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Spectral Analysis of Partial Differential Equations

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Abstract. In this workshop talks on discrete spectra, including Lieb-Thirring estimates and properties of resonances were presented. The connection between continuous and discrete spectra was discussed. Various characteristics of continuous spectra in the context of random or magnetic operators were investigated. The Liouville Theorem, the problem of absolute continuity, and the classical problem of homogenization for perodic operators were treated.

Mathematics Subject Classification (2000): 35xx, 46xx, 47xx, 81xx .

Introduction by the Organisers

The goal of the workshop was to bring together specialists working in various branches of spectral theory with applications to solid state physics, superconductivity, quantum mechanics etc. The meeting was attended by more than 45 participants from Europe, Japan, Russia, South America and US. During the five days 26 talks were delivered. A special care was taken that apart from the recognized experts in the field, young participants also had an opportunity to speak about their results. The Wednesday morning session, preceding the traditional afternoon hike, consisted of talks of survey nature, which was appreciated by all.

There were several major themes in the workshop. One was the study of discrete spectra, including Lieb-Thirring estimates, properties of resonances. A substantial number of talks was concerned with the connection between the continuous and discrete spectra. These include, in particular, the study of the so-called trace formulas. The investigation of various characteristics of the continuous spectra (e.g. density of states, spectral shift function) was featured in a number of talks in the context of random or magnetic operators. A variety of new results were also reported on the theory of periodic operators. They concerned the Liouville Theorem, the problem of absolute continuity, and the classical problem of homogenization.

A relatively low number of talks gave the participants an opportunity for discussions in small groups outside the scheduled lecture time. It is hoped that these contacts will result in further collaboration.

It is our pleasure to thank the administration and staff of the *Mathematisches Forschungsinstitut Oberwolfach* for creating comfortable and genuinely inspiring atmosphere, which facilitated the work of the organisers and contributed to the success of the workshop.

Workshop: Spectral Analysis of Partial Differential Equations

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Abstracts

A Fermi Golden Rule at thresholds

Arne Jensen
(joint work with Gheorghe Nenciu)

We describe our main results in the form of an example. Consider a Schrödinger operator

$$H = -\Delta + V \quad \text{on } L^2(\mathbf{R}^3),$$

where we assume that $V \in C_0^\infty(\mathbf{R}^3)$. The essential spectrum is $[0,\infty)$, and is purely absolutely continuous. There may be a finite number of negative eigenvalues, and an eigenvalue at zero. We assume here that 0 is a non-degenerate eigenvalue, with normalized eigenfunction Ψ_0. We study what happens to this eigenvalue under small perturbations. Let $W \in C_0^\infty(\mathbf{R}^3)$. To avoid the case that 0 becomes a negative discrete eigenvalue, we introduce

Assumption (A1)**.** $b = \langle \Psi_0, W\Psi_0 \rangle > 0.$

The we consider

$$H(\varepsilon) = H + \varepsilon W, \quad \varepsilon > 0.$$

We show that the zero eigenvalue becomes a resonance, in the time-dependent sense introduced by A. Orth [5]. In order to formulate the main result we need some further results and assumptions.

In the resolvent $R(z) = (H-z)^{-1}$ we change the variable to $\kappa = -i\sqrt{z}$, $\operatorname{Im} z > 0$, $\operatorname{Re}\kappa \geq 0$. It is well-known (see [2]) that we have an asymptotic expansion as $\kappa \to 0$

$$R(-\kappa^2) = \frac{1}{\kappa^2}P_0 + \sum_{j=-1}^{N} \kappa^j G_j + \mathcal{O}(\kappa^{N+1}),$$

valid in the topology of the weighted spaces, $\mathcal{B}(L^s(\mathbf{R}^3), L^{-s}(\mathbf{R}^3))$, for s sufficiently large, depending on N.

We can now formulate the next essential assumption.

Assumption (A2)**.** *There exists an odd integer $\nu \geq -1$, such that*

$$g_\nu = \langle \Psi_0, WG_\nu W\Psi_0 \rangle \neq 0, \quad G_j = 0, \quad j = -1, 1, 3, \ldots, \nu - 2.$$

Our main result can then be formulated as follows.

Theorem. *There exists $\varepsilon_0 > 0$ such that*

$$\langle \Psi_0, e^{-itH(\varepsilon)}\Psi_0 \rangle = e^{-it\lambda(\varepsilon)} + \delta(\varepsilon, t), \quad t > 0, \quad 0 < \varepsilon < \varepsilon_0.$$

Here

$$|\delta(\varepsilon, t)| \leq C\varepsilon^{p(\nu)} |\ln \varepsilon|^\iota,$$

where $\iota = 1$ *for* $\nu = -1, 1$, *and zero otherwise. We write* $p(\nu) = \min\{2, (2+\nu)/2\}$. *We have* $\lambda(\varepsilon) = x_0(\varepsilon) - i\Gamma(\varepsilon)$, *with the expansions*

$$x_0(\varepsilon) = b\varepsilon(1 + \mathcal{O}(\varepsilon)),$$
$$\Gamma(\varepsilon) = -i^{\nu-1} g_\nu b^{\nu/2} \varepsilon^{2+(\nu/2)}(1 + \mathcal{O}(\varepsilon)),$$

as $\varepsilon \to 0$.

We note that $-i^{\nu-1}g_\nu > 0$. Our main result holds in an abstract setting, where we assume the existence of an asymptotic expansion of the type above for the resolvent of H.

The result shows how the Fermi Golden Rule has to be modified, to get the lifetime of the resonance. Notice that in the usual case of perturbation of an eigenvalue embedded in the continuum proper, the coupling constant dependence for the imaginary part is ε^2, whereas we have $\varepsilon^{2+(\nu/2)}$, $\nu \geq -1$ and odd. All possible values of ν can be shown to occur in explicit examples.

It is possible to compute g_ν explicitly. In the example under consideration we have the following result. Take Ψ_0 real-valued, and let

$$X_j = \int_{\mathbf{R}^3} \Psi_0(x) V(x) x_j dx, \quad j = 1, 2, 3.$$

Assume that at least one $X_j \neq 0$. Then $\nu = -1$, and we have

$$g_{-1} = \frac{b^2}{12\pi}(X_1^2 + X_2^2 + X_3^2).$$

Resonances can also be defined as poles of a meromorphic continuation of the resolvent $R(z)$ is a suitable sense. The problem of perturbation of a threshold eigenvalue was studied by B. Baumgartner [1] in a two channel setting, using meromorphic continuation. He also gave heuristics for the modification of the Fermi Golden Rule, which agrees with our main theorem.

In [4] we give complete results, and, based on the resolvent expansions in [3], we give a large number of examples of both one channel and two channel Schrödinger operators satisfying our assumptions.

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Homogenization of periodic differential operators in $\mathbb{R}^d$ as a spectral threshold effect

M. Sh. Birman
(joint work with T. A. Suslina)

In [1], the *spectral approach* to homogenization problems for one class of selfadjoint elliptic matrix second order differential operators is systematically developed. On the basis of the Floquet-Bloch decomposition, it is shown that homogenization is a *threshold effect* near the bottom of the spectrum. In what follows, we explain this point of view and discuss the typical results. The results presented in Section 4 are new.

1. Let $\Gamma \subset \mathbb{R}^d$ be a lattice, and let Ω be the cell of Γ. We use the notation $\mathfrak{G} = L_2(\mathbb{R}^d;\mathbb{C}^n)$, $\mathfrak{G}_* = L_2(\mathbb{R}^d;\mathbb{C}^m)$, $\mathbf{D} = -i\nabla$. It is assumed that $m \geq n$. Let h be an $(m \times m)$-matrix-valued Γ-periodic function in $\mathbb{R}^d$ such that $h, h^{-1} \in L_\infty(\mathbb{R}^d)$. We put $g = h^*h$. Let $b(\boldsymbol{\xi})$, $\boldsymbol{\xi} \in \mathbb{R}^d$, be an $(m\times n)$-matrix-valued linear homogeneous function such that $\operatorname{rank} b(\boldsymbol{\xi}) = n$ for $\boldsymbol{\xi} \neq 0$. Then

$$\alpha_0 \mathbf{1}_n \leq b(\boldsymbol{\theta})^* b(\boldsymbol{\theta}) \leq \alpha_1 \mathbf{1}_n, \quad |\boldsymbol{\theta}| = 1, \quad 0 < \alpha_0 \leq \alpha_1 < \infty.$$

We consider the first order differential operator $hb(\mathbf{D}) = \mathcal{X} : \mathfrak{G} \to \mathfrak{G}_*$; $\operatorname{Dom} \mathcal{X} = H^1(\mathbb{R}^d;\mathbb{C}^n)$. Here H^1 is the Sobolev space. Then the operator $\mathcal{A}(g) = \mathcal{X}^*\mathcal{X} = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})$ is selfadjoint in $\mathfrak{G}$.

Our main object is the operator family $\mathcal{A}_\varepsilon(g) = \mathcal{A}(g^\varepsilon)$, where $g^\varepsilon(\mathbf{x}) = g(\varepsilon^{-1}\mathbf{x})$, $\varepsilon > 0$. We study the behavior of solutions of the equation

$$\mathcal{A}_\varepsilon(g)\mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon = \mathbf{F}, \quad \mathbf{F} \in \mathfrak{G}, \tag{1}$$

as $\varepsilon \to 0$.

2. The following definition of the constant effective matrix g^0 is standard for the homogenization theory. Let $\mathbf{C} \in \mathbb{C}^m$, and let $\mathbf{w}$ be a weak Γ-periodic solution of the equation

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\mathbf{w} + \mathbf{C}) = 0. \tag{2}$$

Then g^0 is defined by the relation

$$g^0\mathbf{C} = |\Omega|^{-1} \int_\Omega g(\mathbf{x})(b(\mathbf{D})\mathbf{w} + \mathbf{C})\, d\mathbf{x}.$$

The effective matrix satisfies the estimates

$$|\Omega| \left(\int_\Omega (g(\mathbf{x}))^{-1}\, d\mathbf{x} \right)^{-1} = \underline{g} \leq g^0 \leq \overline{g} = |\Omega|^{-1} \int_\Omega g(\mathbf{x})\, d\mathbf{x}.$$

Theorem 1. *We have*

$$\|(\mathcal{A}(g) + \varepsilon^2 I)^{-1} - (\mathcal{A}(g^0) + \varepsilon^2 I)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}} \leq C\varepsilon^{-1}, \quad 0 < \varepsilon \leq 1, \tag{3}$$

where the constant C depends only on Γ, α_0, α_1, $\|h\|_{L_\infty}$, $\|h^{-1}\|_{L_\infty}$.

The estimate (3) is of *threshold nature*, since we consider the resolvent in point $(-\varepsilon^2)$, i. e., near the bottom of the spectrum. (We have $\inf \operatorname{spec} \mathcal{A}(g) = 0$.)

3. Along with equation (1), we consider the *homogenized* eqution $\mathcal{A}(g^0)\mathbf{u}_0 + \mathbf{u}_0 = \mathbf{F}$. By traditional means of homogenization theory, it is easily proved that $\mathbf{u}_\varepsilon$ tends to $\mathbf{u}_0$ weakly in $H^1(\mathbb{R}^d; \mathbb{C}^n)$. Using the spectral approach, we prove the following result, which complements this statement essentially.

Theorem 2. *Let C be the constant from inequality* (3)*. Then*

$$\|(\mathcal{A}_\varepsilon(g) + I)^{-1} - (\mathcal{A}(g^0) + I)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}} \leq C\varepsilon, \quad 0 < \varepsilon \leq 1. \tag{4}$$

Apparently, estimates of the form (4) are new for homogenization theory. In fact, estimates (3) and (4) are equivalent. Indeed, let T_ε be the unitary scale transformation in $\mathfrak{G}$: $(T_\varepsilon \mathbf{u})(\mathbf{x}) = \varepsilon^{d/2}\mathbf{u}(\varepsilon\mathbf{x})$. Then

$$(\mathcal{A}_\varepsilon(g) + I)^{-1} = \varepsilon^2 T_\varepsilon^* (\mathcal{A}(g) + \varepsilon^2 I)^{-1} T_\varepsilon.$$

The operator $\mathcal{A}(g^0)$ satisfies similar identity, but $(g^0)^\varepsilon = g^0$.

Note that using the scale transformation is possible only for the estimates in the operator norm. For the study of convergence of different types, this method does not work.

4. In homogenization theory, adding appropriate *correction term* of order ε to $\mathbf{u}_0$, one obtains more accurate approximation for $\mathbf{u}_\varepsilon$. This correction term contains some rapidly oscillating factors. This way is also possible in L_2-theory. Here we present the corresponding result for the simplest case, namely, for the operator $\mathcal{A}(g) = \mathbf{D}^* g(\mathbf{x})\mathbf{D} = -\operatorname{div} g(\mathbf{x})\nabla$ (now $n = 1$, $m = d$, $b(\boldsymbol{\xi}) = \boldsymbol{\xi}$). Let $v_j(\mathbf{x})$, $j = 1, \ldots, d$, be the Γ-periodic solution of the equation $\mathbf{D}^* g(\mathbf{x})(\mathbf{D}v_j(\mathbf{x}) + \mathbf{e}_j) = 0$ such that $\int_\Omega v_j(\mathbf{x})\, d\mathbf{x} = 0$. Here $\{\mathbf{e}_j\}$, $j = 1, \ldots, d$, is the standard basis in $\mathbb{R}^d$. By $\Lambda(\mathbf{x})$ we denote the matrix-row $\{v_1(\mathbf{x}), v_2(\mathbf{x}), \ldots, v_d(\mathbf{x})\}$. Then $\Lambda(\mathbf{x})$ is Γ-periodic. We put $\Lambda^\varepsilon(\mathbf{x}) = \Lambda(\varepsilon^{-1}\mathbf{x})$, and consider the operator $Z_\varepsilon : \mathfrak{G} \to \mathfrak{G}$,

$$Z_\varepsilon = \Lambda^\varepsilon \mathbf{D}(\mathcal{A}(g^0) + I)^{-1}.$$

Theorem 3. *For the operator $\mathcal{A}(g) = \mathbf{D}^* g(\mathbf{x})\mathbf{D}$ under the above assumptions we have*

$$\|(\mathcal{A}_\varepsilon(g) + I)^{-1} - (\mathcal{A}(g^0) + I)^{-1} - \varepsilon(Z_\varepsilon + Z_\varepsilon^*)\|_{\mathfrak{G}\to\mathfrak{G}} \leq C_*\varepsilon^2, \quad 0 < \varepsilon \leq 1, \tag{5}$$

where C_* depends only on Γ, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$.

Remarks. 1) In homogenization theory, the traditional correction term is Z_ε. However, in L_2-theory, in order to obtain the precise estimate (5), we have to take the symmetric expression $(Z_\varepsilon + Z_\varepsilon^*)$. In the case where the columns of $g(\mathbf{x})$ are divergence free, we have $Z_\varepsilon = 0$ and then

$$\|(\mathcal{A}_\varepsilon(g) + I)^{-1} - (\mathcal{A}(g^0) + I)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}} \leq C_*\varepsilon^2, \quad 0 < \varepsilon \leq 1.$$

2) The estimate similar to (5) is true for the matrix operators $\mathcal{A}(g)$ defined in Section 1, if $m = n$. Apparently, in the general case one more additional summand should be added in the correction term.

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Finite Casimir energy for the electromagnetic field in a cavity

G.M. Graf

(joint work with F. Bernasconi, D. Hasler)

We present a Hilbert space formulation [3] of the *classical* Maxwell equations in a cavity $\Omega \subset \mathbb{R}^3$. In a preliminary Hilbert space $L^2(\Omega, \mathbb{C}^3)$ of (complex-valued) vector fields on Ω we define the dense subspaces

$$\mathcal{R} = \{\mathbf{V} \in L^2(\Omega, \mathbb{C}^3) \mid \operatorname{rot} \mathbf{V} \in L^2(\Omega, \mathbb{R}^3)\}\ ,$$
$$\mathcal{R}_0 = \{\mathbf{V} \in \mathcal{R} \mid \langle \mathbf{U}, \operatorname{rot} \mathbf{V}\rangle = \langle \operatorname{rot} \mathbf{U}, \mathbf{V}\rangle, \forall \mathbf{U} \in \mathcal{R}\}$$

and the (closed) operator $R = \operatorname{rot}$ with domain $\mathcal{D}(R) = \mathcal{R}_0$. Its adjoint is $R^* = \operatorname{rot}$ with $\mathcal{D}(R^*) = \mathcal{R}$. We remark that R, resp. R^*, is also the closure of rot defined on smooth vector fields $\mathbf{V}$ with boundary condition $\mathbf{V}_\parallel = 0$ on the smooth boundary $\partial\Omega$, resp. without boundary conditions. Similarly, gradients ∇, $\tilde{\nabla}$: $L^2(\Omega) \to L^2(\Omega, \mathbb{C}^3)$ can be defined with domains $\mathcal{D}(\nabla) = \{\varphi \in L^2(\Omega) \mid \nabla\varphi \in L^2(\Omega),\ \varphi = 0 \text{ on } \partial\Omega\}$, resp. $\mathcal{D}(\tilde{\nabla})$ without the last boundary condition. Clearly, $\operatorname{Ran} \nabla \subset \operatorname{Ker} R$, $\operatorname{Ran} \tilde{\nabla} \subset \operatorname{Ker} R^*$, so that

$$\begin{aligned} \operatorname{Ran} R^* &\subset (\operatorname{Ker} R)^\perp \subset (\operatorname{Ran} \nabla)^\perp =: \mathcal{H}\ , \\ \operatorname{Ran} R &\subset (\operatorname{Ker} R^*)^\perp \subset (\operatorname{Ran} \tilde{\nabla})^\perp =: \mathcal{H}'\ . \end{aligned} \tag{1}$$

Therefore the Maxwell operator

$$M = \begin{pmatrix} 0 & \mathrm{i}R^* \\ -\mathrm{i}R & 0 \end{pmatrix} = M^*$$

on $L^2(\Omega, \mathbb{C}^3) \oplus L^2(\Omega, \mathbb{C}^3)$ restricts to the invariant subspace $\mathcal{H} \oplus \mathcal{H}'$, which is the *physical* Hilbert space for electromagnetic fields $(\mathbf{E}, \mathbf{B})$. Indeed, the spaces

$$\begin{aligned} \mathcal{H} &= \{\mathbf{E} \in L^2(\Omega, \mathbb{C}^3) \mid \operatorname{div} \mathbf{E} = 0\}\ , \\ \mathcal{H}' &= \{\mathbf{B} \in L^2(\Omega, \mathbb{C}^3) \mid \operatorname{div} \mathbf{B} = 0,\ \mathbf{B}_\perp = 0 \text{ on } \partial\Omega\} \end{aligned} \tag{2}$$

consist of divergence free fields and the Maxwell equations can be written as

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = M\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}. \tag{3}$$

The usual boundary conditions $\mathbf{E}_\parallel = 0$, $\mathbf{B}_\perp = 0$ on the ideally conducting shell $\partial\Omega$ are accounted for through $\mathcal{D}(R)$, resp. $\mathcal{H}'$.

Remark. In [2] we defined M as an operator on $\mathcal{H} \oplus \mathcal{H}$. The difference consists of fields $(\mathbf{E}, \mathbf{B}) = (0, \nabla\psi)$ with ψ harmonic, and hence of (infinitely many) zero modes of M, which are irrelevant to the Casimir energy, see below.

We discuss the heat kernel traces for $M^2 = \mathrm{diag}(R^*R, RR^*)$,

$$\mathrm{Tr}_{\mathcal{H}}(\mathrm{e}^{-tM^2}) = \sum_k \mathrm{e}^{-t\omega_k^2} \cong \sum_{n=0}^{\infty} a_n t^{\frac{n-3}{2}} , \qquad (t \downarrow 0) , \tag{4}$$

(and similarly for $\mathcal{H}'$ with coefficients a'_n), where ω_k^2 are the eigenvalues of R^*R on $\mathcal{H}$, resp. RR^* on $\mathcal{H}'$. They come in pairs, except for zero modes, and correspond to a single oscillator mode $\omega_k > 0$ for (3). The coefficients a_n are known, see e.g. [6, 4], for general operators of Laplace type. The direct application of such results is prevented by the divergence constraint in $\mathcal{H}$ and $\mathcal{H}'$, see (2).

Let $L_{ab} = (\nabla_{\mathbf{e}_a}\mathbf{e}_b, \mathbf{n})$, $(a, b = 1, 2)$, be the second fundamental form on the boundary $\partial\Omega$ with inward normal $\mathbf{n}$ and local orthonormal frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}$. We denote by $|\Omega|$ the volume of Ω and set $f[\partial\Omega] = \int_{\partial\Omega} f(y)\mathrm{d}y$, where $\mathrm{d}y$ is the (induced) Euclidean surface element on $\partial\Omega$. The corresponding Laplacian on $\partial\Omega$ is denoted by ∇^2.

Theorem. [2] *Let $\Omega \subset \mathbb{R}^3$ be an open, connected domain with compact closure and smooth boundary $\partial\Omega$. Then*

$$\begin{aligned}
a_0 &= 2(4\pi)^{-\frac{3}{2}}|\Omega| , \qquad a_1 = 0 , \qquad a_2 = -\frac{4}{3}(4\pi)^{-\frac{3}{2}}(\mathrm{tr}L)[\partial\Omega] , \\
a_3 &= \frac{1}{64}(4\pi)^{-1}\big(3(\mathrm{tr}L)^2 + 28\det L\big)[\partial\Omega] , \\
a_4 &= \frac{16}{315}(4\pi)^{-\frac{3}{2}}\big(2(\mathrm{tr}L)^3 - 9\mathrm{tr}L\cdot\det L\big)[\partial\Omega] , \\
a_5 &= \frac{1}{122880}(4\pi)^{-1}\big(2295(\mathrm{tr}L)^4 - 12440(\mathrm{tr}L)^2\det L + \\
&\qquad + 13424(\det L)^2 + 1200\mathrm{tr}L\cdot\nabla^2\mathrm{tr}L\big)[\partial\Omega] .
\end{aligned}$$

The coefficients a'_n are the same, except for $n = 3$, where

$$a'_3 = \frac{1}{64}(4\pi)^{-1}\big(3(\mathrm{tr}L)^2 - 36\det L\big)[\partial\Omega] + 1 .$$

By the Gauss-Bonnet theorem we have $a_3 - a'_3 = (4\pi)^{-1}(\det L)[\partial\Omega] - 1 = \sum_{i=1}^n (1 - g_i) - 1 = (n-1) - \sum_{i=1}^n g_i$, where $g_1, g_2, \ldots, g_n$ are the genera of the n connected components of $\partial\Omega$. This equals the difference in the numbers of electrostatic, $n-1$, and magnetostatic, $\sum_{i=1}^n g_i$, modes.

Sketch of proof. The transversal modes of the electromagnetic field, together with their unphysical, longitudinal counterparts in $\mathrm{Ran}\,\nabla$ and $\mathrm{Ran}\,\tilde{\nabla}$, see (1), are the eigenfunctions of the Laplacian acting on unconstrained vector fields, to which existing heat kernel expansions may be applied. The spurious contribution so introduced is essentially that of the Laplacian on scalar fields. Alternatively, consider the Laplacian of the de Rham complex of a 3-manifold with boundary. The electric and magnetic fields are then associated to forms of degree $p = 1$ and $p = 2$ respectively. In this correspondence transverse modes are associated with coexact, resp. exact forms, which permit to map longitudinal modes to forms of degree $p = 0$ and $p = 3$.

We apply the Theorem to the Casimir effect of the *quantum* field. To this end we retain: (i) The Weyl term a_0 is proportional to the volume of the cavity; (ii) $a_1 = 0$; (iii) a_2, a_4 are odd in the second fundamental form of the boundary; (iv) the asymptotic series (4) may be differentiated w.r.t. t. For the purpose of this discussion we simply define the Casimir energy by the mode summation method, see e.g. [1]. We shall observe that it is finite – a conclusion drawn in [1], but questioned in [9]. We do not however address the issue [7] of whether this definition is the most appropriate physically, nor do we compare it with others, based e.g. on the local energy density.

We enclose the cavity $\Omega \subset \mathbb{R}^3$ in a large ball Ω_0 and compare the vacuum energy of the electromagnetic field in the domains $\Omega \cup (\Omega_0 \setminus \overline{\Omega})$ with that of the reference domain Ω_0. Each eigenmode of either configuration contributes a zero-point energy $\omega_k/2$, resp. $\omega_k^0/2$. As a regulator for the eigenfrequencies $\omega_k = \lambda_k^{1/2}$, we choose $\mathrm{e}^{-\gamma\lambda_k}$, $(\gamma > 0)$. The corresponding definition of the Casimir energy is

$$E_C = \frac{1}{2} \lim_{\Omega_0 \uparrow \mathbb{R}^3} \lim_{\gamma \downarrow 0} \Big(\sum_k \lambda_k^{\frac{1}{2}} \mathrm{e}^{-\gamma\lambda_k} - \sum_k (\lambda_k^0)^{\frac{1}{2}} \mathrm{e}^{-\gamma\lambda_k^0} \Big) .$$

We now show that the limit $\gamma \downarrow 0$ is finite. (The subsequent limit $\Omega_0 \uparrow \mathbb{R}^3$ also exists.) Using $\lambda_k^{1/2} = -\pi^{-1/2} \int_0^\infty \mathrm{d}t\, t^{-1/2} \mathrm{d}(\mathrm{e}^{-t\lambda_k})/\mathrm{d}t$ we obtain

$$\begin{aligned} \sum_k \lambda_k^{\frac{1}{2}} \mathrm{e}^{-\gamma\lambda_k} &\approx -\sum_{n=0}^{4} \frac{n-3}{2\sqrt{\pi}} a_n \int_0^\delta dt\, t^{-\frac{1}{2}} (t+\gamma)^{\frac{n-5}{2}} \\ &\approx \frac{2}{\sqrt{\pi}} a_0 \gamma^{-2} + \frac{\sqrt{\pi}}{2} a_1 \gamma^{-\frac{3}{2}} + \frac{1}{\sqrt{\pi}} a_2 \gamma^{-1} + 0 \cdot a_3 \gamma^{-\frac{1}{2}} + \frac{1}{2\sqrt{\pi}} a_4 \log\gamma , \end{aligned}$$

where $\delta > 0$ is arbitrary, but fixed, and "$\approx$" means up to bounded terms as $\gamma \downarrow 0$. Hence a finite E_C requires that a_0, a_1, a_2, a_4 (but not necessarily a_3!) agree for $\Omega \cup (\Omega_0 \setminus \overline{\Omega})$ and for the reference domain Ω_0 [8], [5]. By the Theorem this is so for a_0 and a_1, but also for a_2, a_4 as the contribution from the two sides of $\partial\Omega$ cancel.

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The band-edge behavior of the density of surface states

Frédéric Klopp
(joint work with Werner Kirsch)

This talk is devoted to the integrated density of surface states for a simple discrete model of surface random operators (see e.g. [3, 1, 2, 7]). We study the asymptotic behavior of this quantity near the edges of the spectrum of the random model. The results are taken from [5, 6].
On $\mathbb{Z}^d$ ($d = d_1 + d_2$, $d_1 > 0$, $d_2 \geq 3$), we consider random Hamiltonians of the form

$$H_\omega = -\Delta + V_\omega \tag{1}$$

where

(H0): Let H be a translational invariant Jacobi matrix with exponential off-diagonal decay that is $H = ((h_{\gamma-\gamma'}))_{\gamma,\gamma'\in\mathbb{Z}^d}$ such that,
- $h_{-\gamma} = \overline{h_\gamma}$ for $\gamma \in \mathbb{Z}^d$ and for some $\gamma \neq 0$, $h_\gamma \neq 0$.
- there exists $c > 0$ such that, for $\gamma \in \mathbb{Z}^d$,

$$|h_\gamma| \leq \frac{1}{c} e^{-c|\gamma|}. \tag{2}$$

(H1): V_ω is a random potential concentrated on the sub-lattice $\mathbb{Z}^{d_1} \times \{0\} \subset \mathbb{Z}^d$ of the form

$$V(\gamma_1, \gamma_2) = \begin{cases} \omega_{\gamma_1} & \text{if } \gamma_2 = 0, \\ 0 & \text{if } \gamma_2 \neq 0. \end{cases}, \gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2} = \mathbb{Z}^d. \tag{3}$$

and $(\omega_{\gamma_1})_{\gamma_1 \in \mathbb{Z}^{d_1}}$ is a family of non trivial i.i.d. bounded random variables.
The operator H_ω is bounded for almost every ω. It is ergodic. So we know there exists Σ the almost sure spectrum of H_ω (see e.g. [4, 8]). Note that the Σ_0 contains the spectrum of H.

Remark 1. *An interesting case which can be brought back to a Hamiltonian of the form* (1) *with H and V_ω as above is the following.*
Consider Γ, a sub-lattice of $\mathbb{Z}^d$ obtained in the following way $\Gamma = G(\{0\} \times \mathbb{Z}^{d_2})$ where G is a matrix in $GSL_d(\mathbb{Z})$, the d-dimensional special linear group over $\mathbb{Z}$, i.e. the multiplicative group of invertible matrices with coefficients in $\mathbb{Z}$ and unit determinant. One easily shows that the random operator

$$H_\omega(\Gamma) = -\frac{1}{2}\Delta + \sum_{\gamma\in\Gamma} \omega_\gamma \Pi_\gamma$$

(where Π_γ is the projector onto the vector $\delta_\gamma \in \ell^2(\mathbb{Z}^d)$) is unitarily equivalent to $H + V_\omega$ where V_ω is defined in (3) *for h chosen appropriately (see* [5]*).*

For H_ω as in (1) and satisfying (H0) and (H1), one defines the integrated density of surface states (the IDSS in the sequel), say N_s , in the following way (see e.g. [3, 1, 2, 7]): for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, we set

$$(\varphi'', N_s) = \mathbb{E}(\mathrm{tr}(\Pi_1[\varphi(H_\omega) - \varphi(-\Delta)]\Pi_1)) \tag{4}$$

where Π_1 is the orthogonal projector on the subspace $\mathbb{C}\delta_0 \otimes \ell^2(\mathbb{Z}^{d_2}) \subset \ell^2(\mathbb{Z}^d)$. Here δ_0 denotes the vector with components $(\delta_{0j})_{j\in\mathbb{Z}^{d_1}}$.
We normalize N_s so that it vanishes below Σ. In [5], we prove that the function N_s is continuous.
We now present our results on the behavior of N_s near the lower edge of Σ (the study near the upper edge is the same). To fix ideas, assume that $0 = \inf \Sigma$.

Definition 2. *We say that E, an edge (or boundary) of the spectrum of H_ω, is stable if it is an edge of the spectrum of $H + tV_\omega$ for all $t \in [0, 1]$. If an edge is not stable, we call it a fluctuation edge.*

Let ω_- be the infimum of the support of the random variables $(\omega_{\gamma_1})_{\gamma_1}$. Let $h(\theta)$ be the real analytic function

$$h(\theta) = \sum_{\gamma\in\mathbb{Z}^d} h_\gamma e^{i\gamma\theta}.$$

One checks

Proposition 3 ([5])**.** *Write $h(\theta) = h(\theta_1, \theta_2)$ where $\theta = (\theta_1, \theta_2)$, $\theta_1 \in \mathbb{T}^{d_1}$, $\theta_2 \in \mathbb{T}^{d_2}$. Then, 0 is a stable spectral edge if and only if ω_- satisfies condition*

$$1 + \omega_- I_\infty \geq 0 \text{ where } I_\infty := \sup_{\theta_1\in\mathbb{T}^{d_1}} \int_{\mathbb{T}^{d_2}} \frac{1}{h(\theta_1, \theta_2)} d\theta_2 \tag{5}$$

1. The fluctuation edges

We now assume that $\inf \sigma(H) > 0$. In this case, we consider a effective operator $\tilde{H}$ which acts on $\ell^2(\mathbb{Z}^{d_1})$. In Fourier representation this operator is multiplication by the function $\tilde{h}$ given by:

$$\tilde{h}(\theta_1) = \left(\int_{\mathbb{T}^{d_2}} \frac{1}{h(\theta_1, \theta_2)}\, d\theta_2\right)^{-1} \tag{6}$$

We either suppose:

- **(H2):** the function $h : \mathbb{T}^d \to \mathbb{R}$ admits a unique minimum; it is quadratic non-degenerate.

or we assume the weaker hypothesis:

- **(H2'):** the function $\tilde{h} : \mathbb{T}^d \to \mathbb{R}$ is not constant.

Let P_0 be the common distribution of the random variables $(\omega_{\gamma_1})_{\gamma_1}$ defining the potential (3). We assume:

- **(H3):** P_0 is not trivial and $P_0([\omega_-, \omega_- + \varepsilon)) \geq \varepsilon^k/k$ for some $k > 0$.

We prove

Theorem 4 ([5]). *If (H0) – (H2) and (H3) are satisfied then*

$$\lim_{E\searrow 0} \frac{\ln|\ln(N_s(E))|}{\ln E} = -\frac{d_1}{2}.$$

We have an additional result for low dimension of the surface:

Theorem 5 ([5]). *Assume (H0) – (H2') and (H3) hold. If $d_1 = 1$ then*

$$\lim_{E\searrow 0} \frac{\ln|\ln(N_s(E))|}{\ln E} = -\lim_{E\searrow 0} \frac{\ln(n(E-\omega_-))}{\ln E} \tag{7}$$

where $n(E)$ is the integrated density of states for $\tilde{H}$.
If $d_2 = 2$, then

$$\lim_{E\searrow 0} \frac{\ln|\ln(N_s(E))|}{\ln(E} < 0. \tag{8}$$

Both limits (7) and (8) can be computed in terms of the Taylor series of $\tilde{h}$ at its minima (see [5] for details).

2. The stable edges when $d_2 \in \{1,2\}$

In this case, we prove

Theorem 6. *Assume (H0) and (H2) hold. Assume, moreover, that 0 is a stable spectral edge for H_ω. Then,*

- *if $d_2 = 1$: $N_s(E) \underset{E\to 0^+}{\sim} \frac{Vol(\mathbb{S}^{d_1-1})\cdot C(h)}{d_1(d_1+2)(2\pi)^{d_1}} E^{1+d_1/2}$;*
- *if $d_2 = 2$: $N_s(E) \underset{E\to 0^+}{\sim} \frac{2\,Vol(\mathbb{S}^{d_1-1})\cdot C(h)}{d_1(d_1+2)(2\pi)^{d_1}} \frac{E^{1+d_1/2}}{|\log E|}$.*

Here, the constant $C(h)$ depends only of the Hessian of h at its minimum.

The striking feature is that to first order these asymptotics are independent of the random potential. The reason for this is that the asymptotics of integrated density of surface states for a constant surface potential near a stable edge does not depend on the value of the potential (to leading order).

3. The stable edges when $d_2 \geq 3$

We assume that

(H3): for almost every θ_1, the function $\theta_2 \mapsto h(\theta_1, \theta_2)$ is not constant.

Consider the embedding $U_2 : \ \ell^2(\mathbb{Z}^{d_1}) \to \ell^2(\mathbb{Z}^d)$ defined by $v = U_2(u)$ where

$$v_{\gamma_1,\gamma_2} = u_{\gamma_1}\delta_{\gamma_2,0} \quad \text{for} \quad u = (u_{\gamma_1})_{\gamma_1\in\mathbb{Z}^{d_1}}. \tag{9}$$

The embedding U_2 is a partial isometry as $U_2^* U_2 = Id$ on $\ell^2(\mathbb{Z}^{d_1})$. One proves that, under assumptions (H0) – (H3), for almost every ω, the operator $U_2^* H U_2 + V_\omega$ is

positive and the operator $\mathbb{E}((U_2^* H U_2 + V_\omega)^{-1})$ is bounded and positive.
Define V_{eff} to be the operator

$$V_{\text{eff}} = \left[\mathbb{E}((U_2^* H U_2 + V_\omega)^{-1})\right]^{-1} - U_2^* H U_2$$

acting on $\ell^2(\mathbb{Z}^{d_1})$. One proves that the operator V_{eff} acts as a convolution. Let $\theta_1 \mapsto v_{\text{eff}}(\theta_1)$ be the symbol of this operator (i.e. the operator is conjugated to multiplication by this function using the discrete Fourier transform). The function v_{eff} is real analytic on the torus $\mathbb{T}^{d_1}$. Note that the strict convexity of $x \mapsto 1/x$ for $x > 0$ implies that, for $\theta_1 \in \mathbb{T}^{d_1}$, $\omega_- < v_{\text{eff}}(\theta_1) < \mathbb{E}(\omega_0)$.
Our main result is

Theorem 7 ([6]). *Assume that* 0 *is a stable edge. Under the assumptions (H0) – (H3), one has*

- *if* $v_{\mathit{eff}}(0) \neq 0$*, then*

$$N(E) = \frac{C}{\sqrt{Det Q}} \frac{v_{\mathit{eff}}(0)}{1 + v_{\mathit{eff}}(0) \cdot I} \cdot E^{d/2}(1 + o(1)) \quad \text{as} \quad E \to 0^+,$$

- *if* $v_{\mathit{eff}}(0) = 0$*, then*

$$N(E) = o(E^{d/2}) \quad \text{as} \quad E \to 0^+.$$

where

- C *is a constant depending only on* d_1 *and* d_2*;*
- Q *is the Hessian matrix of* h *at* 0 *and* $I = \displaystyle\int_{\mathbb{T}^{d_2}} \frac{1}{h(0,\theta_2)} d\theta_2$.

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Recent results on singular spectrum of Schrödinger operators

Alexander Kiselev

In the recent years, there has been significant interest and progress in studying spectral types of one-dimensional Schrödinger operators with slowly decaying potentials and Stark (constant electric field) operators with rough potentials. Many of the new results concern the operators which can have rich and subtle spectral structure, such as dense imbedded point spectrum, singular continuous spectrum imbedded in the absolutely continuous, or singular continuous spectrum of fixed Hausdorff dimension. New results often involved new technology, such as use of fairly advanced Fourier analysis for studying the asymptotic behavior of solutions or a fruitful interaction of spectral theory methods and methods developed by orthogonal polynomials community. This brief note reviews just a small piece of the big picture consisting of a couple of recent results of the author, partly in collaboration with Michael Christ. The references are far from complete - rather fairly sketchy given the format of the note.

Let us define

$$H_V = -\frac{d^2}{dx^2} + V(x) \tag{1}$$

to be a Schrödinger operator defined on half-axis $\mathbb{R}^+ = (0, \infty)$ with, say, Dirichlet boundary condition at the origin. Let us also denote modified wave operators

$$\Omega^m_\pm f = \lim_{t \to \mp\infty} e^{itH_V} e^{-itH_0 \pm iW(H_0^{1/2}, \mp t)} f$$

for all $f \in L^2(\mathbb{R}^+)$, where existence of the limit has to be established. Here W is given by

$$W(\lambda, t) = -(2\lambda)^{-1} \int_0^{2\lambda t} V(s)\, dx.$$

Theorem 1. *Assume that the potential $V \in L^p$ with $p < 2$. Then there exist modified wave operators $\Omega^m_\pm$. If $\int_0^x V(s)\, ds$ has a finite limit as x goes to infinity, usual Möller wave operators exist. Moreover, for a.e. k there exist a solution $u(x, k)$ with WKB-type asymptotic behavior as $x \to \infty$:*

$$u(x, k) = \exp(ikx - \frac{i}{2k} \int_0^x V(s))\, ds)(1 + o(1)). \tag{2}$$

This theorem appeared in [2]. Classical results on one-dimensional Schrödinger operators with decaying potentials gave L^1 condition. The proof is based on studying the asymptotic behavior of solutions based on almost everywhere convergence results for the multilinear integral operators. The theorem does not hold for $p > 2$ [13, 9]. The theorem is conjectured to be true for $V \in L^2$, but this case is open, and, at least as far as the asymptotic behavior (2) is concerned, presumably very hard. The solution is likely to be related to a nontrivial extension of celebrated Carleson theorem on a.e. convergence of the Fourier series of an L^2 function. It is known, however, that the absolute continuity of the spectrum persists for

$p = 2$. This sharp result is due to Deift and Killip [3], who employ a sum rule to control the spectrum. In the higher dimensions, the slowly decaying perturbations are much less understood. The conjecture of Barry Simon, which is also put forward as one of his fifteen "twenty first century" problems in Schrödinger operators [14], states that the absolutely continuous spectrum is preserved as far as $\int |V(x)|^2(1+|x|)^{-d+1}\,dx < \infty$. However, the best general result available is still a classical short range result of Agmon. There are some recent results under mild additional conditions on the oscillation of potential [4, 10], and an interesting result of Bourgain in random case [1].

While in the situation of Theorem 1, the absolutely continuous spectrum fills the whole real axis, the crucial difference with the short range case is that the singular spectrum can also be very rich. Dense imbedded point spectrum is possible due to the results of Naboko and Simon. The set of singular energies (which we define as a Lebesgue measure zero set where the asymptotic behavior (2) fails) can have any Hausdorff dimension ≤ 1. For $p = 2$, the singular part of the spectral measure can be pretty much arbitrary modulo some normalization conditions, as follows from work of Killip and Simon (see [7] for the discrete case). One of the "twenty first century" problems of Barry Simon has asked whether potentials satisfying $|V(x)| \leq C(1+|x|)^{-\alpha}$ for $\alpha > 1/2$ can lead to imbedded singular continuous spectrum. Controlling imbedded singular continuous spectrum is difficult, since there is no simple criteria to establish its existence, and the typical approach of proving there is some spectra which cannot be neither pure point nor absolutely continuous [15] does not work. First important progress has been achieved by Denisov [5], who proved that if $V \in L^2$, the singular continuous spectrum may appear (with further beautiful and complete results of Killip and Simon). Our next two theorems provide a sharp answer to the question of a decay rate for which singular continuous spectrum may appear [8].

Theorem 2. *For any function $h(x) \to \infty$, there exists a potential $V(x)$ such that $|V(x)| \leq \frac{h(x)}{1+x}$ and the singular continuous spectrum of the operator H_V is not empty.*

The theorem also provides, up to the best of my knowledge, the first example where the wave operators exist and cohabit with imbedded singular continuous spectrum, thus leading to the lack of asymptotic completeness. The proof of this theorem is fairly involved; it is based on approximation by operators having imbedded eigenvalues and careful study of the asymptotic behavior of the solutions to establish control over the weights the spectral measure assigns to these eigenvalues. Generalized Prüfer transform and analysis of oscillatory integrals with the nonlinear dependence of phase on the argument function play a key role.

Theorem 3. *If $|V(x)| \leq \frac{C}{1+|x|}$ for some constant C, then the singular continuous spectrum of the operator H_V is empty.*

This theorem shows that the critical threshold is the Coulomb rate of decay and so the construction of the previous theorem is sharp. The proof of the absence of the singular continuous spectrum for potentials decaying at the Coulomb rate is based on the analysis of approximations where the potential is cut off at a finite scale. The main difficulty lies in the fact that there can be a singular set where the derivative of the spectral measure is infinite and which would be large enough to support the singular continuous spectrum (and can even be dense in $(0, \infty)$!). Therefore, one cannot use some sort of standard resolvent estimates. The technique used involves Gilbert-Pearson subordinacy theory, analysis of the singular set using Fourier transform methods, and a general approximation lemma proved in [6].

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Dispersive estimates for Schrödinger equations

Kenji Yajima

We consider the time decay in L^p spaces of solutions of the initial value problem for three dimensional Schrödinger equations

$$i\partial_t u = (-\Delta + V(x))u, \quad u(0) = \phi \in L^2(\mathbf{R}^3). \tag{1}$$

We assume that the potentials $V(x)$ decay faster than $C\langle x\rangle^{-5/2-\varepsilon}$ at infinity. The operator $H = -\Delta + V$ in the right of (1) is selfadjoint in $L^2(\mathbf{R}^3)$ and the solution of (1) is uniquely given by $u(t) = e^{-itH}\phi$. Let P_c be the orthogonal projection to the continuous spectral subspace for H. Then, $e^{-itH}P_c\phi$ is a scattering solution of (1) and it is now well known that it satisfies the so called L^p-L^q estimates

$$\|e^{-itH}P_c u\|_p \le C_p t^{-3\left(\frac{1}{2}-\frac{1}{p}\right)}\|u\|_q, \quad u \in L^2 \cap L^q \tag{2}$$

for $1 \le q \le 2 \le p \le \infty$, $1/p + 1/q = 1$, provided that 0 is not an eigenvalue nor a resonance of H (Goldberg-Schlag ([5]), see also [7], [1], [15], [15], [16], [13], [10], [12] for earlier and related works). This implies Strichartz inequality and it has been a very useful and important tool for studying linear and nonlinear Schrödinger equations (see e.g. [8]). It is also known that (2) cannot hold for all $2 \le p \le \infty$ if H is of exceptional type as it would contradict the local decay estimate of Jensen-Kato[6] or Murata[9].

In this paper, we analyze the behavior as $t \to \pm\infty$ of scattering solutions of (1) in L^p spaces when 0 is an eigenvalue or/and a resonance of H. We show how (2) is violated and propose a new estimate which replaces (2). To state the main results we introduce some notation. For $1 \le p, q \le \infty$, $L^{p,q}$ is the Lorentz space with the norm $\|u\|_{p,q}$. For $\gamma \in \mathbf{R}$, $L^2_\gamma = L^2(\mathbf{R}^3, \langle x\rangle^{2\gamma}dx)$ is the weighted L^2 space. We write $R_0(z) = (H_0 - z)^{-1}$ and $R(z) = (H - z)^{-1}$ for the resolvents of $H_0 = -\Delta$ and H respectively. For $\lambda \in \mathbf{C}$

$$G_0(\lambda)u(x) = \frac{1}{4\pi}\int \frac{e^{i\lambda|x-y|}}{|x-y|}u(y)dy. \tag{3}$$

We have $R_0(\lambda^2) = G_0(\lambda)$ for $\Im\lambda > 0$. The integral kernel of $G_0(\lambda)$ is an entire function of $\lambda \in \mathbf{C}$ and, using its derivatives at $\lambda = 0$, we define

$$D_j u(x) = \frac{1}{4\pi j!}\int |x-y|^{j-1}u(y)dy, \quad j = 0, 1, \dots, \tag{4}$$

so that $G_0(\lambda) = D_0 + i\lambda D_1 + (i\lambda)^2 D_2 + \cdots$ at least formally.

For any $1/2 < \gamma < \beta - 1/2$, the operator $D_0 V$ is of Hilbert-Schmidt type in $L^2_{-\gamma}$ and we denote the null space of $1 + D_0 V$ by $\mathcal{M}$. The space $\mathcal{M}$ is finite dimensional and is independent of $1/2 < \gamma < \beta - 1/2$. All $\phi \in \mathcal{M}$ satisfy the stationary Schrödinger equation $-\Delta\phi(x) + V(x)\phi(x) = 0$ and the converse is also true for $\phi \in L^2_{-\frac{3}{2}}$. The eigenspace $\mathcal{E}$ of H with eigenvalue 0 is therefore a subspace of $\mathcal{M}$. The function $\phi \in \mathcal{M}$ is in $\mathcal{E}$ if and only if $\langle V, \phi\rangle = 0$ and $\mathrm{codim}_{\mathcal{M}}\mathcal{E} \le 1$. The sesquilinear form $-(u, Vv)$ is an inner product in $\mathcal{M}$.

Definition 1. *We say H or V is of generic type if $\mathcal{M} = \{0\}$ and is of exceptional type otherwise. H is of exceptional type of the first kind if $\mathcal{M} \neq \{0\}$ and $\mathcal{E} = 0$; of the second kind if $\mathcal{E} = \mathcal{M} \neq \{0\}$; and of the third kind if $\{0\} \subset \mathcal{E} \subset \mathcal{M}$ with strict inclusions. A function $\phi \in \mathcal{M} \setminus \mathcal{E}$ is called a resonance of H.*

Any resonance $\phi(x)$ satisfies $\phi(x) - C|x|^{-1} \in L^2$ for a constant $C \neq 0$ and that $\phi \in \mathcal{E}$ may decay as slowly as $C\langle x\rangle^{-2}$. We write P_0 for the orthogonal projection in L^2 onto $\mathcal{E}$.

When H is of exceptional type of the third kind, we let $\phi_1 \in \mathcal{M}$ be a (uniquely determined) resonance such that $\langle V, \phi_1\rangle > 0$, $-\langle \phi_1, V\phi_1\rangle = 1$ and $-\langle \phi_1, V\phi_j\rangle = 0$ for all $\phi_j \in \mathcal{E}$ and define the *canonical resonance* by $\varphi(x) = \phi_1(x) + P_0 V D_2 V \phi_1(x)$. Using $\varphi(x)$, set $a = 4\pi i |\langle V, \varphi\rangle|^{-2}$ and $\zeta(t,x) = e^{i\frac{x^2}{4t}}\varphi(x)$. We define

$$\mu(t,x) = \frac{i}{|x|}\int_0^1 (e^{\frac{i|x|^2}{4t}} - e^{\frac{i\theta^2|x|^2}{4t}})d\theta; \tag{5}$$

$\mu(t)$ is multiplication with $\mu(t,x)$ and $f \otimes g$ is the rank one operator defined by integral kernel $f(x)g(y)$ (not $f(x)\overline{g(y)}$).

Definition 2. *We define the operators $R(t)$ and $S(t)$ respectively by*

$$R(t) = \frac{ae^{-i\frac{3\pi}{4}}}{\sqrt{\pi t}}\zeta(t,x) \otimes \zeta(t,x), \tag{6}$$

$$S(t) = \frac{e^{-i\frac{3\pi}{4}}}{\sqrt{\pi t}}\left(-iP_0 V D_3 V P_0 + \mu(t) D_2 V P_0 + P_0 V D_2 \mu(t)\right). \tag{7}$$

When H is of exceptional type of the first or the second kind, we use the same notation, setting, of course, $S(t) = 0$ or $R(t) = 0$ respectively.

We remark that $\zeta(t,x) - \varphi(x)$ and $\mu(t,x)$ are both bounded by

$$C \min\left(\frac{1}{\sqrt{t}}, \frac{1}{|x|}, \frac{|x|}{|t|}\right). \tag{8}$$

As $\phi \in \mathcal{E}$ satisfy $\int V(x)\phi(x)dx = 0$, $(D_2 V\phi)(x)$ are bounded and, if $\{\phi_2, \dots, \phi_d\}$ is an orthonormal basis of $\mathcal{E}$ and $w_j(t,x) = \mu(t,x)(D_2 V\phi_j)(x)$, $j = 2, \dots, d$, then $w_j(t,x)$ are bounded by (8) and $S(t)$ may be written in the form

$$\frac{e^{\frac{i\pi}{4}}}{\sqrt{\pi t}}\left(\sum_{j,k=2}^{d} a_{jk}\phi_j \otimes \phi_k + \sum_{j=2}^{d}(w_j(t) \otimes \phi_j + \phi_j \otimes w_j(t))\right).$$

Theorem 3. *Let V satisfy $|V(x)| \leq C\langle x\rangle^{-\beta}$ for some $\beta > 11/2$. Suppose that H is of exceptional type. Then the following statements are satisfied:*

(i) *Estimate (2) holds when $3/2 < q \leq 2 \leq p < 3$ and $1/p + 1/q = 1$.*
(ii) *(2) holds when L^3 and $L^{\frac{3}{2}}$ are respectively replaced by $L^{3,\infty}$ and $L^{\frac{3}{2},1}$.*
(iii) *When $3 < p \leq \infty$ and $1 \leq q < 3/2$ are such that $1/p + 1/q = 1$, there exists a constant C_{pq} such that for any $u \in L^2 \cap L^q$*

$$\left\|\left(e^{-itH}P_c - R(t) - S(t)\right)u\right\|_p \leq C_{pq} t^{-3\left(\frac{1}{2}-\frac{1}{p}\right)}\|u\|_q. \tag{9}$$

If H is of exceptional type of the first kind, theorem holds under the condition $|V(x)| \le C\langle x\rangle^{-\beta}$ with $\beta > 9/2$.

Theorem 4. *Let V satisfy $|V(x)| \le C\langle x\rangle^{-\beta}$ for some $\beta > 11/2$. Suppose that H is of exceptional type. Then, for $3 < p \le \infty$ and $1 \le q < 3/2$ such that $1/p + 1/q = 1$, there exists a constant C such that*

$$\|e^{-itH}P_c u\|_p \le Ct^{-3(\frac{1}{2}-\frac{1}{p})}(\|u\|_q + \|\langle x\rangle^{\frac{6}{q}-5}u\|_1) \tag{10}$$

for any $u \in L^2 \cap L^q$ which satisfies $\langle \phi, u\rangle = 0$ for all $\phi \in \mathcal{M}$ and $\langle x\rangle^{\frac{6}{q}-5}u \in L^1$. If H is of exceptional type of the first kind, the same statement holds under a weaker decay condition $|V(x)| \le C\langle x\rangle^{-\beta}$ with $\beta > 9/2$.

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Classical and Quantum Mechanics for a Particle in a Long-Range Magnetic Field

Ira Herbst

This talk is about some work of Horia Cornean, Erik Skibsted, and I in progress [CHS2] concerning the dynamics of a charged particle moving in a plane subject to a magnetic field which is homogeneous of degree -1. The work the talk is drawn from also deals with electric forces with the same homogeneity, but for simplicity, here we set the electric potential equal to zero. We analyze the classical and quantum dynamics of this system for large time with the objective to prove asymptotic completeness in quantum mechanics with some simple appropriate approximate dynamics.

Thus consider a magnetic field of the form

$$B = \frac{b(\theta)}{r},$$

where (r, θ) are the polar coordinates of a point in R^2. We always assume that b is smooth (and periodic of period 2π). Introducing the velocities

$$\rho = \frac{dr}{dt},$$

$$\eta = \frac{rd\theta}{dt},$$

and the new time τ given by

$$\frac{d\tau}{dt} = \frac{1}{r},$$

we can write the equations of motion of the particle in a reduced phase space as

$$\frac{d\rho}{d\tau} = \eta(\eta + b(\theta)),$$

$$\frac{d\eta}{d\tau} = -\rho(\eta + b(\theta)),$$

$$\frac{d\theta}{d\tau} = \eta.$$

Introducing the angle ϕ by

$$\rho = \sqrt{2E}\sin\phi,$$

$$\eta = \sqrt{2E}\cos\phi,$$

where E is the conserved kinetic energy, the first two equations above become

$$\frac{d\phi}{d\tau} = \sqrt{2E}\cos\phi + b(\theta),$$

which shows that the reduced classical phase space at energy E is a 2-torus. Note that r can be found once ρ is known:

$$r = r_0 e^{\int_0^\tau \rho(\tau')d\tau'}.$$

The case of $b < 0$ was treated in [CHS], where it was shown that in classical mechanics, above a certain energy E_d there is an attracting periodic orbit on the torus which attracts all orbits except for another periodic orbit which only lives for a finite amount of real time. Asymptotic completeness was proved in quantum mechanics above E_d using a semiclassical approximate dynamics based on this attracting periodic orbit. Below E_d nothing is known. But in the case where b is a non-zero constant the Hamiltonian has dense point spectrum [CFKS] below E_d.

We consider below mostly the classical mechanics of the model and only mention any difficulties that arise in quantum mechanics. One of these difficulties arises immediately when we consider the classical observable

$$A_1 = \rho - \int_0^\theta b(\theta')d\theta'.$$

Note that

$$A_1(\tau_2) - A_1(\tau_1) = \int_{\theta(\tau_1)}^{\theta(\tau_2)} \eta(\tau)^2 d\tau.$$

The problem with this observable is that unless the "flux" $\int_0^{2\pi} b(\theta)d\theta = 0$, it is not a function on the torus (but rather on a covering space of the torus), so it does not have a good quantization. Let

$$A_2 = -\rho\eta b(\theta).$$

Then for bounded E, we have that for C large enough

$$\frac{d(CA_1 + A_2)}{d\tau} \geq Eb^2 + \eta^2.$$

Let us assume that the flux is ≤ 0, and that b has zeros but all are non-degenerate. Then with some additional work, it follows that either

(1)
$$\theta(\tau) \to \infty,$$

or

(2)
$$\lim_{\tau\to\infty} [\theta(\tau) - \theta_0]^2 + \eta(\tau)^2 = 0,$$

where $b(\theta_0) = 0$.

Consider the fixed point in (2) where at $\tau = \infty$, $\rho = \sqrt{2E}$. Then this classical channel has a corresponding quantum channel if and only if the fixed point is a sink on the torus [HS2] (this corresponds to $b'(\theta_0) > 0$). Otherwise the fixed point has a corresponding stable manifold, but there are no states for which θ approaches θ_0 in quantum mechanics. The fixed points in (2) for which $\rho = -\sqrt{2E}$ correspond to orbits which hit the origin in finite time and have no analog in quantum mechanics. The $\rho = +\sqrt{2E}$ quantum channels may be described by an approximate dynamics as in [HS1] and asymptotic completeness proved, at least if the energy is high enough.

If (1) obtains, then aside from a finite set of energies, if the orbit does not collapse at the origin, it is attracted to a periodic orbit in the reduced phase space. There is also a corresponding channel in quantum mechanics. Asymptotic completeness can be proved there using a simple semiclassical approximate dynamics as in [CHS].

The analysis in [CHS2] consists first of a detailed description of the classical dynamics of this model. This alone is very non-trivial. Where quantum observables with positive Heisenberg derivatives are available to prove appropriate smoothness estimates, they are used. But there seem to be situations where not enough of these observables are available, and we supplement the analysis with "propagation of decay" estimates as in the usual propagation of singularities theorems. This idea (with x and ξ reversed) of using propagation of singularity theorems was introduced into scattering theory by Melrose in [M] and used for example in [HMV] in work closely related to [HS1].

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Some variational principles for relativistic energy functionals

Jean-Marie Barbaroux

(joint work with V. Bach, M. Esteban, W. Farkas, B. Helffer, E. Séré and H. Siedentop)

We give here some connections between two models describing the energy of a system of relativistic particles in the field of a pointwise fixed nucleus: The Dirac-Fock equations [4, 7], derived from the so-called Dirac-Fock functional $\mathcal{E}^{DF}$, and the electron/positron field functional $\mathcal{E}^{e^-/e^+}$ (see (2) below) derived from a simple no photon QED formal Hamiltonian, in the generalized Hartree-Fock approximation [1, 3].

The Hamiltonien for one electron in the field of a nucleus of charge eZ is given by the Coulomb-Dirac operator

$$D_Z := \boldsymbol{\alpha}\cdot\frac{1}{i}\nabla + m\beta - e^2\frac{Z}{|\mathbf{x}|} \quad \text{on } \mathfrak{H} := L^2(\mathbb{R}^3)\otimes\mathbb{C}^4, \tag{1}$$

where e^2 is the Sommerfeld fine structure constant, and α and β are the 4×4 Dirac matrices. Here, we assume $e^2Z \in [0, \sqrt{3}/2)$. For $Z = 0$, D_0 is the free Dirac operator. In the following we will also need the Coulomb-Dirac operator written in another system of units:

$$D_c := c\boldsymbol{\alpha}\cdot\frac{1}{i}\nabla + mc^2\beta - \frac{1}{|\mathbf{x}|}\ ,$$

where c is the speed of light. Let $\mathfrak{H}_+$ be a closed subspace of $\mathfrak{H}$, and define Λ_+ to be the orthogonal projection onto $\mathfrak{H}_+$, $\Lambda_- := 1-\Lambda_+$ and $\mathfrak{H}_- := \Lambda_-\mathfrak{H} = (\mathfrak{H}_+)^\perp$. We construct the following variational sets

$$\begin{aligned} S(\mathfrak{H}_+) &= \{\gamma\in\mathfrak{S}_1(\mathfrak{H}) \mid \gamma=\gamma^*,\ \mathrm{tr}(|D_0|^{\frac12}|\gamma|\,|D_0|^{\frac12})<\infty,\ -\Lambda_-\le\gamma\le\Lambda_+,\}\ ,\\ S_N(\mathfrak{H}_+) &= \{\gamma\in S(\mathfrak{H}_+) \mid \mathrm{tr}\gamma = N\}\ ,\\ T_N(\mathfrak{H}_+) &= \{\gamma\in S(\mathfrak{H}_+) \mid \mathrm{tr}\gamma = N,\ \Lambda_-\gamma\Lambda_+ = 0\}\ ,\end{aligned}$$

where $\mathfrak{S}_1(\mathfrak{H})$ denotes the space of trace class operators on $\mathfrak{H}$. For $\gamma\in S(\mathfrak{H}_+)$, $\mathrm{tr}\gamma$ is the charge of the system (corresponding to an electronic charge $-e\mathrm{tr}\gamma$). The electron/positron field functional we consider is

$$\mathcal{E}^{e^-/e^+}: \begin{array}{lcl} S(\mathfrak{H}_+) & \to & \mathbb{R} \\ \gamma & \mapsto & \mathrm{tr}(D_Z\gamma) + \frac{e^2}{2}\int\frac{\overline{\rho_\gamma(\mathbf{x})}\rho_\gamma(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{x}d\mathbf{y} - \frac{e^2}{2}\int\frac{\overline{\gamma(x,y)}\gamma(x,y)}{|\mathbf{x}-\mathbf{y}|}dxdy \end{array} \tag{2}$$

where $x=(\mathbf{x},\sigma)$ and $y=(\mathbf{y},\tau)$ are in $\mathbb{R}^3\times\{1,2,3,4\}$, $\gamma(x,y)$ is the kernel of γ and $\rho_\gamma(\mathbf{x}) = \sum_{\sigma=1}^4\gamma((\mathbf{x},\sigma)\,,\,(\mathbf{x},\sigma))$.

Our first result states that without any constraints on the charge, the most stable projection Λ_+ , i.e., the one yielding the highest ground state energy, is given by the projection onto the positive spectral subspace of the Coulomb-Dirac operator.

We denote by $\mathfrak{T}$ the set of all closed subspace $\mathfrak{H}_+$ of $\mathfrak{H}$ such that the orthogonal projections $\Lambda_\pm$ onto $\mathfrak{H}_\pm$ leave $\mathcal{D}(D_Z)$ invariant.

Theorem 1. [1] *Consider D_Z with values of $e, Z\ge 0$ such that $e^2\le 4(1-2e^2Z)/\pi$. We have*

$$\sup_{\mathfrak{H}_+\in\mathfrak{T}}\ \inf_{\gamma\in S(\mathfrak{H}_+)}\mathcal{E}^{e^-/e^+}(\gamma) = \inf_{\gamma\in S(\chi_{(0,+\infty)}(D_Z))}\mathcal{E}^{e^-/e^+}(\gamma) = \mathcal{E}^{e^-/e^+}(0) = 0. \tag{3}$$

Moreover, the supremum in (3) *is attained only for $\mathfrak{H}_+ = \chi_{(0,+\infty)}(D_Z)$.*

We now discuss the case of systems with fixed total charge $N\in\mathbb{N}$. For that purpose, we first need to define Dirac-Fock operators. For $\delta\in F := \{\delta\in\mathfrak{S}_1(\mathfrak{H}) \mid \delta =$

δ^*, $\mathrm{tr}(|D_0|^{1/2}\,|\delta|\,|D_0|^{1/2}) < \infty\}$, we construct the associated Dirac-Fock operator $D^{(\delta)}$ as

$$\begin{aligned} D^{(\delta)}\psi(x) &= D_Z\psi(x) + e^2 W^{(\delta)}\psi(x) \\ &= D_Z\psi(x) + e^2 \int \frac{\rho_\delta(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y}\ \psi(x) + e^2 \int \frac{\delta(x,y)\psi(y)}{|\mathbf{x}-\mathbf{y}|} dy\ , \end{aligned}$$

where $\delta(x,y)$ is the kernel of δ and $\rho_\delta(\mathbf{x}) = \sum_{\sigma=1}^4 \delta((\mathbf{x},\sigma);(\mathbf{x},\sigma))$. Here, $W^{(\delta)}$ is the mean field Dirac-Fock potential created by the N electrons in the state δ.

We define the associated one-electron space

$$\mathfrak{H}_+^{(\delta)} = \Lambda_+^{(\delta)}\mathfrak{H}\ ,$$

where

$$\Lambda_+^{(\delta)} = \chi_{(0,+\infty)}(D^{(\delta)})\ .$$

As argued in [6], the equality (3) suggests to explore a max-min variational problem similar to (3) in the case of atomic systems with prescribed electronic charge $e(Z-N)$, in order to find the ground state energy:

$$\sup_{\delta\in F}\ \inf_{\gamma\in T_N(\mathfrak{H}_+^{(\delta)})} \mathcal{E}^{e^-/e^+}(\gamma). \tag{4}$$

The next result shows the existence of solutions for the minimization procedure in (4), and gives the properties of the minimizers.

Theorem 2. [3] *Let $0 \le \delta \in F$ and assume $Z \ge N \in \mathbb{N}$ such that*

$$e^2\pi(N+1/4)/(1-2e^2Z-4e^2N) < 1.$$

Then $\mathcal{E}^{e^-/e^+}|_{T_N(\mathfrak{H}_+^{(\delta)})}$ has a minimizer in $T_N(\mathfrak{H}_+^{(\delta)})$, and each minimizer γ^0 is equal to the spectral projection onto the N first eigenvalues of the projected Dirac-Fock operator $\Lambda_+^{(\delta)}D^{(\gamma^0)}\Lambda_+^{(\delta)}$: there exist $\varphi_1^0, \varphi_2^0, \ldots, \varphi_N^0$ in $\Lambda_+^{(\delta)}\mathfrak{H}\cap\big(H^{1/2}(\mathbb{R}^3)\otimes\mathbb{C}^4\big)$, normalized and orthogonal, and $(\epsilon_i^0)_{i=1,\ldots,N}$ in $(0,m)$ such that

$$\gamma^0 = \sum_{i=1}^N |\varphi_i^0\rangle\,\langle\varphi_i^0|\ ,$$

and

$$\Lambda_+^{(\delta)}D^{(\gamma^0)}\Lambda_+^{(\delta)}\varphi_i^0 = \epsilon_i^0\varphi_i^0,\quad i=1,\ldots,N\ ,$$

where $(\epsilon_i^0)_{i=1,\ldots,N}$ are the N lowest eigenvalues in $(0,\,m)$ of $\Lambda_+^{(\delta)}D^{(\gamma^0)}\Lambda_+^{(\delta)}$.

We can now compare the max-min procedure (4) with the solutions obtained in [4, 5, 7] by solving the Dirac-Fock equations. We discuss here the nonrelativistic limit case, i.e., with D_Z replaced by D_c, $c \ll 1$, and for $\delta\in F$, $D^{(\delta)} = D_c + e^2W^{(\delta)}$.

Let $\lambda_1 < \lambda_2 < \ldots$ be the ordered (positive) eigenvalues of the Coulomb-Dirac operator D_c and let $N_i := \dim(\mathrm{Ker}(D_c-\lambda_i))$ be the dimension of the associated eigenspaces.

Theorem 3. [2]*[Close to the linear closed shells case]*

Let N be the number of electrons, and assume $c \gg 1$ and $e^2 \ll 1$. If $N = \sum_{i=1}^{K} N_i$ (closed shells), then the variational problem (4) is attained by the self-consistent pair (γ^0, γ^0), where $\gamma^0 = \sum_{i=1}^{N} |\varphi_i^0\rangle\langle\varphi_i^0|$, with

$$\Lambda_+^{(\gamma^0)} = \chi_{(0,+\infty)}(D^{(\gamma^0)}) \ ,$$

and

$$\left(D_c + e^2 W^{(\gamma^0)}\right)\varphi_i^0 = \epsilon_i^0 \varphi_i^0, \quad \epsilon_i^0 \in (0, m), \quad i = 1, \cdots, N \ ,$$

i.e., the N-uple $(\varphi_1^0, \cdots, \varphi_N^0)$ is solution of the self-consistent Dirac-Fock equations. Moreover, it is the ground state solution of the Dirac-Fock equations in the sense that it yields the smallest Dirac-Fock energy among the solutions of the Dirac-Fock equations: for any solution $(\psi_1, \cdots, \psi_N)$ of the self-consistent Dirac-Fock equations, we have

$$\sum_{i=1}^{N}(\psi_i, D_c\psi_i)+\frac{e^2}{2}\sum_{i\neq j}\left(\int \frac{|\psi_i(x)|^2|\psi_j(y)|^2}{|\mathbf{x}-\mathbf{y}|}d\mathbf{x}d\mathbf{y}-\int \frac{\psi_i(x)\overline{\psi_j(y)\psi_i(y)}\psi_j(x)}{|\mathbf{x}-\mathbf{y}|}dxdy\right)$$
$$\leq \sum_{i=1}^{N}(\varphi_i^0, D_c\varphi_i^0) + \frac{e^2}{2}\sum_{i\neq j}\Big(\int \frac{|\varphi_i^0(x)|^2|\varphi_j^0(y)|^2}{|\mathbf{x}-\mathbf{y}|}d\mathbf{x}d\mathbf{y}$$
$$- \int \frac{\varphi_i^0(x)\overline{\varphi_j^0(y)\varphi_i^0(y)}\varphi_j^0(x)}{|\mathbf{x}-\mathbf{y}|}dxdy\Big) = \mathcal{E}^{e^-/e^+}(\gamma^0) \ .$$

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On the mathematical model of the irreversible quantum graph

Mikhail Solomyak

Some time ago the physicist Uzy Smilansky suggested a mathematical model which he called "Irreversible quantum graph". In this model an interaction between the Laplacian on a metric graph Γ and the harmonic oscillator in an "outer space" is studied. The interaction is introduced by means of the boundary condition of a specific type. This condition involves the coupling parameter $\alpha \geq 0$ which expresses the strength of interaction. For $\alpha = 0$ the interaction is absent.

In the mathematical language the problem consists in the study of the spectral properties of a self-adjoint operator $\mathbf{A}_\alpha$ in the Hilbert space $L^2(\Gamma \otimes \mathbb{R})$. For simplicity, we consider the case when $\Gamma = \Gamma_d$, i.e. the star graph with d edges, each of infinite length, emanating from the only vertex o, the root of the tree. The operator is defined by the differential expression

$$\mathcal{A}U(x,q) = -U''_{xx} + \frac{1}{2}(-U''_{qq} + q^2 U), \qquad x \in \Gamma \setminus \{o\},\ q \in \mathbb{R},$$

and the condition

$$[U'_x](o,q) = \alpha q U(o,q), \qquad \forall q \in \mathbb{R}.$$

Here $[U'_x]$ stands for the combination of derivatives appearing in the classical Kirchhoff condition.

On the first glance, this can be reduced to a typical problem of Perturbation Theory for operators defined via their quadratic forms. However, the perturbation turns out to be too strong: it is only bounded but not compact with respect to the unperturbed quadratic form. For this reason, the standard approaches do not apply, and the character of results is rather unusual. Their most important feature is a "phase transition" at the value $\alpha = d/\sqrt{2}$ of the parameter: the spectral properties of the operator A_α for $\alpha\sqrt{2} < d$ and for $\alpha\sqrt{2} > d$ are quite different.

For $\alpha = 0$ separation of variables shows that

$$\sigma(\mathbf{A}_0) = \sigma_{a.c.}(\mathbf{A}_0) = [1/2, \infty);$$

$$\mathfrak{m}_{a.c.}(\lambda; \mathbf{A}_0) = dn \qquad \text{for } \lambda \in (n - 1/2, n + 1/2),\ n \in \mathbb{N}.$$

Here $\mathfrak{m}_{a.c.}(\lambda; .)$ stands for the multiplicity function for a self-adjoint operator.

The following results describe the picture for $\alpha > 0$.

1. Let $0 < \alpha\sqrt{2} < d$. Then the operator $\mathbf{A}_\alpha$ is positive definite;

$$\sigma_{a.c.}(\mathbf{A}_\alpha) = \sigma_{a.c.}(\mathbf{A}_0) = [1/2, \infty),$$

and the similar equality is satisfied for the multiplicity function.

The operator has no eigenvalues $\geq 1/2$. The spectrum on $(0, 1/2)$ is non-empty and finite. The number $N_-(1/2; \mathbf{A}_\alpha)$ of eigenvalues satisfies the asymptotic formula

$$N_-(1/2; \mathbf{A}_\alpha) \sim \frac{1}{4\sqrt{2(\mu(\alpha) - 1)}},\ \mu(\alpha) = \frac{d}{\alpha\sqrt{2}} \qquad \text{as } \alpha\sqrt{2} \nearrow d$$

2. Let $\alpha\sqrt{2} \geq d$. Then the operator has no eigenvalues;

$$\sigma_{a.c.}(\mathbf{A}_\alpha) = \begin{cases} [0, \infty), & \alpha\sqrt{2} = d; \\ \mathbb{R}, & \alpha\sqrt{2} > d. \end{cases}$$

$$\mathfrak{m}_{a.c.}(\lambda; \mathbf{A}_\alpha) = 1 + \mathfrak{m}_{a.c.}(\lambda; \mathbf{A}_0).$$

The results for $\alpha\sqrt{2} \geq d$ were obtained in cooperation with S.N. Naboko.

So, at $\alpha\sqrt{2} = d$ the point spectrum disappears and a new branch of $\sigma_{a.c.}$ arises.

For the proof we use the variational techniques (case $\alpha\sqrt{2} < d$) and the techniques of operator-valued analytic functions (case $\alpha\sqrt{2} \geq d$). In both cases Jacobi matrices arise and play the decisive role in the analysis.

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A Lieb–Thirring Inequality and an Isoperimetric Problem for Closed Curves Curves in $\mathbb{R}^2$

Rafael D. Benguria

(joint work with Michael Loss)

The Lieb–Thirring inequalities [8] play a crucial role in the proof of the stability of matter [9]. Let $H = -\Delta + V$ be the Schrödinger operator acting on $L^2(\mathbb{R}^n)$, $n \geq 1$ and denote by $e_1 \leq e_2 \leq \cdots < 0$ the negative eigenvalues of H. The Lieb–Thirring inequalities are given by

$$\sum_{j\geq 1} |e_j|^\gamma \leq L_{\gamma,n} \int_{\mathbb{R}^n} V_-(x)^{\gamma+n/2}\, dx, \tag{1}$$

where $V_-(x) \equiv \max(-V(x), 0)$ is the negative part of the potential. The above inequalities hold for $\gamma \geq 1/2$ when $n = 1$, for $\gamma > 0$ when $n = 2$, and for $\gamma \geq 0$ for $n \geq 3$. The sharp constants for the Lieb–Thirring are known for any $n \geq 1$ when $\gamma \geq 3/2$ and also in the case $n = 1$, $\gamma = 1/2$. See e.g., [6] and references therein for the best constants to date. The sharp constants for the one dimensional Lieb–Thirring inequalities with exponent $\gamma \in (1/2, 3/2)$ are still not known. Lieb and Thirring have conjectured [10] that the sharp constants for this range of exponents should be attained by potentials having only one bound state, and therefore,

$$L_{\gamma,1} \equiv L^1_{\gamma,1} = \frac{1}{\sqrt{\pi}} \frac{1}{\gamma - 1/2} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1/2)} \left(\frac{\gamma - 1/2}{\gamma + 1/2}\right)^{\gamma+1/2} \tag{2}$$

([7, 10]).

We have recently shown [1] that there is a connection between this conjecture for $\gamma = 1$ and $n = 1$ and an (still open) isoperimetric inequality for smooth, closed curves, with positive curvature in $\mathbb{R}^2$.

Let's denote by C a smooth closed curve in the plane, of length 2π, with positive curvature $\kappa(s)$, and let

$$H(C) \equiv -\frac{d^2}{ds^2} + \kappa^2 \tag{3}$$

acting on $L^2(C)$ with periodic boundary conditions. Here s denotes arc–length. Let $\lambda_1(C)$ the lowest eigenvalue of $H(C)$. It has been conjectured (see e.g., [3, 4]), that

$$\lambda_1(C) \geq 1, \tag{4}$$

with equality for a one parameter family of curves that includes the circle.

In recent years several authors have obtained isoperimetric inequalities for the lowest eigenvalues of a variant of $H(C)$. Consider the Schrödinger operator

$$H_g(C) \equiv -\frac{d^2}{ds^2} + g\kappa^2 \tag{5}$$

defined on $L^2(C)$ with periodic boundary conditions. As before, C denotes a closed curve in $\mathbb{R}^2$ with positive curvature κ, and length 2π. If $g < 0$, the lowest eigenvalue of $H_g(C)$, say $\lambda_1(g, C)$ is uniquely maximized when C is a circle [2]. When $g = -1$, the second eigenvalue, $\lambda_2(-1, C)$ is uniquely maximized when C is a circle [5]. If $0 < g \leq 1/4$, $\lambda_1(g, C)$ is uniquely minimized when C is a circle [3]. It is an open problem to determine the curve C that minimizes $\lambda_1(g, C)$ in the cases, $1/4 < g \leq 1$, and $g < 0, g \neq -1$. If $g > 1$ the circle is not a minimizer for $\lambda_1(g, C)$ (see, e.g., [3, 4] for more details on the subject).

Our main result [1] is the following theorem:

Theorem 1. *Suppose that the Schrödinger operator $H = -d^2/dx^2 + V$, acting on $L^2(\mathbb{R})$, has only two negative eigenvalues, say $e_1 < e_2 < 0$. Then, if the isoperimetric inequality (4) holds, we have*

$$|e_1| + |e_2| \leq L^1_{1,1} \int_{\mathbb{R}} V_-(x)^{3/2}\, dx. \tag{6}$$

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Accurate estimates for magnetic bottles in connection with superconductivity

B. Helffer

(joint work with S. Fournais)

In this talk, which refers to [FoHe2], we consider a magnetic Schrödinger operator with Neumann boundary conditions in a smooth, bounded domain Ω. We are interested in finding an accurate description of the eigenvalues near the bottom of the spectrum. In particular, we will improve estimates given in [HeMo] in the case of constant magnetic field.

Apart from its intrinsic mathematical interest, this question is important for applications to superconductivity. Precise knowledge of the lowest eigenvalues of this magnetic Schrödinger operator is crucial for a detailed description of the nucleation of superconductivity (on the boundary) for superconductors of Type II and for accurate estimates of the critical field H_{C_3}. We refer the reader to the works of Bernoff-Sternberg [BeSt] who are the first to propose the main conjecture on the basis of formal constructions of quasimodes, Lu-Pan [LuPa1, LuPa2, LuPa3] and Del Pino-Felmer-Sternberg [PiFeSt] for further discussion of this subject.

The domain $\Omega \subset \mathbb{R}^2$ is supposed to be smooth, bounded and simply connected. Points (x_1, x_2) in $\mathbb{R}^2$ are denoted by x. At each point x of the boundary, we denote by $\nu(x)$ the interior unit normal vector to the boundary of Ω. We define the magnetic Neumann operator $\mathcal{H}$ by

$$\mathcal{D}(\mathcal{H}) \ni u \mapsto \mathcal{H}u = \mathcal{H}_{h,\Omega}u = (-ih\nabla_x - A(x))^2 u \,. \tag{1}$$

Here $A(x) = (-x_2/2, x_1/2)$, so that $\operatorname{curl} A = 1$, and the domain $\mathcal{D}(\mathcal{H})$ of the operator $\mathcal{H}$ is defined by

$$\mathcal{D}(\mathcal{H}) = \left\{ u \in H^2(\Omega) \,\middle|\, \nu \cdot (-ih\nabla_x - A(x))u\big|_{\partial\Omega} = 0 \right\} .$$

The case of the half-plane, $\Omega = \mathbb{R} \times \mathbb{R}_+$, will be important for fixing notations and determining the main term of the asymptotics. After a gauge transformation and a partial Fourier transformation, we get,when $h = 1$, the family of models on the half-line:

$$H^{N,\xi} = D_t^2 + (t+\xi)^2 \;, \tag{2}$$

on $L^2(\mathbb{R}_+)$ and with Neumann boundary conditions at $t = 0$. Let $\hat{\mu}^{(1)}(\xi)$ be the lowest eigenvalue of $H^{N,\xi}$. Then $\xi \mapsto \hat{\mu}^{(1)}(\xi)$ has a unique minimum Θ_0 attained at a point that we will denote by ξ_0. The corresponding unique positive, normalized eigenfunction of H^{N,ξ_0} will be denoted by u_0. We also introduce :

$$C_1 = \frac{u_0^2(0)}{3} \;. \tag{3}$$

The main result in [FoHe2] gives the asymptotic expansion of the lowest eigenvalues of $\mathcal{H}$.

Theorem 1. *Suppose that Ω is a smooth bounded domain, that its curvature $\partial\Omega \ni s \mapsto \kappa(s)$ at the boundary has a unique maximum,*

$$\kappa(s) < \kappa(s_0) =: k_{\max} \;, \text{ for all } s \neq s_0 \;, \tag{4}$$

and that the maximum is non-degenerate, i.e.

$$k_2 := -\kappa''(s_0) \neq 0 \;. \tag{5}$$

Then, for all $n \in \mathbb{N}^*$, there exists a sequence $\{\zeta_j^{(n)}\}_{j=1}^{\infty} \subset \mathbb{R}$ (which can be calculated recursively to any order) such that the n-th eigenvalue of $\mathcal{H}$ $\mu^{(n)}(h)$ admits the following asymptotic expansion, when $h \searrow 0$,

(6)

$$\mu^{(n)}(h) \sim \Theta_0 h - C_1 k_{\max} h^{3/2} + C_1 \Theta_0^{1/4} \sqrt{\tfrac{3k_2}{2}}(2n-1)h^{7/4} + h^{15/8} \sum_{j=0}^{\infty} h^{j/8} \zeta_j^{(n)} \;.$$

Remark 2.
Previous results on the bottom of the spectrum of $\mathcal{H}_{h,\Omega}$ were obtained in [HeMo], *who gave the two first terms in the expansion of $\mu^{(1)}(h)$ (see* [HeMo, Theorems 10.3 and 11.1]*):*

$$\mu^{(1)}(h) = \Theta_0 h - k_{\max} C_1 h^{3/2} + \mathcal{O}(h^{5/3}) \;. \tag{7}$$

Remark 3. *If the uniqueness condition in* (4) *is replaced by the assumption that there is a finite number of maxima (for which* (5) *is assumed to hold), we expect the existence of sequences of eigenvalues $z^{(n)}(h)$ corresponding to each maximum.*

For applications to bifurcations from the normal state in superconductivity it seems important to calculate the splitting between the ground and first excited states of $\mathcal{H}(h)$. Let us define

$$\Delta(h) = \mu^{(2)}(h) - \mu^{(1)}(h) \;. \tag{8}$$

Corollary 4.
Under the hypothesis of the theorem, $\Delta(h)$ admits the following asymptotics :

$$\Delta(h) \sim C_1 \Theta_0^{1/4} \sqrt{6k_2} h^{7/4} + h^{15/8} \sum_{j=0}^{\infty} h^{j/8} \xi_j \ . \tag{9}$$

where $\xi_j = \zeta_j^{(2)} - \zeta_j^{(1)}$.

The case where Ω is a disc has been analyzed in great detail in [BaPhTa], using the radial symmetry to reduce the problem to ordinary differential equations. In this case the splitting $\Delta(h)$ turns out to become zero for a sequence of values of h tending to 0. This is a complication in the analysis of bifurcation. Thus, in some sense, the more 'generic' situation considered here has a nicer property.

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Wave Front Set for Solutions to Schrödinger Equations

Shu Nakamura

In this talk, we discuss the wave front set for solutions to Schrödinger equation with variable coefficients. It is well-known that the propagation speed of the wave front set of solutions to Schrödinger equation is infinite, and hence we cannot expect the usual propagation theorem such as for the solutions to the wave equation. Instead, relations between the decay property of the initial condition and the wave front set of solutions have been studied, which is generally called (microlocal) smoothing properties. Here we present a different formulation, which is closer to the "propagation of singularity theorem", at least in the spirit.

Part of results we discuss is joint work with André Martinez and Vania Sordoni (Bologna University).

We consider a Schrödinger equation:

$$\frac{d}{dt}u(t) = -iHu(t), \qquad u(0) = u_0 \in L^2(\mathbb{R}^d)$$

on $L^2(\mathbb{R}^d)$, where $d \geq 1$, and H is the Schrödinger operator defined by

$$H = \frac{1}{2}\sum_{i,j=1}^{d} D_j a_{jk}(x) D_k + V(x), \qquad D_j = -i\frac{\partial}{\partial x_j}.$$

We suppose the coefficients $\{a_{ij}(x)\}$ and the potential $V(x)$ satisfy the following conditions:

Assumption A. $a_{ij}(x), V(x) \in C^\infty(\mathbb{R}^d;\mathbb{R})$ for $i,j = 1,\dots,d$, and there exist $\mu > 0$, and $C_\alpha > 0$ for each $\alpha \in \mathbb{Z}^d_+$ such that

$$|\partial_x^\alpha (a_{ij}(x) - \delta_{ij})| \leq C_\alpha \langle x\rangle^{-\mu-|\alpha|}, \qquad |\partial_x^\alpha V(x)| \leq C_\alpha \langle x\rangle^{2-\mu-|\alpha|},$$

for $x \in \mathbb{R}^d$. Moreover, H is elliptic, i.e., $\det(a_{ij}(x)) \neq 0$ for each $x \in \mathbb{R}^d$.

Then it is well-known that H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$. We denote the unique self-adjoint extension by the same symbol H. Thus, by the Stone theorem, $u(t) = e^{-itH}u_0$ is the solution to the Schrödinger equation with the initial condition $u(0) = u_0$. We consider the following quite basic PDE-type problem:

Problem: Describe the singularity of $u(t)$ in terms of u_0.

We use the notion of the *wave front set* to describe singularity of solutions to the Schrödinger equation (see [6] Section X.10, or [8] Section VI.1 for the definition). We denote the wave front set of $u \in \mathcal{D}'(\mathbb{R}^d)$ by $WF(u) \subset \mathbb{R}^{2d}$.

Let us recall the propagation of singularity theorem for the wave equation. It shows that the propagation of the wave front set for the solutions to the wave equation is described by the geometric optics. We note the analogue of the geometric optics for Schrödinger equation is the classical mechanics, and the relationship is given by the WKB analysis. However, the WKB theory describe the semiclassical behavior of the solution, and it does not give any information about the singularity of solutions, at least directly. As we will see, the *high energy* classical mechanics gives us the description of the singularity of the solutions, and it is closely related to the scattering theory of the classical mechanicss.

We denote the symbol of the kinetic energy part by $p(x, \xi)$, i.e.,

$$p(x, \xi) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j, \quad x, \xi \in \mathbb{R}^d.$$

We denote the solution to the Hamilton equation:

$$\frac{d}{dt} y(t) = \frac{\partial p}{\partial \xi}(y(t), \eta(t)), \quad \frac{d}{dt} \eta(t) = -\frac{\partial p}{\partial x}(y(t), \eta(t))$$

with initial condition $y(0) = x$, $\eta(0) = \xi$ by $y(t; x, \xi)$ and $\eta(t; x, \xi)$.

Definition 1: $(x, \xi) \in \mathbb{R}^{2d}$ is said to be *backward nontrapping* if $|y(t; x, \xi)| \to +\infty$ as $t \to -\infty$.

We say H is a *short-range perturbation* of $H_0 = -\frac{1}{2}\triangle$ (or simply short-range type) if Assumption A is satisfied with $\mu > 1$. In this case, if (x, ξ) is backward nontrapping, then it is well-known that there exists $(x_-, \xi_-) \in \mathbb{R}^{2d}$ such that

$$|y(t; x, \xi) - (x_- + t\xi_-)| \to 0 \quad \text{as } t \to -\infty.$$

Namely, the classical trajectory $y(t; x, \xi)$ approaches to a free motion $x_- + t\xi_-$ as $t \to -\infty$. The map:

$$S : (x, \xi) \mapsto (x_-, \xi_-)$$

is the classical (inverse) wave operator.

Theorem 1 ([5]) *Suppose Assumption A with $\mu > 1$, and suppose $(x, \xi) \in \mathbb{R}^{2d}$ is backward nontrapping. Let $u(t) = e^{-itH} u_0$ with $u_0 \in L^2(\mathbb{R}^d)$, and let $t > 0$. Then*

$$(x, \xi) \in WF(u(t)) \quad \Longleftrightarrow \quad (x_-, \xi_-) \in WF(e^{-itH_0} u_0).$$

If the metric is flat, i.e., if $H = -\frac{1}{2}\triangle + V(x)$, then Theorem 1 implies that $WF(u(t)) = WF(e^{-itH_0} u_0)$. This observation suggests that $e^{itH_0} e^{-itH}$ is a pseudo-differential operator, and in fact we can prove it. This result and its generalization to non-flat case will be discussed in a forthcoming paper.

Recently, Hassel and Wunsch [2] have obtained different characterization of the wave front set of solutions to Schrödinger equations using the *quadratic scattering wave front set*. The setting and the formulation are quite different, and the relationship is not clear to the author.

If the perturbation is long-range type, i.e., if $0 < \mu \le 1$, then the above theorem does not hold in general, and we need to replace the free propagator e^{-itH_0} by a different Fourier multiplier, quite similar to one appearing in the long-range scattering theory. This part is still in progress, and we do not discuss here. We have somewhat weaker result, which we discuss in the following. We recall that the classical motion not necessarily approaches to a free motion, but the asymptotic momentum $\xi_- := \lim_{t\to-\infty} \eta(t;x,\xi)$ does exists if the trajectory is nontrapping. We introduce the following notion of the wave front set:

Definition 2: Let $u \in \mathcal{S}'(\mathbb{R}^d)$. We say $(x,\xi) \in \mathbb{R}^{2d} \setminus 0$ is not in the *homogeneous wave front set* of u if there exists $a \in C_0^\infty(\mathbb{R}^{2d})$ such that $a(x,\xi) \neq 0$ and $\|a(hx, hD_x)u\|_{L^2} = O(h^N)$ as $h \to +0$ with any N. We denote $(x,\xi) \notin HWF(u)$ if this condition is satisfied, and denote the complement by $HWF(u)$.

Theorem 2 ([4]) *Suppose Assumption A with $\mu > 0$, and suppose $(x,\xi) \in \mathbb{R}^{2d}$ is backward nontrapping. Let $t_0 > 0$. If $(-t_0\xi_-, \xi_-) \notin HWF(u_0)$, then $(x_0, \xi_0) \notin WF(u(t_0))$.*

The microlocal smoothing property of Craig, Kappeler and Strauss [1] follows from Theorem 2. (In fact our result is more general, since they considered short-range case only.) It is also related to a work by Wuncsh [9], though the relationship is not clear to the author. A similar theorem also holds for the analytic wave front set under the assumption of the analyticity of the coefficients. This result is proved by a recent joint work with Martinez and Sordoni [3], and it is a generalization of results by Robbiano and Zuily [7].

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Spectral shift function for self-adjoint operators without spectral gaps

D. R. Yafaev

The concept of the spectral shift function first appeared in the work of I. M. Lifshits [7] in connection with the quantum theory of crystals. A mathematical theory of the SSF was soon thereafter constructed by M. G. Kreĭn in [5]. One of his results can be formulated in the following way. Let H_0 and H be self-adjoint operators with a trace class (denoted $\mathfrak{S}_1$) difference $V = H - H_0$. Then there exists a function$\xi(\lambda) = \xi(\lambda; H, H_0)$, $\xi \in L_1(\mathbb{R})$, known asthe spectral shift function such that the trace formula

$$\operatorname{Tr}\Big(f(H) - f(H_0)\Big) = \int_{-\infty}^{\infty} \xi(\lambda) f'(\lambda) d\lambda, \quad \xi(\lambda) = \xi(\lambda; H, H_0), \tag{1}$$

holds at least for all functions $f \in C_0^\infty(\mathbb{R})$. A relatively detailed presentation of the theory of the SSF can be found in [3] or [8].

If the operators H_0 and H have a common spectral gap, then the trace formula automatically remains true for a much wider class of the operators V. If, for instance, $\lambda = 0$ is a common regular point of the operators H_0 and H and $H^{-m} - H_0^{-m} \in \mathfrak{S}_1$ for some integer odd m, then the trace formula (1) for the pair H_0, H can be deduced from that for the pair H_0^{-m}, H^{-m} (see [8], for details).

A connection between scattering theory and the theory of the SSF was found by M. Sh. Birman and M. G. Kreĭn in [1]. Actually, they showed that the scattering matrix $S(\lambda; H, H_0)$ minus the identity operator I belongs to the trace class and

$$\det S(\lambda; H, H_0) = e^{-2\pi i \xi(\lambda; H, H_0)} \tag{2}$$

for almost all λ (from the core of the spectrum of the operator H_0).

Our goal is to extend the theory of the spectral shift function to the case where only the difference of some powers of the resolvents of self-adjoint operators belongs to the trace class. The main result is given by the following

Theorem 1. *Let, for a pair of self-adjoint operators H_0 and H, the assumption*

$$(H - z)^{-m} - (H_0 - z)^{-m} \in \mathfrak{S}_1$$

hold for some positive odd integer m and all z with $\Im z \neq 0$. Suppose that a function $f(\lambda)$ has two bounded derivatives and

$$\partial^\alpha (f(\lambda) - f_0 \lambda^{-m}) = O(|\lambda|^{-m-\epsilon-\alpha}), \quad \alpha = 0, 1, 2, \quad \epsilon > 0,$$

where the constant f_0 is the same for $\lambda \to \infty$ and $\lambda \to -\infty$. Then

$$f(H) - f(H_0) \in \mathfrak{S}_1$$

and there exists a function $\xi(\lambda; H, H_0)$ satisfying the condition

$$\int_{-\infty}^{\infty} |\xi(\lambda; H, H_0)| (1 + |\lambda|)^{-m-1} d\lambda < \infty$$

such that the trace formula (1) is true. Moreover, for the corresponding scattering matrix $S(\lambda; H, H_0)$, the operator $S(\lambda; H, H_0) - I \in \mathfrak{S}_1$ and the relation (2) holds for almost all λ.

Note that in the case $m = 1$ Theorem 1 reduces to a well-known result of M. G. Kreĭn [6]. Somewhat different general conditions for the existence of the spectral shift function were given by L. S. Koplienko [4].

Our proof of Theorem 1 relies on its reduction to the special case $m = 1$ with the help of the theory of Double Operator Integrals developed by M. Sh. Birman and M. Z. Solomyak (see, e.g., [2]).

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Uniform Magnetic Lieb-Thirring inequalities

László Erdős
(joint work with Jan Philip Solovej)

Lieb-Thirring inequalities refer to estimates that bound moments of negative eigenvalues of Schrödinger type operators in terms of the external fields. They play a fundamental role in various results concerning localized many-fermion systems. Most notably, the ground state energy of the many-body Hamiltonian in many cases is related to the sum of the negative eigenvalues of an effective one-body Hamiltonian. Among other useful applications, Lieb-Thirring inequalities stand behind the most effective and elegant proofs of stability of matter. They also serve as a powerful apriori estimate for the semiclassical analysis of the many-fermion ground state.

We focus on the particular case of *magnetic Lieb-Thirring (MLT) inequalities*. They estimate moments of negative eigenvalues $e_1(H) \leq e_2(H) \leq \ldots \leq 0$ of the Pauli operator

$$H := [\boldsymbol{\sigma} \cdot (-i\nabla + \mathbf{A})]^2 + V \tag{1}$$

on $L^2(\mathbf{R}^3, \mathbf{C}^2)$ with a vector potential $\mathbf{A}$, magnetic field $\mathbf{B} := \nabla \times \mathbf{A}$ and external potential V. Here $\boldsymbol{\sigma} \cdot \mathbf{v} = \sigma^1 v_1 + \sigma^2 v_2 + \sigma^3 v_3$, $\mathbf{v} \in \mathbf{R}^3$, and $\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices. Unlike in the nonmagnetic case, where the *optimal form* of the estimates is well-established and the remaining main challenge is to find the *optimal constants*, the magnetic case is much less understood.

For a constant magnetic field $\mathbf{B}$, the inequality

$$\sum_j |e_j(H)| \le (const)\Big(\int_{\mathbf{R}^3} |\mathbf{B}|[V]_-^{3/2} + \int_{\mathbf{R}^3} [V]_-^{5/2}\Big)\ , \tag{2}$$

proven in [LSY], is optimal, apart from the constant, where $[a]_- := -\min\{0, a\}$ denotes the negative part of a. It has seemed to be reasonable to conjecture that (2) also holds for an arbitrary magnetic field. However, such a naive generalization fails for two reasons.

Firstly, even when $\mathbf{B}$ has constant direction in $\mathbf{R}^3$, (2) can be correct only if $|\mathbf{B}(x)|$ is replaced by an effective field strength, $B^*(x)$, obtained by averaging $|\mathbf{B}|$ locally on the magnetic lengthscale, $|\mathbf{B}|^{-1/2}$.

Secondly, the existence of the celebrated Loss-Yau zero modes [LY] contradicts (2). Indeed, for certain magnetic fields with nonconstant direction the Dirac operator $\mathcal{D} := \boldsymbol{\sigma}\cdot(-i\nabla + \mathbf{A})$ has a nontrivial L^2-kernel. In this case a small potential perturbation of $\mathcal{D}^2$ shows that $\sum_j |e_j(H)|$ behaves as $\int n(x)[V(x)]_- \mathrm{d}x$, i.e. it is linear in $[V]_-$. Here $n(x)$ is the density of zero modes, $n(x) = \sum_j |u_j(x)|^2$, where $\{u_j\}$ is an orthonormal basis in $\mathrm{Ker}\,\mathcal{D}$. Thus an extra term linear in $[V]_-$ must be added to (2) and $n(x)$ has to be estimated.

Let

$$H(h, b) := [\boldsymbol{\sigma}\cdot(-ih\nabla + b\mathbf{A})]^2 + V$$

be the Pauli operator with the semiclassical parameter h and a field strength parameter b. The semiclassical formula for the sum of the negative eigenvalues, i.e. the asymptotic formula for $\sum_j |e_j(H(h, b))|$ as $h \to 0$, behaves linearly in the field strength for a constant magnetic field ([LSY]). This fact suggests that $\sum_j |e_j(H)|$ may be bounded by an expression that grows only with the first power of $|\mathbf{B}|$ even for nonconstant magnetic fields.

Our goal is to establish such MLT estimates with as few technical assumptions on $\mathbf{B}$ as possible and no technical assumptions on V. The density $n(x)$ has a dimension $(length)^{-3}$. Since $|\mathbf{B}|$ has dimension $(length)^{-2}$, we need to introduce an extra lengthscale to be able to bound $n(x)$ by the magnetic field. We will therefore make assumptions on certain derivatives of the field.

In almost all previous Lieb-Thirring bounds, the density $n(x)$ was estimated by a function that behaves quantitatively as $|\mathbf{B}(x)|^{3/2}$. This power was reduced to $5/4$ in [ES-I] with a further unnatural $V \in W^{1,1}$ assumption on the potential. A worse power, $17/12$, was obtained in [BFG] but without further assumptions on the potential.

For our theorem we assume that $\mathbf{B}(x) \neq 0$ for all $x \in \mathbf{R}^3$, i.e. the unit vectorfield $\mathbf{n} := \mathbf{B}/|\mathbf{B}|$ is well defined. We also assume that the vectorfield $\mathbf{n}$ satisfies the following global regularity condition

$$L_{\mathbf{n}}^{-1} := \sum_{\gamma=1}^{5} \|\nabla^\gamma \mathbf{n}\|_\infty^{1/\gamma} < \infty\ . \tag{3}$$

For any $L > 0$, $x \in \mathbf{R}^3$ we also define

$$B_L^*(x) := \sup\{|\mathbf{B}(y)| \ : \ |y-x| \le L\} \ + L \cdot \sup\{|\nabla \mathbf{B}(y)| \ : \ |y-x| \le L\} \ .$$

Theorem 1 (Magnetic Lieb-Thirring inequality)**.** [ES-IV] *For any* $0 < L \le L_{\mathbf{n}}$, *the sum of the negative eigenvalues,* $e_1(H) \le e_2(H) \le \ldots \le 0$, *of* H *satisfies*

$$\sum_j |e_j(H)| \le (const)\Big(L^{-1}\int (B_L^* + L^{-2})[V]_- + \int B_L^*[V]_-^{3/2} + \int [V]_-^{5/2}\Big) \ . \tag{4}$$

The density of Loss-Yau zero modes is estimated by

$$n(x) \le L^{-1}(B_L^*(x) + L^{-2}) \ .$$

The linear power of $|\mathbf{B}|$ in the estimate reflects the basic fact that the space with a magnetic field cannot be considered isotropic: the quantum motion parallel with the magnetic field behaves differently than the transversal one. The magnetic field affects only the two-dimensional transversal motion. All MLT estimates that yield $|\mathbf{B}|^{3/2}$ behaviour neglect this geometric fact by simply comparing the magnetic problem with a three dimensional nonmagnetic one, usually via a diamagnetic inequality that loses the anisotropic feature of the problem. The typical estimate is of the form

$$\mathcal{D}^2 \ge b^{-1}\mathcal{D}^2 = b^{-1}[(-i\nabla + \mathbf{A})^2 + \boldsymbol{\sigma}\cdot\mathbf{B}] \ , \tag{5}$$

where $b := \|\mathbf{B}\| \gg 1$ is some (local) norm of $\mathbf{B}$. The kinetic energy is scaled down so that the dangereous $\boldsymbol{\sigma}\cdot\mathbf{B}$ becomes bounded uniformly in b. The magnetic Laplacian can then be controlled by the nonmagnetic Laplacian, $-\Delta$, but the factor b^{-1} now affects all three coordinates, yielding a scaling of $b^{3/2}$. The key to our theorem is to separate the parallel and transversal motions and use a crude estimate similar to (5) only in the two-dimensional transversal kinetic energy.

Our theorem uses only natural assumptions on V and it gives the correct (linear) dependence on the field strength $|\mathbf{B}|$. However, the original magnetic field $\mathbf{B}$ is replaced by an effective field $B_L^* + L^{-2}$ that involves the global L^∞-norm of $\mathbf{n}$. In particular the estimate (4) is sensitive to the behavior of the magnetic field far away from the support of $[V]_-$. Hence the irregular behaviour of $\mathbf{n}$ far away from the support of $[V]_-$ renders our estimate large despite that it should not substantially influence the negative spectrum.

In a separate work [ES-III] we also proved a magnetic Lieb-Thirring bound that enjoys a *locality property*. More precisely, the constant $L_{\mathbf{n}}$ was replaced by a function $L(x)$ describing the local variation lengthscale of the magnetic field. The precise definition is somewhat complicated, but it depends only locally on $\mathbf{B}$. In particular the inverse lengthscale $L^{-1}(x)$ can be bounded by $c\delta^{-1}$ if $\mathbf{B}$ vanishes in a δ-neighborhood of x.

The proof of this second theorem is much more involved. The complications are due to the lack of effective offdiagonal bounds on the resolvent of the Pauli operator, $(\mathcal{D}^2 + E)^{-1}(x,y)$. For constant magnetic field, the resolvent decays on the magnetic lengthscale $B^{-1/2}$ in the direction perpendicular to the field:

$$(\mathcal{D}^2 + E)^{-1}(x,y) \sim e^{-cB(x_\perp - y_\perp)^2}$$

but similar estimate is unknown for a general field. This problem is closely related to the poorly understood structure of the Loss-Yau zero modes.

It is amusing to note that it was a substantial endeavour to show that a zero mode may exist at all [LY], and that multiple zero modes may also occur [ES-II]. On the other hand, it seems also quite difficult to give an upper bound on their number in terms of the first power of the field strength.

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Old and New Tales about Lifshitz Tails

Werner Kirsch

We consider random Schrödinger operators $H_\omega = H_0 + V_\omega$ with V_ω either of alloy type or of Poisson type.

By alloy type we mean potentials of the form

$$V_\omega(x) = \sum_{i\in\mathbb{Z}^d} q_i(\omega) f(x-i) \tag{1}$$

with independent identically distributed random variables q_i. For the Poisson model the potential V_ω is given by

$$V_\omega(x) = \sum f(x - \xi_i(\omega)) \tag{2}$$

where the $\{\xi_i\}$ are Poisson distributed random points.

In both cases the function f, also called the single site potential, has to decay fast enough at infinity to ensure convergence of the series (1), e.g.

$$|f(x)| \le \frac{c}{1+|x|^\alpha} \tag{3}$$

with $\alpha > d$, $|x|$ large.

We study the integrated density of states $N(E)$ for these operators. $N(E)$ is defined as a thermodynamic limit in the following way: Let Λ_L be the cube of side length L around the origin and restrict H_ω to $\ell^2(\Lambda_L)$ with appropriate boundary conditions (Dirichlet, say). The corresponding operator H_L has a purely discrete sprectrum. Let us denote its eigenvalues by $E_1(H_L) \leq E_2(H_L) \leq \cdots$, repeated according to multiplicity.

For any E we set $N_L(E) = \frac{1}{L^d} \#\{E_n(H_L) \leq E\}$.

Under very weak conditions on V_ω it is known that N_L converges (for all but countably many E at least) to a nonrandom limit $N(E)$, the integrated density of states.

The physicist Lifshitz [1] observed in 1964 that the low energy behavior of $N(E)$ of random potentials is drastically different from the one for periodic potentials. In fact, Lifshitz argued that in the ordered (i.e. periodic) case

$$N(E) \sim (E - E_0)^{\frac{d}{2}} \tag{4}$$

as $E \searrow E_0 = \inf \sigma(H_{per})$. For random operators Lifshitz obtained

$$N(E) \sim e^{-c(E-E_0)^{-\frac{d}{2}}} \tag{5}$$

as $E \searrow E_0$. This super exponential decay is nowadays called Lifshitz tail behavior.

Lifshitz´ arguments for his results were convincing but not mathematically rigorous.

The first mathematical proof of (5) was given by Donsker and Varadhan in [2].

Their proof relies on the machinery of the Donsker-Varadhan large deviations results. For their proof to work Donsker and Varadhan need that the single-site potential decays fast enough, namely:

$$|f(x)| \leq \frac{c}{1 + |x|^\alpha} \tag{6}$$

with $\alpha > d + 2$.

Pastur [3] proved that for slower decay, i.e.

$$f(x) \sim \frac{c}{1 + |x|^\alpha} \tag{7}$$

with $d < \alpha < d + 2$ the Lifshitz behavior (5) is changed to

$$N(E) \sim e^{-c(E-E_0)^{-\frac{d}{\alpha-d}}}. \tag{8}$$

In the eighties the so called Dirichlet-Neumann-bracketing technique was used to prove Lifshitz tails (as in (5) or in (8)) for a greater variety of random potentials

([4], [5], [6]). This technique is much simpler than the Donsker-Varadhan method. It is also much closer to the original physical arguments by Lifshitz.

Recently, in [7] single-site potentials with anisotropic decay were considered.

Suppose that $x = (x_1, x_2) \quad x_1 \in \mathbb{R}^{d_1},\ x_2 \in \mathbb{R}^{d_2}$ and

$$|f(x)| \sim \frac{c}{1 + |x_1|^{\alpha_1} + |x_2|^{\alpha_2}}. \tag{9}$$

We define $\gamma_i = \frac{d_i}{d_k}$ and $\gamma = \gamma_1 + \gamma_2$. Then

$$N(E) \sim e^{-c(E-E_0)^{-\eta}} \tag{10}$$

where

$$\eta = \max(\frac{d_1}{2}, \frac{\gamma_1}{1-\gamma}) + \max(\frac{d_2}{2}, \frac{\gamma_2}{1-\gamma}). \tag{11}$$

If we introduce a constant magnetic field into the Hamiltonian the Lifshitz behavior is qualitatively changed.

For example for $d = 2$ and $f(x) \sim \frac{c}{1+|x|^\alpha}$ it was proved [8] that

$$N(E) \sim e^{-c(E-E_0)^{-\frac{d}{\alpha-d}}}. \tag{12}$$

even if $\alpha > d + 2$. L. Erdös [9] proved that for compactly supported $f \quad N(E)$ decays polynomially. For $d = 3$ see ([10], [11]).

Finally, we mention that Lifshitz tails may also exist at internal band edges ([12], [13]) as well as for random surface potentials ([14], [15]).

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Liouville theorems on abelian coverings of compact manifolds

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(joint work with Yehuda Pinchover, Technion, Israel)

The classical Liouville theorem claims that any harmonic function in $\mathbb{R}^n$ of a polynomial growth is in fact a polynomial. In particular, the space of all harmonic functions that grow not faster than $C(1+|x|)^N$, is of finite dimension $\binom{n+N}{N} - \binom{n+N-2}{N-2}$. Analogously, the space of holomorphic function in $\mathbb{C}^n$ of same growth consists of holomorphic polynomials of order N. The problem of extending this result to more general elliptic operators and/or to Laplace-Beltrami operators on general Riemannian manifolds of non-negative Ricci curvature was suggested in work of S. T. Yau [14]. One is interested in finite dimensionality of the spaces of solutions of a prescribed polynomial growth, estimates of (or even formulas for) their dimensions, and structure of these solutions (see [3, 7, 8] and references therein). Yau's conjecture on validity of the Liouville theorem for Riemannian manifolds of non-negative Ricci curvature was proven in full generality in [3] (see a description of previous partial results in [7, 8]).

An amazing case was discovered in [1, 13], where divergence form periodic elliptic equations $Lu = -\sum_{1\leq i,j\leq n} (a^{i,j}(x)u_{x_i})_{x_j} = 0$ were considered. It was shown that the space of solutions of polynomial growth of order at most N of $Lu = 0$ has the same dimension as the space of harmonic polynomials of the same rate of growth. Any such solution is representable in the Floquet form $v(x) = \sum_{j=(j_1,\dots,j_n)\in\mathbb{Z}^n_+} x^j p_j(x)$, with periodic coefficients $p_j(x)$.

The natural questions to ask are: Is it important that the operator is of divergence form? What can be said about more general periodic (including higher order and matrix) equations? Is it possible to determine for a given periodic elliptic equation whether the Liouville theorem holds? How crucial is the usage in [1, 13] of homogenezation theory tools (which automatically restrict the class of equations)? Can these results be generalized for abelian coverings of compact manifolds?

Some partial answers to these questions were obtained in simultaneous papers [5, 9]. In [9], the results were generalized to second order periodic operators without lower order terms. At the same time, [5] contained a necessary and sufficient condition for the validity of the Liouville theorem for a periodic elliptic operator in $\mathbb{R}^n$, as well as (in most cases implicit) description of the dimensions of the corresponding spaces of solutions.

Simultaneously, an activity has existed of studying Liouville theorems for holomorphic functions on complex analytic manifolds (see e.g., [10, 11, 12]). In particular, one asks whether Liouville theorems for holomorphic functions hold for coverings of compact analytic manifolds. One of the results in [10] shows that the space of such bounded functions is finite-dimensional on nilpotent coverings of compact complex analytic manifolds. It was not clear whether one could say the same about the spaces of functions of given polynomial growth, except in the Kähler case [2].

The talk describes the results of [6] that clarify these issues for elliptic equations and systems (including overdetermined ones) on abelian coverings of compact Riemannian manifolds and holomorphic functions on abelian coverings of compact complex manifolds. The crucial techniques used come from the Floquet theory and are related to spectral notions common to the solid state physics.

Let $X \overset{G}{\mapsto} M$ be an abelian covering of a compact d-dimensional Riemannian manifold M with an abelian deck group G (w.l.o.g. one can assume $G = \mathbb{Z}^n$). Let P be an elliptic G-periodic operator on X, with sufficiently smooth coefficients. For any character χ of G, one can consider a "twisted" version $P(\chi)$ of P on M that acts in sections of the linear bundle on M determined by χ (it is the push-down of P considered on χ-automorphic functions on X). In "normal" cases, the spectra of all $P(\chi)$ are discrete. The spectrum of $P(\chi)$ as a multiple-valued function of the character χ is called in solid state physics the *dispersion curve* or *dispersion relation* of P. The *Fermi surface* F of P is the set of unitary characters χ s.t. $Pu = 0$ has a non-zero χ-automorphic solution (i.e., F is the zero level set of the dispersion relation).

We say that the Liouville theorem holds to an order N for $Pu = 0$, if the space $V_N(P)$ of solutions of the equation with a bound $|u(x)| \leq C(1 + \rho(x))^N$ is finite dimensional. Here $\rho(x)$ is the distance of $x \in X$ from a fixed point $x_0 \in X$.

The theorem below describes our main results for the elliptic case.

Theorem 1

(1) *If Liouville theorem for the equation $Pu = 0$ holds to an order $N \geq 0$, it holds to any order.*
(2) *In order for the Liouville theorem to hold, it is necessary and sufficient that the Fermi surface F consists of finitely many points (this essentially means that one should expect the Liouville theorem to hold only when zero is at an edge of the spectrum of P).*

(3) *If the Liouville theorem holds, then under some genericity condition on the operator P, the dimension of the space $V_N(P)$ can be computed explicitly in terms of the first non-zero term of the Taylor expansion of the dispersion curve near its zeros.*

(4) *Under the same conditions, one can describe a constant coefficient ('homogenized') linear differential operator $\Lambda(D)$ on $\mathbb{R}^n$, such that there is a one-to-one correspondence between polynomial solutions of $\Lambda v = 0$ on $\mathbb{R}^n$ and polynomially growing solutions of $Pu = 0$ on X.*

Similar results hold for overdetermined elliptic systems, including Cauchy-Riemann $\bar{\partial}$ operators. Here one obtains in particular the following

Theorem 2 *Let $X \to M$ be an abelian covering of a compact complex analytic manifold M and X be equipped with a periodic with respect to the deck group Riemannian metric. Then for any N the space of holomorphic functions on X of the polynomial growth of order N is finite dimensional. All such functions are polynomials of a fixed finite set of holomorphic functions.*

The proofs of the results dependent upon the techniques of Floquet theory [4].

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Homogenization problem for the stationary periodic Maxwell system

T. A. Suslina

1. We study the homogenization problem for the stationary periodic Maxwell system in the small period limit. There is a vast literature on this problem. In particular, it was discussed in the books [1,2]. However, known results provide only the weak convergence of solutions. We report on the new results [5,6] about approximations of the solutions in the $L_2(\mathbb{R}^3)$-norm. The results are based on the abstract operator theory approach developed in [3,4].

2. Statement of the problem. Let Γ be a lattice in $\mathbb{R}^3$, and let Ω be the cell of Γ. Suppose that the permittivity $\eta(\mathbf{x})$ and the permeability $\mu(\mathbf{x})$ are Γ-periodic measurable (3×3)-matrix-valued functions in $\mathbb{R}^3$ with real entries, and

$$c_0\mathbf{1} \le \eta(\mathbf{x}) \le c_1\mathbf{1}, \quad c_0\mathbf{1} \le \mu(\mathbf{x}) \le c_1\mathbf{1}, \quad \mathbf{x}\in\mathbb{R}^3, \quad 0 < c_0 \le c_1 < \infty. \tag{1}$$

Here $\mathbf{1}$ is the identity matrix. We put $\mathfrak{G} = L_2(\mathbb{R}^3;\mathbb{C}^3)$. By $\mathfrak{G}(\eta^{-1}) = L_2(\mathbb{R}^3;\mathbb{C}^3;\eta^{-1})$ we denote the "weighted" space with the norm $(\eta^{-1}\mathbf{f},\mathbf{f})_{\mathfrak{G}}^{1/2}$. The space $\mathfrak{G}(\mu^{-1}) = L_2(\mathbb{R}^3;\mathbb{C}^3;\mu^{-1})$ is defined in a similar way. We denote $J = \{\mathbf{f}\in\mathfrak{G} : \operatorname{div}\mathbf{f} = 0\}$. In what follows, $\mathbf{u}$ and $\mathbf{v}$ stand for the electric and magnetic field intensity, respectively, $\mathbf{w} = \eta\mathbf{u}$ is the electric displacement vector, and $\mathbf{z} = \mu\mathbf{v}$ is the magnetic induction vector. We represent the Maxwell operator $\mathcal{M} = \mathcal{M}(\eta,\mu)$ in terms of $\mathbf{w}$ and $\mathbf{z}$ assuming that they are divergence free. Then $\mathcal{M}$ acts in the space $J\oplus J$ and is defined by the relations

$$\begin{gathered}\mathcal{M}(\eta,\mu) = \begin{pmatrix} 0 & i\operatorname{rot}\mu^{-1} \\ -i\operatorname{rot}\eta^{-1} & 0\end{pmatrix}, \\ \operatorname{Dom}\mathcal{M}(\eta,\mu) = \{(\mathbf{w},\mathbf{z}):\ \mathbf{w}\in J,\ \mathbf{z}\in J,\ \operatorname{rot}\eta^{-1}\mathbf{w}\in\mathfrak{G},\ \operatorname{rot}\mu^{-1}\mathbf{z}\in\mathfrak{G}\}.\end{gathered} \tag{2}$$

The operator $\mathcal{M}$ is selfadjoint in $J\oplus J$ treated as a subspace of $\mathfrak{G}(\eta^{-1})\oplus\mathfrak{G}(\mu^{-1})$.

Let ε be a parameter. We denote $\eta^\varepsilon(\mathbf{x}) = \eta(\varepsilon^{-1}\mathbf{x})$, $\mu^\varepsilon(\mathbf{x}) = \mu(\varepsilon^{-1}\mathbf{x})$. Consider the family of operators $\mathcal{M}_\varepsilon = \mathcal{M}(\eta^\varepsilon,\mu^\varepsilon)$. *Our goal is to study the behavior of the resolvent* $(\mathcal{M}_\varepsilon - iI)^{-1}$ *as* $\varepsilon\to 0$. Consider the equation

$$(\mathcal{M}_\varepsilon - iI)\begin{pmatrix}\mathbf{w}_\varepsilon \\ \mathbf{z}_\varepsilon\end{pmatrix} = \begin{pmatrix}\mathbf{q} \\ \mathbf{r}\end{pmatrix}, \quad \mathbf{q},\mathbf{r}\in J. \tag{3}$$

The corresponding intensities are given by $\mathbf{u}_\varepsilon = (\eta^\varepsilon)^{-1}\mathbf{w}_\varepsilon$ and $\mathbf{v}_\varepsilon = (\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon$. We are interested in the behavior of all four fields $\mathbf{u}_\varepsilon$, $\mathbf{v}_\varepsilon$, $\mathbf{w}_\varepsilon$, $\mathbf{z}_\varepsilon$ as $\varepsilon\to 0$.

3. Results. It is useful to represent the solution components as the sums $\mathbf{w}_\varepsilon = \mathbf{w}_\varepsilon^{(q)} + \mathbf{w}_\varepsilon^{(r)}$, $\mathbf{z}_\varepsilon = \mathbf{z}_\varepsilon^{(q)} + \mathbf{z}_\varepsilon^{(r)}$, where the pair $\mathbf{w}_\varepsilon^{(q)}$, $\mathbf{z}_\varepsilon^{(q)}$ is the solution of (3) with $\mathbf{r} = 0$ and the pair $\mathbf{w}_\varepsilon^{(r)}$, $\mathbf{z}_\varepsilon^{(r)}$ is the solution of (3) with $\mathbf{q} = 0$. The fields $\mathbf{u}_\varepsilon$ and $\mathbf{v}_\varepsilon$ are represented in a similar way. For "half of the fields", namely, *for* $\mathbf{v}_\varepsilon^{(r)}$, $\mathbf{z}_\varepsilon^{(r)}$ *and* $\mathbf{u}_\varepsilon^{(q)}$, $\mathbf{w}_\varepsilon^{(q)}$ *we obtain uniform approximations in the* $\mathfrak{G}$*-norm.*

These approximations are of precise order with respect to parameter ε. For the remaining fields $\mathbf{v}_\varepsilon^{(q)}$, $\mathbf{z}_\varepsilon^{(q)}$ and $\mathbf{u}_\varepsilon^{(r)}$, $\mathbf{w}_\varepsilon^{(r)}$ we still have only weak convergence (which was known before).

Consider the case where $\mathbf{q} = 0$ in detail. Then equation (3) takes the form

$$\left.\begin{aligned} \mathbf{w}_\varepsilon^{(r)} = \operatorname{rot}\,(\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon^{(r)}, \quad \operatorname{div}\mathbf{z}_\varepsilon^{(r)} = 0, \\ \operatorname{rot}\,(\eta^\varepsilon)^{-1}\mathbf{w}_\varepsilon^{(r)} + \mathbf{z}_\varepsilon^{(r)} = i\mathbf{r}, \quad \operatorname{div}\mathbf{w}_\varepsilon^{(r)} = 0. \end{aligned}\right\} \tag{4}$$

Accordingly,

$$\mathbf{u}_\varepsilon^{(r)} = (\eta^\varepsilon)^{-1}\mathbf{w}_\varepsilon^{(r)}, \quad \mathbf{v}_\varepsilon^{(r)} = (\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon^{(r)}. \tag{5}$$

The results are formulated in terms of the "homogenized" Maxwell system and the "correction" Maxwell system. Let μ^0 be the "effective" matrix (e. g., see [1,2]) corresponding to the elliptic operator $-\operatorname{div}\mu(\mathbf{x})\nabla$. Recall the definition of the (constant positive) matrix μ^0. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the standard ortonormal basis in $\mathbb{R}^3$, and let $\Phi_j \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$, $j = 1, 2, 3$, be a Γ-periodic solution of the equation $\operatorname{div}\mu(\mathbf{x})(\nabla\Phi_j(\mathbf{x}) + \mathbf{e}_j) = 0$. By $\widetilde{\mu}(\mathbf{x})$ we denote the matrix with columns $\mu(\mathbf{x})(\nabla\Phi_j(\mathbf{x}) + \mathbf{e}_j)$, $j = 1, 2, 3$. Then $\mu^0 = |\Omega|^{-1}\int_\Omega \widetilde{\mu}(\mathbf{x})\,d\mathbf{x}$. The effective matrix η^0 corresponding to the operator $-\operatorname{div}\eta(\mathbf{x})\nabla$ is defined in a similar way. We put $\mathcal{M}^0 = \mathcal{M}(\eta^0, \mu^0)$. Let $(\mathbf{w}_0^{(r)}, \mathbf{z}_0^{(r)})$ be the solution of the "homogenized" Maxwell system

$$(\mathcal{M}^0 - iI)\begin{pmatrix} \mathbf{w}_0^{(r)} \\ \mathbf{z}_0^{(r)} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{r} \end{pmatrix}. \tag{6}$$

We put

$$\mathbf{u}_0^{(r)} = (\eta^0)^{-1}\mathbf{w}_0^{(r)}, \quad \mathbf{v}_0^{(r)} = (\mu^0)^{-1}\mathbf{z}_0^{(r)}. \tag{7}$$

Now we describe the "correction" Maxwell system. Let $F(\mathbf{x})$ be the matrix with columns $\nabla\Phi_j(\mathbf{x})$, $j = 1, 2, 3$. Note that $F(\mathbf{x})$ is a Γ-periodic matrix-valued function with zero mean value. We denote $F^\varepsilon(\mathbf{x}) = F(\varepsilon^{-1}\mathbf{x})$. Let $\mathcal{P}_0$ be the orthogonal projection in $\mathfrak{G}((\mu^0)^{-1})$ onto J. We put $\mathbf{r}_\varepsilon = \mathcal{P}_0(F^\varepsilon)^*\mathbf{r}$. Then $\mathbf{r}_\varepsilon \in H^{-1}(\mathbb{R}^3; \mathbb{C}^3)$ and $\operatorname{div}\mathbf{r}_\varepsilon = 0$. Let $(\widetilde{\mathbf{w}}_\varepsilon^{(r)}, \widetilde{\mathbf{z}}_\varepsilon^{(r)})$ be the solution of the "correction" Maxwell system

$$(\mathcal{M}^0 - iI)\begin{pmatrix} \widetilde{\mathbf{w}}_\varepsilon^{(r)} \\ \widetilde{\mathbf{z}}_\varepsilon^{(r)} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{r}_\varepsilon \end{pmatrix}. \tag{8}$$

We put

$$\widetilde{\mathbf{v}}_\varepsilon^{(r)} = (\mu^0)^{-1}\widetilde{\mathbf{z}}_\varepsilon^{(r)}. \tag{9}$$

Note that the fields $\widetilde{\mathbf{w}}_\varepsilon^{(r)}$, $\widetilde{\mathbf{z}}_\varepsilon^{(r)}$, $\widetilde{\mathbf{v}}_\varepsilon^{(r)}$ weakly tend to zero in $\mathfrak{G}$. The reason is that the right-hand side $\mathbf{r}_\varepsilon$ in (8) contains the factor F^ε, which weakly tends to zero in $L_{2,\mathrm{loc}}(\mathbb{R}^3)$ (by the "mean value property").

Our main result (as applied to the case $\mathbf{q} = 0$) is the following theorem.

Theorem. *Suppose that Γ-periodic matrix-valued functions $\eta(\mathbf{x})$, $\mu(\mathbf{x})$ satisfy conditions* (1)*. Let $(\mathbf{w}_\varepsilon^{(r)}, \mathbf{z}_\varepsilon^{(r)})$ be the solution of system* (4) *with $\mathbf{r} \in J$, and let $\mathbf{u}_\varepsilon^{(r)}, \mathbf{v}_\varepsilon^{(r)}$ be defined by* (5)*. Supose that $(\mathbf{w}_0^{(r)}, \mathbf{z}_0^{(r)})$ is the solution of system* (6)*,*

and let $\mathbf{u}_0^{(r)}, \mathbf{v}_0^{(r)}$ *be defined by* (7). *Suppose that* $(\widetilde{\mathbf{w}}_\varepsilon^{(r)}, \widetilde{\mathbf{z}}_\varepsilon^{(r)})$ *is the solution of system* (8), *and let* $\widetilde{\mathbf{v}}_\varepsilon^{(r)}$ *be defined by* (9). *Then the following assertions hold.*
1°. *For the magnetic intensity* $\mathbf{v}_\varepsilon^{(r)}$ *we have the approximation*

$$\|\mathbf{v}_\varepsilon^{(r)} - (\mathbf{1} + F^\varepsilon)(\mathbf{v}_0^{(r)} + \widetilde{\mathbf{v}}_\varepsilon^{(r)})\|_{\mathfrak{G}} \le C\varepsilon \|\mathbf{r}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{10}$$

2°. *As* $\varepsilon \to 0$, $\mathbf{v}_\varepsilon^{(r)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{v}_0^{(r)}$, *and* $\operatorname{rot} \mathbf{v}_\varepsilon^{(r)}$ *weakly tends in* $\mathfrak{G}$ *to* $\operatorname{rot} \mathbf{v}_0^{(r)}$.
3°. *For the magnetic induction vector* $\mathbf{z}_\varepsilon^{(r)}$ *we have the approximation*

$$\|\mathbf{z}_\varepsilon^{(r)} - (\mathbf{1} + G^\varepsilon)(\mathbf{z}_0^{(r)} + \widetilde{\mathbf{z}}_\varepsilon^{(r)})\|_{\mathfrak{G}} \le C\varepsilon \|\mathbf{r}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1, \tag{11}$$

where $G(\mathbf{x}) := \widetilde{\mu}(\mathbf{x})(\mu^0)^{-1} - \mathbf{1}$ *is a* Γ*-periodic matrix-valued function with zero mean value, and* $G^\varepsilon(\mathbf{x}) = G(\varepsilon^{-1}\mathbf{x})$.
4°. *As* $\varepsilon \to 0$, $\mathbf{z}_\varepsilon^{(r)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{z}_0^{(r)}$.
5°. *As* $\varepsilon \to 0$, *the electric field intensity* $\mathbf{u}_\varepsilon^{(r)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{u}_0^{(r)}$. *For* $\operatorname{rot} \mathbf{u}_\varepsilon^{(r)} = i\mathbf{r} - \mathbf{z}_\varepsilon^{(r)}$ *we have approximation in the* $\mathfrak{G}$*-norm* (*see* (11)).
6°. *As* $\varepsilon \to 0$, *the electric displacement vector* $\mathbf{w}_\varepsilon^{(r)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{w}_0^{(r)}$.

The case where $\mathbf{r} = 0$ can be treated in a similar way. For the fields $\mathbf{u}_\varepsilon^{(q)}$ and $\mathbf{w}_\varepsilon^{(q)}$ we obtain approximations in the $\mathfrak{G}$-norm similar to (10), (11), while for $\mathbf{v}_\varepsilon^{(q)}$, $\mathbf{z}_\varepsilon^{(q)}$ we have only the weak convergence.

In the case where permeability is constant: $\mu = \mu^0$, the results simplify. In this case the solutions of the "correction" system (8) are trivial, and we have

$$\|\mathbf{v}_\varepsilon^{(r)} - \mathbf{v}_0^{(r)}\|_{\mathfrak{G}} \le C\varepsilon \|\mathbf{r}\|_{\mathfrak{G}}, \quad \|\mathbf{z}_\varepsilon^{(r)} - \mathbf{z}_0^{(r)}\|_{\mathfrak{G}} \le C\varepsilon \|\mathbf{r}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{12}$$

For $\mu(\mathbf{x}) = \mathbf{1}$ this result has been obtained before in [4]. If $\eta = \eta^0$, then for $\mathbf{u}_\varepsilon^{(q)}$, $\mathbf{w}_\varepsilon^{(q)}$ we have similar results.

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Bound States and Essential Spectrum

David Damanik

Given a Schrödinger operator $H_V = -\Delta + V$ in $L^2(\mathbb{R}^d)$ or $h_V = \Delta + V$ in $\ell^2(\mathbb{Z}^d)$, a basic problem is to study the discrete spectrum and the essential spectrum. In recent years, several papers have uncovered a surprising connection between these two parts of the spectrum.

We first consider operators with empty discrete spectrum. The following theorem was shown by Killip and Simon [5]:

Theorem 1. *Suppose $\mathcal{H} = \ell^2(\mathbb{Z})$. Then $\sigma(h_V) \subseteq [-2, 2]$ implies $V \equiv 0$.*

In particular, every potential that does not vanish identically must produce spectrum outside of the free spectrum, $[-2, 2]$. Note that this result holds without any apriori assumption on V. The proof of Theorem 1 given in [5] relies on sum rules and is, to some extent, a by-product of their general study culminating in a characterization of all (half-line) Jacobi matrices that are Hilbert-Schmidt perturbations of the free operator in terms of properties of the spectral measure.

A more direct and elementary proof based on suitable choices of test functions was given in [1]. Moreover, it was possible to extend the result to two dimensions:

Theorem 2. *Suppose $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ with $d = 1$ or 2. Then $\sigma(h_V) \subseteq [-2d, 2d]$ implies $V \equiv 0$.*

It was also shown in [1] that $\sigma_{\text{ess}}(h_V) \subseteq [-2d, 2d]$ implies $V \to 0$. Both statements fail in dimensions $d \geq 3$.

On the half-line, the following example was discussed in [1]. Consider the operator h_V in $\ell^2(\mathbb{Z}_+)$ with potential $V(n) = (-1)^n/n$. Then, h_V has spectrum $[-2, 2]$. This shows that not only can $V \equiv 0$ fail under the assumption $\sigma(h_V) \subseteq [-2, 2]$ for half-line operators, it is also not immediately clear what one can say about the spectral type inside $[-2, 2]$.

Half-line operators were studied in [2], where the following theorem was proven:

Theorem 3. *Suppose $\mathcal{H} = \ell^2(\mathbb{Z}_+)$. Then $\sigma(h_V) \subseteq [-2, 2]$ implies $\sigma_{\text{sing}}(h_V) = \emptyset$.*

The main steps in the proof of Theorem 3 are as follows: First map the spectral measure to the unit circle via $E = z + z^{-1}$ and find relations between the potential and the Verblunsky coefficients of the associated measure. Then use this connection to find bounds on the potential. Finally, use these bounds to show that there cannot be any embedded singular spectrum. For example, it is shown that under the assumption $\sigma(h_V) \subseteq [-2, 2]$, V may be written as

$$V(n) = W(n) - W(n-1) + Q(n),$$

where

$$Q \in \ell^1 \text{ and } \sum_{n=1}^{N} n W(n)^2 \leq \frac{1}{4} \log N + C.$$

The continuum case is also studied in [2]. Note that the unitary $[U\phi](n) = (-1)^n \phi(n)$ conjugates h_{-V} and $-h_V$. Therefore, $\sigma(h_V) \subseteq [-2, 2]$ is equivalent to the two conditions $\sigma(h_{\pm V}) \subseteq [-2,\infty)$. Thus, the following theorem from [2] is the natural continuum analogue of Theorem 3:

Theorem 4. *Suppose $\mathcal{H} = L^2(\mathbb{R}_+)$ and $V \in \ell^\infty(L^2)$. Then $\sigma(H_{\pm V}) \subseteq [0,\infty)$ implies $\sigma_{\text{sing}}(H_V) = \emptyset$.*

Given this observation, it natural to ask whether the $\ell^2(\mathbb{Z})$ and $\ell^2(\mathbb{Z}^2)$ results have continuum analogues. This is indeed the case, as shown in [3]:

Theorem 5. *Let $d = 1$ or 2. Suppose that $Q \in L^2_{\text{loc}}(\mathbb{R}^d)$ and the operator H_Q has a bounded positive ground state. If $V \in L^2_{\text{loc}}(\mathbb{R}^d)$ and both $H_{Q\pm V}$ are bounded below by the ground state energy of H_Q, then $V \equiv 0$.*

For $Q \equiv 0$, this gives the continuum analogue of Theorem 2 (choose $\psi \equiv 1$ as the bounded positive ground state), but it also applies to periodic Q, for example.

Thus, one has a good understanding of cases without bound states. If a perturbation introduces only finitely many bound states, one may still hope for strong restrictions on V and the spectral type inside the essential spectrum. In fact, Theorems 3 and 4 extend quite easily to the case of finitely many bound states. Thus, on the half-line, finiteness of the number of bound states implies the absence of embedded singular spectrum, as shown in [2]. For operators on the line, the corresponding results were shown in [3]. The problem is open in two dimensions.

The following example shows that an extension to operators with infinitely many bound states could be involved. On the half-line, the operator with Wigner-von Neumann-type potential $V(n) = (1+\varepsilon)(-1)^n/n + O(1/n^2)$, $\varepsilon > 0$, has an embedded eigenvalue and the discrete eigenvalues decay exponentially; see [3]. In particular, the finiteness of bound state moments is not sufficient to exclude embedded singular spectrum. On the other hand, positive results are obtained in [4]. For example, if the p-th bound state moment is finite, then the embedded singular spectrum must be supported on a set of Hausdorff dimension $4p$.

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Properties of Coulombic wavefunctions and their electron density

Maria Hoffmann-Ostenhof
(joint work with Søren Fournais, Thomas Hoffmann-Ostenhof and Thomas Østergaard Sørensen)

Let H be the non-relativistic Schrödinger operator of an N-electron atom with nuclear charge Z and the nucleus fixed in the origin, given by

$$H = -\Delta + V = \sum_{j=1}^{N}(-\Delta_j - \frac{Z}{|x_j|}) + \sum_{1\le i<j\le N} \frac{1}{|x_i - x_j|} \tag{1}$$

where $x_j \in \mathbb{R}^3$, $1 \le j \le N$, are the electron coordinates and the Δ_j the associated Laplacians. Let $\psi \in L^2(\mathbb{R}^{3N})$ be an eigenfunction of H with eigenvalue E. By a classical result of Kato [5], ψ is locally Lipschitz and real analytic away from the singularities of the potential V.

The main result in [3] is a representation result for electronic wavefunctions of atoms and molecules which is stated here for simplicity for the atomic case:

Theorem
Let ψ be as above and let

$$\mathcal{F} = e^{F_2+F_3}$$

with

$$F_2 = -\frac{Z}{2}\sum_{i=1}^{N}|x_i| + \frac{1}{4}\sum_{1\le i<j\le N}|x_i - x_j|$$

and

$$F_3 = c_0 Z \sum_{1\le i<j\le N} \langle x_i|x_j\rangle \ln(|x_i|^2 + |x_j|^2), \quad c_0 = \frac{2-\pi}{12\pi}.$$

Then

$$\psi = \mathcal{F}\Phi \text{ with } \Phi \in C^{1,1}(\mathbb{R}^{3N}).$$

From this and earlier results of the present authors certain properties of ψ and the associated 1-electron density

$$\rho(x) = \int_{\mathbb{R}^{3N-3}} |\psi|^2(x, x')dx', \; x \in \mathbb{R}^3$$

can be shown (work in progress): In 1957 Kato, [5], analyzed the behaviour of ψ in an averaged sense near two particle coalescence points, (Kato's cusp conditions). Generalizations of such "cusp properties" are investigated. By a cusp condition (resp. property) we will understand a condition a solution ψ has to satisfy near a point, where the potential in (1) is singular. In an L^∞-sense such properties concerning second order partial derivatives of ψ are given in [3]. In progress is work on cusp conditions for the 1-electron density ρ. It can be shown that for some $\overrightarrow{c} \in \mathbb{R}^3$

$$\frac{\partial\rho(r\omega)}{\partial r}\Big|_{r=0} = \lim_{r\downarrow 0}\frac{\partial}{\partial r}\rho(r\omega) = -Z\rho(0) + \langle\omega|\overrightarrow{c}\rangle$$

where $x = r\omega$, so that $\omega \in \mathbb{S}^2$.

Other investigations concern the regularity properties of ρ, respectively, of the spherically averaged density $\widetilde{\rho} = \widetilde{\rho}(r)$. It is known [1, 2] that ρ is smooth and even real analytic away from the origin. An open question is the regularity of $\widetilde{\rho}$ near the origin $\mathcal{O}$. So far only $C^2([0,\infty))$ was known, [4], and this can be extended, via apropriate estimates derived in [3], to C^3.

Another interesting question is whether ρ is strictly positive in $\mathbb{R}^3$. Of course this cannot be true in general, since it fails for excited states of the Hydrogen atom. It is well known that the mathematical groundstate ψ satisifies $|\psi| > 0$ in $\mathbb{R}^{3N}$ and therefore the associated density ρ is strictly positive.

We investigate the spherically averaged density associated to a groundstate of an atom in some symmetry subspace and are going to show that it is strictly positive and we will also give an explicit lower bound to $\rho(0)$. (Note that for these considerations we use the symmetrized (physical) density ρ instead of the one defined above.)

Whether $\widetilde{\rho}(r)$ is monotonically decreasing is an open problem for decades. This monotonicity is expected to hold for groundstate densities, but not known even for the bosonic case like the Helium groundstate in spite of overwhelming numerical evidence. So far it is only known in a sufficiently small neighborhood of the origin and sufficiently far away from the nucleus.

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On a Magnetic Hardy Inequality in The Waveguide

Hynek Kovařík

(joint work with Denis Borisov and Tomas Ekholm)

It is well known that the classical Hardy inequality fails to hold in dimensions one and two. This is closely related to the fact, that arbitrarily small attractive potential perturbation produces at least one negative eigenvalue of the Schrödinger operator on $L^2(\mathbb{R}^d)$, $d = 1, 2$. As a consequence, the threshold of the spectrum of the Laplacian in the so called quantum waveguide, i.e. in a two-dimensional strip Ω with Dirichlet boundary conditions, is unstable under any local enlargement or bending of the waveguide, see [BGRS], [EŠ], [GJ].

On the other hand, in 1999 Laptev and Weidl proved a modified version of the Hardy inequality in $\mathbb{R}^2$ for the quadratic form of a magnetic Schrödinger operator

$$\text{Const} \int_{\mathbb{R}^2} \frac{|u(x)|^2}{1+|x|^2}\, dx \;\leq\; \int_{\mathbb{R}^2} |(-\mathrm{i}\nabla + A)u(x)|^2\, dx, \tag{1}$$

and gave a sharp result for the case of Aharonov-Bohm field. See [LW] for details. This work was later extended in [B] to multiple Aharonov-Bohm magnetic potentials, see also [EL]. Recently another generalization of the result by Laptev and Weidl was obtained in [BLS].

In our model we study the spectrum of the magnetic Schrödinger operator $(-\mathrm{i}\nabla + A)^2$ on $L^2(\Omega)$ with $\Omega = \mathbb{R} \times (0, \pi))$. Essential difference to the case treated in [LW] is that due to the Dirichlet boundary conditions the spectrum starts from 1. Consequently inequality (1) becomes trivial. Therefore we shall subtract the threshold of the spectrum and prove a Hardy-inequality in the form

$$\text{Const} \int_{\mathbb{R}\times(0,\pi)} \frac{|u(x)|^2}{1+x_1^2}\, dx \;\leq\; \int_{\mathbb{R}\times(0,\pi)} \big(|(-\mathrm{i}\nabla + A)u(x)|^2 - |u(x)|^2\big)\, dx, \tag{2}$$

for all u in the magnetic Sobolev space $H^1_{0,A}(\mathbb{R}\times(0,\pi))$. We prove this inequality for the magnetic Schrödinger operator with a locally bounded field.

As an application of inequality (2) we show that the threshold of the spectrum of the corresponding magnetic Schrödinger operator is stable under local geometrical perturbations of the waveguide as well as under local perturbations of the boundary conditions. In the first case it is shown that a sufficiently weak enlargement of the waveguide, depending on the magnetic field, will not produce any discrete spectrum of the operator $(-\mathrm{i}\nabla + A)^2$. In a similar way we note that the discrete spectrum of $(-\mathrm{i}\nabla + A)^2$ in a mildly curved waveguide stays empty as long as the corresponding curvature and its first derivative are small enough.

In the second model we consider a situation where the Dirichlet boundary condition is switched to magnetic Neumann on a fixed segment of the length $2l$ of the boundary of Ω. Such a perturbation is stronger than the geometrical perturbations of the boundary mentioned above and therefore a different approach is needed in order to establish the desired stability result. Using a similar integral inequality to (2) we are able to show that it suffices to prove the non-existence of discrete eigenvalues of the one-dimensional Schrödinger operator

$$A = -\frac{d^2}{dx^2} + V,$$

where V is a sum of the purely attractive potential well of the width $2l$ and a small, but fixed positive potential. We conclude that the discrete spectrum of A and consequently also the discrete spectrum of of the corresponding magnetic Schrödinger operator stays empty provided l is small enough.

This talk has been based on two papers, [EK] and [BEK], obtained in the collaboration with T.Ekholm and D.Borisov, T.Ekholm respectively.

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Spectral asymptotics for the Landau Hamiltonian and logarithmic capacity

Alexander Pushnitski

(joint work with Nikolai Filonov)

Consider the operator

$$H_0 = \left(-i\frac{\partial}{\partial x} + \frac{B}{2}y\right)^2 + \left(-i\frac{\partial}{\partial y} - \frac{B}{2}x\right)^2 \quad \text{in } L^2(\mathbb{R}^2, dx\,dy),$$

where $B > 0$ is the strength of the constant magnetic field. The spectrum of H_0 consists of the eigenvalues (known as Landau levels) $\Lambda_q = B(2q+1)$, $q = 0, 1, 2, \ldots$; each of these eigenvalues has infinite multiplicity.

Let $V \geq 0$ be a perturbation potential such that $V \in L^\infty(\mathbb{R}^2)$, and $\Omega = \operatorname{supp}(V)$ is compact. We consider the spectrum of the operators $H_\pm = H_0 \pm V$. It is well known that $\sigma_{ess}(H_\pm) = \sigma_{ess}(H_0) = \cup_{q=0}^\infty \Lambda_q$. Moreover, for any q the eigenvalues of H_+ can accumulate to Λ_q only from the right, and eigenvalues of H_- can accumulate to Λ_q only from the left.

Let us enumerate the eigenvalues of H_- in $(-\infty, \Lambda_0)$ in the ascending order (counting multiplicities):

$$\lambda_1^- \leq \lambda_2^- \leq \cdots \leq \lambda_n^- \leq \cdots < \Lambda_0.$$

Similarly, let us enumerate the eigenvalues of H_+ in (Λ_0, Λ_1) in the descending order (counting multiplicities):

$$\Lambda_0 < \cdots \leq \lambda_n^+ \leq \cdots \leq \lambda_2^+ \leq \lambda_1^+ < \Lambda_1.$$

For a bounded Borel set $C \subset \mathbb{R}^2$, we denote by $\operatorname{Cap} C$ the logarithmic capacity of C (see [2]). Denote

$$\rho(V) = \operatorname{Cap}\Omega, \quad \Omega = \operatorname{supp} V,$$

$$\rho_-(V) = \inf\{\operatorname{Cap} C \mid C \subset \mathbb{R}^2 \text{ is a bounded Borel set}, \int_{\mathbb{R}^2\setminus C} V(x,y)dx\,dy = 0\}.$$

Clearly, $\rho_-(V) \le \rho(V)$.

Theorem *Assume that $\rho_-(V) = \rho(V)$. Then one has the asymptotics:*

$$(*) \qquad \lambda_n^\pm - \Lambda_0 = \pm\frac{1}{n!}\left(\frac{B}{2}\rho(V)^2\right)^{n+o(n)}, \quad n \to \infty.$$

Remarks

(1) The asymptotics $(*)$ should be understood as

$$\log(\pm(\lambda_n^\pm - \Lambda_0)n!) = n\log\left(\frac{B}{2}\rho(V)^2\right) + o(n), \quad n \to \infty.$$

(2) An elementary calculation shows that the asymptotics $(*)$ is equivalent to

$$\lambda_{n+k}^\pm - \Lambda_0 = \pm\frac{1}{n!}\left(\frac{B}{2}\rho(V)^2\right)^{n+o(n)}, \quad n \to \infty,$$

for any integer k.

(3) We also have a way of treating the eigenvalue asymptotics near higher Landau levels Λ_q, $q \ge 1$, but at present this construction requires more restrictive assumptions on V and slightly more technical arguments, so we do not include it in this preliminary report.

For $t > 0$, let us define $N_-(t)$ as the total number of eigenvalues of H_- (counting multiplicities) in the interval $(-\infty, \Lambda_0 - t)$. Similarly, for $0 < t < 2B$, let us define $N_+(t)$ as the total number of eigenvalues of H_+ in $(\Lambda_0 + t, \Lambda_1)$. An elementary calculation shows that $(*)$ is equivalent to

$$N_\pm(t) = \frac{|\log t|}{(\log|\log t|)^2}\left(\log|\log t| + \log\log|\log t| + \log\left(\frac{B}{2}\rho(V)^2\right) + 1 + o(1)\right),$$

as $t \to +0$. In the papers [4] and [3], the asymptotics

$$N_\pm(t) = \frac{|\log t|}{(\log|\log t|)}(1 + o(1)), \quad t \to +0$$

was obtained.

The proof is based on the following ideas. First, as in the papers [4] and [3], we reduce the question to the asymptotics of the eigenvalues of the auxiliary operator P_0VP_0. Here P_0 is the spectral projection of H_0, corresponding to the first Landau level Λ_0. Next, the eigenvalues of P_0VP_0 are identified with the singular numbers of the embedding $F \subset L^2(\mathbb{R}^2, \tilde{V}(x,y)dx\,dy)$, where F is the so-called Fock class, and $\tilde{V}(x,y) = V(x,y)e^{-x^2-y^2}$. Using the techniques of [6], the singular numbers of this embedding are then expressed in terms of the asymptotics for a certain sequence of orthogonal polynomials. Finally, application of the "regularity criteria" of [5] yields the required result.

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Gaussian extremizers for the Strichartz inequality in one and two dimensions

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(joint work with Vadim Zharnitsky)

We show that in dimension one and two the only maximizers for the homogenous Strichartz inequality for the free Schrödinger evolution are Gaussians.

More precisely, let u be the solution to the free Schrödinger equation

$$i\partial_t u = \Delta u \tag{1}$$

with initial condition $u(0) = f \in L^2(\mathbb{R}^2)$. It is, of course, given by

$$u(t,x) = (e^{-it\Delta} f)(x) \tag{2}$$

where $e^{-it\Delta}$ is defined, for example, by the functional calculous. Since, for fixed time t, $e^{-it\Delta}$ is a unitary operator on $L^2(\mathbb{R}^d)$, one immediately sees that $u \in L_t^\infty(L^2(\mathbb{R}^d))$. But, in fact, due to the dispersive nature of the free Schrödinger equation, the solution u, as a function of space-time, obeys the stronger L^p-bound

$$\|u\|_{L^p(\mathbb{R}\times\mathbb{R}^d)} \leq S_d \|f\|_{L^2(\mathbb{R}^d)} \tag{3}$$

where $p = p(d) = 2 + \frac{4}{d}$. This was first shown by Strichartz [6] who followed the L^p restriction proof of Stein-Tomas. Later simplified proofs were given by Ginibre and Velo [4], see also [2, 7].

The sharp value of S_d, i.e., the quantity

$$S_d = \sup_{f\neq 0,} \frac{\|u\|_{L^p(\mathbb{R}\times\mathbb{R}^d)}}{\|f\|_{L^2(\mathbb{R}^d)}} \tag{4}$$

has been unknown until very recently. In fact, even the existence of maximizers for (4), that is, functions $f_* \neq 0$ such one has equality in (4),

$$S_d = \frac{\|e^{-it\Delta} f_*\|_{L^p(\mathbb{R}^{d+1})}}{\|f_*\|_{L^2(\mathbb{R})}}, \tag{5}$$

has been only recently established. By using an elaborate application of Lions' concentration compactness method, Markus Kunze showed in [5] that (4) has a maximizer in one dimension. His proof does not, however, provide any explicit information about the maximizer nor the value of S_1. The reason why even the existence of maximizers has not been known until recently is the invariance of the Strichartz inequality under the rather large group of Galilei transformations and scaling. This makes the usual existence proof for maximizers via minimizing sequences very hard, since they can very easily converge weakly to zero. The usual method to circumvent this is the concentration compactness principle, however, in this setting it has to be used twice, first in Fourier space, then in real space.

Very recently, Damiano Foschi [3] gave a proof of the Strichartz inequality in one and two dimensions, which yields the sharp constant. He showed

Theorem 1 (Foschi 2004, [3]). *The sharp constants for the Strichatz inequality in one and two dimensions are $S_1 = 12^{-1/12}$ and $S_2 = 2^{-1/2}$, respectively. Moreover, if the initial condition f is given by a Gaussian, then one has equality in the Strichartz inequality.*

However, the existence of non-Gaussian maximizers was not ruled out in [3]. The main purpose of this note is to give an simple argument which shows that at least in one and two dimensions the *only maximizers* in the Strichartz inequality are Gaussians. More precisely, we have the following

Theorem 2 (Gaussian maximizers). *Let $d = 1$ or 2. The function $f_* \in L^2(\mathbb{R}^d)$ is a maximizer for the Strichartz inequality (3), that is, (4) holds, if and only if f_* is a Gaussian. More precisely, there exists $A \in \mathbb{C}$, $\lambda > 0$, $\mu \in \mathbb{R}$, $a \in \mathbb{R}^d$, and $b \in \mathbb{C}^d$ such that*

$$f_*(x) = A\, e^{(-\lambda+i\mu)|x-a|^2+b\cdot x}. \tag{6}$$

The key for our proof is the following representation theorem. It shows that the Strichartz estimate follows from a simple bound on a *linear* operator and, moreover, gives a geometric criterion for the maximizer in the Strichartz inequality. For $f \in L^2(\mathbb{R}^d)$, denote by $f \otimes f$ be the usual tensor product, $\mathbb{R}^d \times \mathbb{R}^d \ni x = (x_1, x_2) \to f \otimes f(x) := f(x_1) f(x_2)$. Similarly for the triple tensor product $f \otimes f \otimes f$. Furthermore, let $P_1 : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ be the orthogonal projection operator onto the subspace consisting of functions $F \in L^2(\mathbb{R}^3)$ which are symmetric under rotations of $\mathbb{R}^3$ keeping the $(1,1,1)$ direction fixed. And similarly, let $P_2 : L^2(\mathbb{R}^4) \to L^2(\mathbb{R}^4)$ be the orthogonal projection operator onto functions $F \in L^2(\mathbb{R}^3)$ which are symmetric under rotations of $\mathbb{R}^4$ fixing both the $(1,0,1,0)$ and $(0,1,0,1)$ direction. With this, we have

Theorem 3. *Let $f \in L^2(\mathbb{R}^d)$.*

a) In dimension one,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |e^{-it\Delta} f(x)|^6 \, dxdt = \frac{1}{2\sqrt{3}} \langle \hat{f}\otimes\hat{f}\otimes\hat{f}, P_1(\hat{f}\otimes\hat{f}\otimes\hat{f})\rangle_{L^2(\mathbb{R}^3)}$$

b) In dimension two,

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} |e^{-it\Delta} f(x)|^4 \, dxdt = \frac{1}{4} \langle \hat{f}\otimes\hat{f}, P_2(\hat{f}\otimes\hat{f})\rangle_{L^2(\mathbb{R}^4)},$$

where $\hat{f}$ is the (space) Fourier transform of f.

One immediately gets the sharp Strichartz inequality, using that any projection operator operator is bounded by the identity. One also sees that, in order to have equality in the Strichartz inequality, the function $\hat{f}\otimes\hat{f}\otimes\hat{f}$ must be in the range of P_1 in dimension one, and similarly for the two-dimensional case. In other words, for any one-dimensional maximizer f of the Strichartz inequality, the function $\hat{f}\otimes\hat{f}\otimes\hat{f}$ is invariant under rotations of $\mathbb{R}^3$ which keep the $(1,1,1)$ direction fixed. Similarly, for any two-dimensional maximizer f, the function $\hat{f}\otimes\hat{f}$ is invariant under rotations of $\mathbb{R}^4$ which keep both the $(1,0,1,0)$ and $(0,1,0,1)$ directions fixed. This is obviously the case if $\hat{f}$, and hence f, is a Gaussian, and a simple proof, mimicked after a result by Carlen [1], shows that this geometric condition forces f to be a Gaussian.

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Some Results on the Spectra of Periodic Landau Operators

Daniel M. Elton

The *periodic Landau operator* on $\mathbb{R}^d$ is the magnetic Schrödinger operator

$$H_{B,V} = (D - A)^2 + V, \tag{1}$$

where $D = -i\nabla$, V is a periodic potential (with respect to some lattice Λ) and the magnetic field $B = \nabla\times A$ is constant. In this talk we are interested in spectral problems related to $H_{B,V}$ when $d = 2, 3$. For technical convenience we take $B =$

$(0, 0, \beta)$ with $\beta > 0$ when $d = 3$; when $d = 2$ this reduces to $B = \beta$. We will also assume that the lattice of periods is $\Lambda = (2\pi\mathbb{Z})^d$.

1. Reduction of the operator

To study the spectrum $\sigma(H_{B,V})$ we firstly use a metaplectic transformation to replace (1) with a unitarily equivalent operator; when $d = 2$ we get

$$D_x^2 + (\beta x)^2 + \mathrm{Op}^w\big(V(x + y/\beta, -\eta - \xi/\beta)\big) \tag{1}$$

acting in $L^2(\mathbb{R}^2)$, where $\mathrm{Op}^w(p)$ denotes the Weyl-quantised pseudo-differential operator (on $\mathbb{R}^2$) with symbol $p(x, y, \xi, \eta)$. In general (1) is a harmonic oscillator with a free variable, perturbed by an oscillatory pseudo-differential operator.

Although V and B are periodic functions, $H_{B,V}$ is not a periodic operator (owing to the presence of the magnetic potential A in (1)). The Bloch (or Floquet) techniques commonly used for periodic spectral problems are not directly available to study the spectrum $\sigma(H_{B,V})$. A partial remedy is available via a symmetry group consisting of "magnetic Bloch-transformations"; however this is only useful under the *flux rationality assumption*:

$$\beta = |B| = \frac{p}{2\pi q} \quad \text{for some } p, q \in \mathbb{N}. \tag{FR}$$

Under this condition, $H_{B,V}$ is unitarily equivalent to a direct integral

$$\int_{[0,1)}^{\oplus} dk_1 \int_{S^1}^{\oplus} dk_2\, \mathcal{H}(k_1, k_2) \tag{2}$$

with fibre operator $\mathcal{H}(k_1, k_2) = (D_x^2+(\beta x)^2)\otimes\mathcal{I}_p+\mathcal{A}(k_1, k_2)$ acting on $\bigoplus_{j=0}^{p-1} L^2(\mathbb{R})$; the potential V has become a $p \times p$ matrix of oscillatory pseudo-differential operators $\mathcal{A}(k_1, k_2)$. Since the spectrum of $(D_x^2 + (\beta x)^2) \otimes \mathcal{I}_p$ consists of discrete eigenvalues (the eigenvalues of the 1-dimensional harmonic oscillator, each with multiplicity p), we obtain a band gap picture for $\sigma(H_{B,V})$ (the bands are simply the ranges of the eigenvalues of $\mathcal{H}(k_1, k_2)$ considered as functions of the parameters k_1, k_2).

The above discussion modifies in the obvious way for the case $d = 3$.

2. Dimension $d = 3$

When $V \equiv 0$ a straightforward calculation shows $\sigma(H_{B,0}) = [\beta, \infty)$ and this spectrum is purely absolutely continuous. Under the assumption of flux rationality, the addition of a periodic potential does not alter the broadest features of $\sigma(H_{B,V})$; the spectrum remains purely absolutely continuous (this can be proved using the standard Thomas approach employed for $-\Delta + V$). Furthermore the spectrum contains at most finitely many gaps (the "Bethe-Sommerfeld conjecture"):

Theorem 1. *Suppose* (FR) *holds and* V *satisfies the regularity condition*

$$\sum_{m\in\mathbb{Z}^3} |m|^{\delta}|\widehat{V}_m| < +\infty$$

for some $\delta > 0$ *(where* $\widehat{V}_m$ *denote the Fourier coefficients of* V*). Then there exists* $\Gamma \in \mathbb{R}$ *such that* $[\Gamma, \infty) \subseteq \sigma(H_{B,V})$*; in particular,* $\sigma(H_{B,V})$ *contains only finitely many gaps. Furthermore,* Γ *depends continuously on the lattice* Λ *and on* $\beta = |B|$*.*

See [1]; previously the result was obtained for any sufficiently small bounded V in [3].

There do not appear to be any general results on $\sigma(H_{B,V})$ in the case of non-rational flux.

3. Dimension $d = 2$

It is well known that $\sigma(H_{B,0})$ consists of the discrete eigenvalues $\beta(2n-1)$, $n \in \mathbb{N}$, each of which has infinite multiplicity (the Landau levels). The presence of a non-zero potential V smears the Landau level $\beta(2n-1)$ into a region of spectrum contained within the interval

$$I^n_{\beta,V} = \beta(2n-1) + \widehat{V}_0 + C_{\beta,V}\, n^{-1/4}[-1,1].$$

In particular, $\sigma(H_{B,V})$ contains infinitely many gaps for any $\beta \neq 0$ and V.

The character of the spectrum $\sigma(H_{B,V}) \cap I^n_{\beta,V}$ depends critically on the rationality of the flux. Under condition (FR), the form of the direct integral (2) makes it clear that (for sufficiently large n) $\sigma(H_{B,V}) \cap I^n_{\beta,V}$ will consist of p (possibly overlapping and/or degenerate) bands. The existence of eigenvalues for $V \not\equiv 0$ has not been fully resolved, although it appears the spectral bands are non-degenerate at least for generic V ([5]).

The study of $\sigma(H_{B,V}) \cap I^n_{\beta,V}$ in the case of non-rational flux has been undertaken in various limiting regimes (strong and weak magnetic fields were considered in [4]); in particular, it has been found that, after suitable normalisation, the limiting spectrum $\sigma(H_{B,V}) \cap I^n_{\beta,V}$ can be described by a Harper type operator. In this line, the following new result has been obtained for the large energy limit $n \to \infty$.

Theorem 2. *For all* $n \gg 1$ *there exists a neighbourhood* Ω_n *of* $C_{\beta,V}[-1,1] \subset \mathbb{C}$ *and a holomorphic family of oscillatory pseudo-differential operators on* $L^2(\mathbb{R})$*,* $Q_n(\mu)$*,* $\mu \in \Omega_n$*, such that* $\lambda \in \sigma(H_{B,V}) \cap I^n_{\beta,V}$ *iff* $0 \in \sigma(Q_n(\mu))$*, where* $\lambda = \beta(2n-1) + \widehat{V}_0 + n^{-1/4}\mu$*. Furthermore, as* $n \to \infty$

$$Q_n(\mu) = \mathrm{Op}^w\big(W_n(x, \xi/\beta)\big) - \mu + O(n^{-1/4}\ln n),$$

where W_n *is the periodic function given by*

$$W_n(x,\xi) = \frac{(2\beta)^{1/4}}{\sqrt{\pi}} \sum_{m \in \mathbb{Z}^2 \setminus 0} e^{i(m_1 x + m_2 \xi)} \frac{\widehat{V}_m}{\sqrt{|m|}} \cos\Big(\frac{\sqrt{2}\,|m|}{\sqrt{\beta}}\sqrt{n} - \frac{\pi}{4}\Big).$$

In particular, in the limit $n \to \infty$ the normalised spectrum $\sigma(H_{B,V}) \cap I^n_{\beta,V}$ is given as the spectrum of the operator $\mathrm{Op}^w(W_n(x,\xi/\beta))$. This operator is in general of Harper type; in particular, for the potential $V(x,y) = \cos(x) + \cos(y)$ we get

$$\mathrm{Op}^w\big(W_n(x,\xi/\beta)\big) = \frac{(2\beta)^{1/4}}{\sqrt{\pi}} \cos\Big(\sqrt{2n/\beta} - \frac{\pi}{4}\Big)\big(\cos(x) + \cos(D/\beta)\big),$$

which is a scaled version of the standard Harper operator (with parameter $1/\beta$).

In [4] it is shown that for certain V and large irrational magnetic fluxes β, the spectrum $\sigma(H_{B,V}) \cap I^n_{\beta,V}$ is a Cantor set (as is the case for the limiting Harper type operator). It is anticipated that similar results should be attainable for the large energy limit $n \to \infty$ (and probably for the weak electric field limit $V \to 0$).

The methods used to obtain Theorem 2 lead to an asymptotic formula for the eigenvalues of a harmonic oscillator perturbed by a (quasi-)periodic potential; these asymptotics are unusual in the sense that the leading order term contains an oscillatory factor, knowledge of which leads to the recovery of "half" the Fourier coefficients of V (see [2]).

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Spectral Shift Function for Magnetic Schrödinger Operators

Georgi Raikov

Let $H_0 := (i\nabla + A)^2 - b$ be the 3D magnetic Schrödinger operator essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$. Here $A = \left(-\frac{bx_2}{2}, \frac{bx_1}{2}, 0\right)$ is a magnetic potential which generates the constant magnetic field $B = \text{curl } A = (0,0,b)$, $b > 0$. It is well-known that $\sigma(H_0) = \sigma_{\rm ac}(H_0) = [0,\infty)$ (see e.g. [1]), where $\sigma(H_0)$ stands for the spectrum of H_0, and $\sigma_{\rm ac}(H_0)$ for its absolutely continuous spectrum. Moreover, the so-called Landau levels $2bq$, $q \in \mathbb{Z}_+$, play the role of thresholds in $\sigma(H_0)$.
Further, assume that the function V satisfies

$$(1) \quad V \not\equiv 0, \quad V \in C(\mathbb{R}^3), \quad 0 \le V(\mathbf{x}) \le c_0(1+|\mathbf{x}|)^{-m}, \quad m > 3, \quad \mathbf{x} \in \mathbb{R}^3.$$

On the domain of H_0 define the operator $H_\pm := H_0 \pm V$ so that the electric potential $\pm V$ has a fixed sign. For every $E < \inf \sigma(H_\pm)$ we have $(H_\pm - E)^{-1} - (H_0 - E)^{-1} \in S_1$ where S_1 denotes the trace class. Hence, there exists a unique function $\xi = \xi(\cdot; H_\pm, H_0) \in L^1(\mathbb{R}; (1+E^2)^{-1}dE)$ vanishing identically on $(-\infty, \inf\sigma(H_\pm))$, such that *the Lifshits-Krein trace formula*

$$\text{Tr}\,(f(H_\pm) - f(H_0)) = \int_{\mathbb{R}} \xi(E; H_\pm, H_0) f'(E) dE$$

holds for each $f \in C_0^\infty(\mathbb{R})$ (see [7, Chapter 8]). The function $\xi(\cdot; H_\pm, H_0)$ called *the spectral shift function* (SSF) for the operator pair $(H_\pm, H_0)$, is well defined on $\mathbb{R} \setminus 2b\mathbb{Z}_+$, bounded on every compact subset of $\mathbb{R} \setminus 2b\mathbb{Z}_+$, and continuous on

$\mathbb{R} \setminus \{2b\mathbb{Z}_+ \cup \sigma_{\rm pp}(H_\pm)\}$ where $\sigma_{\rm pp}(H_\pm)$ is the set of the eigenvalues of $H_\pm$ (see [2]). In this talk based on the results of [3], we will discuss the asymptotic behaviour as $\lambda \to 0$ of $\xi(2bq + \lambda; H_\pm, H_0)$, the parameters $b > 0$ and $q \in \mathbb{Z}_+$ being fixed.
Let $h_0 := \left(i\frac{\partial}{\partial x_1} - \frac{bx_2}{2}\right)^2 + \left(i\frac{\partial}{\partial x_2} + \frac{bx_1}{2}\right)^2 - b$ be the Landau Hamiltonian essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$. It is well-known that $\sigma(h_0) = \cup_{q=0}^\infty \{2bq\}$, and each eigenvalue $2bq$, $q \in \mathbb{Z}_+$, has infinite multiplicity (see e.g. [1]). For $q \in \mathbb{Z}_+$ denote by $p_q = p_q(b)$ the orthogonal projection onto the eigenspace Ker $(h_0 - 2bq)$.
Assume that (1) holds. For $X_\perp := (x_1, x_2) \in \mathbb{R}^2$ set $W(X_\perp) := \int_{\mathbb{R}} V(X_\perp, x_3)\, dx_3$. Then the Toeplitz-type operator $p_q W p_q : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ satisfies $0 \le p_q W p_q \in S_1$ and $\operatorname{rank} p_q W p_q = \infty$ for each $q \in \mathbb{Z}_+$.
If $T = T^*$ is a compact operator, we denote by $n_+(s;T)$ the number of the eigenvalues of T lying on the interval (s, ∞), $s > 0$, and counted with the multiplicities.

Theorem 1. [3, Theorem 3.1] *Assume that V satisfies (1). Fix $b > 0$ and $q \in \mathbb{Z}_+$. Then for each $\varepsilon \in (0,1)$ we have*

$$\xi(2bq - \lambda; H_+, H_0) = O(1), \quad \lambda \downarrow 0, \tag{2}$$

$$-n_+((1-\varepsilon)2\sqrt{\lambda}; p_q W p_q) + O(1) \le$$
$$\xi(2bq - \lambda; H_-, H_0) \le$$
$$-n_+((1+\varepsilon)2\sqrt{\lambda}; p_q W p_q) + O(1), \quad \lambda \downarrow 0. \tag{3}$$

Estimate (2) shows that $\xi(2bq - \lambda; H_+, H_0)$ remains bounded while, since rank $p_q W p_q = \infty$, estimate (3) implies that $\xi(2bq - \lambda; H_-, H_0) \longrightarrow -\infty$ as $\lambda \downarrow 0$.
Suppose that V satisfies (1). For $\lambda \ge 0$ define the matrix-valued function

$$\mathbb{W}_\lambda = \mathbb{W}_\lambda(X_\perp) := \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}, \quad X_\perp \in \mathbb{R}^2,$$

where

$$w_{11} := \int_{\mathbb{R}} V(X_\perp, x_3) \cos^2(\sqrt{\lambda} x_3) dx_3, \quad w_{22} := \int_{\mathbb{R}} V(X_\perp, x_3) \sin^2(\sqrt{\lambda} x_3) dx_3,$$

$$w_{12} = w_{21} := \int_{\mathbb{R}} V(X_\perp, x_3) \cos(\sqrt{\lambda} x_3) \sin(\sqrt{\lambda} x_3) dx_3.$$

Then the operator $p_q \mathbb{W}_\lambda p_q : L^2(\mathbb{R}^2)^2 \to L^2(\mathbb{R}^2)^2$ satisfies $0 \le p_q \mathbb{W}_\lambda p_q \in S_1$ and $\operatorname{rank} p_q \mathbb{W}_\lambda p_q = \infty$ for each $q \in \mathbb{Z}_+$ and $\lambda \ge 0$.

Theorem 2. [3, Theorem 3.2] *Assume that (1) holds. Fix $b > 0$ and $q \in \mathbb{Z}_+$, Then for each $\varepsilon \in (0,1)$ we have*

$$\pm\frac{1}{\pi}\mathrm{Tr}\,\arctan\,(((1 \pm \varepsilon)2\sqrt{\lambda})^{-1} p_q \mathbb{W}_\lambda p_q) + O(1) \le$$
$$\xi(2bq + \lambda; H_\pm, H_0) \le$$
$$\pm\frac{1}{\pi}\mathrm{Tr}\,\arctan\,(((1 \mp \varepsilon)2\sqrt{\lambda})^{-1} p_q \mathbb{W}_\lambda p_q) + O(1), \quad \lambda \downarrow 0.$$

Since $\operatorname{rank} p_q \mathbb{W}_\lambda p_q = \infty$, Theorem 2 implies that $\xi(2bq+\lambda; H_\pm, H_0) \longrightarrow \pm\infty$ as $\lambda \downarrow 0$.
The main tool used in the proofs of Theorems 1 and 2 is the representation of the SSF due to A. Pushnitski (see [4]).
Combining Theorems 1 and 2 with some results on the eigenvalue asymptotics for compact Toeplitz-type operators obtained in [5] and [6], we can deduce more explicit asymptotic formulae describing the behaviour as $\lambda \to 0$ of $\xi(2bq+\lambda; H_\pm, H_0)$ under generic assumptions about the decay of the electric potential at infinity. Roughly speaking, these assumptions concern the cases where W admits a power-like decay at infinity, W decays exponentially, or the support of W is compact.

Corollary 3. [3, Corollaries 3.1, 3.2] *Let* (1) *hold. Fix* $b > 0$ *and* $q \in \mathbb{Z}_+$.
i) Assume that $W \in C^1(\mathbb{R}^2)$, *and*

$$W(X_\perp) = w_0(X_\perp/|X_\perp|)|X_\perp|^{-\alpha}(1+o(1)), \quad |X_\perp| \to \infty,$$
$$|\nabla W(X_\perp)| \le c_1(1+|X_\perp|)^{-\alpha-1}, \quad X_\perp \in \mathbb{R}^2,$$

with $\alpha > 2$, $0 \le w_0 \in C(\mathbb{S}^1)$, *and* $w_0 \not\equiv 0$. *Then we have*

$$\xi(2bq-\lambda; H_-, H_0) = -\psi_\alpha(2\sqrt{\lambda})\,(1+o(1)), \quad \lambda \downarrow 0,$$
$$\xi(2bq+\lambda; H_\pm, H_0) = \pm\frac{1}{2\cos(\pi/\alpha)}\,\psi_\alpha(2\sqrt{\lambda})\,(1+o(1)), \quad \lambda \downarrow 0,$$

where $\psi_\alpha(s) := s^{-2/\alpha}\frac{b}{4\pi}\int_{\mathbb{S}^1} w_0(t)^{2/\alpha}dt$, $s > 0$.
ii) Assume that $W \in L^\infty(\mathbb{R}^2)$, *and*

$$\ln W(X_\perp) = -\mu|X_\perp|^{2\beta}(1+o(1)), \quad |X_\perp| \to \infty,$$

with some $\mu > 0$, *and* $\beta > 0$. *Suppose in addition that* V *satisfies the estimate*

$$V(X_\perp, x_3) \le c_2(1+|X_\perp|)^{-m_\perp}(1+|x_3|)^{-m_3}, \quad X_\perp \in \mathbb{R}^2,\ x_3 \in \mathbb{R}, \tag{4}$$

with $m_\perp > 2, m_3 > 2$. *Then we have*

$$\xi(2bq-\lambda; H_-, H_0) = -\varphi_\beta(2\sqrt{\lambda})\,(1+o(1)), \quad \lambda \downarrow 0,$$
$$\xi(2bq+\lambda; H_\pm, H_0) = \pm\frac{1}{2}\,\varphi_\beta(2\sqrt{\lambda})\,(1+o(1)), \quad \lambda \downarrow 0,$$

where

$$\varphi_\beta(s) := \begin{cases} \frac{b}{2\mu^{1/\beta}}|\ln s|^{1/\beta} & \text{if} \quad 0 < \beta < 1, \\ \frac{1}{\ln(1+2\mu/b)}|\ln s| & \text{if} \quad \beta = 1, \\ \frac{\beta}{\beta-1}(\ln|\ln s|)^{-1}|\ln s| & \text{if} \quad 1 < \beta < \infty, \end{cases} \qquad s \in (0, e^{-1}).$$

iii) Finally, assume that $W \in L^\infty(\mathbb{R}^2)$, $\operatorname{supp} W$ *is compact, and there exists a constant* $c > 0$ *such that* $W \ge c$ *on an open non-empty subset of* $\mathbb{R}^2$. *Suppose in addition that* V *satisfies* (4) *with* $m_\perp > 2, m_3 > 2$. *Then we have*

$$\xi(2bq-\lambda; H_-, H_0) = -\varphi_\infty(2\sqrt{\lambda})\,(1+o(1)), \quad \lambda \downarrow 0,$$
$$\xi(2bq+\lambda; H_\pm, H_0) = \pm\frac{1}{2}\,\varphi_\infty(2\sqrt{\lambda})\,(1+o(1)), \quad \lambda \downarrow 0,$$

where

$$\varphi_\infty(s) := (\ln|\ln s|)^{-1}|\ln s|, \quad s \in (0, e^{-1}).$$

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On the Laplacian in the halfspace with a periodic boundary condition

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The characteristic feature of Schrödinger operators that are periodic with respect to some, but not all directions is the appearance of surface states, see [1] and the references in [5], [6]. On physical grounds one expects that these states are not bound but correspond to additional channels of scattering, i.e., that the spectrum of the corresponding operator is purely absolutely continuous. We are only aware of [2], [3], [4], [5] dealing with this problem.

Here we follow [4] and study spectral and scattering properties of the Laplacian

$$H^{(\sigma)}u = -\Delta u \qquad \text{on } \mathbb{R}^{d+1}_+ := \{(x,y) \in \mathbb{R}^d \times \mathbb{R} : y > 0\}$$

together with a boundary condition of the third type

$$\frac{\partial u}{\partial \nu} + \sigma u = 0 \qquad \text{on } \mathbb{R}^d \times \{0\}$$

with a $(2\pi\mathbb{Z})^d$-periodic function $\sigma : \mathbb{R}^d \to \mathbb{R}$. Under the condition

$$\sigma \in L_{q,loc}(\mathbb{R}),\ q > 1,\ \text{if } d = 1, \qquad \sigma \in L_{d,loc}(\mathbb{R}^d) \text{ if } d \geq 2, \tag{1}$$

$H^{(\sigma)}$ can be defined as a self-adjoint operator in $L_2(\mathbb{R}^{d+1}_+)$ by means of the lower semibounded and closed quadratic form

$$\int_{\mathbb{R}^{d+1}_+} |\nabla u(x,y)|^2 \, dxdy + \int_{\mathbb{R}^d} \sigma(x)|u(x,0)|^2 \, dx, \qquad u \in H^1(\mathbb{R}^{d+1}_+).$$

Note that $H^{(\sigma)}$ can be viewed as a Schrödinger-type operator with singular potential $\sigma(x)\delta(y)$ describing the interaction of a quantum-mechanical particle with the surface of a crystal.
We investigate the scattering with respect to the Neumann Laplacian $H^{(0)}$.

Theorem 1. *Assume that σ satisfies* (1). *Then the wave operators*

$$W_\pm^{(\sigma)} := s - \lim_{t\to\pm\infty} \exp(itH^{(\sigma)}) \exp(-itH^{(0)})$$

exist and satisfy $\mathcal{R}(W_+^{(\sigma)}) = \mathcal{R}(W_-^{(\sigma)})$.

If σ is non-negative, we obtain a rather complete result.

Theorem 2. *Assume that σ satisfies* (1) *and $\sigma(x) \geq 0$ for a.e. $x \in \mathbb{R}^d$. Then the wave operators $W_\pm^{(\sigma)}$ are unitary and satisfy*$H^{(\sigma)} = W_\pm^{(\sigma)} H^{(0)} W_\pm^{(\sigma)*}$.

However, in general the wave operators will *not* be complete due to the existence of *surface states*, i.e., states that are localized near the boundary for all time. These states correspond to bands in the spectrum of $H^{(\sigma)}$. A sufficient condition for $\sigma\left(H^{(\sigma)}(k)\right) \cap (-\infty, 0) \neq \emptyset$ is

$$\int_{(-\pi,\pi)^d} \sigma(x) \, dx \leq 0, \qquad \sigma \not\equiv 0.$$

It is natural to ask whether the spectrum of $H^{(\sigma)}$ is still absolutely continuous in this situation.

Theorem 3. *Assume that σ satisfies* (1) *if $d \leq 4$ and $\sigma \in L_{2(d-2),loc}(\mathbb{R}^d)$ if $d \geq 5$. Then the operator $H^{(\sigma)}$ has purely absolutely continuous spectrum.*

Hence surface states correspond to *additional channels of scattering.*

Let us explain some of the mathematical ideas involved. By means of Floquet theory we represent $H^{(\sigma)}$ as a direct integral

$$\int_{[-\frac{1}{2},\frac{1}{2}]^d} \oplus H^{(\sigma)}(k) \, dk$$

with operators $H^{(\sigma)}(k)$ acting in $L_2(\Pi)$ where $\Pi := (-\pi,\pi)^d \times \mathbb{R}_+$ is a halfcylinder. The investigation of the operator $H^{(\sigma)}$ reduces to the study of the fibers $H^{(\sigma)}(k)$. Note that the fundamental domain Π is unbounded, so the operators $H^{(\sigma)}(k)$ have continuous spectrum. This part can be studied by scattering theory. To prove the absolute continuity of the spectrum of $H^{(\sigma)}$ we cannot (directly) apply the Thomas approach, since eigenvalues of $H^{(\sigma)}(k)$ may be embedded in the continuous spectrum. We "separate" them from the remaining spectrum by characterizing them, in the spirit of the Birman-Schwinger principle, as parameters λ for which a pseudo-differential operator $B^{(\sigma)}(\lambda, k)$ on the boundary $(-\pi, \pi)^d \times \{0\}$ has eigenvalue 0. The latter operator has discrete spectrum and can be handled by Thomas' method.

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A Multidimensional Trace Formula

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(joint work with S. Naboko and O. Safronov)

Let us consider the equation

$$Hu = -\Delta u + Vu = k^2 u, \tag{1}$$

where $V \in C_0^\infty(\mathbb{R}^3)$ and $\text{supp}V \subset \{x : c_1 < |x| < c_2\}$, $c_1, c_2 > 0$. By using the unitary transformation U from $L^2((0,\infty), dr; L^2(\mathbb{S}^2))$ to $L^2((0,\infty), r^2 dr; L^2(\mathbb{S}^2))$,

$$v(t,\theta) = Uu(r,\theta) = r^{-1}u,$$

we reduce the study of (1) the operator $\tilde{H}$ in $L^2((0,\infty), dr; L^2(\mathbb{S}^2))$

$$\tilde{H}v = -\partial^2_{rr}v + \frac{B}{r^2}v + Vv = k^2 v, \tag{2}$$

where B is the Laplace-Beltrami operator in $L^2(\mathbb{S}^2)$.

We now consider the equation

$$-f''_{rr}(r,\theta,k) + \frac{B}{r^2}f(r,\theta,k) + Vf(r,\theta,k) = k^2 f(r,\theta,k), \tag{3}$$

subject to the initial condition

$$f(r,\theta,k) = e^{-ikr}, \qquad 0 < r < c_1. \tag{4}$$

It can be shown that there are "scattering" coefficients a and b such that

$$f(r,\theta,k) = a(\theta,k)e^{-ikr}\Big(1 + O(|kr|^{-1})\Big) + b(\theta,k)e^{ikr}\Big(1 + O(|kr|^{-1})\Big), \quad r \to \infty. \tag{5}$$

If $f(r,\theta,k),\ k \in \mathbb{C},\ r \geq 1$, is a solution of the differential equation (3) then it satisfies the integral equation

$$f(r,\theta,k) = e^{-ikr} - \frac{1}{2ik}\int_0^r \Big(e^{-ik(r-t)} - e^{ik(r-t)}\Big)\Big(V(t,\theta) + \frac{B}{r^2}\Big) f(t,\theta,k)\,dt. \tag{6}$$

Substituting $\psi(r,\theta,k) = e^{ikr}f(r,\theta,k)$ we obtain

$$\psi(r,\theta,k) = 1 - \mathcal{K}\,\psi(r,\theta,k) = \int_0^r K(r,t,k)\psi(t,\theta,k)\,dt, \tag{7}$$

where by $\mathcal{K}$ we denote the integral operator whose operator valued symbol is equal to

$$K(r,t,k) = \frac{(1 - e^{2ik(r-t)})}{2ik}\Big(V(t,\cdot) + \frac{B}{r^2}\Big). \tag{8}$$

Solving the Volterra equation (6) we obtain the series

$$\psi(r,\theta,k) = 1 + \sum_{j=1}^{\infty} \int_{r\geq t_1\geq\cdots\geq t_m\geq 0}\cdots\int \prod_{q=1}^{j} K(t_{l-1},t_l,k)\,dx_1\cdots dx_j \cdot 1\,.$$

This series is convergent pointwise and, in particular, $\psi(r,\theta,k) \equiv 1$ if $0 \leq r \leq c_1$. The function $\psi(r,\theta,k)$ is smooth and also analytic with respect to $k \in \mathbb{C}\setminus\{0\}$. Indeed, since the kernel $K(r,t,k)$ is analytic in k, we obtain

$$\frac{\partial}{\partial\bar{k}}\psi(r,\theta,k) = -\int_0^r K(r,t,k)\frac{\partial}{\partial\bar{k}}\psi(t,\theta,k)\,dt.$$

Therefore $\partial\psi(r,\theta,k)/\partial\bar{k}$ satisfies a homogeneous Volterra integral equation and hence it identically equal to zero.

The Volterra equation (6) can be rewritten as

$$\begin{aligned} f(r,\theta,k) = e^{-ikr}\Big[1 - \frac{1}{2ik}\int_0^r V(t,\theta,k)\,dt \\ -\frac{1}{2ik}\int_0^r \Big(V(t,\theta,k) + \frac{B}{r^2}\Big)(\psi(t,\theta) - 1)\,dt\Big] \\ +\frac{e^{ikr}}{2ik}\Big[\int_0^r e^{-2ikt}\,V(t,\theta,k)\,dt + \int_0^r e^{-2ikt}\,\Big(V(t,\theta,k) + \frac{B}{r^2}\Big)(\psi(t,\theta) - 1)\,dt\Big]. \end{aligned} \tag{9}$$

Comparing (9) with (5) we see that

$$a(\theta,k) = 1 - \frac{1}{2ik}\int_0^r V(t,\theta,k)\,dt \tag{10}$$

$$-\frac{1}{2ik}\int_0^r \Big(V(t,\theta,k)+\frac{B}{r^2}\Big)(\psi(t,\theta)-1)\,dt$$

(11) $$b(\theta,k)=\frac{1}{2ik}\int_0^r e^{-2ikt}\,V(t,\theta,k)\,dt$$

$$+\frac{1}{2ik}\int_0^r e^{-2ikt}\Big(V(t,\theta,k)+\frac{B}{r^2}\Big)(\psi(t,\theta)-1)\,dt.$$

Note that for a fixed k_0, Im $k_0>0$, if we assume that the function f is from the class $L^2((0,\infty)\times\mathbb{S}^2)$, then $a(\theta,k_0)$ is equal to zero identically in θ. This implies that $a(\theta,k_0)\equiv 0$ if k_0 is an eigenvalue of the operator (1).

Let $\varkappa_j$, $j=1,\dots,J$, be zeros of the function $\int_{\mathbb{S}^2} a(\theta,k)\,d\theta$ in the upper half plane. We obtain a version of Buslaev-Faddeev-Zakharov trace formula, see [1] and [2].

Theorem 1. *Let V be a $C_0^\infty(\mathbb{R}^3)$ and supp $V\subset\{x:\, c_1<|x|<c_2\}$, $c_1,c_2>0$. Then the following trace formula holds true*

$$\sum_j \varkappa_j^3+\frac{3}{2\pi}\int_{-\infty}^{\infty} k^2\log\Big|\int_{\mathbb{S}^2} a(\theta,k)\,d\theta\Big|\,dk$$

$$=\frac{3}{16}\int_{\mathbb{S}^2}\int_0^\infty\Big\{\Big|\int_0^r \nabla_\theta V(t,\theta)\,dt\Big|^2 r^{-2}+V^2(r,\theta)\Big\}\,dr\,d\theta.$$

When proving the theorem we use an approach developed in [3], where the authors have considered trace formulae with operator valued potentials and their applications. Similar ideas have been also used in [4] when proving absolute continuity of the spectrum of Schrödinger operators with oscillating potentials.

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