) := \frac{1}{c_n} s_{+*}j_*[M]$ and gets

$$r_n(\gamma(M)) = \frac{1}{c_n} < b_n, s_{+*}j_*[M] >= vol(M).$$

This proves Theorem 1. (For noncompact hyperbolic manifolds of finite volume, the proof becomes more involved because one has to work with the relative fundamental class.) □

The second (Hodge-theoretic) construction, which we did not have the time to discuss, uses a motivic interpretation of the scissors congruence groups. It associates to each simplex a mixed Hodge-Tate structure such that the volume of the simplex is the period of a certain framing. The Dehn-invariant, considered as a coproduct on the scissors congruence group, corresponds to the well-known coproduct $\tilde{\nu}$ on the Hopf algebra of all framed mixed Hodge-Tate structures. A hyperbolic manifold gives an element in the scissors congruence group, which is in the kernel of the Dehn-invariant. Hence the corresponding Hodge-Tate structure is in the kernel of $\tilde{\nu}$. But, according to Beilinson, $ker\left(\tilde{\nu}\right) \cong K_*\left(\overline{\mathbb{Q}}\right) \otimes \mathbb{Q}$ and the period (i.e. the volume) corresponds to the Borel regulator. ([Go5], section 3)

Finally, we explain how Theorem 1 connects hyperbolic volume to polylogarithms.

Zagier's conjecture states that $K_{2n-1}\left(F\right) \otimes \mathbb{Q}$ is isomorphic to the Bloch group $\mathcal{B}_n\left(F\right)$ and that, under this isomorphism, the Borel regulator $r_{2n-1} : K_{2n-1}\left(F\right) \otimes \mathbb{Q} \to \mathbb{R}$ corresponds to the single-valued polylogarithm $P_n : \mathcal{B}_n\left(F\right) \to \mathbb{R}$.

In view of Theorem 1, Zagier's conjecture would imply that volumes of odd-dimensional hyperbolic manifolds are related to polylogarithms as follows.

Conjecture 1. *: Let M^{2n-1} be a 2n-1-dimensional hyperbolic manifold of finite volume. Then*

$$vol\left(M\right) = \sum_i P_n\left(z_i\right)$$

for some $z_i \in \overline{\mathbb{Q}}$ with $\delta_n\left(\sum_i z_i\right) = 0 \in B_{n-1}\left(\overline{\mathbb{Q}}\right) \otimes \overline{\mathbb{Q}}^*$.

Conjecture 1 is well-known in the case of 3-manifolds. Indeed, if Δ is an ideal geodesic simplex in $\mathbb{H}^3$ with vertices $v_0, v_1, v_2, v_3 \in \partial_\infty \mathbb{H}^3 = \mathbb{C}P^1$, then $vol\left(\Delta\right) = P_2\left(r\left(v_0, v_1, v_2, v_3\right)\right)$, where r denotes the cross ratio. Assuming an ideal geodesic triangulation on M, this gives the first equality of conjecture 1. The second equality is true because $\delta_2\left(z_i\right)$ factors over the Dehn invariant D_3 of the geodesic simplex with vertices $\infty, 0, 1, z_i$. (Precisely, if we define a map $\lambda : \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z} \to \Lambda^2\mathbb{C}$ by $\lambda\left(r \otimes \theta\right) = e^r \wedge e^{i\theta}$, then $\delta_2 \sim \lambda D_3$.) If we have a geodesic triangulation of the closed manifold M, then the angles around any edge add up to 2π, hence the Dehn invariants add up to 0. This implies $\sum_i \delta_2\left(z_i\right) = 0$.

Conjecture 1 is also known in the case of 5-manifolds, which follows from the results in [Go3].

Analytic torsion

Christopher Deninger

The aim of the lecture was to explain the higher analytic torsion forms of Bismut and Lott [Bi-Lo] and to prove that for S^1-bundles these can be expressed in terms of higher polylogarithms.

1 Consider a smooth fibre bundle $\pi : M \to B$ with compact connected fibres $Z_b = \pi^{-1}(b)$. For simplicity assume that the vertical tangent bundle TZ is oriented. For a flat vector bundle F on M with a Hermitian metric h^F let $H^p(Z, F\,|_Z)$ be the flat vector bundle on B corresponding to $R^p\pi_* F$. The Riemann–Roch–Grothendieck theorem for smooth submersions is the following assertion:

Theorem 1 ([Bi-Lo])**.** *For every odd $k \geq 1$ we have an equality*

$$\sum_p (-1)^p c_k(H^p(Z, F\,|_Z)) = \int_Z e(TZ) c_k(F) \quad \textit{in } H^k(B, \mathbb{R}) \, .$$

Here the c_k are the characteristic classes of Kamber and Tondeur for flat bundles. They have a "Chern–Weil" type description as cohomology classes of certain closed forms $c_k(F, h^F)$ and $c_k(H^p(Z, F\,|_Z), h^p)$ where h^p is an induced Hermitian metric on $H^p(Z, F\,|_Z)$.

Now let us fix a connection on $M \xrightarrow{\pi} B$, i.e. a horizontal complement $T^H M$ in TM to TZ and a Riemannian metric g^{TZ} on TZ. They induce a connection ∇^{TZ} on TZ and the metrics h^p mentioned above.

Bismut and Lott construct $(k-1)$-forms $\mathcal{T}_{k-1} = \mathcal{T}_{k-1}(T^H M, g^{TZ}, h^F)$ in $\mathcal{A}^{k-1}(\mathcal{B})$, the higher torsion forms, and prove the following result which implies theorem 1:

Theorem 2 ([Bi-Lo])**.** *In $\mathcal{A}^k(B)$ we have*

$$d\mathcal{T}_{k-1} = \int_Z e(TZ, \nabla^{TZ}) c_k(F, h^F) - \sum_p (-1)^p c_k(H^p(Z, F\,|_Z), h^p) \, .$$

For $k = 1$ they also prove that $\mathcal{T}_0(b)$ is the Ray–Singer analytic torsion of the h^{Z_b}-Laplacian $\Delta^{\bullet}_{Z_b}$ corresponding to the de Rham complex of Z_b twisted by $F\,|_{Z_b}$

$$\mathcal{T}_0(b) = \frac{1}{2} \sum_p (-1)^p p \log \det{}' \Delta^p_{Z_b} \, .$$

This explains the name "higher analytic torsion forms" for the $\mathcal{T}_{k-1}$.

2 Let W be the infinite dimensional bundle over B whose sheaf of sections is $\pi_*(\Lambda T^* Z \otimes F)$, i.e. $W_b = \mathcal{A}(Z_b, F\,|_{Z_b})$. Using the natural isomorphism

$$\mathcal{A}(M, F) \cong \mathcal{A}(B, W)$$

the connection d_M on $\mathcal{A}(M, F)$ becomes a superconnection d_M on $\mathcal{A}(B, W)$ with respect to the total $\mathbb{Z}/2$-grading of $\mathcal{A}(B, W)$. It decomposes into three terms

$$d_M = d^Z + \nabla^W + i_T$$

according to the $\mathbb{Z}$-grading of $\mathcal{A}(B,W)$ coming from the one on $\Lambda^\bullet T^*Z$. Here $d_M^0 = d^Z$ is the exterior derivative along the fibres, $d_M^1 = \nabla^M$, for a vector field U on B is the Lie derivative with respect to the horizontal lift U^H of U and $i_T(U,U')$ is interior multiplication by $T(U,U') = -\mathrm{pr}^{TZ}[U^H, U'^H]$.

3 The metrics g^{TZ} and h^F give a (L^2-)metric h^W on W. For $t > 0$ consider the rescaled flat superconnection

$$C'_t = t^{N/2} d^M t^{-N/2}$$

where N is the number operator, i.e. $N = p$ in degree p. Set

$$D_t = \frac{1}{2}(C_t'^* - C'_t) \in \mathcal{A}^-(B, \mathrm{End}W)\ .$$

Then $-D_t^2$ equals the curvature C_t^2 of the superconnection $C_t = \frac{1}{2}(C_t'^* + C_t)$ on W. Set $f(z) = z\exp(z^2)$, so that $f'(z) = g(z^2)$ where $g(z) = (1+2z)\exp z$ and introduce the even real form

$$f^\wedge(C'_t, h^W) = \mathrm{Tr}_s\left(\frac{N}{2} g(D_t^2)\right) \quad \text{in } \mathcal{A}(B)\ .$$

Here $-D_t^2$ is a fibrewise elliptic operator. Hence $g(D_t^2)$ is a fibrewise trace class operator and the fibrewise supertrace Tr_s is defined. Under the simplifying conditions $H(Z, F\,|_Z) = 0$ and $\chi(Z) = 0$ the higher torsion form $\mathcal{T}$ corresponding to g is defined as the integral

$$\mathcal{T} = -\int_0^\infty f^\wedge(C'_t, h^W)\frac{dt}{t}\ .$$

Nontrivial asymptotic estimates for $t \to 0, \infty$ are required to show that the integral converges. In general, suitable correction terms have to be subtracted. For the components $\mathcal{T}_{k-1}$ of $\mathcal{T}$ Bismut and Lott then prove theorem 2.

4 Now let P be a principal $U(1)$-bundle over $\mathcal{B}$ with a connection and curvature form $\Omega \in \mathcal{A}^2(P, i\mathbb{R})^{\text{basic}} = \mathcal{A}^2(B, i\mathbb{R})$. Give S^1 the $U(1)$-action by $g \cdot z = g^r z$ for some r with $|r| > 1$ and set $M = P \times_{U(1),r} S^1$. Fix $1 \neq \zeta \in \mu_r$ and let F_{S^1} be the line bundle on S^1 corresponding to $\pi_1(S^1) = \mathbb{Z} \to \mathbb{C}^*, \nu \mapsto \zeta^\nu$. Then F_{S^1} lifts to a line bundle F on M and using Fourier theory one finds:

Theorem 3 ([Bi-Lo]). *In $\mathcal{A}(B)$ we have:*

$$\mathcal{T} = \frac{1}{2}\sum_{j=0}^{\infty} \frac{(2j+1)!}{2^{2j}(j!)^2}\left(\sum_{m\neq 0}\frac{\zeta^m}{m^j|m|}\right)\left(\frac{r\Omega}{2\pi}\right)^j\ .$$

Thus for j even, $\mathrm{Re}\,\mathrm{Li}_{j+1}(\zeta)$ appears and for j odd $\mathrm{Im}\,\mathrm{Li}_{j+1}(\zeta)$.

A higher version of Reidemeister torsion was introduced by Igusa and Klein, see e.g. [Ig] which yields the same formula (after passing to cohomology). A topological proof of a stronger version of theorem 1 was given by Dwyer, Weiss and Williams.

Physics

Dirk Kreimer

We introduced one-particle irreducible Feynman graphs as basic combinatorial objects which allow to define a pre-Lie algebra $(\mathcal{L}, \star)$ of graph insertions:

$$\Gamma_1 \star \Gamma_2 = \sum_{\Gamma} n(\Gamma_1, \Gamma_2, \Gamma)\Gamma$$

where $n(\Gamma_1, \Gamma_2, \Gamma)$ gives the number of ways of shrinking Γ_2 to a point in Γ such that Γ_1 is obtained.

The corresponding graded Lie algebra $(\mathcal{L}, [,])$ obtained by antisymmetrizing $\star$ to a bracket $[,]$ has a universal enveloping algebra $U(\mathcal{L})$. It hence allows to define dually a Hopf algebra of graph decompositions on one-particle irreducible graphs

$$\Delta(\Gamma) = \sum_{\gamma} \gamma \otimes \Gamma/\gamma.$$

The notion of powercounting was then introduced together with the Feynman rules $\phi : H \to V$,

$$\phi(\Gamma) = \int \prod_{\text{edges } e} \frac{d^D k_e}{\text{quadric}(k_e)} \prod_{\text{vertices } v} \delta\left(\sum_{e \text{ incident to } v} k_e \right).$$

It was shown that these Feynman rules suffer from short-distance singularities. Interpreting them as characters $H \to V$ into a suitable Rota–Baxter algebra (V, R) then makes renormalization self-evident. Defining the counterterm character

$$S_R^{\phi} = -R \circ m \circ [S_R^{\phi} \otimes (\phi \circ P)]\Delta$$

allows to define the algebraic Birkoff decomposition $\phi_- = S_R^{\phi}$,

$$\phi_+ = m \circ [\phi_- \otimes \phi]\Delta = [\text{id} - R] \circ m \circ [S_R^{\phi} \otimes (\phi \circ P)]\Delta,$$

using the projection P into the augmentation ideal. These algebraic structures are very similar to the ones observed in the polylog, see [Kr2].

Emphasis was given then to primitive elements γ in the Hopf algebra,

$$\Delta(\gamma) = \gamma \otimes e + e \otimes \gamma,$$

abundantly provided by superficially divergent graphs which have no divergent subgraph. Those are characterized by having a residue which determines their contribution to the renormalization group flow.

A summary of the known results [Kr1] was given emphasizing the empirical fact that these residues map a graph to multiple zeta values

$$\zeta(s_1, \ldots, s_k) = \sum_{n_1 < \ldots < n_k} \frac{1}{n_1^{s_1} \ldots n_k^{s_k}},$$

in a manner which assigns increasing transcendental weight to increasing non-planarity in the graph γ. One of the few proven facts here is the rationality of the contributions of the most planar graphs (of rainbow or ladder topology).

A discussion of the peculiarities of gauge theories in this context ended the talk, formulating the conjecture that the quenched β-function is a series in the fine-structure constant with rational coefficients.

Etale realization

Denis Vogel

In our talk we described the construction of Soulé elements, see [So2]. We fix an odd prime number p and a number field F. We denote its ring of integers by $\mathcal{O}_F$, and let $A = \mathcal{O}_F[1/p]$. Define F_n to be $F(\mu_{p^n})$ and A_n the integral closure of A in F_n. We fix a generator (ζ_{p^n}) of $\varprojlim \mu_{p^n}$ and let

$$\beta_n : \mu_{p^n} \to K_2(A_n, \mathbb{Z}/p^n\mathbb{Z})$$

denote the morphism given by the composition of

$$\mu_{p^n} \cong \pi_2(BA_n^*, \mathbb{Z}/p^n\mathbb{Z})$$

with

$$\pi_2(BA_n^*, \mathbb{Z}/p^n\mathbb{Z}) \to \pi_2(BGL(A_n)^+, \mathbb{Z}/p^n\mathbb{Z}) = K_2(A_n, \mathbb{Z}/p^n\mathbb{Z}).$$

Let

$$N_n : K_{2r-1}(A_n, \mathbb{Z}/p^n\mathbb{Z}) \to K_{2r-1}(A, \mathbb{Z}/p^n\mathbb{Z})$$

be the norm map of K-theory. Let $(\omega_n) \in \varprojlim A_n^*$, where the transition maps are induced by the usual norm map. The elements

$$N_n(\omega_n \beta_n(\zeta_{p^n})^{r-1}) \in K_{2r-1}(A, \mathbb{Z}/p^n\mathbb{Z})$$

are compatible and give rise to an element in $K_{2r-1}(A, \mathbb{Z}_p)$. Here we have interpreted ω_n as an element of $K_1(A_n) = A_n^*$ and made use of the product in K-theory with coefficients. A similar construction may be carried out in etale cohomology. Here we consider the element

$$(cores_{F_n/F}\omega_n \otimes (\zeta_{p^n})^{r-1})_n \text{ in } H^1(SpecA, \mathbb{Z}_p(r))$$

where ω_n is considered as an element of $H^1(SpecA_n, \mathbb{Z}/p^n\mathbb{Z}(1))$. Using the chern class map

$$K_{2r-1}(A, \mathbb{Z}_p) \to H^1(SpecA, \mathbb{Z}_p(r))$$

we may compare the above constructions, and it turns out that they coincide up to an integral factor. For $F = \mathbb{Q}(\zeta_m)$ where $m = qp^{n_0}$ with$(p, q) = 1$ the above elements with $\omega_n = 1 - \zeta_{qp^n}$ are called Soulé elements. A variant of these elements can be obtained by the Deligne torsor, see [De]: It is given by the compatible system of $\mathbb{Z}/p^n\mathbb{Z}(r)$-torsors P_{m,k,p^n} over $Spec\mathbb{Z}[\zeta_m][1/p]$ given at the fibers by

$$P_{m,k,p^n,\zeta} = \sum_{\alpha^{p^n} = \zeta_m} (\alpha^m)^{\otimes(r-1)} T_{p^n}(\alpha)$$

where $T_{p^n}(\alpha)$ denotes the $\mathbb{Z}/p^n\mathbb{Z}(1)$-torsor of the p^n-th roots of α. By a theorem of Beilinson, see [Wi1], Thm 4.5, these elements appear as values of torsion sections of the l-adic polylogarithm.

Motivic version of the classical Polylogarithms

Ayoub Joseph

We show that the mixed Hodge variation pol_H and the ℓ-adic sheaf pol_ℓ are realization of a same motivic object $pol_\mathcal{M}$ which live in the abelian category $\mathbf{MTM}(\mathbb{U})$ of mixed Tate motives over $\mathbb{U} = \mathbb{P}^1 - \{0, 1, \infty\}$.

(1) Categories of motives.
(2) Construction of $\mathcal{L}og_\mathcal{M}$ and $pol_\mathcal{M}$.
(3) Comparison with the realizations.

1. Categories of motives

Given a scheme X one has the Voevodsky's category $\mathbf{DM}(X)$ of triangulated motives over X (see [Vo]). Recall that objects of $\mathbf{DM}(X)$ are $\mathbb{G}\mathrm{m}$-spectra of complexes $(A^k_\bullet)_k$ where A^k_j are smooth X-schemes locally of finite type[1] and the differentials $A^k_{j+1} \to A^k_j$ as well as the assembly maps $\mathbb{G}\mathrm{m} \wedge A^k_j \to A^{k+1}_j$ are given by some kind of finite correspondences which behave well under composition. We put $\mathbb{Z}_X(1)[1] = [\mathrm{id}_X \to \mathbb{G}\mathrm{m}_X]$ where $\mathbb{G}\mathrm{m}_X$ is in degree zero. For every $n \in \mathbb{Z}$ we define the Tate object $\mathbb{Z}_X(n)$ by the usual formula and for any $A \in \mathbf{DM}(X)$ we put $A(n) = A \otimes \mathbb{Z}_X(n)$.

The Voevodsky's categories $\mathbf{DM}(X)$ like the Saito's categories of mixed Hodge modules ([Sa]) have the full Grothendieck formalism of the six operations. What we don't (yet) have in $\mathbf{DM}(X)$ is a motivic t-structure. Such a t-structure should play the role of the canonical t-structures in the classical theories (ℓ-adic sheaves and mixed Hodge modules...); in particular it's heart should contain at least the Tate objects $\mathbb{Q}(n)$. The existence of such a t-structure is very related to the Beilinson-Soulé Vanishing conjecture:

Conjecture: For every smooth scheme X over a field k, the motivic cohomology groups $H^p(X, \mathbb{Z}(q))$ vanish for $p < 0$.

Where $H^p(X, \mathbb{Z}(q))$ is defined to be the group $\hom_{\mathbf{DM}(k)}([X], \mathbb{Z}(q)[p])$. Unfortunately this conjecture remains wide open... It is only known in some very special cases: for example X the spectrum of a number field[2]. In particular if we restrict ourself to the sub-category $\mathbf{DTM}(U) \subset \mathbf{DM}(U)$ generated (as a triangulated category) by the Tate objects $\mathbb{Z}_U(n)$ for U a subscheme (open or closed) of $\mathbb{P}^1_\mathbb{Q}$ we got the:

Theorem: The category $\mathbf{DTM}(U)$ can be equipped with a motivic t-structure. The heart of this t-structure is the abelian category of mixed Tate motives $\mathbf{MTM}(U)$ generated by the $\mathbb{Z}_U(n)$.

Recall that $\mathbb{U}$ is $\mathbb{P}^1 - \{0, 1, \infty\}$. Our main result will be the construction of a pro-object $pol_\mathcal{M}$ in $\mathbf{MTM}(\mathbb{U})$.

[1]Infinite disjoint union of smooth varieties are allowed.
[2]This a consequence of Borel work on the K-theory of number fields.

2. Construction of $\mathcal{L}og_{\mathcal{M}}$ and $pol_{\mathcal{M}}$

We adapt here the construction given in [HW1] to the motivic context. We first consider the Kummer mixed Tate motive $\mathcal{K} \in \mathbf{MTM}(\mathbb{G}\mathrm{m})$. It fits naturally in an exact sequence in $\mathbf{MTM}(\mathbb{G}\mathrm{m})$:

$$0 \longrightarrow \mathbb{Q}(1) \longrightarrow \mathcal{K} \longrightarrow \mathbb{Q}(0) \longrightarrow 0$$

Or equivalently in a distinguished triangle in $\mathbf{DTM}(\mathbb{G}\mathrm{m})$:

$$\mathbb{Q}(1) \longrightarrow \mathcal{K} \longrightarrow \mathbb{Q}(0) \xrightarrow{e} \mathbb{Q}(1)[1]$$

Thus $\mathcal{K}$ is uniquely (up to a unique isomorphism!) determined by the morphism e which can be constructed using the diagonal morphism $\mathbb{G}\mathrm{m} \longrightarrow \mathbb{G}\mathrm{m} \times \mathbb{G}\mathrm{m}$ considered as a morphism of $\mathbb{G}\mathrm{m}$-schemes.

Definition: $\mathcal{L}og^N_{\mathcal{M}} = \mathrm{Sym}^N(\mathcal{K})$ and $\mathcal{L}og_{\mathcal{M}}$ is the projective system $(\mathcal{L}og^{N+1}_{\mathcal{M}} \to \mathcal{L}og^N_{\mathcal{M}})_N$.

Note that we have exact sequences:

$$0 \longrightarrow \mathbb{Q}(N) \longrightarrow \mathcal{L}og^N_{\mathcal{M}} \longrightarrow \mathcal{L}og^{N-1}_{\mathcal{M}} \longrightarrow 0$$

Let us denote $(\mathcal{L}og_{\mathcal{M}})_{|\mathbb{U}}$ the pull-back of $\mathcal{L}og_{\mathcal{M}}$ by the inclusion $\mathbb{U} \subset \mathbb{G}\mathrm{m}$. The polylogarithmic mixed Tate motive will be defined as an extension:

$$0 \longrightarrow (\mathcal{L}og_{\mathcal{M}})_{|\mathbb{U}} \longrightarrow pol_{\mathcal{M}} \longrightarrow \mathbb{Q}(0) \longrightarrow 0$$

or equivalently as an element of $\mathcal{E}xt^1(\mathbb{Q}(0), (\mathcal{L}og_{\mathcal{M}})_{|\mathbb{U}})$. The main technical result is the identification of this ext-group with $\mathbb{Q}$ which allows us to make the definition:

Definition: $pol_{\mathcal{M}}$ correspond to 1 by the identification (still to be proven): $\mathcal{E}xt^1(\mathbb{Q}(0), (\mathcal{L}og_{\mathcal{M}})_{|\mathbb{U}}) = \mathbb{Q}$.

For the computation of our ext-group we consider the following commutative diagram:

$$\begin{tikzcd} \mathbb{U} \arrow[r, "j"] \arrow[dr, "q"'] & \mathbb{G}\mathrm{m} \arrow[d, "p"] & 1 \arrow[l, "i"'] \arrow[dl, equal] \\ & \mathbb{Q} & \end{tikzcd}$$

so we can write:

$$\begin{aligned} \mathcal{E}xt^1(\mathbb{Q}(0), (\mathcal{L}og_{\mathcal{M}})_{|\mathbb{U}}) &= \hom_{\mathbf{DM}(\mathbb{U})}(q^*\mathbb{Q}, j^*\mathcal{L}og_{\mathcal{M}}[+1]) \\ &= \hom_{\mathbf{DM}(\mathbb{Q})}(\mathbb{Q}, p_* j_* j^* \mathcal{L}og_{\mathcal{M}}[+1]) \end{aligned}$$

The last equality comes from adjunction. Next we invoke the distinguished triangle:

$$i_* i^! \mathcal{L}og_{\mathcal{M}} \longrightarrow \mathcal{L}og_{\mathcal{M}} \longrightarrow j_* j^* \mathcal{L}og_{\mathcal{M}} \longrightarrow i_* i^! \mathcal{L}og_{\mathcal{M}}[+1]$$

The computation then splits into two parts:

- $p_* \mathcal{L}og_{\mathcal{M}} = \mathbb{Q}(-1)[-1]$.

- $i^! \mathcal{L}og_{\mathcal{M}} = i^* \mathcal{L}og_{\mathcal{M}}(-1)[-2] = \prod_{k\geq 0} \mathbb{Q}(k-1)[-2]$.

Which gives the exact sequence:

$$\hom(\mathbb{Q}, \mathbb{Q}(-1)[-1]) \to \mathcal{E}xt^1 \to \hom(\mathbb{Q}, \prod_{k\geq 0} \mathbb{Q}(k-1)) \to \hom(\mathbb{Q}, \mathbb{Q}(-1))$$

It is clear that the first and the last groups are zero. Thus we get our identification:

$$\mathcal{E}xt^1 = \hom(\mathbb{Q}(0), \prod_{k\geq 0} \mathbb{Q}(k-1)) = \hom(\mathbb{Q}(0), \mathbb{Q}(0)) = \mathbb{Q} \tag{1}$$

3. Compatibility with the realizations

We concentrate here on the Hodge realization: the ℓ-adic case is relatively easier. We assume that we have a realization functor from $\mathbf{DM}(X)$ to the category of Saito's mixed Hodge modules $\mathbf{MHM}(X)$ over a $\mathbb{C}$-scheme X[3]. This realization functor should be compatible with the six operations. On the other hand the computation carried out in the previous section can also be done in the context of mixed Hodge variations. In particular we get an element $pol'_{\mathcal{H}}$ in :

$$\mathcal{E}xt^1_{\mathbf{MHM}(\mathbb{U})}(\mathbb{Q}(0), (\mathcal{L}og)_{|\mathbb{U}}) = Q$$

and the realization of $pol_{\mathcal{M}}$ is exactly $pol'_{\mathcal{H}}$. So in order to prove that the Hodge realization of $pol_{\mathcal{M}}$ gives the classical polylogarithmic variation of mixed Hodge structure, we have to identify the class of the extension $pol_{\mathcal{H}}$ with the class of 1 (under the identification 1). To do this we recall that $pol_{\mathcal{H}}$ was associated to the following pro-matrix (see [BD2]):

$$\begin{pmatrix} 1 & 0 & 0 & .. \\ -\mathrm{li}_1 & 2\pi\mathbf{i} & 0 & .. \\ -\mathrm{li}_2 & (2\pi\mathbf{i})\log & (2\pi\mathbf{i})^2 & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Denoting again $pol_{\mathcal{H}}$ the class of this extension in $\mathcal{E}xt^1_{\mathbf{MHV}(\mathbb{U})}(\mathbb{Q}(0), (\mathcal{L}og)_{|\mathbb{U}})$ and using the injectivity of the map: $\mathcal{E}xt^1_{\mathbb{U}}(\mathbb{Q}(0), (\mathcal{L}og)_{|\mathbb{U}}) \longrightarrow \mathcal{E}xt^1_{\mathbb{U}}(\mathbb{Q}(0), \mathcal{K}_{|\mathbb{U}})$, one sees that it suffices to prove that the image of $pol_{\mathcal{H}}$ coïncide with the image of the Kummer torsor over $\mathbb{A}^1_{\mathbb{Q}} - \{1\}$ under: $\mathcal{E}xt^1_{\mathbb{U}}(\mathbb{Q}(0), \mathbb{Q}(1)) \longrightarrow \mathcal{E}xt^1_{\mathbb{U}}(\mathbb{Q}(0), \mathcal{K}_{|\mathbb{U}})$. This means that we have to show that the 2×2-sub-matrix:

$$\begin{pmatrix} 1 & 0 \\ -\mathrm{li}_1 & 2\pi\mathbf{i} \end{pmatrix}$$

defines the expected Kummer torsor. This is obvious.

[3]Such a realisation functor has not been constructed yet!

p-adic polylogarithms

Frank Herrlich

The defining power series of the polylogarithm

$$\sum_{n=1}^{\infty} \frac{z^n}{n^k}$$

has rational coefficients and thus can also be seen as a series over $\mathbb{C}_p$. As such it converges for $|z| < 1$. As in the complex setting, the question of analytic continuation arises. Therefore we begin with a very brief review of

1. p-adic analytic functions

A function $f : U \to \mathbb{C}_p$ on an open subset U of $\mathbb{C}_p^n$ is called *locally analytic* if it can locally be represented by a convergent power series. Since U can be covered by disjoint open balls, there are "too many" locally analytic functions, and a more rigid notion is required:

Definition and Properties
a. $T_n := \mathbb{C}_p \ll X_1, \ldots, X_n \gg = \{\sum_{\nu=(\nu_1,\ldots,\nu_n)} a_\nu X^\nu : |a_\nu| \to 0\}$ is the *Tate algebra* of convergent power series on $D_n := \{x \in \mathbb{C}_p^n : |x_i| \le 1, i = 1, \ldots, n\}$.
b. The set of maximal ideals of T_n is $\mathrm{Spm}\,(T_n) = D_n$.
c. For $f = \sum a_\nu x^\nu \in T_n$, $||f|| := \max |a_\nu|$ is the *Gauß norm.*
d. Any $\mathbb{C}_p$-algebra $A = T_n/I$ for a closed ideal I of T_n is an *affinoid algebra.*
e. $X := \mathrm{Spm}(A)$ is an *affinoid domain*, $A(X) := A$ its algebra of *analytic functions.*
f. A inherits the Gauß norm from T_n, $A^0 := \{f \in A : ||f|| \le 1\}$ is a subalgebra, $A^{00} := \{f \in A : ||f|| < 1\}$ an ideal therein; finally the *reduction* $\bar{X} := \mathrm{Spec}(\bar{A})$ with $\bar{A} := A^0/A^{00}$ is an affine variety over $\bar{\mathbb{F}}_p$.

The last property is the reason for the name "affinoid domain".
The crucial point now is to define a Grothendieck topology on X, in particular to carefully single out the *admissible coverings*: For an affinoid domain X, this are precisely the finite coverings by affinoid subdomains. Then one can define the structure sheaf on an affinoid domain, glue affinoid domains to more general (rigid) analytic spaces, and finally arrives at

Definition
A function $f : X \to \mathbb{C}_p$ on an analytic space X is *analytic* if there is an admissible covering $\{X_i\}_i$ by affinoid domains X_i such that $f|_{X_i}$ is analytic for all i.

For a systematic introduction to p-adic (rigid) analytic geometry see e.g. [FP].

Example The affinoid subdomains of $\mathbb{P}^1(\mathbb{C}_p)$ are "disks with holes", i.e. the sets of the form $X = \overline{B(a,r)} - \bigcup_{i=1}^n B(a_i, r_i)$. The corresponding affinoid algebra is $\mathbb{C}_p \ll \frac{z-a}{r}, \frac{r_1}{z-a_1}, \ldots, \frac{r_n}{z-a_n} \gg$.

Unfortunately, there is no analytic function on $\mathbb{C}_p - \{1\}$ that agrees on the open unit disk with the polylogarithm series $\sum \frac{z^n}{n^k}$.

2. Coleman's integration theory

To define the p-adic polylogarithm we shall use the differential relation

$$\mathrm{Li}_k'(z) = \frac{1}{z}\,\mathrm{Li}_{k-1}(z) \quad (k \geq 1) \quad (\text{with } \ \mathrm{Li}_0(z) = \frac{z}{1-z}).$$

For Li_1, i.e. the logarithm, we can use the functional equation to show that there is a unique locally analytic function $\log : \mathbb{C}_p^* \to \mathbb{C}_p$ which satisfies $\log(xy) = \log(x) + \log(y)$, $\log(p) = 0$, and $-\log(1-z) = \sum \frac{z^n}{n}$ for $|z| < 1$.

Inductively we find locally analytic functions ℓ_k as primitives of $\ell_{k-1}\frac{dz}{z}$; Coleman's integration (explained in [Br]; see also [Co] for more details) provides a unique choice of the local constants, guided by "Dwork's principle" of using the Frobenius structure. The variant of Coleman's method that we use for the polylogarithms works as follows:

$X/\mathbb{C}_p$ is a smooth projective curve with good reduction $Y/\bar{\mathbb{F}}_p$, $\mathrm{sp}\colon X \to Y$ the specialisation map, $S \subset X(\mathbb{C}_p)$ a finite set, $U = X - S$ (as analytic space), $W = \mathrm{sp}^{-1}(Y(\bar{\mathbb{F}}_p) - \bar{S})$ the "underlying" affinoid. Then $U - W = \bigcup_{s\in S} C_s$ with punctured disks $C_s \cong \{z \in \mathbb{C}_p : 0 < |z| < 1\}$. Coleman shows that a Frobenius on Y (over some finite field of definition) can be lifted (not uniquely) to a Frobenius morphism on W, and that for r sufficiently close to 1 it can be extended to a morphism $\varphi : U_r \to U$ on a "Frobenius neighborhood" $U_r := W \cup \bigcup_{s\in S} C_{s,r}$, where $C_{r,s}$ is the subset of C_s corresponding to $\{z \in \mathbb{C}_p : r < |z| < 1\}$.
We apply this to the situation $X = \mathbb{P}^1_{\mathbb{C}_p}$ and $S = \{1, \infty\}$. Then $W = \{z \in \mathbb{C}_p : |z| \leq 1, |z-1| \geq 1\}$; as Frobenius we can take $\varphi(z) = z^p$; U_r is a Frobenius neighborhood for $r > p^{-\frac{1}{p-1}} = |\zeta_p - 1|$. A key observation concerning Frobenius is

Remark On every residue class B of W, a suitable power φ^m of φ acts as a contraction, i.e. φ^m has a unique fixed point ε_B s.t. $\varphi^{mn}(x) \to \varepsilon_B$ for all $x \in B$.

Since $H^1_{dR}(U) \cong H^1_{dR}(U_r)$, φ induces an endomorphism φ^* of $H^1_{dR}(U)$ (in our special situation, φ^* is multiplication by p). Let P_1 denote the characteristic polynomial of φ^*.

Theorem For r sufficiently close to 1 there is for each $\omega \in \Omega^1_U$ a locally analytic function f_ω, unique up to an additive constant, satisfying:

(1) $df_\omega = \omega$
(2) $P_1(\varphi^*)(f_\omega)$ is analytic on U_r
(3) $f_\omega|_{C_s} \in A(C_s)[\log f : f \in A(C_s)^\times]$ for each $s \in S$.

Note that the $P_1(\varphi^*)$ is zero as an endomorphism of $H^1_{dR}(U)$, but in (2) it is applied to a locally analytic function. For the proof of the theorem see [Br, 2.2.1]. We can iterate this procedure by defining inductively

$$A_k(U) := A_{k-1}(U) + \sum_{\omega \in A_{k-1}(U)\otimes\Omega^1_U} f_\omega A_{k-1}(U) \qquad (k \geq 1), \quad A_0(U) = A(U)$$

and obtain the same statement for $\omega \in A_k(U) \otimes_{A(U)} \Omega^1_U$, with (2) replaced by

(2_k) $P_{k+1}(\varphi^*)(f_\omega) \in A_k(U_r)$, where P_k is the characteristic polynomial of φ^* on $H^1_{dR}(A_{k-1}(U))$.

Applying this result to the situation described above we obtain

Corollary There are unique locally analytic functions $\ell_k \in A_k(\mathbb{C}_p - \{1\})$, called *p-adic polylogarithms*, satisfying $\ell_0(z) = \frac{z}{1-z}$, $\ell_k(0) = 0$, $\ell'_k(z) = \frac{1}{z}\ell_{k-1}(z)$, and $P_k(\varphi^*)(\ell_k) \in A_{k-1}(U_r)$ for all $r > p^{-\frac{1}{p-1}}$.

3. Special values of L-functions

Let $\chi : (\mathbb{Z}/d\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character of conductor d and $L(s,\chi)$ the associated L-function, which for $d = 1$ is the Riemann zeta function and in all other cases is an analytic function on $\mathbb{C}$, given by the power series $\sum_{n=1}^\infty \frac{\chi(n)}{n^s}$ for $\mathrm{Re}(s) > 1$. Its values at integer arguments are known classically:

(a) $L(1-n,\chi) = -\frac{1}{n}B_{n,\chi}$ for $n \geq 1$, where $B_{n,\chi} \in \mathbb{Q}(\chi)$ are the generalized Bernoulli numbers.

(b) $L(k,\chi) = \frac{1}{d}g(\chi,\zeta)\sum_{a=1}^d \bar{\chi}(a)\mathrm{Li}_k(\zeta^{-a})$ for $k \geq 1$, where ζ is a primitive d-th root of unity and $g(\chi,\zeta) = \sum_{a=1}^d \chi(a)\zeta^a$ is the Gauß sum.

Since the $B_{n,\chi}$ are algebraic numbers whose p-adic values behave nicely, (a) can be used to define a $\mathbb{C}_p$-valued function $L_p(s,\chi)$ on $\mathbb{Z}_p$ interpolating these numbers; this is the *Kubota-Leopoldt L-function.* For an approach via p-adic measures see [Ko].

The proof of (b) is elementary: on both sides the defining power series can be used to calculate the value, and it is an easy exercise on Gauß sums that the coefficients for $\frac{1}{n^k}$ are equal. Nothing of this holds for the analogous p-adic functions. Nevertheless, (b) has a p-adic counterpart:

Theorem (Leopoldt for $k = 1$, Coleman for $k \geq 2$)

$$L_p(k,\chi\omega^{1-k}) = (1 - \frac{\chi(p)}{p})\frac{1}{d}g(\chi,\zeta)\sum_{a=1}^d \bar{\chi}(a)\ell_k(\zeta^{-a}),$$

where $\omega : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{C}_p^\times$ is the Teichmüller character.

For the proof see [Ko, 1,2] and [Co, 7]. It relies on Koblitz's representation of $L_p(s,\chi)$ as an integral w.r.t. a certain p-adic measure $d\mu_z$ and Coleman's formula

$$\ell_k(z) - p^k\ell_k(z^p) = \int_{X^*} x^{-k} d\mu_z.$$

4. Interpretations

Like in the complex case, the values of the p-adic polylogarithm appear in the description of extension classes of $\mathbb{Q}_p$-vector spaces with Hodge filtration. More generally, there is an interpretation of the p-adic polylogarithm as an object in the

p-adic analogue of the category of variations of mixed Hodge structures. For this and related topics, see e.g. [Ba] and the references therein.

Multiple zeta values and multiple polylogarithms

Francis C.S. Brown

Multiple zeta values are real numbers which generalise the values of the Riemann zeta function at integers. The standard relations, or double shuffle products, are conjectured to yield all algebraic relations between these numbers. By introducing the multiple polylogarithms, one obtains a natural functional model which generalises the classical polylogarithms, and gives the multiple zeta values on taking regularised values at 1.

1. Multiple zeta values. Let $s_1, \dots, s_k \in \mathbf{N}$, and suppose that $s_1 \geq 2$. The *multiple zeta value* $\zeta(s_1, \dots, s_k)$ is the real number defined by the convergent sum

$$\zeta(s_1, \dots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \dots n_k^{s_k}} \ .$$

These numbers were first studied by Euler, and have recently resurfaced in the study of knot invariants, quantum field theory, and in Grothendieck-Teichmüller theory. One aim is to describe all the algebraic relations between multiple zeta values (MZVs) over the rational numbers. The first, algebraic, part of the problem is to write down all possible relations. The so-called standard relations consist of two quadratic and one linear relation, but there are many other natural relations [IK]. The other, transcendental, half of the problem is to show that these are the only relations that are satisfied. The only results that are known in this direction are essentially that π is transcendental (Lindemann), that $\zeta(3)$ is irrational (Apéry), and that infinitely many of the values $\zeta(2n+1)$ are linearly independent over $\mathbf{Q}$ (Rivoal [Ri]). It is still not known whether $\zeta(5)$ is irrational or not.

The standard relations can be described in terms of two product structures on a certain algebra. Let $X = \{x_0, x_1\}$ denote an alphabet on two letters, and let $\mathbf{Q}\langle X \rangle$ denote the free non-commutative $\mathbf{Q}$-algebra on the symbols x_0 and x_1 with the concatenation product. We introduce two further products on the subalgebra $\mathfrak{H}^1 = \mathbf{Q}1 \oplus \mathbf{Q}\langle X \rangle x_1$ which is generated by all words in x_0 and x_1 which end in x_1. For each $i \geq 1$, we write $y_i = x_0^{i-1} x_1$. Then $\mathfrak{H}^1$ is also generated by the symbols y_i. The shuffle product, written $ш : \mathfrak{H}^1 \times \mathfrak{H}^1 \to \mathfrak{H}^1$, and stuffle product, written $\star : \mathfrak{H}^1 \times \mathfrak{H}^1 \to \mathfrak{H}^1$ are then defined inductively as follows:

$$\begin{array}{rcl} w \,ш\, 1 = 1 \,ш\, w = w \ , & \text{and} & w \star 1 = 1 \star w = w \ , \\ x_i w \,ш\, x_j w' & = & x_i (w \,ш\, x_j w') + x_j (x_i w \,ш\, w') \ , \\ y_k w \star y_\ell w' & = & y_k (w \star y_\ell w') + y_\ell (y_k w \star w') + y_{k+\ell} (w \star w') \ , \end{array}$$

for all $w, w' \in \mathfrak{H}^1$, all $i, j \in \{0, 1\}$, and all $k, \ell \geq 1$.

If we set $\mathfrak{H}^0 = \mathbf{Q}1 \oplus x_0\mathbf{Q}\langle X\rangle x_1$, then there is a well-defined linear map

$$\begin{aligned} \zeta : \mathfrak{H}^0 &\rightarrow \mathbf{R} , \\ \zeta(y_{s_1} \dots y_{s_r}) &= \zeta(s_1, \dots, s_r) , \end{aligned}$$

where $\zeta(1)$ is defined to be 1. One can prove that $\mathfrak{H}^0$ is closed under ш and $\star$, and that ζ is a homomorphism for both products. In other words,

$$\begin{aligned} \zeta(w_1 ш w_2) &= \zeta(w_1)\,\zeta(w_2) , \\ \zeta(w_1 \star w_2) &= \zeta(w_1)\,\zeta(w_2) \quad \text{for all} \quad w_1, w_2 \in \mathfrak{H}^0. \end{aligned}$$

These are the shuffle and stuffle relations respectively. Finally, one proves that $x_1 \star w - x_1 ш w \in \mathfrak{H}^1$ actually lies in $\mathfrak{H}^0$ for all $w \in \mathfrak{H}^0$, and that

$$\zeta(x_1 \star w - x_1 ш w) = 0 .$$

It is conjectured that all algebraic relations over $\mathbf{Q}$ satisfied by the multiple zeta values are generated by the previous three identities [Wa].

If d_k denotes the dimension of the $\mathbf{Q}$-vector space spanned by the multiple zeta values of weight k (*i.e.* the set of $\zeta(w)$, where $w \in X^*$ has k symbols), then extensive computer calculations confirm this conjecture and suggest that $d_k = d_{k-2} + d_{k-3}$. By motivic arguments, it has recently been shown by Terasoma [Te] and Goncharov [Go6, Go7] (see also [DG]), that the dimensions d_k are bounded above by the expected quantity.

2. Multiple polylogarithms. Let $s_1, \dots, s_k \in \mathbf{N}$. If $z_i \in \mathbf{C}$ such that $|z_i| < 1$, then the multiple polylogarithm is defined by the power series

$$\mathrm{Li}_{s_1,\dots,s_k}(z_1, \dots, z_k) = \sum_{n_1 > n_2 > \dots > n_k \geq 1} \frac{z_1^{n_1} \dots z_k^{n_k}}{n_1^{s_1} \dots n_k^{s_k}} .$$

If $s_1 \geq 2$, then the sum converges for $|z_i| = 1$, and we have $\mathrm{Li}_{s_1,\dots,s_k}(1, \dots, 1) = \zeta(s_1, \dots, s_k)$. These functions have been studied extensively by Goncharov in relation to periods of variation of mixed Tate motives [Go6, Go7].

One can show that the multiple polylogarithms appear naturally as iterated integrals on the moduli space $\mathcal{M}_{0,n}$ of Riemann spheres with n ordered marked points, where $n \geq 4$. In fact, by applying Chen's general theory (*c.f.* the fourth talk) to these spaces, one can show that the set of all homotopy-invariant iterated integrals on $\mathcal{M}_{0,n}$ defines a Hopf algebra of multi-valued functions in which the set of multiple polylogarithms in $k = n - 3$ variables is strictly contained. In the case $n = 4$, we can identify $\mathcal{M}_{0,4}$ with the projective line minus three points $\mathbf{P}^1\backslash\{0, 1, \infty\}$. The shuffle product formula for iterated integrals immediately yields the shuffle relations for the multiple zeta values on taking a regularised limit at 1.

The multiple polylogarithms also satisfy relations which generalise the stuffle product, the simplest example of which is the identity

$$\mathrm{Li}_{s_1}(z_1)\mathrm{Li}_{s_2}(z_2) = \mathrm{Li}_{(s_1,s_2)}(z_1, z_2) + \mathrm{Li}_{(s_2,s_1)}(z_2, z_1) + \mathrm{Li}_{s_1+s_2}(z_1 z_2),$$

which follows directly from the power series definition above. On taking a regularised limit as the variables z_i tend to 1, one retrieves all three standard relations for the multiple zeta values.

One can also consider the values of multiple polylogarithms at roots of unity. These numbers will automatically inherit the double shuffle relations, but will also satisfy distribution relations. The latter relations come from an identity on the level of the multiple polylogarithm functions which directly generalises the distribution relations for the classical polylogarithms. The dimensions of the $\mathbf{Q}$-vector spaces spanned by these numbers are conjecturally related to the K-theory of cyclotomic fields ([DG, Go4]).

Zagier's Conjecture: Statement and motivic interpretation

Marco Hien

Let F be an algebraic number field with discriminant d_F and let r_1, resp. r_2, denote the number of real, resp. pairs of complex embeddings of F. Let Σ_- be the set of real embeddings and Σ_+ be the union of Σ_- and a complete set of pairwise non-conjugate complex embeddings. These sets have order $n_- = r_2$ and $n_+ = r_1 + r_2$ respectively. It is conjectured, that the value of the Dedekind zeta-function ζ_F of F at a natural number $k \geq 2$ can be expressed by the single-valued polylogarithm functions P_k (as introduced in the first talk). More precisely:

Conjecture (Zagier). *Let $k \in \mathbb{N}$, $k \geq 2$. If k is even, there exist elements $x_1, \dots, x_{n_-} \in F$, such that*

$$\zeta_F(k) \in \pi^{k \cdot n_+} \cdot |d_F|^{-1/2} \cdot \det\big((P_k(\sigma x_1))_\sigma, \dots, (P_k(\sigma x_{n_-}))_\sigma\big) \cdot \mathbb{Q}^\times \ ,$$

where σ runs through Σ_-. A similar statement holds for odd k after interchanging all occuring signs $\pm$ in the indices.

The conjecture is known to be true for $k = 2$ ([Za2]) and $k = 3$ ([Go4]). In general it could be deduced from a conjectural description of the algebraic K-groups $K_{2k-1}(F) \otimes \mathbb{Q}$ as follows:

For any $k \geq 1$, one conjectures the existence of

- a $\mathbb{Q}$-vector space L^k,
- a map $\{\}_k : F^\times \setminus \{1\} \to L^k$, whose image generates L^k over $\mathbb{Q}$,
- a homomorphism $d_k : L^k \to \bigwedge^2(\oplus_{l=1}^{k-1} L^l)$ and
- a monomorphism $\varphi_k : \ker(d_k) \hookrightarrow K_{2k-1}(F) \otimes \mathbb{Q}$,

such that for any embedding $\sigma : F \hookrightarrow \mathbb{C}$, the composition with the Borel regulator map $\mathrm{reg}_B^\sigma : K_{2k-1}(f) \otimes \mathbb{Q} \to i^{k-1}\mathbb{R}$ is given by polylogarithms as follows:

$$\mathrm{reg}_B^\sigma(\varphi_k(\sum_\alpha \lambda_\alpha \{x_\alpha\}_k)) = -\sum_\alpha \lambda_\alpha \cdot P_k(\sigma x_\alpha)$$

for all $\sum_\alpha \lambda_\alpha \{x_\alpha\}_k \in \ker(d_k)$. This is referred to as the *weak Zagier Conjecture*. Under the additional assumption that φ_k is surjective, the above conjecture on the special values of ζ_F follows using Borel's Theorem on the image of the regulator map for number fields [Bo1].

Following Beilinson and Deligne, the weak Zagier Conjecture allows a motivic interpretation. One assumes to have a good category of mixed Tate motives over the number field F, namely a Tannakian category $T(F)$ over $\mathbb{Q}$ with a fixed invertible object $\mathbb{Q}(1)$, such that every object has a weight filtration, where the graded object consists of a direct sum of tensor powers $\mathbb{Q}(k) := \mathbb{Q}(1)^{\otimes k}$ of $\mathbb{Q}(1)$. Additionally, one assumes to have a realisation functor into the category of $\mathbb{Q}$-variations of mixed Hodge structures on $F(\mathbb{C})$ and an isomorphism

$$K_{2k-1}(F) \otimes \mathbb{Q} \xrightarrow{\sim} \mathrm{Ext}^1(\mathbb{Q}, \mathbb{Q}(k)) \ .$$

The composition of the latter with the regulator into Deligne cohomology, in this case $\mathrm{reg} : K_{2k-1}(F) \otimes \mathbb{Q} \to \mathrm{Ext}^1_{F(\mathbb{C})}(\mathbb{Q}, \mathbb{Q}(k))$, should coincide with the map given by the realisation functor. In analogy to the case of mixed Hodge structures, there should exist a projective system of objects $\mathrm{pol}^{(N)}$ in the category of mixed Tate motives over $\mathbb{P}^1 - \{0, 1, \infty\}$ with similar properties.

In [BD2], Beilinson and Deligne proof the weak Zagier Conjecture assuming the existence of such a category including the polylogarithm objects. The main ingredient is the Lie algebra of the pro-unipotent part U of the Tannaka group G of $T(F)$, i.e. the algebraic group scheme G over $\mathbb{Q}$ whose finite-dimensional representations determine $T(F)$ up to equivalence of categories in the usual sense of Tannakian theory. The data required in the weak Zagier Conjecture can be derived from the action of this Lie algebra on the system $\mathrm{pol}^{(N)}$.

The value $\zeta_F(3)$ after A. Goncharov

Florin Nicolae

For $k = 1, 2, 3$ let $Li_k(z)$ be the k-th polylogarithm. Define

$$\mathcal{L}_3(z) := Re(Li_3(z) - \log|z| \cdot Li_2(z) + \frac{1}{3}\log^2|z| \cdot Li_1(z)).$$

This is a single-valued, real-analytic function on the complex projective line without three points $0, 1, \infty$, continuous at $0, 1, \infty$ with the values

$$\mathcal{L}_3(0) = \mathcal{L}_3(\infty) = 0, \mathcal{L}_3(1) = \sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta_{\mathbb{Q}}(3).$$

Let F be an arbitrary algebraic number field; d_F the discriminant of F; r_1, resp. r_2, the number of real, resp. complex places, so $[F : \mathbb{Q}] = r_1 + 2r_2$; and $\{\sigma_j\}$ the set of all possible embeddings of F in $\mathbb{C}$ $(1 \leq j \leq r_1 + 2r_2)$ numbered so that $\bar{\sigma}_{r_1+l} = \sigma_{r_1+r_2+l}$.

Let $\mathbb{Z}[\mathbb{P}^1_F \setminus \{0, 1, \infty\}]$ be the free abelian group, generated by symbols $\{x\}$, where $x \in \mathbb{P}^1_F \setminus \{0, 1, \infty\}$. The function $\mathcal{L}_3$ defines a homomorphism

$$\mathbb{Z}[\mathbb{P}^1_F \setminus \{0, 1, \infty\}] \to \mathbb{R}$$

$$\sum n_i\{x_i\} \mapsto \sum n_i \mathcal{L}_3(x_i).$$

Denote by $R_2(F)$ the subgroup of $\mathbb{Z}[\mathbb{P}^1_F \setminus \{0,1,\infty\}]$ generated by the expressions

$$\{x\} - \{y\} + \{\frac{x}{y}\} - \{\frac{1-y^{-1}}{1-x^{-1}}\} + \{\frac{1-y}{1-x}\},$$

where $x \neq y$. Set

$$B_2(F) := \mathbb{Z}[\mathbb{P}^1_F \setminus \{0,1,\infty\}]/R_2(F).$$

Consider the homomorphism

$$\delta : \mathbb{Q}[\mathbb{P}^1_F \setminus \{0,1,\infty\}] \to B_2(F)_{\mathbb{Q}} \otimes F^*_{\mathbb{Q}}$$

$$\{x\} \mapsto \{x\}_2 \otimes x$$

($\{x\}_2$ is the projection of $\{x\}$ onto $B_2(F)$.)

Theorem. Let $\zeta_F(s)$ be the Dedekind zeta-function of F. There exist

$$y_1, \ldots, y_{r_1+r_2} \in \mathrm{Ker}\delta$$

such that

$$\zeta_F(3) = \pi^{3r_2} \cdot |d_F|^{-\frac{1}{2}} \cdot \det(\mathcal{L}_3(\sigma_j(y_i))_{1 \le i,j \le r_1+r_2}).$$

The elliptic polylogarithm I and II

David Blottière and François Brunault

Introduction

The aim of these two talks is to introduce an elliptic analogue of Zagier's conjecture. We follow the article of J. Wildeshaus [Wi3].

In the first one, we define a formalism for an elliptic polylogarithm in a general setting. Then we prove that such an elliptic polylogarithm exists in the context of admissible and graded polarizable variations of mixed Hodge structures (VMHS). After having explained the notion of polylogarithm value at torsion points of a complex elliptic curve, we give an expression for this in terms of Eisenstein-Kronecker series.

In the second talk, we state an elliptic analogue of Zagier's conjecture. Broadly speaking, it gives a recipe for constructing nonzero elements in specific motivic cohomology groups attached to an elliptic curve, and predicts that their images under Beilinson's regulator are polylogarithms. We explain how this is implied by the existence of a suitable category of smooth motivic sheaves, admitting amongst other things the formalism of an elliptic polylogarithm.

1. The elliptic polylogarithm I - Hodge realization

1.1. **A formalism for an elliptic polylogarithm.** In this section, we define the notion of a formalism for an elliptic polylogarithm as a system of data which satisfies seven axioms. We note that there is another but equivalent way to define the elliptic polylogarithm in the l-adic or Hodge context (e.g. [Ki1, 1.1]). Our approach is useful for discussing Zagier's conjecture for elliptic curves.

We fix some data. Let S be a connected base scheme and F a field of characteristic 0. For each quasi projective and smooth scheme B over S, we have a F-linear abelian category $\mathcal{T}(B)$ with an associative, commutative and unitary (we write $F(0)$ for the neutral element) tensor product, such that $B \mapsto \mathcal{T}(B)$ is natural in a contravariant way.

These data satisfy the following seven axioms.

(A) For B connected, $\mathcal{T}(B)$ is a neutral abelian F-linear Tannakian category and for $f : B_1 \to B_2$, $f^* : \mathcal{T}(B_2) \to \mathcal{T}(B_1)$ is exact.

(B) $\mathcal{T}(B)$ is a tensor category with weights ([Wi3, Def. 2.4]).

(C) There is an object $F(1)$ of rank 1 and weight -2 in $\mathcal{T}(S)$. For a scheme B over S, we still denote by $F(1)$ the pullback of $F(1)$ by the structural morphism of B. For $V \in \mathcal{T}(B)$, $V(1) := V \otimes F(1)$.

(D) For any elliptic curve $\pi : \mathcal{E} \to B$, there exists an object $R^1\pi_*F$ of rank 2 and weight 1 in $\mathcal{T}(B)$, and an isomorphism $\cup_\pi : \overset{2}{\Lambda}\, R^1\pi_*F \xrightarrow{\sim} F(-1)$ (the dual of $F(1)$) compatible with base change.

(E) For any elliptic curve $\pi : \mathcal{E} \to B$, we have a morphism $[\ ] : \mathcal{E}(B) \otimes_{\mathbb{Z}} F \to \mathrm{Ext}^1_{\mathcal{T}(B)}(F(0), R^1\pi_*F(1))$.

(F) For any elliptic curve $\pi : \mathcal{E} \to B$, form the base change $pr_1 : \mathcal{E} \times_B \mathcal{E} \to \mathcal{E}$, and consider $[\Delta] \in \mathrm{Ext}^1_{\mathcal{T}(\mathcal{E})}(F(0), \pi^* R^1\pi_*F(1))$ and $\mathrm{Sym}^{N-1}[\Delta]$. $(\mathrm{Sym}^{N-1}[\Delta])_{N\geq 1}$ forms a projective system (the transition morphisms are induced by $[\Delta] \to F(0)$), the logarithmic pro-sheaf $\mathcal{L}og$ on $\mathcal{E}$. Moreover there exists a projective system (the (small) polylogarithmic extension)
$\mathcal{P}ol = (\mathcal{P}ol^N)_{N\in\mathbb{N}} \in \varprojlim_{N\in\mathbb{N}} \mathrm{Ext}^1_{\mathcal{T}(\widetilde{\mathcal{E}})}(\pi^* R^1\pi_*F(1)_{|\widetilde{\mathcal{E}}}, \mathrm{Sym}^{N-1}[\Delta](1)_{|\widetilde{\mathcal{E}}})$
where $\widetilde{\mathcal{E}}$ is the complement of the zero section in $\mathcal{E}$, such that
$$\mathcal{P}ol^1 \in \mathrm{Ext}^1_{\mathcal{T}(\widetilde{\mathcal{E}})}(F(0), (\pi^* R^1\pi_*F(1)_{|\widetilde{\mathcal{E}}})^\vee(1))$$
coincides with the pushout of $[\Delta]_{|\widetilde{\mathcal{E}}}$ under the isomorphism induced by $\cup_\pi$, $R^1\pi_*F(1)_{|\widetilde{\mathcal{E}}} \xrightarrow{\sim} (R^1\pi_*F(1)_{|\widetilde{\mathcal{E}}})^\vee(1)$, $v \mapsto \cup_\pi(v,\ \ )$.

(G) Let $\psi : \mathcal{E}_1 \to \mathcal{E}_2$ be an isogeny between two elliptic curves over B. Then $\psi^*\mathcal{L}og_{\mathcal{E}_2} = \mathcal{L}og_{\mathcal{E}_1}$.

1.2. **An elliptic polylogarithm for VMHS.** We consider the following data. Let $S := \mathrm{Spec}(\mathbb{C})$, $F = \mathbb{Q}$ or $\mathbb{R}$, $\mathcal{T}(B) := \mathrm{VMHS}_F(B)$. Then $f : B_1 \to B_2$ induces

$f^* : \mathrm{VMHS}_F(B_2) \to \mathrm{VMHS}_F(B_1)$ given by the pullback at the level of local systems. We explain briefly why the axioms (A) - (G) are fulfilled in this setting.

(A) Let $b \in B(\mathbb{C})$. To $\mathbb{V} \in \mathrm{VMHS}_F(B)$, one associates $\overline{\mathbb{V}}_b$, where $\overline{\mathbb{V}}$ is the local system underlying to $\mathbb{V}$. This defines a fibre functor.
(B) $\mathbb{V} \in \mathrm{VMHS}_F(B)$ has a weight filtration compatible with $\otimes$.
(C) $F(1) \in \mathrm{MHS}_F$ is first Tate's twist.
(D) Consider the (topological) first higher direct image under π of the constant sheaf F on $\mathcal{E}(\mathbb{C})$. Its fibre at $b \in B(\mathbb{C})$ is $H^1(\mathcal{E}_b(\mathbb{C}), F)$. By classical Hodge theory, $H^1(\mathcal{E}_b(\mathbb{C}), F)$ is equipped with a pure Hodge structure of weight 1. The collection $(H^1(\mathcal{E}_b(\mathbb{C}), F))_{b\in B(\mathbb{C})}$ forms an object of $\mathrm{VMHS}_F(B)$ of rank 2 and weight 1 which is, by definition, $R^1\pi_*F$. $\cup_\pi$ is induced by the fibrewise cup product.
(E) The map [ ] is constructed by using Saito's theory of mixed Hodge modules ([Wi3, 3.2]).
(F) Let $\widetilde{\pi} : \widetilde{\mathcal{E}} \to B$ be the restriction of π to $\widetilde{\mathcal{E}}$. A fundamental property ([Ki1, Prop. 1.1.3 b)]) of the $\mathcal{L}og$ pro-sheaf is

$$R^n\widetilde{\pi}_*\ \mathcal{L}og \ = \begin{cases} \prod\limits_{k>0} \mathrm{Sym}^k(R^1\pi_*F(1))\,(-1) & \text{if } n = 1 \\ 0 & \text{else.} \end{cases}$$

Now, the Leray spectral sequence for $\mathrm{RHom}((R^1\pi_*F(1)),\ \ )\ o\ R\widetilde{\pi}_*$ and weight considerations give an isomorphism :
$Res\ :\ \mathrm{Ext}^1_{\widetilde{\mathcal{E}}}(\widetilde{\pi}^*R^1\pi_*F(1), R^1\widetilde{\pi}_*\ \mathcal{L}og(1)) \xrightarrow{\sim} \mathrm{Hom}_B(R^1\pi_*F(1), R^1\pi_*F(1))$.
We define $\mathcal{P}ol$ by $Res(\mathcal{P}ol) = Id$. The compatibility between $\mathcal{P}ol^1$ and $[\Delta]$ is satisfied ([Wi2, Prop. 2.4, Prop. 2.5]).
(G) For the compatibility of $\mathcal{L}og$ with respect to isogenies, we refer to [Be-Le, 1.2.10.(vi)].

Remark 1. *: The fibre of $\mathcal{L}og$ at $x \in \widetilde{\mathcal{E}}(\mathbb{C})$ can be described in terms of relative homology groups (*[Le, 2.4.4]*).*

1.3. **Values of the polylogarithm at torsion points (real coefficients).** Let $(E/\mathbb{C}, 0)$ be an elliptic curve over $\mathbb{C}$, x a nonzero torsion point of E and $F = \mathbb{R}$. In this context, $R^1\pi_*F(1) = H^1(E(\mathbb{C}), \mathbb{R})(1) =: H_1$. First, we precise the notion of value of the polylogarithm at x. It is obvious that $0^*\mathcal{L}og = \prod\limits_{k\geq 0} \mathrm{Sym}^k H_1$. Using (G), we prove $x^*\mathcal{L}og = \prod\limits_{k\geq 0} \mathrm{Sym}^k H_1$. So $x^*\mathcal{P}ol$ lies in $\mathrm{Ext}^1_{MHS_\mathbb{R}}(\mathbb{R}(0), \prod\limits_{k\geq 0} \mathrm{Sym}^k H_1 \otimes H_1^\vee\ (1))$. The k-th value of $\mathcal{P}ol$ at x, $[x^*\mathcal{P}ol]^k$, is the pushout of $x^*\mathcal{P}ol$ under the composition of the contraction map
$\prod\limits_{k\geq 0} \mathrm{Sym}^k H_1 \otimes H_1^\vee\ (1) \to \prod\limits_{k\geq 0} \mathrm{Sym}^{k-1} H_1(1)$ with the projection on the $(k-2)$-th factor. So $[x^*\mathcal{P}ol]^k \in \mathrm{Ext}^1_{MHS_\mathbb{R}}(\mathbb{R}(0), \mathrm{Sym}^{k-2} H_1\ (1))$.

Let $V \in MHS_{\mathbb{R}}$, V of weight ≤ -1. Then we have an isomorphism

$$\overline{V \otimes \mathbb{R}(-1)} \xrightarrow{\sim} \overline{V}_{\mathbb{C}}/\overline{V} \xrightarrow{\sim} \mathrm{Ext}^1_{MHS_{\mathbb{R}}}(\mathbb{R}(0), V)$$

which associates to $h \in \overline{V \otimes \mathbb{R}(-1)} \subset \overline{V}_{\mathbb{C}}$ the following 1-extension : we put the diagonal weight and Hodge filtrations on $\mathbb{C} \oplus \overline{V}_{\mathbb{C}}$ and we take $< 1-h, \overline{V} >_{\mathbb{R}} \subset \mathbb{C} \oplus \overline{V}_{\mathbb{C}}$ as real structure. Applying this result to $V = \mathrm{Sym}^{k-2} H_1$ (1) one identifies $\mathrm{Sym}^{k-2}\overline{H_1}$ and $\mathrm{Ext}^1_{MHS_{\mathbb{R}}}(\mathbb{R}(0), \mathrm{Sym}^{k-2} H_1(1))$.

Now, we introduce the Eisenstein-Kronecker series. Fix an isomorphism $\eta : E(\mathbb{C}) \to \mathbb{C}/L$ where L is a lattice in $\mathbb{C}$ and let $\omega(L) := \eta^* dz$. Recall the definition of the Pontryagin product : $(z, \gamma)_L = \exp(\pi . Vol(L)^{-1} . (z\overline{\gamma} - \overline{z}\gamma))$, $z \in \mathbb{C}/L$, $\gamma \in L$. The Eisenstein-Kronecker series $K_{a,b,L} : \mathbb{C} - L \to \mathbb{C}$, for $a, b \geq 1$ is defined by

$$K_{a,b,L}(z) := \sum_{\gamma \in L - \{0\}} \frac{(z, \gamma)_L}{\gamma^a \overline{\gamma}^b}.$$

We are now able to give an explicit formula for $[x^* \mathcal{P}ol]^k$ viewed as an element of $\mathrm{Sym}^{k-2}\overline{H_1}$.

Theorem. [Wi2, Prop. 1.3, Cor. 4.10 (a)]

(1) *For* $k \geq 2$, $G_{E,k}(x) := \sum_{a+b=k-2} K_{a+1,b+1,L}(\eta(x))\, \omega(L)^a \overline{\omega(L)}^b$, *which is an element of* $\mathrm{Sym}^{k-2}\overline{H_1}_{\mathbb{C}}$, *lies actually in* $\mathrm{Sym}^{k-2}\overline{H_1}$ *and does not depend on any choice.*

(2) $[x^* \mathcal{P}ol]^k = G_{E,k}(x)$.

2. The elliptic polylogarithm II - Zagier's conjecture

We begin by stating the so-called weak version of Zagier's conjecture for elliptic curves. It is of inductive nature : there is a statement for each $k \geq 2$, and the k-th step can only be formulated if all previous steps are true.

We will use the following notation for motivic cohomology. For any scheme X and any integers $i, j \in \mathbb{Z}$, we put

$$H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) := K^{(j)}_{2j-i}(X),$$

the j-th Adams eigenspace of Quillen's K-group tensorized with $\mathbb{Q}$.

Let K be a number field and B a smooth, quasi-projective, connected scheme over K or $\mathcal{O}_K$. Let $\pi : \mathcal{E} \to B$ be an elliptic curve. For any integer $k \geq 2$, we wish to construct explicit elements in

$$H^{k-1}_{\mathcal{M}}(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1))_{\mathrm{sgn}},$$

where $\mathcal{E}^{(k-2)} := \ker(\sum : \mathcal{E}^{k-1} \to \mathcal{E})$ and the subscript $(\cdot)_{\mathrm{sgn}}$ denotes the signature-eigenspace determined by the action of the symmetric group $\mathcal{S}_{k-1}$ on $\mathcal{E}^{(k-2)}$.

For any $k \geq 1$, let $\mathcal{L}_k^\sharp$ be the $\mathbb{Q}$-vector space with basis elements $(\{s\}_k^\sharp, s \in \mathcal{E}(B), s \neq 0)$. Let

$$\begin{aligned} \phi_1 : \mathcal{L}_1^\sharp &\to \mathcal{E}(B) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \{s\}_1^\sharp &\mapsto s \otimes 1. \end{aligned}$$

Put $\mathcal{L}_1 := \mathcal{L}_1^\sharp / \ker \phi_1 \cong \mathcal{E}(B) \otimes_{\mathbb{Z}} \mathbb{Q}$ and define $\{s\}_1 :=$ class of $\{s\}_1^\sharp = s \otimes 1$.

Conjecture. *There exist quotients $\mathcal{L}_k$ of $\mathcal{L}_k^\sharp$ (for all $k \geq 2$) with the following properties. Denoting the class of $\{s\}_k^\sharp$ in $\mathcal{L}_k$ by $\{s\}_k$, we define the differential*

$$\begin{aligned} d_k : \mathcal{L}_k^\sharp &\to \mathcal{L}_{k-1} \otimes_{\mathbb{Q}} \mathcal{L}_1 \\ \{s\}_k^\sharp &\mapsto \{s\}_{k-1} \otimes \{s\}_1. \end{aligned}$$

Then there exists a homomorphism

$$\phi_k : \ker d_k \to H_{\mathcal{M}}^{k-1}(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1))_{\mathrm{sgn}}$$

such that

(1) *ϕ_k is compatible with base change $B' \to B$ and with isogenies $\psi : \mathcal{E}_1 \to \mathcal{E}_2$ satisfying $\ker \psi \subset \mathcal{E}_1(B)$.*

(2) *($B = \mathrm{Spec} K$). Let r_∞ be the regulator of Deligne and Beilinson*

$$r_\infty : H_{\mathcal{M}}^{k-1}(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1))_{\mathrm{sgn}} \to \left(\bigoplus_{\sigma : K \hookrightarrow \mathbb{C}} \mathrm{Sym}^{k-2} H_{\mathrm{B}}^1(\mathcal{E}^\sigma(\mathbb{C}), \mathbb{R}(1)) \right)^+,$$

where H_{B}^1 indicates Betti cohomology and $(\cdot)^+$ denotes the fixed subspace for the complex conjugation acting both on the set $\{\sigma : K \hookrightarrow \mathbb{C}\}$ and on the coefficients $\mathbb{R}(1)$. Then

$$r_\infty(\phi_k(S)) = \left(G_{\mathcal{E}^\sigma(\mathbb{C}),k}(S^\sigma)\right)_\sigma \qquad \textit{for all } S \in \ker d_k,$$

where $G_{\mathcal{E}^\sigma(\mathbb{C}),k}(S^\sigma)$ is defined by linearity.

(3) *($B = \mathrm{Spec} K$). This condition, which we do not give explicitely here, is an integrality criterion. It gives a necessary and sufficient condition on $S \in \ker d_k$ in order that $\phi_k(S)$ belongs to the integral subspace of motivic cohomology (this $\mathbb{Q}$-subspace is defined using the Néron model of $\mathcal{E}$).*

If the conjecture at step k is true, define $\mathcal{L}_k := \mathcal{L}_k^\sharp / \ker \phi_k$ and go to step $k+1$.

Remark 2. *Condition (2) ensures that the homomorphism ϕ_k isn't trivial.*

Remark 3. *If $s \in \mathcal{E}(B)$ is a nonzero torsion point of $\mathcal{E}$, then $\{s\}_k^\sharp$ belongs to $\ker d_k$, as it can be seen from the definition of $\mathcal{L}_1$. Thus (rational) torsion points of $\mathcal{E}$ always yield elements of the motivic cohomology group of interest.*

Remark 4. *We have the following chain of inclusions*

$$\ker \phi_k \subset \ker d_k \subset \mathcal{L}_k^\sharp.$$

The group on the left should come from the functional equations of the elliptic polylogarithm. The quotient $\ker d_k / \ker \phi_k$ can be identified, via ϕ_k, with a subspace of $H_{\mathcal{M}}^{k-1}(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1))_{\mathrm{sgn}}$. This subspace is strict in general (there are elliptic curves with trivial Mordell-Weil group).

We now briefly indicate how the formalism of an elliptic polylogarithm allows us to interpret the conjecture in a convenient way.

By Jannsen's lemma [Ja, Lemma 9.2], the target space of the regulator map r_∞ is given by the $(\cdot)^+$ part of the following 1-extension group

$$\mathrm{Ext}^1_{\mathrm{VMHS}(B)}(\mathbb{R}(0), \mathrm{Sym}^{k-2} V_{2,\mathbb{R}}(1))$$

where $V_{2,\mathbb{R}} := R^1\pi_*\mathbb{R}(1)$ is pure of weight -1 and rank 2. We hope that the motivic cohomology groups we are interested in are described by similar Ext^1-groups in a suitable category $\mathcal{T}(B)$ of smooth motivic sheaves over B. More precisely, we require that

(1) $\mathcal{T}(B)$ satisfies axioms (A) - (G).
(2) Put $V_2 := R^1\pi_*\mathbb{Q}(1)$. Then, there is a canonical isomorphism

$$\mathrm{Ext}^1_{\mathcal{T}(B)}(\mathbb{Q}(0), \mathrm{Sym}^{k-2} V_2(1)) \cong H_{\mathcal{M}}^{k-1}(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1))_{\mathrm{sgn}}$$

which is compatible with base change $B' \to B$ and with isogenies $\mathcal{E}_1 \to \mathcal{E}_2$.
(3) There is an (exact) Hodge realization $\mathcal{T}(B) \to \mathrm{VMHS}(B \otimes_{\mathbb{Q}} \mathbb{C})$ which is compatible with axioms (A) - (G) and with r_∞.

Since we work in a category with the formalism of an elliptic polylogarithm, we use the existence of $\mathcal{P}ol \in \mathcal{T}(\widetilde{\mathcal{E}})$ to construct the map ϕ_k. For any section $s \in \mathcal{E}(B)$, $s \neq 0$, we consider $s^*\mathcal{P}ol \in \mathcal{T}(B)$. Using the Tannakian formalism, it turns out that suitable formal linear combinations of $s^*\mathcal{P}ol$ (varying s) yield extensions in $\mathrm{Ext}^1_{\mathcal{T}(B)}(\mathbb{Q}(0), \mathrm{Sym}^{k-2} V_2(1))$. In order to carry out this task, one considers the graded $\mathbb{Q}$-vector space underlying $s^*\mathcal{P}ol$. It is equipped with a Lie algebra representation which can be described by a pro-matrix. The coefficients of the latter give the desired extensions.

Thus the existence of a "good" category of smooth motivic sheaves implies the elliptic Zagier conjecture. For the details we refer to [Wi3].

Finally we give the known results on the conjecture. We restrict to the case where $\mathcal{E}$ is an elliptic curve defined over a number field K. In the case $k = 2$, the weak version of Zagier's conjecture is already proved in [Wi3]. In the case $k = 3$ and $K = \mathbb{Q}$, it has been proved by Goncharov and Levin [GL], together with a certain surjectivity property of ϕ_k. In the case where $k = 3$ and K is any number field, the conjecture and the surjectivity property have been proved by Rolshausen and Schappacher [RS].

Applications to Special Values of L-functions I

Ralf Gerkmann

The Tamagawa number conjecture due to S. Bloch and K. Kato, as stated in [Bl-Ka] for the first time, relates the integral values of the L-function of a Chow motive M to cohomological data of M. This generalizes a large number of prominent conjectures from number theory, for example the Birch-Swinnerton-Dyer conjecture on abelian varieties or the cohomological Lichtenbaum conjecture. For a special class of motives related to abelian number fields, this conjecture was recently proved by A. Huber and G. Kings, and independently by D. Burns and C. Greither in [BG]. This talk and the subsequent one are based on the paper [HK] of the first two authors. Their aim is to give a rough overview on the proof and its relation to polylogarithms.

Remember that a *Chow motive* M over $\mathbb{Q}$ with coefficients in a number field E is given by a triple (X, q, r) where $X/\mathbb{Q}$ is a smooth projective variety, q is an algebraic correspondence inducing an idempotent map on the K-groups of X, and r is an integer. The *realizations* of M are E-vector spaces coming from Betti and de Rham cohomology, and $E_p := E \otimes_{\mathbb{Q}} \mathbb{Q}_p$-modules coming from étale cohomology. For all primes p, let $X(p)$ denote the reduction mod p of the variety X. By considering the Frobenius action on étale cohomology of $X(p)$ for all p, and by forming an Euler product, we obtain the L-function $L(M, s)$ of the motive M. It is conjectured to enjoy a number of desirable properties, like convergence on some half-plane, analytic continuation and functional equation. The Bloch-Kato conjecture has a precise formulation for motives in arbitrary dimension. In this talk, however, we restrict to the case of *Artin motives*, i.e. special 0-dimensional motives. These are quotients of $h^0(F) = (\mathrm{Spec}(F), \mathrm{id}, 0)$, where F is a number field. If F is abelian over $\mathbb{Q}$, we also call them *Dirichlet motives*.

The *motivic cohomology groups* of a motive M, denoted by $H^i_{\mathcal{M}}(\mathbb{Z}, M)$ with $i = 0, 1$, are defined by the Chow groups and the K-groups of the underlying variety. These groups, together with the de Rham and Betti realization, are used to construct the so-called *fundamental line* $\Delta_f(M)$. The natural map

$$r_\infty : \mathrm{H}^1_{\mathcal{M}}(\mathbb{Z}, M(r)) \to M_B(1-r)^+_{\mathbb{R}}$$

relating motivic cohomology and Betti realization, called the *Beilinson regulator*, allows to define a canonical isomorphism $\iota_\infty : \Delta_f(M(r)) \otimes_{\mathbb{Q}} \mathbb{R} \cong E \otimes_{\mathbb{Q}} \mathbb{R} =: E_\infty$.

On the other hand, there is a *p-adic regulator map*

$$r_p : \mathrm{H}^1_{\mathcal{M}}(\mathbb{Z}, M(r)) \to \mathrm{H}^1(\mathbb{Z}[1/p], M_p(r))$$

defined by C. Soulé relating motivic cohomology with p-adic étale cohomology of $\mathrm{Spec}(\mathbb{Z}[1/p])$. Based on the theory of p-adic Galois representations due to Fontaine, Messing and others, this regulator is used to define a canonical isomorphism

$$\iota_p : \Delta_f(M(r)) \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\cong} \det_{E_p} R\Gamma_c(\mathbb{Z}[1/p], M_p(r))$$

The object on the right hand side in the derived category of E_p-modules is étale cohomology with compact support. Its determinant has two important properties: First, every Galois stable $\mathcal{O}_{E_p}$-sublattice of $M_p(r)$ gives rise to a free $\mathcal{O}_{E_p}$-submodule which is independent of the particular choice of the lattice; here $\mathcal{O}_{E_p}$ denotes the ring of integers in E_p. Secondly, there is a canonical duality isomorphism with usual étale cohomology

$$\det_{E_p} R\Gamma_c(\mathbb{Z}[1/p], M_p(r)) \otimes_{E_p} \det_{E_p} M_p(r)^+ = \det_{E_p} R\Gamma(\mathbb{Z}[1/p], M_p^\vee(1-r))$$

that maps the integral substructures given by a stable lattice onto each other.

With the maps ι_∞ and ι_p for all primes p being constructed, the Bloch-Kato conjecture can now be stated as follows: Let M be an Artin motive, $r \in \mathbb{Z}$ and let

$$L(M,r)^* = \lim_{s \to r} \frac{L(M,s)}{(s-r)^g} \qquad g := \text{order of } L(M,s) \text{ at } r$$

Then there is an element $\delta \in \Delta_f(M(r))$ such that ι_∞ maps $\delta \otimes L(M,r)^*$ onto 1, and such that ι_p maps $\delta \otimes 1$ onto the canonical $\mathcal{O}_{E_p}$-sublattice of $R\Gamma_c(\mathbb{Z}[1/p], M_p(r))$ mentioned above. The main result in [HK] asserts that this conjecture holds for Dirichlet motives up to units in $\mathcal{O}_E[\frac{1}{2}]$. There is also an equivariant version of the Bloch-Kato conjecture which considers motives over $\mathbb{Q}$ with coefficients in $E[\mathrm{Gal}(K|\mathbb{Q})]$ instead of E. This version becomes important when relating the Bloch-Kato conjecture to Iwasawa theory. Let us now give overview on the proof of the main theorem in [HK]. The essential steps are the following:

First the Bloch-Kato conjecture is proved directly for $r = 0$ and $M = h^0(F)$ using the classical analytic class number formula for the field F. Then the compatibility of the Bloch-Kato conjecture with the functional equation of $L(M,s)$ is used in order to derive the result for $r = 1$ and $M = h^0(F)^\vee$.

In the next step the Bloch-Kato conjecture in the equal parity case is reformulated in terms of Euler systems, i.e. collections of elements $c_r(\zeta_N)(\chi) \in H^1(\mathbb{Z}[\zeta_m][1/p], T_p(\chi)(r))$ in the étale cohomology, with coefficients in $T_p(\chi)(r)$ given by the stable $\mathcal{O}_{E_p}$-lattice associated to the motives of the Dirichlet characters χ modulo N. (Here "'equal parity"' means that the parities of the character χ and $r \in \mathbb{Z}$ coincide.) The connection to Euler systems is established via the polylogarithm: On the one hand, the Dirichlet L-series is a linear combination of Hurwitz zeta functions, and the Taylor coefficients of these zeta functions are essentially values of the polylogarithm. On the other hand, the $c_r(\zeta_N)(\chi)$ are sent by the p-adic exponential map onto elements in the de Rham realization which can also be described by values of the polylogarithm. Using these interrelations, one can show that the Bloch-Kato conjecture holds if and only if the elements $c_r(\zeta_N)(\chi)$ generate the $\mathcal{O}_{E_p}$-sublattice in the determinant of étale cohomology.

The Bloch-Kato conjecture in its reformulated version is closely related to the Iwasawa main conjecture for Iwasawa modules attached to Dirichlet characters. Starting with the Bloch-Kato conjecture for $h^0(F)$, this relation is used to derive, step by step, both conjectures in all but finitely many cases. For details, we refer to the following talk by M. Witte. In the more delicate situations (equal parity and $r < 0$), one additionally needs certain explicit elements $b_k(\zeta_N)$ in the motivic cohomology that are related to the polylogarithm and to the Euler system by the regulator maps. At each stage, the unequal parity cases are derived from equal parity via the compatibility with the functional equation. Finally, the Iwasawa main conjecture in the unequal parity case is used in order to handle the remaining finite number of cases.

The talk is concluded by giving some details on the first part of the proof, the class number formula case. Remember that the Betti realization M_B of any motive M has a natural integral substructure, which we denote by T_B. Now let $M = h^0(F)$, where F is a number field. The classical regulator in the Dirichlet unit theorem is given by the volume

$$R_\infty(F) = \mathrm{vol}(\ (T_B^\vee \otimes \mathbb{R})^+/(r_\infty(\mathcal{O}_F^\times/\mu(F)) \oplus s_\infty(\mathbb{Z}))\ ) \tag{1}$$

where r_∞ is the Beilinson regulator, $s_\infty : H^0_{\mathcal{M}}(\mathbb{Z}, h^0(F)) \to (M_B^\vee)^+ \otimes \mathbb{R}$ is the cycle class map and $\mu(F)$ is the subgroup of $\mathcal{O}_F^\times$ given by roots of unity. Let $\delta \in \Delta_f(M)$ be an element such that $\delta \otimes \zeta_F(0)^* = 1$. By the classical analytic class number formula, we have

$$\zeta_F(0)^* = -\frac{R_\infty(F) h_F}{\sharp\mu(F)} \qquad h_F := \text{class number of } F$$

Combining this with (1), we obtain $\delta\mathbb{Z} = \det\ \mathrm{Cl}(\mathcal{O}_F) \otimes \det^{-1}\mathcal{O}_F^\times \otimes \det^{-1} T_B^+$. It remains to check that $\delta \otimes 1$ is a generator of the integral structure in $\det_{E_p}\ R\Gamma_c(\mathbb{Z}[1/p], M_p)$. The main ingredients to achieve this are a Poitou-Tate localization sequence which relates the groups $\mathrm{Cl}(\mathcal{O}_F)$ and $\mathcal{O}_F^\times$ with étale cohomology, and the global duality isomorphism from above that connected usual cohomology and cohomology with compact support.

Application to Special Values of L-Functions II

Malte Witte

The purpose of this talk is to give some insight into the proof of the Bloch-Kato conjecture for Dirichlet motives given in [HK]. It is a direct continuation of the previous talk by Ralf Gerkmann, in which the conjecture was formulated. Up to slight modifications we continue to use the same notation. An excellent survey of the conjecture and the proof can also be found in [Ki2].

Let $h(\chi)(r)$ be the r-th Tate twist of the Dirichlet motive associated to the character $\chi : \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \longrightarrow \mathbb{C}^*$. In the following we focus on the equal parity case, i.e. we consider only those characters with $\chi(-1) = (-1)^r$, and we exclude the special case $r = 0$. The compatibility of the Bloch-Kato conjecture under the functional equation can then be used to show that the conjecture is also valid for the unequal parity case (excluding $r = 1$). The remaining cases have to be treated separately.

Consider the fundamental line $\Delta_f(\chi)(r)$ for the Dirichlet motive $h(\chi)(r)$ with coefficients in a number field E. The ∞-part of the Bloch-Kato conjecture states that there is a $\delta_r \in \Delta_f(\chi)(r)$ such that $\delta_r L(\chi, r)^*$ is mapped to 1 under the isomorphism

$$\Delta_f(\chi)(r) \otimes_{\mathbb{Q}} \mathbb{R} \cong E \otimes_{\mathbb{Q}} \mathbb{R}.$$

The Hurwitz functional equation links the values $L(\chi, r)^*$ with values of the polylogarithms Li_{1-r}. For $r \geq 1$ the functions Li_{1-r} are in fact rational, while the motivic cohomology groups are trivial. In this case, an explicit description of δ_r can be obtained by an elementary calculation from the Hurwitz functional equation and the comparison isomorphism between Betti and deRham cohomology (see [HK, proof of Thm. 3.3.2]). For $r < 0$ the same task can be achieved by the argument of [Be] (with corrections in [Ne] and [Es]) that proves the Beilinson conjecture for number fields. Here, the values of the polylogarithms play again a decisive role.

In both cases one can also describe the image of δ_r under the isomorphism

$$\Delta_f(\chi)(r) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \det_{E_p} \mathbf{R}\Gamma_c(\operatorname{Spec} \mathbb{Z}[1/p], j_*(h(\chi)(r))_p)$$

(where $j : \operatorname{Spec} \mathbb{Q} \longrightarrow \operatorname{Spec} \mathbb{Z}[1/p]$ is the natural inclusion, $E_p = E \otimes \mathbb{Q}_p$, and $(h(\chi)(r))_p$ denotes the p-adic realization of the motive) for any odd prime p. For $r < 0$, this follows from the main result of [BD2], respectively [HW2]; for $r \geq 1$ the description is essentially obtained from Kato's explicit reciprocity law ([Ka, Thm. 5.12]).

It is more convenient first to rephrase the p-part of the Bloch-Kato conjecture using global Poitou-Tate duality. With an appropriate choice of a lattice $T_p(\chi^{-1})$ the statement then reads as follows:

Theorem ([HK], Thm. 3.3.2 and Thm. 5.2.3)**.** *The p-part of the Bloch-Kato conjecture for all $h(\chi)(r)$ with $\chi(-1) = (-1)^r$ is true if and only if the cyclotomic element $c_{1-r}(1)(\chi^{-1})$ is a generator of*

$$\det^{-1}_{\mathcal{O}_{E_p}} \mathbf{R}\Gamma(\operatorname{Spec} \mathbb{Z}[1/p], j_* T_p(\chi^{-1})(1-r)).$$

The cyclotomic element $c_{1-r}(1)(\chi^{-1}) \in \mathbf{H}^1(\operatorname{Spec} \mathbb{Z}[1/p], j_* T_p(\chi^{-1})(1-r))$ forms the first layer of the Euler system mentioned in the previous talk. It can be viewed as an element of the determinant of $\mathbf{R}\Gamma(\operatorname{Spec} \mathbb{Z}[1/p], j_* T_p(\chi^{-1})(1-r)) \otimes^{\mathbb{L}}_{O_{E_p}} E_p$ over E_p since the first cohomology module of this complex has E_p-rank 1, whereas all other cohomology modules vanish. For $r < 0$ the latter result is a deep theorem by Soulé (cf. [So1]); for $r \geq 1$ this can be proved by more elementary arguments (cf. [HK, Lemma 3.3.1]).

The above formulation of the Bloch-Kato conjecture can then be reduced to a version of the Iwasawa main conjecture by climbing up the tower of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Let $\mathbb{Q}_n$ be the unique subfield of $\mathbb{Q}(\zeta_{p^{n+1}})$ with $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) \cong \mathbb{Z}/p^n\mathbb{Z}$ and denote by $\mathbb{Z}_n$ the ring of integers of $\mathbb{Q}_n$. Set

$$\mathbf{R}\Gamma_{gl}(T_p(\chi^{-1})(1-r)) = \varprojlim \mathbf{R}\Gamma(\operatorname{Spec}\mathbb{Z}_n, j_*T_p(\chi^{-1})(1-r)).$$

This is a complex of modules over the Iwasawa algebra

$$\Lambda = \varprojlim \mathcal{O}_{E_p}[\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})].$$

Note that the only non-vanishing cohomology group of $\mathbf{R}\Gamma_{gl}(T_p(\chi^{-1})(1-r)) \otimes^{\mathbb{L}}_{\Lambda} Q(\Lambda)$ is the first one, having $Q(\Lambda)$-rank 1. On the other hand, the sequence $(c_{1-r}(\zeta_{p^n})(\chi^{-1}))_{n=0}^{\infty}$ defines an element $c_{1-r}(\chi^{-1}) \in \mathbf{H}^1_{gl}(T_p(\chi^{-1})(1-r))$. Hence, one can formulate the following main conjecture of Iwasawa theory in complete analogy to the above Bloch-Kato conjecture for all $r \in \mathbb{Z}$.

Theorem ([HK], Thm. 4.4.1)**.** *Assume* $\chi(-1) = (-1)^r$.
Then $c_{1-r}(\chi^{-1}) \in \det^{-1}_{Q(\Lambda)} \mathbf{R}\Gamma_{gl}(T_p(\chi^{-1})(1-r)) \otimes^{\mathbb{L}}_{\Lambda} Q(\Lambda)$ *is a generator of* $\det^{-1}_{Q(\Lambda)} \mathbf{R}\Gamma_{gl}(T_p(\chi^{-1})(1-r))$.

Since

$$\begin{aligned} \mathbf{R}\Gamma_{gl}(T_p(\chi^{-1})(1-r)) \otimes^{\mathbb{L}}_{\Lambda} \mathcal{O}_{E_p} &= \mathbf{R}\Gamma(\operatorname{Spec}\mathbb{Z}[1/p], j_*T_p(\chi^{-1})(1-r)) \\ \Lambda c_{1-r}(\chi^{-1}) \otimes^{\mathbb{L}}_{\Lambda} \mathcal{O}_{E_p} &= \mathcal{O}_{E_p} c_{1-r}(1)(\chi^{-1}), \end{aligned}$$

this theorem is seen to imply the Bloch-Kato conjecture. So it remains to prove the Iwasawa main conjecture. The argument follows closely Rubin's proof of the classical main conjecture (see [Ru]). Alternatively, one can also deduce the above theorem directly from the result of Mazur and Wiles [MW].

One crucial property of the main conjecture is that the statement is invariant under twists in the following sense. Let

$$\epsilon_{cycl} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathbb{Z}_p^*$$

be the cyclotomic character and write $\epsilon_{cycl} = \epsilon \times \epsilon_{\infty}$ according to the decomposition $\mathbb{Z}_p^* \cong \mu_{p-1} \times 1 + p\mathbb{Z}_p$. Then there is an isomorphism

$$\mathbf{R}\Gamma_{gl}(T_p(\chi^{-1})(1-r)) \otimes^{\mathbb{L}}_{\mathcal{O}_{E_p}} \mathcal{O}_{E_p}(\epsilon_{\infty}) \cong \mathbf{R}\Gamma_{gl}(T_p(\chi^{-1}\epsilon^{-1})(2-r))$$

which maps $c_{1-r}(\chi^{-1})$ onto $c_{2-r}(\chi^{-1}\epsilon^{-1})$.

The twist invariance is used to complete the proof of the main conjecture as follows. First, one considers the case $r = 0$ to show that $\det_{\Lambda} \mathbf{R}\Gamma_{gl}(T_p(\chi^{-1})(1-r))$ is contained in the Λ-sublattice of $\det_{Q(\Lambda)} \mathbf{R}\Gamma_{gl}(T_p(\chi^{-1})(1-r)) \otimes^{\mathbb{L}}_{\Lambda} Q(\Lambda)$ spanned by $c_{1-r}(\chi^{-1})$. This is accomplished by the Euler system methods developed by Kolyvagin, Rubin, Kato, and Perrin-Riou. Then the case $r = 1$ is used to reduce the other inclusion to the class-number case of the Bloch-Kato conjecture, which was treated in the previous talk.

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Reporter: Gabriela Schmithüsen

Participants

Prof. Dr. Joseph Ayoub
ayoub@math.jussieu.fr
61 rue Gabriel Peri
F-75005 Paris VII

Shahram Biglari
Shahram.Biglari@math.uni-leipzig.de
Fakultät für Mathematik/Informatik
Universität Leipzig
Augustusplatz 10/11
04109 Leipzig

Prof. Dr. Spencer Bloch
bloch@math.uchicago.edu
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
USA

David Blottiere
davidb@math.univ-paris13.fr
Departement de Mathematiques
Universite Paris 13
Inst. Galilee
Avenue Jean-Baptiste Clement
F-93430 Villetaneuse

Dr. Manuel Breuning
manuel.breuning@maths.nottingham.ac.uk
School of Mathematical Sciences
University of Nottingham
University Park
GB-Nottingham NG7 2RD

Francis Brown
brown@clipper.ens.fr
298 Rue Pelleport
F-33800 Bordeaux

Francois Brunault
brunault@math.jussieu.fr
Institut de Mathematiques
"Theorie de Nombres"
Case 247
175, rue du Chevaleret
F-75013 Paris

Dr. Lars Brünjes
lars.bruenjes@mathematik.uni-regensburg.de
Lehrstuhl für Mathematik
Universitätsstr. 31
93053 Regensburg

Prof. Dr. Ulrich Bunke
bunke@uni-math.gwdg.de
Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Prof. Dr. Christopher Deninger
deninger@math.uni-muenster.de
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Boas Erez
erez@math.u-bordeaux.fr
Boas.Erez@math.u-bordeaux1.fr
Mathematiques et Informatique
Universite Bordeaux I
351, cours de la Liberation
F-33405 Talence Cedex

Elisenda Feliu i Trijueque
efeliu@ub.es
Facultat de Matematiques
Universitat de Barcelona
Departament d'Algebra i Geometria
Gran Via 585
E-08007 Barcelona

Dr. Dan Fulea
fulea@euklid.math.uni-mannheim.de
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Dr. Ralf Gerkmann
gerkmann@mi.uni-erlangen.de
Mathematisches Institut
Universität Erlangen-Nürnberg
Bismarckstr. 1 1/2
91054 Erlangen

Prof. Dr. Günter Harder
harder@mpim-bonn.mpg.de
harder@math.uni-bonn.de
Mathematisches Institut
Universität Bonn
Beringstr. 1
53115 Bonn

Prof. Dr. Frank Herrlich
Frank.Herrlich@math.uni-karlsruhe.de
herrlich@math.uni-karlsruhe.de
Mathematisches Institut II
Universität Karlsruhe
76128 Karlsruhe

Gabriel Herz
gherz@math.uni-muenster.de
Fachbereich Mathematik
Universität Münster
Einsteinstr. 62
48149 Münster

Marco Hien
marco.hien@mathematik.uni-regensburg.de
NWF-I Mathematik
Universität Regensburg
93040 Regensburg

Dr. Jens Hornbostel
jens.hornbostel@mathematik.uni-regensburg.de
Naturwissenschaftliche Fakultät I
Mathematik
Universität Regensburg
93040 Regensburg

Dr. Ivan Izmestiev
izmestie@math.fu-berlin.de
Institut für Informatik
Freie Universität Berlin
Takustr. 9
14195 Berlin

Prof. Dr. Uwe Jannsen
uwe.jannsen@mathematik.uni-regensburg.de
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg

Christian Kaiser
ck@math.uni-bonn.de
Mathematisches Institut
Universität Bonn
Beringstr. 1
53115 Bonn

Moritz Caspar Kerz
mokerz@students.uni-mainz.de
Lindenstraße 32
60325 Frankfurt a.M.

Prof. Dr. Guido Kings
guido.kings@mathematik.uni-regensburg.de
Lehrstuhl für Mathematik
Universitätsstr. 31
93053 Regensburg

Dr. Bernhard Köck
bk@maths.soton.ac.uk
B.Koeck@soton.ac.uk
School of Mathematics
University of Southampton
Highfield
GB-Southampton, SO17 1BJ

Prof. Dr. Dirk Kreimer
kreimer@ihes.fr
dkreimer@bu.edu
CNRS - I.H.E.S.
Le Bois Marie
35, route de Chartres
F-91440 Bures-sur-Yvette

Dr. Stefan Kühnlein
stefan.kuehnlein@mi2.uni-karlsruhe.de
Mathematisches Institut II
Universität Karlsruhe
76128 Karlsruhe

Dr. Thilo Kuessner
kuessner@math.uni-siegen.de
Universität Siegen
Fachbereich 6 Mathematik
Walter-Flex-Str. 3
57068 Siegen

Prof. Dr. Herbert Kurke
kurke@mathematik.hu-berlin.de
Institut für Mathematik
Humboldt-Universität
Rudower Chaussee 25
10099 Berlin

Prof. Dr. Florence Lecomte
lecomte@math.u-strasbg.fr
Institut de Recherche
Mathematique Avancee
ULP et CNRS
7, rue Rene Descartes
F-67084 Strasbourg Cedex

Mathias Lederer
mlederer@Mathematik.Uni-Bielefeld.DE
Fakultät für Mathematik
Universität Bielefeld
Universitätsstr. 25
33615 Bielefeld

Mehran Mehrdad
mmehrdad@ictp.trieste.it
NWF-I Mathematik
Universität Regensburg
93040 Regensburg

Andrea Miller
miller@mathi.uni-heidelberg.de
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Martin Möller
martin.moeller@uni-essen.de
FB 6 - Mathematik
Universität Duisburg-Essen
Standort Essen
45117 Essen

Dr. Florin Nicolae
nicolae@mathematik.hu-berlin.de
Institut für Mathematik
Humboldt-Universität
Rudower Chaussee 25
10099 Berlin

Louisa Orton
louisa.orton@mathematik.uni-regensburg.de
NWF-I Mathematik
Universität Regensburg
93040 Regensburg

Oliver Petras
opetras@mathematik.uni-mainz.de
Fachbereich Mathematik/Informatik
Johannes-Gutenberg Universität MZ
Staudingerweg 9
55099 Mainz

Dr. Alexander Pilz
pilz@mathematik.uni-siegen.de
Institut für Mathematik
Universität Siegen
Emmy-Noether-Campus
57068 Siegen

Gereon Quick
gquick@math.uni-muenster.de
Fachbereich Mathematik
Universität Münster
Einsteinstr. 62
48149 Münster

Gabriela Schmithüsen
schmithuesen@mathematik.uni-karlsruhe.de
Mathematisches Institut II
Universität Karlsruhe
Englerstr. 2
76131 Karlsruhe

Prof. Dr. Stefan Schröer
schroeer@math.uni-duesseldorf.de
Mathematisches Institut
Heinrich-Heine-Universität
Universitätsstr. 1
40225 Düsseldorf

Dr. Christian Serpe
serpe@uni-muenster.de
serpe@math.uni-muenster.de
SFB 478
Geom. Strukturen in der Mathematik
Universität Münster
Hittorfstr. 27
48149 Münster

Denis Vogel
denis.vogel@mathematik.uni-regensburg.de
NWF-I Mathematik
Universität Regensburg
93040 Regensburg

Stefan Wiech
wiech@math.uni-muenster.de
SFB Geometrische Strukturen in der Mathematik
Einsteinstr. 62
48149 Münster

Malte Witte
mam98jpu@studserv.uni-leipzig.de
Mathematisches Institut
Universität Leipzig
Augustusplatz 10/11
04109 Leipzig

Prof. Dr. Zdzislaw Wojtkowiak
wojtkow@math.unice.fr
Lab. J.-A. Dieudonne
U.M.R. no. 6621 du CNRS
Universite de Nice-Sophia Antipolis
Parc Valrose
F-06108 Nice Cedex 02