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Enveloping Algebras and Geometric Representation Theory

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Abstract.
The study of Enveloping Algebras has undergone a significant and continuous evolution and moreover has inspired a wide variety of developments in many areas of mathematics including Ring Theory, Differential Operators, Invariant Theory, Quantum Groups and Hecke Algebras. The aim of the workshop was to bring together researchers from diverse but highly interrelated subjects to discuss new developments and bring forward the research in this whole area by fostering the scientific interaction.

Mathematics Subject Classification (2000): 17xx, 22xx, 81xx.

Introduction by the Organisers

Since its inception in the early seventies, the study of Enveloping Algebras has undergone a significant and continuous evolution and moreover has inspired a wide variety of developments in many areas of mathematics including Ring Theory, Differential Operators, Invariant Theory, Quantum Groups and Hecke Algebras.

As indicated above, one of the main goals behind this meeting was to bring together a group of participants with a wide range of interests in and around the geometric and the combinatorial side of the representation theory of Lie groups and algebras. We strongly believe that such an approach to representation theory, in particular interaction between geometry and representation theory, will open up new avenues of thought and lead to progress in a number of areas.

This diversity was well reflected in the expertize represented by the conference participants, as well as in the wide range of topics covered. They may broadly be summarized under the following three headings:

(1) The study of the structure and representation theory through
a) Slodowy slices and the Joseph ideal, b) Equivalence of categories of representations and certain geometric counterparts, c) Hall algebras, Quotient schemes and canonical bases, d) Richardson elements and birationality questions, e) free Lie algebras and current algebras, f) Dirac cohomology, g) L-functions and representation theory, h) Gelfand-Zeitlin theory
(2) Combinatorial Aspects of Lie Systems particularly through
a) Saturation problems and Buildings, b) Affine and double Affine Hecke algebras, Cherednik algebras, c) Solvable lattice models
(3) Geometric Structures including
a) Representation theoretic methods in enumerative geometry, b) Multiplicity free symplectic reduction, c) Monodromy actions of braid groups, d) Associated varieties for Lie super algebras, e) affine Grassmannians, f) Schubert varieties and Demazure modules

The single most important development reported in this conference seems to us to be the generalization of the categorical Satake isomorphism to quantum groups at roots of unity, which allows to bring in a whole lot of new geometry into representation theory. It seems likely that it will lead to the proofs of some open conjectures of Lusztig. We were particularly happy to have quite a few very strong younger participants and some highly promising postdocs. We tried to accomodate young postdocs, as much as feasible, to present their work by devoting one afternoon to their (shorter) talks.

Workshop: Enveloping Algebras and Geometric Representation Theory

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Abstracts

Singularities at the boundary of the crown domain

Eric Opdam
(joint work with Bernhard Krötz)

1. The crown Ξ of a Riemannian symmetric space

The crown or Akhiezer-Gindikin domain [1] Ξ of a Riemannian symmetric space $X = G/K$ of noncompact type is the maximal domain in $X_{\mathbb{C}}$ containing $X = G.x_0$ (where $x_0 = e.K_{\mathbb{C}}$ denotes the base point of $X_{\mathbb{C}}$) such that G acts *properly* on Ξ. It has the following explicit description. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to the maximal compact subgroup K, and choose a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$. Let $\Sigma := \Sigma(\mathfrak{g}, \mathfrak{a}) \subset \mathfrak{a}^*$ denote the restricted root system. Then

$$\Xi = G \exp(i\pi\Omega/2).x_0 \subset X_{\mathbb{C}}, \tag{1}$$

where $\Omega \subset \mathfrak{a}$ is given by

$$\Omega = \{Y \in \mathfrak{a} \mid |\alpha(Y)| < 1, \forall \alpha \in \Sigma\}. \tag{2}$$

The complex crown has been subject to active research activities recently (see [1], [3], [5], [6], [7], [8], [9] and references therein).

The crown has the following remarkable universal holomorphic extension property for any admissible spherical representation (H, π) of G ([7], Prop. 4.1). Let $v \in H$ denote a (normalized) spherical vector. Then the orbit map

$$X \ni x = gK \to v^x := \pi(g)v \in H$$

has a holomorphic extension to Ξ. For a generic irreducible spherical admissible representation, the norm of v^x will blow up everywhere along the boundary $\partial(\Xi)$ of Ξ (see [8]). Hence Ξ is a Stein domain (see [8], and the references therein).

In this talk we will discuss the nature of the singularities of the function $x \to \|v^x\|^2$ on Ξ when $x \in \Xi$ approaches a so-called "distinguished" point of $\partial(\Xi)$ (see below).

The original motivation to consider this problem is a method to estimate triple products of Maass forms on a locally symmetric space of the form $\Gamma\backslash X$ by analytic continuation of representations. This method works quite well when X has rank 1 ([12], [2], [7]). At present it is not clear how to extend this method of analytic continuation to the theory of Maass forms in higher rank cases. Nonetheless, we believe that the estimates which we have obtained are of independent interest.

2. The distinguished boundary of Ξ

For the sake of simplicity we will assume from now on that $\mathfrak{g}$ is simple. The set of closed G-orbits in the closure $\overline{\Xi}$ of $\Xi \subset X_{\mathbb{C}}$ is equal to $G\exp(i\pi\overline{\Omega}/2).x_0$. By definition the distinguished boundary $\partial_d(\Xi) \subset \partial(\Xi)$ consists of the union of the closed G-orbits $G.t.x_0$ where $t = \exp(i\pi\omega/2)$ with $\omega \in \partial\Omega$ an extremal point of the closed convex set $\overline{\Omega}$. Hence there are only finitely many G-orbits in $\partial_d\Xi$.

The distinguished boundary is of special interest since bounded continuous functions on $\overline{\Xi}$ which restrict to holomorphic functions on Ξ will assume their maximum in $\partial_d(\Xi)$ (see [5]). For this reason we restrict our attention to the singular behaviour of v^x at the distinguished boundary points.

The first result we will describe in this talk is a unified description of $\partial_d(\Xi) \subset \partial(\Xi)$ of Ξ (also see [5]). Choose positive roots Σ_+ and let $\Sigma^l \subset \Sigma$ denote the reduced root subsystem of inmultiplicable roots. Observe that $\overline{\Omega} = \cup_{w\in W} w(C)$, where C is the fundamental alcove of the affine Weyl group W^a whose affine Dynkin diagram D^a is the affine extension of the Dynkin diagram of Σ^l. Therefore in order to describe $\partial_d(\Xi)$ it suffices to decide which of the extremal points $\omega \neq 0$ of the alcove C are also extremal in $\overline{\Omega}$.

Theorem 1. *We have $\partial_d(\Xi) = \cup_\omega G.\exp(i\pi\omega/2).x_0$ where ω runs over the set of nonzero extremal points of C with the property that the subgraph of D^a obtained by deleting the vertex of D^a which corresponds to ω is connected.*

Corollary 1. *If ω is a minuscule fundamental co-weight then the corresponding orbit $G.t.x_0$ is distinguished. We call the union of such orbits the minuscule boundary $\partial_m(\Xi) \subset \partial_d(\Xi)$.*

3. Lower estimates

Let $\lambda \in \mathfrak{a}^*$ and let (H, π_λ) denote the spherical unitary minimal principal series, with normalized spherical vector v_λ. A fundamental fact which is at the basis of all our estimates for $\|v_\lambda^x\|^2$ is the doubling formula for spherical functions ([7], Th. 4.2), which states that for $t \in \exp(i\pi\Omega/2) := T_\Omega$ and $x = t.x_0$,

$$\|v_\lambda^x\|^2 = \phi_\lambda(t^2.x_0) \tag{3}$$

where ϕ_λ is the (holomorphic extension of the) spherical function $\phi_\lambda(g.x_0) := \langle v_\lambda, \pi(g)v_\lambda\rangle$ on X. In particular, we see that the restriction of ϕ_λ to $A.x_0$ has a holomorphic continuation to $AT_\Omega^2.x_0$. By studying the Harish-Chandra integral representation of ϕ_λ one obtains:

Theorem 2. *Let $t = \exp(i\pi\omega/2)$ be an extremal boundary point of T_Ω, and put $t_\epsilon = \exp(i\pi(1-\epsilon)\omega/2)$. Let G' denote the centralizer in G of $t^4 \in A_{\mathbb{C}}$. Then there exist constants $\epsilon_0 \in (0,1)$ and $R > 0$, $C > 0$ such that*

$$\phi_\lambda(t_\epsilon^2.x_0) \geq C\epsilon^{(\dim G - \dim G')/4} \max_{w\in W} e^{\pi\lambda(w\omega)(1-R\epsilon)} \tag{4}$$

for all $\lambda \in \mathfrak{a}^*$, and for all $\epsilon \in (0, \epsilon_0)$.

4. Upper estimates

The system of differential equations for the restriction of the zonal spherical function ϕ_λ to $A_{\mathbb{C}}.x_0$ has regular singularities at the collection of hyperplanes $a^{2\alpha} = 1$. Let $t = \exp(i\pi\omega/2) \in \partial(T_\Omega)$ where ω is extremal in $\overline{\Omega}$, and let $t_\epsilon = \exp(i\pi\omega_\epsilon/2).x_0$ where $\mathbb{C} \ni \epsilon \to \omega_\epsilon - \omega$ is linear and such that $t_\epsilon^{2\alpha} \neq 1$ if ϵ is in the punctured unit disc. We may and will assume that $\omega, \omega_1 \in C$.

There exists a singular expansion of the following form for ϕ_λ:

$$\phi_\lambda(t_\epsilon^2.x_0) = \sum_i \epsilon^{n_i}(p_{i,0}(\log(\epsilon)) + p_{i,1}(\log(\epsilon))\epsilon + \dots) \tag{5}$$

where the sum over i is finite, the exponents $n_i \in \mathbb{C}$ are distinct modulo $\mathbb{Z}$, and the $p_{i,j}$ are polynomials of bounded degree. We would like to compute the leading exponent n_ω^X in this expansion, and also the degree of the coefficient $p_{n_\omega^X,0}$ of this leading term.

The method we employ is based on Dunkl-Cherednik theory and the theory of hypergeometric functions for root systems (see e.g. [4], [11]). By this theory, the system of differential equations on $A_{\mathbb{C}}.x_0$ allows a flat deformation for which the root multiplicities of X are complex parameters. The local monodromy of this system at $t^2.x_0$ factors through a Hecke algebra of type W_ω, the Weyl group generated by the affine reflections which fix ω. This makes it possible to compute the exponents at $t^2.x_0$ as in [10].

It turns out that $2n_\omega^X \in \mathbb{Z}$ for each ω as above. To each ω we can attach an irreducible representation $\tau = \tau_\omega$ of W_ω such that for all X with fixed restricted root system Σ,

$$n_\omega^X \leq n_\tau(m^X) := b(\tau) - \frac{1}{2}\sum_{\alpha\in\Sigma_{\omega,+}} m_\alpha^X\left(1 - \frac{\chi_\tau(s_\alpha)}{\deg_\tau}\right), \tag{6}$$

where $b(\tau)$ denotes the harmonic birthday degree of τ, and where m^X denotes the root multiplicity function of X. The method also gives an estimate on the degree of the logarithmic term in the leading term of singular expansion, but we will not discuss that here. We conjecture that the inequality $n_\omega^X \leq n_\tau(m^X)$ is an equality for generic $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

The representation $\tau = \tau_\omega$ is determined by the remark that $\tau \otimes \det$ equals the truncated induction of det from the subgroup $W_\omega^f \subset W_\omega$ generated by the reflections of W which fix ω. Therefore it is an easy matter to compute $n_\tau(m^X)$ explicitly in all cases.

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On the primitive spectrum of a free (Lie) algebra

Markus Reineke

1. The primitive spectrum

Let k be an algebraically closed field, and let $A = k\langle x_1, \ldots, x_m\rangle$ be the free algebra in m generators, which can also be viewed as the enveloping algebra $\mathcal{U}(L^{(m)})$ of the free Lie algebra in m generators. We consider the primitive spectrum $\mathrm{Prim}(A)$, which is the set of annihilators of irreducible representations of A with its natural (Jacobson) topology. In particular, we consider for each $d \in \mathbf{N}$ the set $\mathrm{Prim}^d(A)$ of annihilators of d-dimensional irreducible representations of A.

It was proven by M. Artin [1] that $\mathrm{Prim}^d(A) \subset \mathrm{Prim}(A)$ is a locally closed subset, and its induced topology is the Zariski topology for a natural structure of a smooth irreducible k-variety of dimension $(m-1)d^2+1$. To construct this variety, note that $\mathrm{Prim}^d(A)$ is in bijection with the set of isomorphism classes of d-dimensional irreducible representations of A. In this latter realization, $\mathrm{Prim}^d(A)$ can be embedded as an open subset of the quotient variety $M_d(k)^m//\mathrm{PGL}_d(k)$ of the space of m-tuples of $d \times d$-matrices by the action of $\mathrm{PGL}_d(k)$ via simultaneous conjugation. Namely, this open subset is induced by matrix tuples without a common non-trivial invariant subspace.

The aim of the ongoing project which was reported on is to compute global topological invariants of $\mathrm{Prim}^d(A)$. Three results in this direction are given in the following.

2. Counting rational points

In the case where k is an algebraic closure of a finite field $\mathbf{F}_q$, the set of $\mathbf{F}_q$-rational points of the k-variety $\mathrm{Prim}^d(A)$ can be identified with the set of absolutely irreducible d-dimensional representations of A, i.e. those representations S for which the base extension $k \otimes_{\mathbf{F}_q} S$ is still irreducible.

Theorem 1. *There exist recursively defined polynomials $a_d^{(m)}(t) \in \mathbf{Z}[t]$ such that, for all finite fields $\mathbf{F}_q$, the value $a_d^{(m)}(q)$ equals the number of $\mathbf{F}_q$-rational points of $\mathrm{Prim}^d(A)$.*

The recursive formula for these polynomials is: $a_d^{(m)}(t) =$

$$(1-t)\left((t^{(m-1)\binom{d}{2}} \sum_{\lambda \in \Lambda_d} (-1)^{l(\lambda)} \binom{l(\lambda)}{\mu_1(\lambda), \mu_2(\lambda), \dots} \prod_{i=1}^{l(\lambda)} \frac{t^{m\binom{\lambda_i+1}{2}}}{(t^{\lambda_i}-1)\dots(t-1)} \right.$$

$$\left. - \sum \prod_{i,j,k} \frac{1}{\xi_{ijk}!} (\prod_{l=1}^{k} (1-t^{lj}))^{-\xi_{ijk}} \prod_{i,j} \binom{\frac{1}{j}\sum_{r|j} \mu(\frac{j}{r}) a_i^{(m)}(t^r)}{\sum_k \xi_{ijk}} (\sum_k \xi_{ijk})! \right).$$

In this formula, the second sum is over all functions $\xi : \mathbf{N}_+^3 \to \mathbf{N}$ such that $\sum_{i,j,k} ijk\xi_{ijk} = d$ and $\xi \neq \delta_{d,1,1}$. Moreover, $\Lambda_d = \{\text{partitions of } d\}$, $l(\lambda)$=length of λ, $\mu_m(\lambda)$=multiplicity of m in λ and μ=Möbius function.

Example: We have $a_1^{(m)}(t) = t^m$, $a_2^{(m)}(t) = t^{2m}\frac{(t^m-1)(t^{m-1}-1)}{t^2-1}$ and $a_3^{(m)}(t) =$

$$t^{3m+1}\frac{(t^m-1)(t^{2m-2}-1)(t^{3m-2}+t^{2m-2}-t^m-2t^{m-1}-t^{m-2}+t+1)}{(t^3-1)(t^2-1)}.$$

The nature of these polynomials remains mysterious (see however the following section for a special value). The proof is rather indirect: the above recursion is derived from an identity in the Hall algebra [4] of A.

3. Euler characteristic

There is an obvious action of the group k^* on $\mathrm{Prim}^d(A)$, which in fact admits a geometric quotient $\mathbf{P}\mathrm{Prim}^d(A)$.

Theorem 2. *We have $a_d^{(m)}(1) = 0$ and*

$$\left.\frac{a_d^{(m)}(t)}{t-1}\right|_{t=1} = \chi_c(\mathbf{P}\mathrm{Prim}^d(A)) = \frac{1}{d}\sum_{r|d} \mu(\frac{d}{r}) m^r.$$

Here, χ_c denotes Euler characteristic in ℓ-adic cohomology with compact support. Note that the number on the right hand side equals the number of primitive necklaces with d beads of m colours.

This theorem is proven by first generalizing the statement to moduli spaces of irreducible representations of quivers, and calculating the Euler characteristic by localization techniques and induction over all quivers.

No structural interpretation of this result is available at the moment; it should be noted that, by the PBW theorem, the formula in the theorem also describes the dimension of the degree d-part of the free Lie algebra $L^{(m)}$ [3].

4. Hilbert schemes

We consider a variant of the space $\mathrm{Prim}^d(A)$, which is more accessible to cohomology computations (see [2] for the following results). Denote by $\mathrm{Hilb}^d(A)$ the set of left ideals $I \subset A$ such that $\dim_k(A/I) = d$. Again, one can see that this can be given the structure of a smooth, irreducible k-variety of dimension $(m-1)d^2 + d$ by identifying left ideals with pairs consisting of a representation and a cyclic vector. This interpretation also allows to view $\mathrm{Hilb}^d(A)$ as a partial compactification of a projective bundle over $\mathrm{Prim}^d(A)$.

Theorem 3.

(1) *The variety* $\mathrm{Hilb}^d(A)$ *admits a cell decomposition, whose cells are naturally parametrized by* m*-ary trees with* d *nodes.*

(2) *The generating function of Betti numbers*

$$\zeta(q,t) := \sum_{d=0}^{\infty} q^{(m-1)\binom{d}{2}} \sum_i \dim H^i(\mathrm{Hilb}^d(A)) q^{-i/2} t^d$$

is the unique solution in $\mathbf{Q}[q][[t]]$ *to the algebraic* q*-difference equation* $\zeta(q,t) = 1 + t\prod_{i=0}^{m-1} \zeta(q, q^i t)$.

(3) *Defining a discrete random variable* X_d *by*

$$\mathbf{P}(X_d = i) = \frac{\dim H^{(m-1)d(d-1)/2-2i}(\mathrm{Hilb}^d(A))}{\chi(\mathrm{Hilb}^d(A))},$$

the sequence $\sqrt{\frac{8}{m(m-1)}} d^{-\frac{3}{2}} X_d$ *converges to a limit distribution, namely the Airy distribution* [5].

It would be very interesting to have a structural interpretation for the last result in terms of an object naturally associated to A.

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A generalization of the saturation theorem

Michael Kapovich
(joint work with John J. Millson)

The goal of this talk is to prove a generalization of the following result known as the *Saturation Theorem* by A. Knutson and T. Tao [9]:

Theorem 1. *Suppose that α, β, γ are dominant weights of $SL(n,\mathbb{C})$ such that $\alpha+\beta+\gamma$ belongs to the root lattice $Q(R)$ of $SL(n,\mathbb{C})$ and there exists $N \geq 1$ such that*

$$(V_{N\alpha} \otimes V_{N\beta} \otimes V_{N\gamma})^{SL(n,\mathbb{C})} \neq 0.$$

Then

$$(V_{\alpha} \otimes V_{\beta} \otimes V_{\gamma})^{SL(n,\mathbb{C})} \neq 0.$$

An alternative proof of this theorem was given in [2].

Slightly more geometrically one can restate their result as follows:

There exists a convex homogeneous polyhedral cone $D_3 \subset \Delta^3$ such that $(\alpha,\beta,\gamma) \in D_3$ is a point in $P(R)^3$ with $\alpha+\beta+\gamma \in Q(R)$ if and only if

$$(V_{\alpha} \otimes V_{\beta} \otimes V_{\gamma})^{SL(n,\mathbb{C})} \neq 0.$$

Here V_λ is the irreducible finite-dimensional representation of $SL(n,\mathbb{C})$ with the highest weight λ and Δ is the positive Weyl chamber.

Our goal is to generalize this result to the case of complex reductive Lie groups other than $SL(n,\mathbb{C})$.

Let $\underline{G}$ be a split reductive algebraic group over $\mathbb{Z}$, $Q(R^\vee) \subset L \subset P(R^\vee)$ be its cocharacter lattice, W the Weyl group, Δ the Weyl chamber, α_i's simple roots, θ the highest root

$$\theta = \sum_{i=1}^{\ell} m_i \alpha_i$$

and $k_R = LCM(m_1, ..., m_\ell)$ is the *saturation factor*. Let $\underline{G}^\vee$ be the Langlands dual of $\underline{G}$, $G^\vee := \underline{G}(\mathbb{C})$.

Theorem 2 (M. Kapovich, J.J. Millson, [5])**.** *There exists a convex homogeneous polyhedral cone $D_3 \subset \Delta^3$ depending only on W such that for $k = k_R$:*

$$(\alpha,\beta,\gamma) \in D_3 \cap L^3 \textit{ with } \alpha+\beta+\gamma \in Q(R^\vee)$$

if and only if

$$(V_{k\alpha} \otimes V_{k\beta} \otimes V_{k\gamma})^{G^\vee} \neq 0.$$

Equivalently, if $L_\alpha \to O_\alpha, L_\beta \to O_\beta, L_\gamma \to O_\gamma$ denote the line bundles over flag-manifolds corresponding to α, β, γ, and the Mumford quotient

$$O_\alpha \times O_\beta \times O_\gamma // G^\vee$$

is nonempty, then there are nonzero $G^\vee$-invariants of degree $\leq k_R$ in

$$H^0(O_\alpha \times O_\beta \times O_\gamma, L_\alpha \otimes L_\beta \otimes L_\gamma).$$

We have examples which show that in Theorem 2 multiplication by $k = 2$ is needed for all non-simply-laced groups.

For A_ℓ type root system we get $k_R = 1$ and hence Knutson-Tao theorem as a special case.

Conjecture: In Theorem 2 one can take $k = 1$ for simply laced groups and $k = 2$ in the non-simply laced case.

Despite of the algebraic appearance of Theorem 2, its proof is mostly geometric. Below is a geometric interpretation of the polyhedral cone D_3 which appears in Theorem 2.

Let $\mathbb{K}$ be field with a nonarchimedean discrete valuation, X the Bruhat-Tits building associated with $G := \underline{G}(\mathbb{K})$. Let (A, W) denote the affine apartment and affine Weyl group of G, $\Delta \subset A$ a Weyl chamber of the linear part W_o of W which we identify with the stabilizer of a tip o of Δ.

We get Δ-*valued distance function* $d_\Delta(x, y)$ between points $x, y \in A$ by projecting the vector $\overline{xy} = \overline{ot}$ to Δ. Since any two points in X belong to an apartment, we obtain a Δ-valued distance function on X.

Definition. $D_3 = D_3(X)$ is the collection of triples $(\alpha, \beta, \gamma) \in \Delta^3$ such that there exists an (oriented) geodesic triangle $\tau \subset X$ with the Δ-side lengths α, β, γ.

Theorem 3 (M. Kapovich, B. Leeb, J.J. Millson, [6], [7], [8])**.** *1. $D_3(X)$ is a convex polyhedral convex cone in Δ^3.*
2. $D_3(X)$ depends only on the (finite) Weyl group W_o and nothing else.
3. The system of generalized triangle inequalities inequalities *defining $D_3(X)$ can be computed in terms of the "Schubert calculus".*
4. If $\alpha+\beta+\gamma \in Q(R^\vee)$ and $\alpha, \beta, \gamma \in P(R^\vee)$ then we can assume that the triangle τ has vertices at vertices of X.

Part 3 was also established by Berenstein and Sjamaar in [1].

The easier direction in Theorem 2 does not involve multiplication by k_R. The following result was first established in [8], other proofs can be found in [4] and [5]:

Theorem 4. *If*

$$(V_\alpha \otimes V_\beta \otimes V_\gamma)^{G^\vee} \neq 0$$

then $(\alpha, \beta, \gamma) \in D_3(X)$ and, moreover, there exists a triangle τ in X with special vertices.

The opposite direction *converting triangles to tensors* is much harder. The key tool for this is Littelmann's path model. Suppose that $\tilde{\tau} = [x, y, z] \subset X$ is a geodesic triangle with the Δ-side lengths α, β, γ.

We assume that al vertices of $\tilde{\tau}$ are special points of X and project$\tilde{\tau}$ to a *broken triangle* $\tau \subset \Delta$ via a retraction $X \to \Delta$. The broken triangle τ has two geodesic sides with Δ-length α, γ and one broken side p whose Δ-length is β. We show that p partly satisfies the axioms of an LS path from [10], however in general it does not satisfy the *unit distance* condition. This is analogous to some of the results of [3].

We then replace the geodesic side $[y, z]$ in $\tilde{\tau}$ with a PL path $\tilde{p}$ contained in the 1-skeleton of an apartment in X. We project the resulting polygon $\tilde{P}$ to $P \subset \Delta$ and use the multiplication by k_R to ensure that:

(a) All vertices of a kP are special.

(b) Therefore, the image of the path $\tilde{p}$ satisfies a generalized LS condition which in turn suffices to find a fixed vector in the triple tensor product.

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Brylinski-Kostant Filtrations and Graded Injectivity

Anthony Joseph

Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be a triangular decomposition of a complex semisimple Lie algebra. Set $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$, $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$. Choose $e \in \mathfrak{n}$, $h \in \mathfrak{h}$, $f \in \mathfrak{n}^-$ such that (e, h, f) is a principal tds. For all weights μ, ν, let $V(\mu)$ denote the simple finite dimensional $U(\mathfrak{g})$ module with extreme weight μ and $V(\mu)_\nu$ its subspace of weight ν. The (left) Brylinski-Kostant (or simply, BK) filtration of $V(\mu)$ is defined through $\mathcal{F}^m V(\mu)_\nu = \{a \in V(\mu)_\nu \mid e^{m+1}a = 0\}$. It defines a q-character $ch_q V(\mu)$ which refines the Weyl character. Inspired by the work of Kostant concerning generalized exponents, Brylinski calculated the coefficient of e^ν in $ch_q V(\mu)$ for all ν dominant, using notably a geometric result of Broer. It turned out that this was just Lusztig's q-analogue (for that part) of $ch\ V(\mu)$. In [JLZ] a rather complicated though purely algebraic proof was given and furthermore $ch_q V(\mu)$ was completely determined. This method also depended on Broer's result; which was later refined [J1] to show that $F_{\mathfrak{h}}(Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), S(\mathfrak{g}/\mathfrak{b}) \otimes \mathbb{C}_{-\chi}))$ is injective in the opposed $\mathcal{O}$ category. (Here $F_{\mathfrak{h}}$ means taking the $\mathfrak{h}$ locally finite part of the module in question.)

The present work gives two simplifications to the above proofs. We also note that $V(\mu)$ admits a right BK filtration which outside $\mathfrak{sl}(2)$ is different to the left one. It leads to a q-character $ch'_q V(\mu)$ which coincides with $ch_q V(\mu)$ for $\mathfrak{sl}(3)$ but is generally different to $ch_q V(\mu)$.

Let $\delta M(\chi)$ denote the $\mathcal{O}$ dual of the Verma module with highest weight χ. Since $M(\chi)$ identifies with $U(\mathfrak{n}^-)$, its $\mathcal{O}$ dual admits a $U(\mathfrak{n})$ bimodule structure compatible with the action of $\mathfrak{h}$. Let π denote the set of simple roots. Let x_α (resp. $\alpha^\vee$) denote the root vector (resp. coroot) corresponding to α and set

$$V(\mu)^\star = \{a \in \delta M(0) \mid x_\alpha^{\alpha^\vee(\mu)+1} a = 0, \ \forall \alpha \in \pi\}.$$

Let $V(\mu)^\star_\nu$ denote the subspace of $V(\mu)^\star$ of weight ν and $\mathbb{C}_\chi$ the one dimensional $\mathfrak{b}$ module of weight χ.

A key observation of the present work [HJ] is that $V(\mu)^\star$ identifies as a right $U(\mathfrak{b})$ module with $V(-\mu)^* \otimes \mathbb{C}_{-\mu}$. Let $v_{-\mu} \in V(-\mu)$ be an extreme vector of weight $-\mu$. Then for μ, χ dominant it is shown that the map $\gamma \mapsto \gamma(v_{-\mu})(1)$ is an isomorphism of $Hom_{U(\mathfrak{g})}(V(\mu)^*, Hom_{U(\mathfrak{b})}(U(\mathfrak{g}), \delta M(-\chi)))$ onto $V(\mu)^\star_{-\mu+\chi} \otimes \mathbb{C}_{-\chi}$ compatible with the right BK filtration on $V^\star_{-\mu+\chi}$ and on $\delta M(-\chi)$. To recover Brylinski's result we note that $gr\ \delta M(-\chi)$ identifies with $S(\mathfrak{g}/\mathfrak{b}) \otimes \mathbb{C}_{-\chi}$ and use the above mentioned injectivity. In [J2] we deduce the latter result by only dimension shifting and a combinatorial property of weights [J2]. (The latter unfortunately does not extend to the parabolic case.)

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Cherednik algebras and Hilbert schemes in characteristic p

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(joint work with Roman Bezrukavnikov, Victor Ginzburg)

Let $c \in \mathbb{Q}$ be a rational number, and $\mathsf{H}_{1,c}(\mathbf{A}_{n-1})$ the rational Cherednik algebra of type $\mathbf{A}_{n-1}$ with parameters $t = 1$ and c that has been considered in [7] (over the ground field of complex numbers).

For all primes $p \gg n$, we can reduce c modulo p. Thus, c becomes an element of the finite field $\mathbb{F}_p$. We let $\overline{\mathbb{F}} = \overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$, and let $\mathsf{H}_c := \mathsf{H}_{1,c}(\mathbf{A}_{n-1}, \overline{\mathbb{F}}_p)$ be the Cherednik algebra, viewed as an algebra over $\overline{\mathbb{F}}_p$. Unlike the case of characteristic zero, the algebra H_c has a large center, called the p-center. The spectrum of the p-center is isomorphic to $[(\mathbb{A}^2)^n/S_n]^{(1)}$, the Frobenius twist of the n-th symmetric power of the plane $\mathbb{A}^2$.

We consider $\mathrm{Hilb}^n \mathbb{A}^2$, the Hilbert scheme (over $\overline{\mathbb{F}}_p$) of n points in the plane, see e.g. [10]. There is a canonical Hilbert-Chow map $\Upsilon : \mathrm{Hilb}^n \mathbb{A}^2 \to (\mathbb{A}^2)^n/S_n$ that induces an algebra isomorphism

$$\Gamma(\mathrm{Hilb}^n \mathbb{A}^2, \mathcal{O}) \cong \overline{\mathbb{F}}\left[(\mathbb{A}^2)^n/S_n\right]. \tag{1}$$

Let $\mathrm{Hilb}^{(1)}$ denote the Frobenius twist of $\mathrm{Hilb}^n \mathbb{A}^2$, a scheme isomorphic to $\mathrm{Hilb}^n \mathbb{A}^2$ and equipped with a canonical Frobenius morphism $\mathsf{Fr} : \mathrm{Hilb}^n \mathbb{A}^2 \to \mathrm{Hilb}^{(1)}$. We introduce an Azumaya algebra $\mathcal{H}_c$ on $\mathrm{Hilb}^{(1)}$ of degree $n! \cdot p^n$ (recall that an Azumaya algebra has degree r if each of its geometric fibers is isomorphic to the algebra of $r \times r$-matrices). For all sufficiently large primes p, we construct a natural algebra isomorphism (a version of the Harish-Chandra isomorphism from [7])

$$\Gamma(\mathrm{Hilb}^{(1)}, \mathcal{H}_c) \xrightarrow{\sim} \mathsf{H}_c. \tag{2}$$

The restriction of this isomorphism to the subalgebra $\Gamma(\mathrm{Hilb}^{(1)}, \mathcal{O})$ yields, via (1), the above mentioned isomorphism between the algebra $\overline{\mathbb{F}}\left[\left((\mathbb{A}^2)^n/S_n\right)^{(1)}\right]$ and the p-center.

More generally, for any $c \in \overline{\mathbb{F}}$, not necessarily an element of $\mathbb{F}_p$, there is an Azumaya algebra on the Calogero-Moser space with parameter $c^p - c$ such that an analogue of isomorphism (2) holds for the Calogero-Moser space instead of the Hilbert scheme. This case is somewhat less interesting since the Calogero-Moser space is affine while the Hilbert scheme is not.

The main idea used in the construction of isomorphism (2) is to compare Nakajima's description of $\mathrm{Hilb}^n \mathbb{A}^2$ by means of *Hamiltonian reduction*, see [10], with a refined version of the construction introduced in [7] describing the *spherical subalgebra* of H_c as a *quantum* Hamiltonian reduction of an algebra of differential operators.

We introduce the following set of rational numbers

$$\mathbb{Q}^{good} = \{c \in \mathbb{Q} \;:\; c \geq 0 \;\&\; c \notin \frac{1}{2} + \mathbb{Z}\}. \tag{3}$$

One of our main results reads

Theorem 1. Fix $c \in \mathbb{Q}^{good}$. Then, there exists a constant $d = d(c)$ such that for all primes $p > d(c)$, the functor $R\Gamma : \; D^b(\mathcal{H}_c\text{-}\mathrm{Mod}) \to D^b(\mathsf{H}_c\text{-}\mathrm{Mod})$ is a triangulated equivalence between the bounded derived categories of sheaves of coherent $\mathcal{H}_c$-modules and finitely generated H_c-modules, respectively, whose inverse is the localisation functor $M \mapsto \mathcal{H}_c \overset{L}{\otimes}_{\mathsf{H}_c} M$.

Moreover, we have $H^i(\mathrm{Hilb}^{(1)}, \mathcal{H}_c) = 0$, $\forall i > 0$.

Now, fix $\xi \in [(\mathbb{A}^2)^n/S_n]^{(1)}$, a point in the Frobenius twist of $(\mathbb{A}^2)^n/S_n$. We write $\mathrm{Hilb}^{(1)}_\xi = \Upsilon^{-1}(\xi)$ for the corresponding fiber of the Frobenius twist of the Hilbert-Chow map, and let $\widehat{\mathrm{Hilb}}^{(1)}_\xi = \widehat{\Upsilon^{-1}(\xi)}$ denote its formal neighborhood, the completion of $\mathrm{Hilb}^{(1)}$ along the subscheme $\mathrm{Hilb}^{(1)}_\xi$.

The theorem below, based on a similar result in [4], says that the Azumaya algebra $\mathcal{H}_c$ *splits* on the formal neighborhood of each fiber of the Hilbert-Chow map, that is, we have the following result:

Theorem 2. For each $\xi \in [(\mathbb{A}^2)^n/S_n]^{(1)}$, there exists a vector bundle $\mathcal{V}_\xi$ on $\widehat{\mathrm{Hilb}}^{(1)}_\xi$ such that one has

$$\mathcal{H}_c|_{\widehat{\mathrm{Hilb}}^{(1)}_\xi} \cong (\mathcal{E}nd\,\mathcal{V}_\xi)^{\mathrm{opp}}.$$

Note that the splitting bundle is not unique; it is only determined up to twisting by an invertible sheaf.

Given ξ as above, let $\mathfrak{m}_\xi$ be the corresponding maximal ideal in the p-center of H_c. Let $\widehat{\mathsf{H}}_{c,\xi}$, resp. $\widehat{\mathcal{H}}_{c,\xi} = \mathcal{H}_c|_{\widehat{\mathrm{Hilb}}^{(1)}_\xi}$, be the $\mathfrak{m}_\xi$-adic completion of H_c, resp. of $\mathcal{H}_c$. We write $D^b(\widehat{\mathsf{H}}_{c,\xi}\text{-}\mathrm{Mod})$, resp. $D^b(\widehat{\mathcal{H}}_{c,\xi}\text{-}\mathrm{Mod})$, for the bounded derived category of finitely-generated complete topological $\widehat{\mathsf{H}}_{c,\xi}$-modules, resp. $\widehat{\mathcal{H}}_{c,\xi}$-modules. On the other hand, let $D^b(\mathsf{Coh}(\widehat{\mathrm{Hilb}}^{(1)}_\xi))$ be the bounded derived category of coherent sheaves on the formal scheme $\widehat{\mathrm{Hilb}}^{(1)}_\xi$.

Fix $c \in \mathbb{Q}^{good}$. Then, for all primes $p > d(c)$, Theorems 1,2 imply the following

Corollary 1. The category $D^b(\widehat{\mathcal{H}}_{c,\xi}\text{-}\mathrm{Mod})$ is equivalent to $D^b(\mathsf{Coh}(\widehat{\mathrm{Hilb}}^{(1)}_\xi))$.

Now let $\xi = 0$ be the zero point in $[(\mathbb{A}^2)^n/S_n]^{(1)}$. The fiber $\mathrm{Hilb}^{(1)}_0$ is isomorphic to the (Frobenius twist of the) punctual Hilbert scheme. This is a projective variety with a natural $\mathbb{G}_m \times \mathbb{G}_m$-action induced from the standard $\mathbb{G}_m \times \mathbb{G}_m$-action on $\mathbb{A}^2$ by dilating the coordinate axes.

The cohomology vanishing in Theorem 2 implies that the vector bundle $\mathcal{V}_0$ is *rigid*, i.e., we have $\mathrm{Ext}^1(\mathcal{V}_0, \mathcal{V}_0) = 0$. From this, one deduces that the vector bundle $\mathcal{V}_0$ can be equipped with a $\mathbb{G}_m \times \mathbb{G}_m$-equivariant structure. This equivariant structure induces a $\mathbb{Z}^2$-grading on (a dense subalgebra of) the algebra $\mathrm{Hom}(\mathcal{V}_0, \mathcal{V}_0) = \Gamma(\widehat{\mathrm{Hilb}}^{(1)}_0, \mathcal{H}_c)$, that is, of the algebra $\widehat{\mathsf{H}}_{c,0}$.

Recall further, see [2], that the algebra H_c contains a canonical $\mathfrak{sl}_2$-triple. Let $\mathsf{h} \in \mathsf{H}_c$ denote the semisimple element of that triple. We obtain

Corollary 2. The algebra $\widehat{\mathsf{H}}_{c,0}$ contains a canonical dense $\mathbb{Z}^2$-graded subalgebra $\oplus_{k,l\in\mathbb{Z}}\, \mathsf{H}^{k,l}$ such that, for any $u \in \mathsf{H}^{k,l}$, we have $\mathsf{h} \cdot u - u \cdot \mathsf{h} = (k - l) \cdot u$.

The category of $\mathbb{Z}^2$-graded modules over that subalgebra may be thought of as a '*mixed version*' of the category $\widehat{\mathsf{H}}_{c,0}\text{-}\mathrm{Mod}$, cf. [1, Definition 4.3.1].

Let $\overline{\mathbb{F}}[S_n]$ denote the group algebra of the Symmetric group on n letters. Write $\mathrm{Irr}(S_n)$ for the set of isomorphism classes of simple $\overline{\mathbb{F}}[S_n]$-modules. This set is labelled by partitions of n, since by our assumptions $\mathsf{char}\,\overline{\mathbb{F}} > n$. In particular, we have the trivial 1-dimensional representation triv, and the sign representation sign.

Let $\overline{\mathsf{H}} := \overline{\mathbb{F}}[[\mathbb{A}^{2n}]]\#S_n$ be the cross-product of S_n with $\mathsf{k}[[\mathbb{A}^{2n}]]$, the algebra of formal power series in $2n$ variables acted on by S_n in a natural way. We consider $D^b(\overline{\mathsf{H}}\text{-}\,\mathrm{Mod})$, the bounded derived category of (finitely-generated) complete topological $\overline{\mathsf{H}}$-modules.

Given a simple $\mathsf{k}[S_n]$-module τ, write $\tau_{\overline{\mathsf{H}}}$ for the corresponding $\overline{\mathsf{H}}$-module obtained by pullback via the natural projection $\overline{\mathsf{H}} = \overline{\mathbb{F}}[[\mathbb{A}^{2n}]]\#S_n \to \overline{\mathbb{F}}[S_n]$, $f \rtimes w \mapsto f(0)\cdot w$. Similarly, let L_τ denote the corresponding simple highest weight H_c-module, the unique simple quotient of the standard H_c-module associated with τ, see [6],[2].

The results of Bridgeland-King-Reid [5] and Haiman [9], see also [4], provide an equivalence of categories

$$\mathrm{BKR}: \; D^b(\mathsf{Coh}(\mathrm{Hilb}\,\mathbb{A}^2)) \xrightarrow{\sim} D^b(\mathsf{k}[\mathbb{A}^{2n}]\#S_n\text{-}\,\mathrm{Mod}), \quad \mathcal{F} \mapsto R\Gamma(\mathrm{Hilb}\,\mathbb{A}^2, \mathcal{P} \overset{L}{\otimes} \mathcal{F}),$$

where $\mathcal{P}$ denotes the *Procesi bundle*, the 'unusual' tautological rank $n!$ vector bundle on $\mathrm{Hilb}\,\mathbb{A}^2$ considered in [9]. Restricting this equivalence to the completion of the zero fiber of the Hilbert-Chow map, and composing with the equivalence of Corollary 1, one obtains the following composite equivalence

$$D^b(\widehat{\mathsf{H}}_{c,0}\text{-}\,\mathrm{Mod}) \xrightarrow{\sim} D^b(\mathsf{Coh}(\widehat{\mathrm{Hilb}}_0^{(1)})) \xrightarrow{\sim} D^b(\overline{\mathsf{H}}\text{-}\,\mathrm{Mod}). \tag{4}$$

We recall that the equivalence of Corollary 1 involves a choice of splitting bundle $\mathcal{V}_0$. This choice may be specified by the following

Conjecture 1. a) One can choose the splitting bundle $\mathcal{V}_0$ in such a way that $\Gamma(\mathrm{Hilb}_0^{(1)}, \mathcal{V}_0(-1)) = L_{\mathsf{sign}}$.

b) With this choice of V_0, the composite equivalence in (4) preserves the natural t-structures, in particular, induces an equivalence $\widehat{\mathsf{H}}_{c,0}\text{-}\,\mathrm{Mod} \xrightarrow{\sim} \overline{\mathsf{H}}\text{-}\,\mathrm{Mod}$, of *abelian* categories, such that L_τ goes to $\tau_{\overline{\mathsf{H}}}$, for any simple S_n-module τ.

At the moment we are able to show, using Morita equivalences established in [8], that the above Conjecture holds for $c = 0, 1, 2, \ldots$. We can also prove that if Conjecture 1 holds for $c = \frac{1}{n} + k$, $k = 1, 2, \ldots$, then $\Gamma(\mathrm{Hilb}_0^{(1)}, \mathcal{V}_0 \otimes \mathrm{BKR}^{-1}(\mathsf{triv}_{\overline{\mathsf{H}}}))$ is an H_c-module obtained by reducing modulo p the unique finite dimensional irreducible H_c-module constructed in [3] in case of characteristic zero. In particular, for $c = \frac{1}{n}$, we expect that $\Gamma(\mathrm{Hilb}_0^{(1)}, \mathcal{V}_0 \otimes \mathrm{BKR}^{-1}(\mathsf{triv}_{\overline{\mathsf{H}}}))$ is a 1-dimensional vector space that supports the trivial representation of the group $S_n \subset \mathsf{H}_c$.

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Kashiwara's crystals, semi-infinite Schubert varieties and enumerative geometry

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(joint work with Michael Finkelberg, Dennis Gaitsgory and others)

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra over $\mathbb{C}$ which we shall assume to be of either finite or affine type. Let also X denote the flag variety of $\mathfrak{g}$; this is a scheme of not necessarily finite type. We also set Λ to be the coroot lattice of $\mathfrak{g}$ and $\Lambda^+ \subset \Lambda$ to be the sub-semigroup of positive coroots.

Given $\theta \in \Lambda^+$ one may consider the space $\mathcal{M}^\theta$ classifying algebraic maps $\mathbb{P}^1 \to X$ of degree θ and sending $\infty \in \mathbb{P}^1$ to some fixed point $x \in X$ (we shall refer to this space as the space of *based* maps). The main purpose of this talk is to exhibit some connections between the geometry of $\mathcal{M}^\theta$'s and the combinatorics (and some representation theory) of the Lie algebra $\mathfrak{g}^\vee$ (this is the "Langlands dual" Lie algebra; by the definition its Cartan matrix is trnasposed to that of $\mathfrak{g}$) and exhibit some application of such constructions to certain questions of enumerative geometry. Here are some more specific examples of such connections.

Let $\mathbb{A}^\theta$ denote the space of all "colored" divisors of the form $\sum \theta_i z_i$ where $\theta_i \in \Lambda^+$ and $z_i \in \mathbb{C}$ such that $\sum \theta_i = \theta$. We construct a natural map $\pi_\theta : \mathcal{M}^\theta \to \mathbb{A}^\theta$. In [3] we show that the set of irreducible components of $\cup_\theta \pi_\theta^{-1}(\theta \cdot 0)$ has a natural structure of a crystal for the Lie algebra $\mathfrak{g}^\vee$; in this way we recover geometrically the canonical crystal $B(\infty)$ (modelling in some sense the Verma module for $\mathfrak{g}^\vee$). While doing this one is forced to introduce certain (partial) compactification of $\mathcal{M}^\theta$ (and also its prabolic analogs). When $\mathfrak{g}$ is of finite type it is called the Drinfeld compactification; it was studied extensively in [4] and [6]. In the affine case it is closely related to the Uhlenbeck compactification of moduli spaces of principal bundles on $\mathbb{P}^2$. In the finite case the singularities of these compactifications are supposed to model the singularities of the (not yet well-defined) semi-infinite Schubert varieties. The local intersection cohomology of these varieties is described in [6], [4] and [3]; it is again described in terms of the Lie algebra $\mathfrak{g}^\vee$. This is used in [1] and [2] in order to give a representation-theoretic explanation of the results of Givental-Kim about quantum cohomology of flag varieties as well as to prove a conjecture of Nekrasov about partition functions of N=2 SUSY gauge theory.

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Gelfand–Zeitlin theory from the perspective of classical mathematics

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(joint work with Nolan Wallach)

Let $M(n)$, for any positive integer n, denote the Lie (and associative) algebra of all complex $n \times n$ complex matrices. Let $P(n)$ be the graded commutative algebra of all polynomial functions on $M(n)$. The symmetric algebra over $M(n)$, as one knows, is a Poisson algebra. Using the bilinear form $(x, y) = -tr\, xy$ on $M(n)$ this may be carried over to $P(n)$, defining on $P(n)$ the structure of a Poisson algebra and hence the structure of a Poisson manifold on $M(n)$. Consequently to each $p \in P(n)$ there is associated a holomorphic vector field ξ_p on $M(n)$ such that

$$\xi_p\, q = [p, q]$$

where $q \in P(n)$ and $[p, q]$ is Poisson bracket.

For any positive integer k put $d(k) = k(k+1)/2$ and let I_k be the set $\{1, \ldots, k\}$. If $m \in I_n$ we will regard $M(m)$ (upper left-hand $m \times m$ corner) as a Lie subalgebra of $M(n)$. As a "classical mechanics" analogue to the Gelfand–Zeitlin commutative subalgebra of the universal enveloping algebra of $M(n)$, let $J(n)$ be the subalgebra of $P(n)$ generated by $P(m)^{Gl(m)}$ for all $m \in I_n$. Then

$$J(n) = P(1)^{Gl(1)} \otimes \cdots \times P(n)^{Gl(n)}$$

In addition $J(n)$ is a Poisson commutative polynomial subalgebra of $P(n)$ with $d(n)$ generators. In fact we can write $J(n) = \mathbb{C}[p_1, \ldots, p_{d(n)}]$ where, for $x \in M(n)$, $p_i(x), i \in I_{d(n)}$, "run over" the elementary symmetric functions of the roots of the characteristic polynomial of x_m, $m \in I_n$. Here and throughout, $x_m \in M(m)$ is the upper left $m \times m$ minor of x. The algebraic morphism

$$\Phi_n : M(n) \to \mathbb{C}^{d(n)} \text{ where } \Phi_n(x) = (p_1(x), \ldots, p_{d(n)}(x)) \tag{1}$$

plays a major role in this paper. Let $\mathfrak{b}_e$ be the $d(n)$-dimensional affine space of all $x \in M(n)$ of the form

$$x = \begin{pmatrix} a_{1\,1} & a_{1\,2} & \cdots & a_{1\,n-1} & a_{1\,n} \\ 1 & a_{2\,2} & \cdots & a_{2\,n-1} & a_{2\,n} \\ 0 & 1 & \cdots & a_{3\,n-1} & a_{3\,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{n\,n} \end{pmatrix} \tag{2}$$

where $a_{i\,j} \in \mathbb{C}$ are arbitrary. Elements $x \in M(n)$ of the form (2) are called Hessenberg matrices. As a generalization of classical facts about companion matrices we prove

Theorem 1. *The restriction*

$$\mathfrak{b}_e \to \mathbb{C}^{d(n)} \tag{3}$$

of the map Φ_n is an algebraic isomorphism.

The real and imaginary parts of a complex number define a lexicographical order in $\mathbb{C}$. For any $x \in M(n)$ and $m \in I_n$ let $E_x(m) = \{\mu_{1\,m}(x), \ldots, \mu_{m\,m}(x)\}$ be the (increasing) ordered m-tuple of eigenvalues of x_m, with the multiplicity as roots of the characteristic polynomial. As a corollary of Theorem 1 one has the following independence (with respect to m) of the eigenvalue sequences $E_x(m)$.

Theorem 2. *For all $m \in I_n$ let $E(m) = \{\mu_{1\,m}, \ldots, \mu_{m\,m}\}$ be an arbitrary m-tuple with values in $\mathbb{C}$. Then there exists a unique $x \in \mathfrak{b}_e$ such that $E(m) = E_x(m)$, up to ordering, for all $m \in I_n$.*

For any $c \in \mathbb{C}^{d(n)}$ let $M_c(n) = \Phi_n^{-1}(c)$ be the "fiber" of Φ_n over c. If $x, y \in M(n)$ then x and y lie in the same fiber if and only if $E_x(m) = E_y(m)$ for all $m \in I_n$.

Remark 3. Theorem 1 implies that Φ_n is surjective and asserts that $\mathfrak{b}_e$ is a cross-section of Φ_n. That is, to any $c \in \mathbb{C}^{d(n)}$ the intersection $M_c(n) \cap \mathfrak{b}_e$ consists of exactly one matrix.

0.2. One of the main results of the present paper, Part I, of a two part paper, concern the properties, of a complex analytic abelian group A of dimension $d(n-1)$ which operates on $M(n)$. One has

Theorem 4. *The algebra $J(n)$ is a maximal Poisson commutative subalgebra of $P(n)$. Furthermore the vector field ξ_p, for any $p \in J(n)$, is globally integrable on $M(n)$, defining an analytic action of $\mathbb{C}$ on $M(n)$. Moreover the fiber $M_c(n)$ is stable under this action, for any $c \in \mathbb{C}^{d(n)}$.*

If $p_{(i)}$, $i \in I_{d(n)}$ is any choice of generators of $J(n)$ then the span of $\xi_{p_{(i)}}$, $i \in I_{d(n)}$, is a commutative $d(n-1)$-dimensional Lie algebra $\mathfrak{a}$ of analytic vector fields on $M(n)$. The Lie algebra $\mathfrak{a}$ integrates to an action of a complex analytic group $A \cong \mathbb{C}^{d(n-1)}$ on $M(n)$. In a sense A is very extensive enlargement of a group, for the case where $\mathbb{R}$ replaces $\mathbb{C}$, introduced in § 4 of [6]. However no diagonalizability, compactness, or eigenvalue interlacing is required for the existence of A. In the complex setting the second statement of Theorem 4 and the existence of the action of A can be deduced from an iteration of Theorem 4.1 in [12]. The proof given in the present paper is independent of the theory supporting Theorem 4.1 in [12] and, among other things, leads to an explicit description of an arbitrary orbit $A\cdot x$ is terms the adjoint action of $n-1$ abelian groups defined by $x \in M(n)$. (see Theorem 5) below.

For any $x \in M(n)$ and $m \in I_n$ let $Z_{x,m} \subset M(m)$ be the (obviously commutative) associative subalgebra generated by x_m and the identity of $M(m)$. Let $G_{x,m} \subset Gl(m)$ be the commutative algebraic subgroup of $Gl(n)$ corresponding to $Z_{x,m}$ when the latter is regarded as a Lie algebra. The orbits of A are described in

Theorem 5. *Let $x \in M(n)$. Consider the following morphism of nonsingular irreducible affine varieties*

$$G_{x,1} \times \cdots \times G_{x,n-1} \to M(n) \tag{5}$$

where for $g(m) \in G_{x,m},\ m \in I_{n-1}$,

$$(g(1),\ldots,g(n-1)) \mapsto Ad\,(g(1)\cdots g(n-1))(x) \tag{6}$$

Then the image of (5) is exactly the A-orbit $A\cdot x$.

If $x \in M(n)$ then $dim\,A\cdot x \leq d(n-1)$. We are particularly interested in orbits of maximal dimension $d(n-1)$. We will now say that x is strongly regular if $(dp_i)_x,\ i \in I_{d(n)}$, are linearly independent.

Theorem 6. *Let $x \in M(n)$. Then the following conditions are equivalent.*

- *(a) x is strongly regular*
- *(b) $A\cdot x$ is an orbit of maximal dimension, $d(n-1)$*
- *(c) $dim\,Z_{x,m} = m,\ \forall m \in I_n,$ and $Z_{x,m} \cap Z_{x,m+1} = 0,\ \forall m \in I_{n-1}$*

Let $M^{sreg}(n) \subset M(n)$ be the Zariski open set of all strongly regular matrices. Note that $M^{sreg}(n)$ is not empty since in fact $\mathfrak{b}_e \subset M^{sreg}(n)$. Theorem 5 for the case where $x \in M^{sreg}(n)$ is especially nice.

Theorem 7. *Let $x \in M^{sreg}(n)$. Then the morphism (5) is an algebraic isomorphism onto its image, the maximal orbit $A\cdot x$. In particular $A\cdot x$ is a nonsingular variety and as such*

$$A\cdot x \cong G_{x,1} \times \cdots \times G_{x,n-1} \tag{7}$$

Let $x \in M(n)$. Motivated by the Jacobi matrices which arise in the theory of orthogonal polynomials on $\mathbb{R}$, we will say that x satisfies the eigenvalue disjointness condition if, for any $m \in I_n$, the eigenvalues of x_m have multiplicity one (so that x_m is regular semisimple in $M(m)$) and, as a set, $E_x(m) \cap E_x(m+1) = \emptyset$ for any $m \in I_{n-1}$. Let $M_\Omega(n)$ be the dense Zariski open set of such $x \in M(n)$. One readily has that $M_\Omega(n) = \Phi_n^{-1}(\Omega(n))$ where $\Omega(n)$ is a dense Zariski open set in $\mathbb{C}^{d(n)}$.

Theorem 8. *One has $M_\Omega(n) \subset M^{sreg}(n)$. In fact if $c \in \Omega(n)$ then the entire fiber $M_c(n)$ is a single maximal A-orbit. Moreover if $c \in \Omega(n)$ and $x \in M_c(n)$ then $G_{x,m}$ is a maximal (complex) torus in $Gl(m)$, for any $m \in I_{n-1}$, so that $M_c(n) = A \cdot x$ is a closed nonsingular subvariety of $M(n)$ and as such*

$$M_c(n) \cong (\mathbb{C}^\times)^{d(n-1)} \tag{8}$$

We apply the results above to establish, with an explicit dual coordinate system, a commutative analogue of the Gelfand-Kirillov theorem for $M(n)$. The function field $F(n)$ of $M(n)$ has a natural Poisson structure and an exact analogue would be to show that $F(n)$ is isomorphic to the function field of a $n(n-1)$-dimensional phase space over a Poisson central rational function field in n variables. Instead we show that this the case for a Galois extension, $F(n, \mathfrak{e})$, of $F(n)$. A immediate candidate for "half" the coordinate system would be the functions p_i, $i \in I_{d(n)}$. However it soon becomes clear that one it is much more appropriate to use the eigenvalue functions of the x_m, $m \in I_n$, rather than elementary symmetric functions (p_i) in these eigenvalues. However one cannot consistently and globally define eigenvalue functions r_i on $M_\Omega(n)$. However this can be done on a covering variety $M_\Omega(n, \mathfrak{e})$ of $M_\Omega(n)$. The covering map

$$\pi_n : M_\Omega(n, \mathfrak{e}) \to M_\Omega(n)$$

is a finite étale morphism admitting Σ_n, isomorphic to the direct product of the symmetric groups S_m, $m \in I_n$, as deck transformations. Poisson bracket lifts to the affine ring $\mathcal{O}(M_\Omega(n, \mathfrak{e}))$ of $M_\Omega(n, \mathfrak{e})$ and to the function field $F(n, \mathfrak{e})$ (a Galois extension of $F(n)$ with Galois group Σ_n) of $M_\Omega(n, \mathfrak{e})$.

The Poisson vector fields ξ_{r_i} on $M_\Omega(n, \mathfrak{e})$ integrate and generate a complex algebraic torus, $A_\mathfrak{r} \cong (\mathbb{C}^\times)^{d(n-1)}$ which operates algebraically on $M_\Omega(n, \mathfrak{e})$) and in fact if $M_\Omega(n, \mathfrak{e}, \mathfrak{b})$ is the π_n inverse image of $\mathfrak{b}_{\mathfrak{e},\Omega} = \mathfrak{b}_e \cap M_\Omega(n)$ in $M_\Omega(n, \mathfrak{e})$, then the map

$$A_\mathfrak{r} \times M_\Omega(n, \mathfrak{e}, \mathfrak{b}) \to M_\Omega(n, \mathfrak{e}), \qquad (b, y) \mapsto b \cdot y$$

is an algebraic isomorphism. The natural coordinate system on $A_\mathfrak{r}$ then carries over to $M_\Omega(n, \mathfrak{e})$ defining functions $s_j \in \mathcal{O}(M_\Omega(n, \mathfrak{e}))$, $j \in I_{d(n-1)}$, when they are normalized so that, for all j, s_j is the constant 1 on $M_\Omega(n, \mathfrak{e}, \mathfrak{b})$. Poisson commutativity $[s_i, s_j] = 0$ follows from a "Lagrangian" property of $\mathfrak{b}_e$ established in [11].

Theorem 9. *The image of the map*

$$M_\Omega(n,\mathfrak{e}) \to \mathbb{C}^{n^2}, \qquad z \mapsto (r_1(z),\ldots,r_{d(n)}(z),s_1(z),\ldots,s_{d(n-1)}(z)) \tag{9}$$

is a Zariski open set Y in $\mathbb{C}^{n^2}$ and (9) is an algebraic isomorphism of $M_\Omega(n,\mathfrak{e})$ with Y. Furthermore one has the following Poisson commutation relations:

$$\begin{aligned} &(1)\ [r_i,r_j]=0,\ i,j\in I_{d(n)}\\ &(2)\ [r_i,s_j]=\delta_{i\,j}\,s_j,\ i\in I_{d(n)},\ j\in I_{d(n-1)}\\ &(3)\ [s_i,s_j]=0,\ i,j\in I_{d(n-1)} \end{aligned} \tag{10}$$

Noting that s_i vanishes nowhere on $M_\Omega(n,\mathfrak{e})$ one has $r_{(i)} \in \mathcal{O}(M_\Omega(n,\mathfrak{e}))$ for $i \in I_{d(n-1)}$ where $r_{(i)} = r_i/s_i$. Replacing r_i by $r_{(i)}$ in (2) one has the more familiar phase space commutation relation $[r_{(i)}, s_j] = \delta_{i\,j}$. As a corollary of Theorem 9 one has the following commutative analogue of the Gelfand-Kirillov theorem.

Theorem 10. *One has a rational function field*

$$F(n,\mathfrak{e}) = \mathbb{C}(r_1,\ldots r_{d(n)},s_1,\ldots,s_{d(n-1)}) \tag{11}$$

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Dirac cohomology for Harish-Chandra modules

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(joint work with Jing-Song Huang, David Renard)

Let G be a connected real reductive Lie group with a Cartan involution Θ and a maximal compact subgroup $K = G^\Theta$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}_0$ of G, and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the complexifications. The conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ is denoted by $X \mapsto \bar{X}$. Let B be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$.

Let $\mathfrak{r}$ be a reductive subalgebra of $\mathfrak{g}$ such that B is nondegenerate on $\mathfrak{r}$. Then B is also nondegenerate on $\mathfrak{s} = \mathfrak{r}^\perp$, and there is an orthogonal decomposition $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$. Examples of $\mathfrak{r}$ include $\mathfrak{k}$ and a Levi subalgebra $\mathfrak{l}$ of any parabolic subalgebra of $\mathfrak{g}$.

Let $C(\mathfrak{s})$ denote the Clifford algebra of $\mathfrak{s}$ with respect to B. This algebra is generated by an orthonormal basis Z_i of $\mathfrak{s}$, with relations $Z_i^2 = 1$ and $Z_i Z_j = -Z_j Z_i$ if $i \neq j$. Kostant's cubic Dirac operator [Ko2] is the element

$$D = D(\mathfrak{g}, \mathfrak{r}) = \sum_i Z_i \otimes Z_i - \frac{1}{2} \otimes \sum_{i<j<k} B([Z_i, Z_j], Z_k) Z_i Z_j Z_k$$

of $U(\mathfrak{g}) \otimes C(\mathfrak{s})$, where $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$. It is easy to see that D is independent of the choice of the orthonormal basis Z_i and $\mathfrak{r}$-invariant for the adjoint action of $\mathfrak{r}$ on both factors of $U(\mathfrak{g}) \otimes C(\mathfrak{s})$.

The adjoint action of $\mathfrak{r}$ on $\mathfrak{s}$ defines a map $\mathfrak{r} \to \mathfrak{so}(\mathfrak{s})$, where $\mathfrak{so}(\mathfrak{s})$ consists of skew symmetric matrices in the basis Z_i. On the other hand, $\mathfrak{so}(\mathfrak{s})$ embeds into $C(\mathfrak{s})$ via $E_{ij} - E_{ji} \mapsto \frac{1}{2} Z_i Z_j$, where E_{ij} denotes the matrix with all entries 0 except the ij entry which is 1. The composition of these maps, $\alpha : \mathfrak{r} \to C(\mathfrak{s})$, defines an embedding $\Delta : \mathfrak{r} \hookrightarrow U(\mathfrak{g}) \otimes C(\mathfrak{s})$ via $\Delta(X) = X \otimes 1 + 1 \otimes \alpha(X)$. We denote $\Delta(\mathfrak{r})$ by $\mathfrak{r}_\Delta$, the universal enveloping algebra of $\mathfrak{r}_\Delta$ by $U(\mathfrak{r}_\Delta)$, and the center of $U(\mathfrak{r}_\Delta)$ by $Z(\mathfrak{r}_\Delta)$.

Two main structural results related to the above setting are:

$$D^2 = \Omega_\mathfrak{g} \otimes 1 - \Omega_{\mathfrak{r}_\Delta} + (||\rho_\mathfrak{g}||^2 - ||\rho_\mathfrak{r}||^2); \tag{1}$$

$$z \in Z(\mathfrak{g}) \;\Rightarrow\; z \otimes 1 = \zeta(z) + Da + aD, \tag{2}$$

where $\Omega_\mathfrak{g}$ and $\Omega_{\mathfrak{r}_\Delta}$ denote the Casimir elements of $Z(\mathfrak{g})$ respectively $Z(\mathfrak{r}_\Delta)$, $\rho_\mathfrak{g}$ and $\rho_\mathfrak{r}$ are the half sums of positive roots for $\mathfrak{g}$ respectively $\mathfrak{r}$, $\zeta(z)$ is in $Z(\mathfrak{r}_\Delta)$, and

a is some $\mathfrak{r}$-invariant element of $U(\mathfrak{g}) \otimes C(\mathfrak{s})$. The equation (1) was first proved in [P] for $\mathfrak{r} = \mathfrak{k}$; the general result is in [Ko2]. A slightly weaker version of (2) was conjectured by Vogan [V1] for $\mathfrak{r} = \mathfrak{k}$ and subsequently proved in [HP1]. The result for general $\mathfrak{r}$ as above is in [Ko3], and there are further generalizations in the setting of noncommutative equivariant cohomology [AM], [Ku].

Vogan's reason for conjecturing (2) was the following. Let X be a $(\mathfrak{g}, K)$-module and let S be a spin module for $C(\mathfrak{s})$. The Dirac operator D then acts on $X \otimes S$, and the Dirac cohomology of X is the $\mathfrak{r}$-module $H_D(X) = \ker(D)/(\mathrm{im}(D) \cap \ker(D))$. Then if $H_D(X) \neq 0$ has $\mathfrak{r}$-infinitesimal character χ, it follows from (2) and the explicit knowledge of ζ that X has $\mathfrak{g}$-infinitesimal character χ. To make sense of this statement, let $\mathfrak{h} \supset \mathfrak{t}$ be Cartan subalgebras of $\mathfrak{g}$ respectively $\mathfrak{r}$, and consider $\mathfrak{t}^* \subset \mathfrak{h}^*$ by extending functionals on $\mathfrak{t}$ by 0 on $\mathfrak{t}^\perp$.

The motivation for studying representations with nonzero Dirac cohomology (in $\mathfrak{r} = \mathfrak{k}$ case) comes from the fact that among them there are many interesting unitary representations. Among these are the discrete series representations, which were constructed using Dirac operators ([P], [AS]), but also some not so well understood representations.

In the following, let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$. Then $\mathfrak{r} = \mathfrak{l}$ and $\mathfrak{s} = \mathfrak{u} \oplus \bar{\mathfrak{u}}$ are as above, and if we pick a basis u_i of $\mathfrak{u}$ with dual basis $u_i^* = -\theta \bar{u}_i \in \bar{\mathfrak{u}}$, we can write

$$D = \left(\sum_i u_i^* \otimes u_i - \frac{1}{4} \otimes \sum_{i,j} [u_i^*, u_j^*] u_i u_j \right) + \left(\sum_i u_i \otimes u_i^* - \frac{1}{4} \otimes \sum_{i,j} [u_i, u_j] u_i^* u_j^* \right).$$

Let C and C^- denote the summands in the above formula. One checks that C^- induces the $\mathfrak{u}$-homology differential on $X \otimes S = X \otimes \bigwedge \mathfrak{u}$ (up to a constant factor -2). Moreover, upon identifying $X \otimes \bigwedge \mathfrak{u}$ with $\mathrm{Hom}_{\mathbb{C}}(\bigwedge \bar{\mathfrak{u}}, X)$, C induces the $\bar{\mathfrak{u}}$-cohomology differential on this space.

If X is finite-dimensional, then $H_D(X) = \ker D = \ker D^2 = \ker C \cap \ker C^-$ can serve as a set of harmonic representatives for $\mathfrak{u}$-homology and $\bar{\mathfrak{u}}$-cohomology. This was essentially shown in [Ko1], since the spin Laplacean used there is $2D^2$. One should note here that there is a Hermitian inner product on $X \otimes S$ making D self-adjoint, and that $H_D(X)$ equals $H^\cdot(\bar{\mathfrak{u}}, X) \cong H_\cdot(\mathfrak{u}, X)$ up to a ρ-shift $\rho(\bar{\mathfrak{u}})$ coming from comparing the spin and adjoint actions of $\mathfrak{l}$ on S.

In [V2], Vogan asked if a similar result can be obtained for an irreducible unitary module X. The purpose of our paper [HPR] is to give a positive answer to this question in certain special cases. We use a positive definite Hermitian form on $X \otimes S$ given as the tensor product of the given form on X with the form on S obtained by extending the form $-B(X, \theta \bar{Y})$ on $\mathfrak{u}$ using the determinant.

Case 1. Suppose $\mathfrak{l} = \mathfrak{k}$; in particular, the pair $(\mathfrak{g}, \mathfrak{k})$ must be Hermitian. Then the adjoint of C is C^-, hence D is skew self-adjoint. Moreover, $X \otimes S$ decomposes as a direct sum of eigenspaces for D^2. It follows that $\ker D^2 = \ker D = \ker C \cap \ker C^-$, that $\ker C^\pm$ is the orthogonal of $\mathrm{im}\, C^\mp$, and that $X \otimes S = \ker D \oplus \mathrm{im}\, C \oplus \mathrm{im}\, C^-$,

$\ker C = \ker D \oplus \operatorname{im} C$ and $\ker C^- = \ker D \oplus \operatorname{im} C^-$, which implies the desired result in this case.

Case 2. Suppose $\mathfrak{l} \subset \mathfrak{k}$. In particular, the pair $\mathfrak{g}$ and $\mathfrak{k}$ must have equal rank. In this case we can write $D(\mathfrak{g}, \mathfrak{l}) = D(\mathfrak{g}, \mathfrak{k}) + \Delta D(\mathfrak{k}, \mathfrak{l})$, where $\Delta : U(\mathfrak{k}) \otimes C(\mathfrak{s} \cap \mathfrak{k}) \to U(\mathfrak{g}) \otimes C(\mathfrak{s})$ is obtained from $\Delta : \mathfrak{k} \to U(\mathfrak{g} \otimes \mathfrak{p})$ mentioned earlier. The summands in this decomposition anticommute, hence $D(\mathfrak{g}, \mathfrak{l})^2 = D(\mathfrak{g}, \mathfrak{k})^2 + \Delta D(\mathfrak{k}, \mathfrak{l})^2$. This operator is not necessarily positive definite, but the related operator $\Box = -D(\mathfrak{g}, \mathfrak{k})^2 + \Delta D(\mathfrak{k}, \mathfrak{l})^2$ is. Using some easy linear algebra of anticommuting operators, we first conclude

Theorem 1. *The cohomology of $D(\mathfrak{g}, \mathfrak{l})$ on $X \otimes S$ is equal to the cohomology of $\Delta D(\mathfrak{k}, \mathfrak{l})$ acting on the cohomology of $D(\mathfrak{g}, \mathfrak{k})$ on $X \otimes S(\mathfrak{p})$ tensored with $S(\mathfrak{s} \cap \mathfrak{k})$.*

One would like to similarly decompose $C(\mathfrak{g}, \mathfrak{l})$ as $C(\mathfrak{g}, \mathfrak{k}) + \Delta C(\mathfrak{k}, \mathfrak{l})$, and $C^-(\mathfrak{g}, \mathfrak{l})$ as $C^-(\mathfrak{g}, \mathfrak{k}) + \Delta C^-(\mathfrak{k}, \mathfrak{l})$, but this does not work in general, as some terms belonging to C respectively C^- get switched by Δ. The switched terms are zero when the pair $(\mathfrak{g}, \mathfrak{k})$ is Hermitian. In this case, we can obtain a Hodge decomposition with $\ker \Box$ as the set of harmonic representatives for $\bar{\mathfrak{u}}$-cohomology and $\mathfrak{u}$-homology. Then we can pass to Dirac cohomology and finally obtain

Theorem 2. *Let $(\mathfrak{g}, \mathfrak{k})$ be a Hermitian pair and let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$ with $\mathfrak{l} \subset \mathfrak{k}$ and $\mathfrak{u} \supset \mathfrak{p}^+$. Then the Dirac cohomology of a unitary $(\mathfrak{g}, K)$-module X with respect to $D(\mathfrak{g}, \mathfrak{l})$ is up to a modular twist by $\rho(\bar{\mathfrak{u}})$ equal to $\bar{\mathfrak{u}}$-cohomology of X or to $\mathfrak{u}$-homology of X.*

Even when they are "equal", the Dirac cohomology and the corresponding nilpotent Lie algebra cohomology still differ. Namely, the Dirac cohomology does not have a $\mathbb{Z}$-grading, only a $\mathbb{Z}_2$-grading. On the other hand, the Dirac cohomology does not depend on the choice of $\mathfrak{u}$ inside of $\mathfrak{s} = \mathfrak{l}^\perp$. Finally, the two kinds of cohomology have different homological properties.

Let us also mention that the statement of Theorem 2 holds for the discrete series representations, even if $(\mathfrak{g}, \mathfrak{k})$ is not Hermitian. On the other hand, for a Levi subalgebra of a parabolic in $\mathfrak{g} = \mathfrak{sp}(4, \mathbb{C})$ corresponding to a short noncompact root, one can calculate that the Dirac cohomology of the even Weil representation of $\mathfrak{sp}(4, \mathbb{R})$ with respect to $D(\mathfrak{g}, \mathfrak{l})$ is strictly larger than the $\bar{\mathfrak{u}}$-cohomology, which is still equal to the $\mathfrak{u}$-homology.

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Enveloping algebras of Slodowy slices and the Joseph ideal

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Let $\Bbbk$ be an algebraically closed field of characteristic 0 and let G be a simple algebraic group over $\Bbbk$. Let $\mathfrak{g} = \operatorname{Lie} G$ and let (e, h, f) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Let $(\,\cdot\,,\,\cdot\,)$ be the G-invariant bilinear form on $\mathfrak{g}$ with $(e, f) = 1$ and define $\chi = \chi_e \in \mathfrak{g}^*$ by setting $\chi(x) = (e, x)$ for all $x \in \mathfrak{g}$. Let $\mathcal{O}_\chi$ denote the coadjoint orbit of χ.

Let H_χ be the enveloping algebra of the Slodowy slice $\mathcal{S}_e$; see [7], [4], [2]. Recall that $H_\chi = \operatorname{End}_{\mathfrak{g}}(Q_\chi)^{\mathrm{op}}$ where Q_χ is a generalised Gelfand–Graev module for $U(\mathfrak{g})$ associated with the $\mathfrak{sl}_2$-triple (e, h, f). The module Q_χ is induced from a one-dimensional module $\Bbbk_\chi$ over a nilpotent subalgebra $\mathfrak{m}_\chi$ of $\mathfrak{g}$ such that $\dim \mathfrak{m}_\chi = \frac{1}{2} \dim \mathcal{O}_\chi$. The subalgebra $\mathfrak{m}_\chi$ is $(\operatorname{ad} h)$-stable, all weights of $\operatorname{ad} h$ on $\mathfrak{m}_\chi$ are negative, and χ vanishes on the derived subalgebra of $\mathfrak{m}_\chi$. The action of $\mathfrak{m}_\chi$ on $\Bbbk_\chi = \Bbbk 1_\chi$ is given by $x(1_\chi) = \chi(x) 1_\chi$ for all $x \in \mathfrak{m}_\chi$.

Let $\mathfrak{z}_\chi$ denote the stabiliser of χ in $\mathfrak{g}$. Let $x_1, \ldots, x_r$ be a basis of $\mathfrak{z}_\chi$ such that $[h, x_i] = n_i x_i$ for some $n_i \in \mathbb{Z}_+$. By [7, Theorem 4.6] to each basis vector x_i one can attach an element $\Theta_{x_i} \in H_\chi$ in such a way that the monomials $\Theta_{x_1}^{i_1} \Theta_{x_2}^{i_2} \cdots \Theta_{x_r}^{i_r}$ with $(i_1, i_2, \ldots, i_r) \in \mathbb{Z}_+^r$ form a basis of H_χ over $\Bbbk$. We say that the monomial $\Theta_{x_1}^{a_1} \Theta_{x_2}^{a_2} \cdots \Theta_{x_r}^{a_r}$ has *Kazhdan degree* $\sum_{i=1}^r a_i(n_i + 2)$ and denote by H_χ^k the span of all monomials as above of Kazhdan degree $\leq k$. Then $\{H_\chi^k \mid k \in \mathbb{Z}_+\}$ is an increasing filtration of the algebra H_χ; see [7, (4.6)]. The corresponding graded algebra $\operatorname{gr} H_\chi$ is a polynomial algebra in $\operatorname{gr} \Theta_{x_1}, \operatorname{gr} \Theta_{x_2}, \ldots, \operatorname{gr} \Theta_{x_r}$ which identifies naturally with the coordinate ring of the special transverse slice $\mathcal{S}_e = e + \operatorname{Ker} \operatorname{ad} f$ endowed with its Slodowy grading. We prove that there exists an associative $\Bbbk[t]$-algebra $\mathcal{H}_\chi$ free as a module over $\Bbbk[t]$ and such that

$$\mathcal{H}_\chi/(t - \lambda)\mathcal{H}_\chi \cong \begin{cases} H_\chi & \text{if } \lambda \neq 0, \\ U(\mathfrak{z}_\chi) & \text{if } \lambda = 0 \end{cases}$$

as $\Bbbk$-algebras. Thus H_χ is a deformation of the universal enveloping algebra $U(\mathfrak{z}_\chi)$.

For $\chi = (e,\,\cdot\,)$ we let $\mathcal{C}_\chi$ denote the category of all $\mathfrak{g}$-modules on which $x - \chi(x)$ acts locally nilpotently for all $x \in \mathfrak{m}_\chi$. Given a $\mathfrak{g}$-module M we set

$$\operatorname{Wh}(M) := \{m \in M \mid x.m = \chi(x)m \ \ (\forall\, x \in \mathfrak{m}_\chi)\}.$$

It should be mentioned here that the algebra H_χ acts on $\mathrm{Wh}(M)$ via a canonical isomorphism $H_\chi \cong \big(U(\mathfrak{g})/N_\chi\big)^{\operatorname{ad}\mathfrak{m}_\chi}$ where N_χ denotes the left ideal of $U(\mathfrak{g})$ generated by all $x-\chi(x)$ with $x\in\mathfrak{m}_\chi$. In the Appendix to [7], Skryabin proved that the functors $V \rightsquigarrow Q_\chi \otimes_{H_\chi} V$ and $M \rightsquigarrow \mathrm{Wh}(M)$ are mutually inverse equivalences between the category of all H_χ-modules and the category $\mathcal{C}_\chi$; see also [4, Theorem 6.1].

Skryabin's equivalence implies that for any irreducible H_χ-module V the annihilator $\mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{H_\chi} V)$ is a primitive ideal of $U(\mathfrak{g})$. By a classical result of Lie Theory, the associated variety $\mathcal{VA}(\mathcal{I})$ of any primitive ideal $\mathcal{I}$ of $U(\mathfrak{g})$ is the closure of a nilpotent orbit in $\mathfrak{g}^*$. Generalising a classical result of Kostant on Whittaker modules we show that for any irreducible H_χ-module V the associated variety of $\mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{H_\chi} V)$ contains the coadjoint orbit $\mathcal{O}_\chi$. In the most interesting case where V is a finite dimensional irreducible H_χ-module we prove that

$$\mathcal{VA}\big(\mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{H_\chi} V)\big) = \overline{\mathcal{O}}_\chi \quad \text{and} \quad \mathrm{Dim}(Q_\chi \otimes_{H_\chi} V) = \frac{1}{2}\dim \overline{\mathcal{O}}_\chi$$

where $\mathrm{Dim}(M)$ is the Gelfand–Kirillov dimension of a finitely generated $U(\mathfrak{g})$-module M. In particular, this implies that for any irreducible finite dimensional H_χ-module V the irreducible $U(\mathfrak{g})$-module $Q_\chi \otimes_{H_\chi} V$ is holonomic.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let Φ be the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. Let $\Pi = \{\alpha_1,\dots,\alpha_\ell\}$ be a basis of simple roots in Φ, and let Φ^+ be the positive system of Φ relative to Π. If $\mathfrak{g}$ is not of type A or C, there is a unique long root in Π linked with the lowest root $-\widetilde{\alpha}$ on the extended Dynkin diagram of $\mathfrak{g}$; we call it β. For $\mathfrak{g}$ of type A_n and C_n we let $\beta=\alpha_n$. Choose root vectors $e_\beta, e_{-\beta}\in\mathfrak{g}$ corresponding to roots β and $-\beta$ such that $(e_\beta,[e_\beta,e_{-\beta}],e_{-\beta})$ is an $\mathfrak{sl}_2$-triple and put $h_\beta=[e_\beta,e_{-\beta}]$.

We investigate the algebra H_χ in the case where $(e,h,f)=(e_\beta,h_\beta,e_{-\beta})$. Then $\mathcal{O}_\chi=\mathcal{O}_{\min}$, the minimal nonzero nilpotent orbit in $\mathfrak{g}^*$. We let H denote the minimal nilpotent algebra H_χ. One of our main objectives is to give a presentation of H by generators and relations.

The action of the inner derivation $\operatorname{ad} h$ gives $\mathfrak{g}$ a short $\mathbb{Z}$-grading

$$\mathfrak{g} \,=\, \mathfrak{g}(-2)\oplus\mathfrak{g}(-1)\oplus\mathfrak{g}(0)\oplus\mathfrak{g}(1)\oplus\mathfrak{g}(2), \qquad \mathfrak{g}(i)=\{x\in\mathfrak{g}\mid [h,x]=ix\}$$

with $\mathfrak{g}(1)\oplus\ \mathfrak{g}(2)$ and $\mathfrak{g}(-1)\oplus\mathfrak{g}(-2)$ being Heisenberg Lie algebras. One knows of course that $\mathfrak{g}(\pm 2)$ is spanned by $e_{\pm\beta}$, that $\mathfrak{z}_\chi(i)=\mathfrak{g}(i)$ for $i=1,2$, and that $\mathfrak{z}_\chi(0)$ coincides with the image of the Lie algebra homomorphism

$$\sharp:\ \mathfrak{g}(0)\longrightarrow\mathfrak{g}(0), \qquad x\mapsto x-\frac{1}{2}(x,h)\,h$$

whose kernel $\Bbbk h$ is a central ideal of $\mathfrak{g}(0)$. The graded component $\mathfrak{g}(-1)$ has a basis $z_1,\dots,z_s,z_{s+1},\dots,z_{2s}$ such that the z_i's with $1\le i\le s$ (resp. $s+1\le i\le 2s$) are root vectors for $\mathfrak{h}$ corresponding to negative (resp. positive) roots, and

$$[z_i,z_j]=[z_{i+s},z_{j+s}]=0, \qquad [z_{i+s},z_j]=\delta_{ij}f, \qquad (1\le i,j\le s).$$

Moreover, in the present case we can choose $\mathfrak{m}_\chi$ to be the span of f and the z_i's with $s+1 \le i \le 2s$, an abelian subalgebra of $\mathfrak{g}$ of dimension $s+1 = \frac{1}{2}\dim \mathcal{O}_{\min}$. We set $z_i^* := z_{i+s}$ for $1 \le i \le s$ and $z_i^* := -z_{i-s}$ for $s+1 \le i \le 2s$.

Let C denote the Casimir element of $U(\mathfrak{g})$ corresponding to the bilinear form $(\,\cdot\,,\,\cdot\,)$. This form is nondegenerate on $\mathfrak{z}_\chi(0)$, hence we can find bases $\{a_i\}$ and $\{b_i\}$ of $\mathfrak{z}_\chi(0)$ such that $(a_i, b_j) = \delta_{ij}$. Set $\Theta_{\mathrm{Cas}} := \sum_i \Theta_{a_i}\Theta_{b_i}$, a central element of the associative subalgebra of H generated by the Lie algebra $\Theta(\mathfrak{z}_\chi(0))$. We can regard C as a central element of H.

By a well-known result of Joseph, outside type A the universal enveloping algebra $U(\mathfrak{g})$ contains a unique completely prime primitive ideal whose associated variety is $\overline{\mathcal{O}}_{\min}$; see [5]. This ideal, often denoted $\mathcal{J}_0$, is known as the Joseph ideal of $U(\mathfrak{g})$.

Theorem 1. *The algebra H is generated by the Casimir element C and the subspaces $\Theta(\mathfrak{z}_\chi(i))$ for $i = 0, 1$, subject to the following relations:*

(i) $[\Theta_x, \Theta_y] = \Theta_{[x,y]}$ *for all* $x, y \in \mathfrak{z}_\chi(0)$;

(ii) $[\Theta_x, \Theta_u] = \Theta_{[x,u]}$ *for all* $x \in \mathfrak{z}_\chi(0)$ *and* $u \in \mathfrak{z}_\chi(1)$;

(iii) C *is central in* H;

(iv) $$[\Theta_u, \Theta_v] = \tfrac{1}{2}(f,[u,v])\big(C - \Theta_{\mathrm{Cas}} - c_0\big) + \tfrac{1}{2}\sum_{i=1}^{2s}\big(\Theta_{[u,z_i]^\sharp}\,\Theta_{[v,z_i^*]^\sharp} + \Theta_{[v,z_i^*]^\sharp}\,\Theta_{[u,z_i]^\sharp}\big)$$
for all $u, v \in \mathfrak{z}_\chi(1)$, *where* c_0 *is a constant depending on* $\mathfrak{g}$.

If $\mathfrak{g}$ is not of type A *then c_0 is the eigenvalue of C on the primitive quotient $U(\mathfrak{g})/\mathcal{J}_0$. If $\mathfrak{g}$ is of type* A_n, $n \ge 2$, *then* $c_0 = -\frac{n(n+1)}{4}$. *If $\mathfrak{g}$ is of type* A_1 *then* $H = \Bbbk[C]$.

We study highest weight modules for the algebra H. Let $\Phi_e = \{\alpha \in \Phi \mid \alpha(h) = 0 \text{ or } 1\}$, and put $\Phi_e^\pm = \Phi_e \cap \Phi^\pm$ where $\Phi^- = -\Phi^+$. For $i = 0, 1$ put $\Phi_{e,i}^\pm = \{\alpha \in \Phi_e^\pm \mid \alpha(h) = i\}$. Note that $\mathfrak{z}_\chi$ is spanned by $\mathfrak{h}_e := \mathfrak{h} \cap \mathfrak{g}(0)^\sharp$, by root vectors e_α with $\alpha \in \Phi_e$, and by e. Let $h_1, \ldots, h_{l-1}$ be a basis of $\mathfrak{h}_e$, and let $\mathfrak{n}^\pm(i)$ be the span of all e_α with $\alpha \in \Phi_{e,i}^\pm$. Clearly, $\mathfrak{n}^+(0)$ and $\mathfrak{n}^-(0)$ are maximal nilpotent subalgebras of $\mathfrak{g}(0)^\sharp$. Let $\{x_1, \ldots, x_t\}$ and $\{y_1, \ldots, y_t\}$ be bases of $\mathfrak{n}^+(0)$ and $\mathfrak{n}^-(0)$ consisting of root vectors for $\mathfrak{h}$. For $1 \le i \le s$ let γ_i (resp. γ_i^*) denote the root of z_i (resp. z_i^*), and put $u_i = [e, z_i]$, $u_i^* = [e, z_i^*]$. Then $\{u_1, \ldots, u_s, u_1^*, \ldots, u_s^*\}$ is a $\Bbbk$-basis of $\mathfrak{z}_\chi(1)$.

Given $\lambda \in \mathfrak{h}_e^*$ and $c \in \Bbbk$ we denote by $J_{\lambda,c}$ the linear span in H of all

$$\prod_{i=1}^{t}\Theta_{y_i}^{l_i}\cdot\prod_{i=1}^{s}\Theta_{u_i}^{m_i}\cdot\prod_{i=1}^{\ell-1}\big(\Theta_{h_i} - \lambda(h_i)\big)^{n_i}\cdot(C-c)^{n_\ell}\cdot\prod_{i=1}^{s}\Theta_{u_i^*}^{r_i}\cdot\prod_{i=1}^{t}\Theta_{x_i}^{q_i}$$

with $\sum_{i=1}^{\ell} n_i + \sum_{i=1}^{t} r_i + \sum_{i=1}^{s} q_i > 0$. Using Theorem 1 we show that $J_{\lambda,c}$ is a left ideal of H. We call the H-module $Z_H(\lambda, c) := H/J_{\lambda,c}$ the *Verma module of level* c corresponding to λ. We show that $Z_H(\lambda, c)$ contains a unique maximal submodule which we denote $Z_H^{\max}(\lambda, c)$. Thus to every $(\lambda, c) \in \mathfrak{h}_e^* \times \Bbbk$ there corresponds an irreducible highest weight H-module $L_H(\lambda, c) := Z_H(\lambda, c)/Z_H^{\max}(\lambda, c)$. It is fairly easy to show that $L_H(\lambda, c) \cong L_H(\lambda', c')$ if and only if $(\lambda, c) = (\lambda', c')$ and that any irreducible finite dimensional H-module is isomorphic to exactly one of $L_H(\lambda, c)$ with λ satisfying a natural integrality condition.

To determine the composition multiplicities of the Verma modules $Z_H(\lambda, c)$ we link them with $\mathfrak{g}$-modules obtained by parabolic induction from Whittaker modules for $\mathfrak{sl}(2)$. Let $\mathfrak{s}_\beta = e_\beta \oplus \Bbbk h_\beta \oplus \Bbbk f_\beta$ and put

$$\mathfrak{p}_\beta := \mathfrak{s}_\beta + \mathfrak{h} + \textstyle\sum_{\alpha\in\Phi^+} e_\alpha, \quad \mathfrak{n}_\beta := \sum_{\alpha\in\Phi^+\setminus\{\beta\}} \Bbbk e_\alpha, \quad \widetilde{\mathfrak{s}}_\beta := \mathfrak{h}_e \oplus \mathfrak{s}_\beta.$$

Let $C_\beta = ef + fe + \frac{1}{2}h^2 = 2ef + \frac{1}{2}h^2 - h$, a central element of $U(\widetilde{\mathfrak{s}}_\beta)$. Given $\lambda \in \mathfrak{h}_e^*$ and $c \in \Bbbk$ we denote by $I_\beta(\lambda, c)$ the left ideal of $U(\mathfrak{p}_\beta)$ generated by $f-1, C_\beta - c$, all $h - \lambda(h)$ with $h \in \mathfrak{h}_e$, and all e_γ with $\gamma \in \Phi^+ \setminus \{\beta\}$. Let $Y(\lambda, c) := U(\mathfrak{p}_\beta)/I_\beta(\lambda, c)$, a $\mathfrak{p}_\beta$-module with the trivial action of $\mathfrak{n}_\beta$. Regarded as an $\mathfrak{s}_\beta$-module, $Y(\lambda, c)$ is isomorphic to a Whittaker module for $\mathfrak{sl}(2, \Bbbk)$. Now define

$$M(\lambda, c) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\beta)} Y(\lambda, c).$$

Recall that each z_i^* with $i \le s$ is a root vector corresponding to $\gamma_i^* = -\beta - \gamma_i \in \Phi^+$. Let $\delta = \frac{1}{2}(\gamma_1^* + \cdots + \gamma_s^*)$ and $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$. Since the restriction of $(\,\cdot\,,\,\cdot\,)$ to $\mathfrak{h}_e$ is nondegenerate, for any $\eta \in \mathfrak{h}_e^*$ there is a unique $t_\eta \in \mathfrak{h}_e$ such that $\varphi = (t_\eta, \,\cdot\,)$. Hence $(\,\cdot\,,\,\cdot\,)$ induces a bilinear form on $\mathfrak{h}_e^*$ via $(\mu, \nu) := (t_\mu, t_\nu)$ for all $\mu, \nu \in \mathfrak{h}_e^*$. Given a linear function $\varphi \in \mathfrak{h}^*$ we denote by $\bar{\varphi}$ the restriction of φ to $\mathfrak{h}_e$.

Theorem 2. *Each $\mathfrak{g}$-module $M(\lambda, c)$ is an object of the category $\mathcal{C}_\chi$. Furthermore, $\mathrm{W}h(M(\lambda, c)) \cong Z_H(\lambda + \bar{\delta}, c + (\lambda + 2\bar{\rho}, \lambda))$ as H-modules.*

Combined with Skryabin's equivalence and the main results of Miličić–Soergel [6] and Backelin [1], Theorem 2 shows that the composition multiplicities of the Verma modules $Z_H(\lambda, c)$ can be computed with the help of certain parabolic Kazhdan-Lusztig polynomials. This confirms in the minimal nilpotent case the Kazhdan-Lusztig conjecture for finite $\mathcal{W}$-algebras formulated by De Voss and van Driel in [3].

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Invariants on multiplicity free symplectic representations

Friedrich Knop

Let G be a connected reductive group (everything over $\mathbb{C}$). We reported on an ongoing project to study Hamiltonian G-actions on symplectic varieties.

More precisely, let X be a smooth G-variety equipped with a G-invariant closed non-degenerate 2-form ω. The action is called Hamiltonian if it admits a moment map, i.e., a G-equivariant morphism $m : X \to \mathfrak{g}^*$ (with $\mathfrak{g} = \operatorname{Lie} G$) such that

$$\omega(\xi, \eta x) = \langle m_*(\xi), \eta\rangle \text{ for all } x \in X, \xi \in T_x(X), \eta \in \mathfrak{g}.$$

For $x \in X$ let $A = \mathfrak{g}x$ be the tangent space of the G-orbit through x. Then ω induces a non-degenerate symplectic form on $\Sigma_x := A^\perp/(A \cap A^\perp)$, the symplectic slice at x. Then an equivariant Darboux theorem in the following form holds:

Theorem 1 ([4], 5.1)**.** *Let X be an affine Hamiltonian G-variety and $Gx \subseteq X$ a closed orbit. Then a formal neighborhood of Gx in X is uniquely determined by the triple (H, Σ, a) where $H = G_x$ is the isotropy group of X, $\Sigma = \Sigma_x$ is the symplectic slice at x, and $a = m(x) \in \mathfrak{g}^*$.*

In the talk, I focused on the case when $H = G$ and $a = 0$, i.e., when x is a fixed point of G and the symplectic slice becomes a local model for the action. Therefore, let from now on X be a finite dimensional representation of G, equipped with a G-invariant symplectic form ω.

In that case one has a non-commutative deformation: let $\mathcal{W}(X)$ be the algebra with generators X and relations $xy - yx = \omega(x, y)$ for all $x, y \in X$. This is the Weyl algebra attached to X. The group G acts on it and we can consider the invariant algebra $\mathcal{W}(X)^G$.

Definition 1. *The symplectic representation is called multiplicity free if $\mathcal{W}(X)^G$ is commutative.*

If U is any G-module then $X = U \oplus X^*$ is a symplectic representation. Moreover, the Weyl algebra $\mathcal{W}(X)$ can be identified with the ring $\mathcal{D}(U)$ of polynomial coefficient differential operators on U. Moreover, $\mathcal{D}(U)^G$ is commutative if and only if the ring of polynomial functions $\mathcal{O}(U)$ is multiplicity free as a G-module. This explains the terminology.

In the case at hand, the moment map is very concrete: let $x \in X$ and $\eta \in \mathfrak{g}$. Then $\langle m(x), \eta\rangle = \frac{1}{2}\omega(\eta x, x)$. Then X is multiplicity free if "most" invariants are pull-backs of invariants on $\mathfrak{g}^*$:

Theorem 2 ([5])**.** *The symplectic representation X is multiplicity free if and only if $\mathcal{O}(X)^G$ is a finite over the ring of pull-backs $m^*\mathcal{O}(\mathfrak{g}^*)^G$.*

Our main result says that multiplicity free symplectic representations are very nice from an invariant theoretic point of view:

Theorem 3 ([5])**.** *Every multiplicity free symplectic representation is cofree, i.e., the ring of invariants is a polynomial ring and the whole ring of functions is a free module over the ring of invariants.*

This was previously known when X is of the form $U \oplus U^*$, i.e., when X is a cotangent bundle (see [2] or [3]). The technique is again to produce sufficiently many "finite" sections of the quotient map. The main technical tool is a symplectic generalization of the local structure theorem of Brion-Luna-Vust [1].

This local structure theorem yields also a very simple algorithm for deciding the multiplicity freeness of a given symplectic representations. Subsequently, we were able to completely classify all multiplicity free symplectic representations in [6]. The following seven cases are arguably the most interesting ones since these are the only series where the dimension of the ring of invariants is unbounded:

(1) $G = Sp(2m) \times SO(p)$, $X = \mathbb{C}^{2m} \otimes \mathbb{C}^p$.
(2) $G = Sp(2m) \times SO(p)$, $X = (\mathbb{C}^{2m} \otimes \mathbb{C}^p) \oplus \mathbb{C}^{2m}$.
(3) $G = GL(m) \times GL(n)$, $X = U \oplus U^*$ with $U = \mathbb{C}^m \otimes \mathbb{C}^n$.
(4) $G = GL(m) \times GL(n)$, $X = U \oplus U^*$ with $U = \mathbb{C}^m \otimes \mathbb{C}^n \oplus \mathbb{C}^n$.
(5) $G = GL(n)$, $X = U \oplus U^*$ with $U = S^2\mathbb{C}^n$.
(6) $G = GL(n)$, $X = U \oplus U^*$ with $U = \bigwedge^2 \mathbb{C}^n$.
(7) $G = GL(n)$, $X = U \oplus U^*$ with $U = \bigwedge^2 \mathbb{C}^n \oplus \mathbb{C}^n$.

Note that cases (1) and (3) are precisely Howe's dual pairs. Cases (3), (5), and (6) are related to certain symmetric spaces of hermitian type.

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Crystal structure on rigged configurations

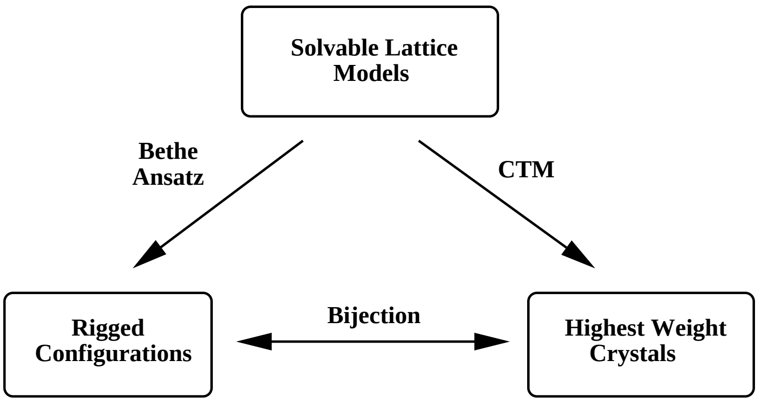
Anne Schilling

There are (at least) two main approaches for solving solvable lattice models or spin chain systems: the Bethe Ansatz [3] and the corner transfer matrix method [2].

In his 1931 paper [3], Bethe solved the Heisenberg spin chain based on the string hypothesis which asserts that the eigenvalues of the Hamiltonian form certain strings in the complex plane as the size of the system tends to infinity. The Bethe Ansatz has been applied to many further models proving completeness of the Bethe vectors. The eigenvalues and eigenvectors of the Hamiltonian are indexed by rigged configuration. However, numerical studies indicate that the string hypothesis is not always true [1].

The corner transfer matrix (CTM) method was introduced by Baxter and labels the eigenvectors by one-dimensional lattice paths. It turns out that these lattice paths have a natural interpretation in terms of Kashiwara's crystal base theory [7], namely as highest weight crystal elements in a tensor product of finite-dimensional crystals.

Even though neither the Bethe Ansatz nor the corner transfer matrix method are mathematically rigorous, they suggest that there should be a bijection between the two index sets, namely rigged configurations on the one hand and highest weight crystal elements on the other hand. This is schematically indicated in the following figure.



For the special case when the spin chain is defined on $V_{\mu_1} \otimes V_{\mu_2} \otimes \cdots \otimes V_{\mu_L}$, where V_{μ_i} is the irreducible $\mathrm{Gl}(n)$ representation indexed by the partition (μ_i) for $\mu_i \in \mathbb{N}$, a bijection between rigged configurations and semi-standard Young tableaux was given by Kerov, Kirillov and Reshetikhin [10, 11]. This bijection was proven and extended to the case when the μ_i are any sequence of rectangles in [12]. The bijection has many amazing properties. For example it takes the cocharge statistics cc defined on rigged configurations to the energy statistics D defined on crystals. Let λ be a partition and $B = B_{\mu_1} \otimes B_{\mu_2} \otimes \cdots \otimes B_{\mu_L}$ a tensor product of crystals where B_{μ_i} is the crystal associated with V_{μ_i}. Denoting the set of rigged configurations of weight λ by $\overline{\mathrm{RC}}(B,\lambda)$ and the set of highest weight crystals by $\overline{\mathcal{P}}(B,\lambda)$, we have the following theorem.

Theorem 1 ([12])**.** *There is a bijection $\Phi : \overline{\mathcal{P}}(B,\lambda) \to \overline{\mathrm{RC}}(B,\lambda)$ such that $D(b) = cc(\Phi(b))$ for all $b \in \overline{\mathcal{P}}(B,\lambda)$.*

In particular, the bijection implies the following identity

$$X_{B,\lambda}(q) := \sum_{b \in \overline{\mathcal{P}}(B,\lambda)} q^{D(b)} = \sum_{(\nu,J) \in \overline{\mathrm{RC}}(B,\lambda)} q^{cc(\nu,J)} =: M_{B,\lambda}(q). \tag{1}$$

Since the sets in (1) are finite, these are polynomials in q. When all μ_i are single row partition, these polynomials are none other than the Kostka–Foulkes polynomials.

The generating function $M_{B,\lambda}(q)$ of rigged configurations leads fermionic formulas. Fermionic formulas are explicit expressions for the partition function of the underlying physical models which reflect the particle structure. For more details regarding the background of fermionic formulas see [8, 9, 6].

On the crystal side, the highest weight crystals are only the tip of the iceberg. Underneath each highest weight crystal element sits a whole structure of elements given by a crystal graph. A natural question that arises is whether one can also extend the rigged configuration side and define a crystal structure on rigged configurations. In this talk we discussed the answer to this question based on unrestricted crystal paths $\mathcal{P}(B,\lambda)$ and an extension of rigged configuration $\mathrm{RC}(B,\lambda)$ as defined in [4]. All details will appear in a forthcoming paper [5]. These results have application to the Bailey lemma, supernomial coefficients, and box-ball systems.

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Quasi–Coxeter algebras, Dynkin diagram cohomology and quantum Weyl groups

Valerio Toledano Laredo

I will begin by reviewing the construction of a flat connection ∇ on the Cartan subalgebra of a complex, simple Lie algebra $\mathfrak{g}$ with simple poles on the root hyperplanes and values in any finite-dimensional $\mathfrak{g}$–module V [4, 5, 6]. This connection, which was obtained in joint work with J. Millson, is a generalisation of the (genus 0) Knizhnik–Zamolodchikov equations to configuration spaces of other Lie types and of Cherednik's rational Dunkl operators for the Weyl group W of $\mathfrak{g}$. Its monodromy gives a one–parameter family of representations of the generalised braid group B_W of type W deforming the action of the (Tits extension of) W on V.

I will then explain how the work of Drinfeld and Kohno on the KZ connection leads one to conjecture that the monodromy of ∇ is described by Lusztig's quantum Weyl group operators and sketch the recent proof of this conjecture [7]. One of its key ingredients is the novel notion of *quasi–Coxeter algebras*, which are to Brieskorn and Saito's Artin groups what Drinfeld's quasi–triangular, quasi–bialgebras are to the classical braid groups. Time permitting, I will motivate their definition by using De Concini and Procesi's compactifications of hyperplane complements which yields, in the case of the Coxeter arrangement of type A_{n-1}, the moduli space $\overline{\mathcal{M}}_{0,n+1}$ of stable, $n+1$–marked curves of genus zero [2, 3].

I will also describe the semi–classical analogue of this conjecture which, through the work of Boalch [1], relates the De Concini–Kac–Procesi action of B_W on the Poisson–Lie group G^* to the isomonodromic deformation of G–connections on the punctured disk having a pole of order 2 at the origin.

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Quantum Groups, the loop Grassmannian, and the Springer resolution

Victor Ginzburg

(joint work with Sergey Arkhipov, Roman Bezrukavnikov)

We establish equivalences of the following three triangulated categories:

$$D_{\text{quantum}}(\mathfrak{g}) \quad\longleftrightarrow\quad D^G_{\text{coherent}}(\widetilde{\mathcal{N}}) \quad\longleftrightarrow\quad D_{\text{perverse}}(\mathsf{Gr}).$$

Here, $D_{\text{quantum}}(\mathfrak{g})$ is the derived category of the principal block of finite dimensional representations of the quantized enveloping algebra (at an odd root of unity) of a complex semisimple Lie algebra $\mathfrak{g}$; the category $D^G_{\text{coherent}}(\widetilde{\mathcal{N}})$ is defined in terms of coherent sheaves on the cotangent bundle on the (finite dimensional) flag manifold for G (= semisimple group with Lie algebra $\mathfrak{g}$), and the category $D_{\text{perverse}}(\mathsf{Gr})$ is the derived category of perverse sheaves on the Grassmannian Gr associated with the loop group $LG^\vee$, where $G^\vee$ is the Langlands dual group, smooth along the Schubert stratification.

The equivalence between $D_{\text{quantum}}(\mathfrak{g})$ and $D^G_{\text{coherent}}(\widetilde{\mathcal{N}})$ is an 'enhancement' of the known expression (due to Ginzburg-Kumar) for quantum group cohomology in terms of nilpotent variety. The equivalence between $D_{\text{perverse}}(\mathsf{Gr})$ and $D^G_{\text{coherent}}(\widetilde{\mathcal{N}})$ can be viewed as a 'categorification' of the isomorphism between two completely different geometric realizations of the (fundamental polynomial representation of the) affine Hecke algebra that has played a key role in the proof of the Deligne-Langlands-Lusztig conjecture. One realization is in terms of locally constant functions on the flag manifold of a p-adic reductive group, while the other is in terms of equivariant K-theory of a complex (Steinberg) variety for the dual group.

The composite of the two equivalences above yields an equivalence between *abelian* categories of quantum group representations and perverse sheaves. A similar equivalence at an even root of unity can be deduced, following Lusztig program, from earlier deep results of Kazhdan-Lusztig and Kashiwara-Tanisaki. Our approach is independent of these results and is totally different (it does not rely on representation theory of Kac-Moody algebras). It also gives way to proving Humphreys' conjectures on tilting $U_q(\mathfrak{g})$-modules, as will be explained in a separate paper.

Richardson elements and birationality questions

Karin Baur
(joint work with Nolan Wallach)

Let G be a semisimple connected, simply-connected algebraic group over $\mathbb{C}$, fix $T \subset B \subset G$ a maximal torus resp. a Borel subgroup. Let P be a parabolic subgroup of G with Levi decomposition $P = M \cdot N$ (Levi factor times the corresponding unipotent radical). We denote the Lie algebras with the corresponding gothic letters (with $\mathfrak{h} = \operatorname{Lie} T$, $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$).

A fundamental theorem of Richardson states that the action of P on the Lie algebra of U has always a dense orbit, called the orbit of *Richardson elements.* In terms of the Lie algebras: there exists an element $x \in \mathfrak{n}$ such that $[\mathfrak{p}, x] = \mathfrak{n}$.

We recall that for P there exists $H \in \mathfrak{h}$ such that $ad(H)$ has integral eigenvalues on $\mathfrak{g}$, giving $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ with $\mathfrak{p} = \oplus_{j\geq 0}\mathfrak{g}_j$, $\mathfrak{m} = \mathfrak{g}_0$ and $\mathfrak{n} = \oplus_{j>0}\mathfrak{g}_j$, the corresponding nilradical.

Motivation 1: Vanishing Theorem of Lynch.

To any $x \in \mathfrak{g}_1$ corresponds a character of the opposite nilradical $\overline{\mathfrak{n}} = \oplus_{j<0}\mathfrak{g}_j$: $\psi = \psi_x : \overline{\mathfrak{n}} \to \mathbb{C}$ via $Y \mapsto iB(x, Y)$ where B is the Killing form.

Let V be any $\mathfrak{g}-$ module. We define an action of $\overline{\mathfrak{n}}$ on V by $\pi_\psi(Y)(v) = Yv - \psi(Y)v$ ($Y \in \overline{\mathfrak{n}}$, $v \in V$) and denote this $\overline{\mathfrak{n}}$-module by $V \otimes \mathbb{C}_{-\psi}$. Assume that

for each $v \in V$ there exists $k = k(v)$ such that $\pi_\psi(Y)^k v = 0$ for all $Y \in \overline{\mathfrak{n}}$. Then the following holds:
Theorem (Lynch [6]).
If x is a Richardson element we have $H^i(\overline{\mathfrak{n}}, V \otimes \mathbb{C}_{-\psi}) = 0$ for all $i > 0$.

Motivation 2: Multiplicity one for Whittaker vectors.
With subscript $\mathbb{R}$ we denote the real forms of the groups, Lie algebras. To $x \in (\mathfrak{g}_1)_\mathbb{R}$ there corresponds a unitary character $\chi : \overline{N}_\mathbb{R} \to S^1$ via $\chi(\exp Y) = e^{iB(x,Y)}$ ($Y \in \overline{\mathfrak{n}}_\mathbb{R}$). Set $I^\infty(\sigma) := \{C^\infty \text{ functions of } Ind_{P_\mathbb{R}}^{G_\mathbb{R}}(\sigma)\}$ (with the C^∞-topology) where (σ, H_σ) is a finite dimensional irreducible representation of $P_\mathbb{R}$. Let $Wh_\chi(I^\infty(\sigma))$ be the space of continuous linear maps T on $I^\infty(\sigma)$ satisfying $T(\pi(\overline{n}))f = \chi(\overline{n})^{-1}T(f)$, $Wh_\chi(I^\infty(\sigma))$ is called the space of generalized Whittaker vectors[1]. Then the following holds:
Theorem (Wallach [8]).
If the associated moment map $m_P : T^*(G/P) \to \mathfrak{g}^*$ is birational onto its image and if furthermore x is a Richardson element, we have

$$\dim Wh_\chi(I^\infty(\sigma)) = \dim H_\sigma.$$

These two results explain why it is important to understand parabolic subgroups P for which there exists a Richardson element in $\mathfrak{g}_1$ and such that $P_x = G_x$ (which is equivalent to the birationality of the induced moment map m_P onto its image).

An important tool in understanding the stabilizers G_x, P_x is a "normal form" for Richardson elements (in $\mathfrak{g}_1$). We give a Richardson element $x \in \mathfrak{g}_1$ with minimal support in R^+ (the positive roots of $\mathfrak{g}$). Minimal in the sense that *supp*(x) has as few roots as possible. E.g. if $P = B$ then *supp*(x) consists of all simple roots. For more general parabolic subalgebras, the support of a general Richardson element contains all roots of $\mathfrak{g}_1$, so is far from being minimal.

We say that the support *supp*(x) is a *simple system of roots* if for any two $\alpha, \beta \in$ *supp*(x), $\alpha - \beta$ is not in $R^+ \setminus R_M^+$, (i.e. is not a root of the subsystem given by *supp*(x)). In the optimal case, *supp*(x) is a simple system of roots.

The Heisenberg parabolic subalgebra of C_2 is an example where there exists a Richardson element x in $\mathfrak{g}_1$ (take for instance $x = x_{\alpha_1}$) but where $P_x \neq G_x$. On the other hand, any parabolic subgroup of GL_n satisfies $P_x = G_x$ (x Richardson) since in that case, G_x is connected. But the parabolic subalgebra of $\mathfrak{sl}_5$ where the simple roots of the Levi factor are α_1 and α_4 has no Richardson element in $\mathfrak{g}_1$.

A theorem of Broer says that if the simple roots of the Levi factor M are pairwise orthogonal and short, then the induced moment map is birational onto its image $G\mathfrak{n}$. Furthermore, $G\mathfrak{n}$ is normal [3].

Let us recall what is already known:

(i) There exist classifications of parabolic subalgebras in simple Lie algebras such that there is a Richardson element in $\mathfrak{g}_1$. Cf. [2], [5] and [6].
(ii) In his paper [7], Hesselink has given a formula for $|G_x/P_x|$ for nilpotent x in the classical case (in terms of the partition of x).

[1] π is the G action by right multiplication, $(\pi(g)f)(x) = f(xg)$

We now present the main class of parabolic subalgebras that have a Richardson element in $\mathfrak{g}_1$ and such that $P_x = G_x$ for Richardson elements. To do so, we introduce the following notion:

Let $\mathfrak{p} = \oplus_{j\geq 0}\mathfrak{g}_j$ be a parabolic subalgebra, where the grading on $\mathfrak{g}$ is given by $H \in \mathfrak{h}$. We say that $\mathfrak{p}$ (or P) is *given by an $\mathfrak{sl}_2$-triple* if there exists a nonzero $x \in \mathfrak{g}_1$ such that the $\mathfrak{sl}_2$-triple of x is $\{x, 2H, y\}$ (for some $y \in \mathfrak{g}$). Note that in that case, x is a Richardson element for $\mathfrak{p}$. In this case we also get the birationality of the moment map:

Theorem (cf. [7] for the classical case).

Let $P \subset G$ be a parabolic subgroup of G. If $\mathfrak{p}$ is given by an $\mathfrak{sl}_2$-triple then $G_x = P_x$.

The natural question we can ask here is the following:

Question:

If there is a Richardson element $x \in \mathfrak{g}_1$ and if $P_x = G_x$ does it follow that P is given by an $\mathfrak{sl}_2$-triple?

The answer is no for type A_n, yes for types B_n, C_n, G_2, F_4, mostly yes for D_n. Conjecturally, in case $G = E_n$, all P with a Richardson element in $\mathfrak{g}_1$ satisfy $G_x = P_x$ (x Richardson). For E_n there are such P that are not given by an $\mathfrak{sl}_2$-triple.

Normal forms of Richardson elements.

Now we describe the normal forms of Richardson elements. We start with type A_n which can already be found [4]. A parabolic subgroup P is given by a dimension vector $d_1, \ldots, d_r$ of the block lengths of the standard Levi factor. One can show that $\mathfrak{p}$ has a Richardson element in $\mathfrak{g}_1$ if and only if $d_1 \leq \cdots \leq d_s \geq \cdots \geq d_r$ (unimodality of (d)). Furthermore, $\mathfrak{p}$ is given by an $\mathfrak{sl}_2$-triple if and only if d is unimodal and $d_i = d_{r+1-i}$ for all i (symmetry of (d)). So the set of parabolic subalgebras given by an $\mathfrak{sl}_2$-triple is much smaller than the ones with a Richardson element x in $\mathfrak{g}_1$ and $P_x = G_x$ (which is no restriction in type A_n).

The first graded part $\mathfrak{g}_1$ consists of the sequence of blocks $\mathfrak{g}_{i,i+1}$ of size $d_i \times d_{i+1}$ on the first superdiagonal. We construct a Richardson element $x(d)$ by choosing an identity matrix of size $d_i \times d_i$ in the upper left corner of $\mathfrak{g}_{i,i+1}$ and zeroes else, as in the first picture below. The support of $x(d)$ forms a simple system of roots[2]. It is of the form $(GL_{\lambda_s}) \times \cdots \times (GL_{\lambda_1})$ where λ is the dual of the partition given by $\{d_1, \ldots, d_r\}$.

Here we illustrate the normal forms for Richardson element for the dimension vectors $d = (2, 3, 1)$ for $\mathfrak{sl}_6$, and $d = (3, 4, 3)$ for $\mathfrak{sp}_{10}$ resp. for $\mathfrak{so}_6$ (To define the other classical groups we use the form given by the skew diagonal matrix J_n resp. by the matrix having J_n in the upper right corner and $-J_n$ in the lower left corner for the special orthogonal resp. the symplectic group). Explicit descriptions of the

[2] i.e. if α and β are roots of the $S := \mathrm{supp}X(d)$ then $\alpha - \beta$ is not a root of S

construction can be found in [1] (also for exceptional Lie algebras).

$$\begin{pmatrix} \cdot & \cdot & 1 & 0 & 0 & \\ \cdot & \cdot & 0 & 1 & 0 & \\ & & \cdot & \cdot & \cdot & 1 \\ & & \cdot & \cdot & \cdot & 0 \\ & & \cdot & \cdot & \cdot & 0 \\ & & & & & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & 1 & & & & & & \\ \cdot & \cdot & \cdot & & & 1 & 0 & & & \\ \cdot & \cdot & \cdot & & & 0 & 1 & & & \\ & & & \cdot & \cdot & \cdot & \cdot & -1 & 0 & \\ & & & \cdot & \cdot & \cdot & \cdot & 0 & -1 & \\ & & & \cdot & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & \cdot & & & -1 \\ & & & & & & & \cdot & \cdot & \cdot \\ & & & & & & & \cdot & \cdot & \cdot \\ & & & & & & & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & 1 & 0 & 0 & 1 & \\ & \cdot & \cdot & \cdot & \cdot & -1 \\ & \cdot & \cdot & \cdot & \cdot & 0 \\ & \cdot & \cdot & \cdot & \cdot & 0 \\ & \cdot & \cdot & \cdot & \cdot & -1 \\ & & & & & \cdot \end{pmatrix}$$

These examples illustrate the phenomenon that roughly speaking for the symplectic Lie algebras, the Richardson element uses small identity matrices of size $\lfloor d_i/2 \rfloor$ resp. $\lceil d_i/2 \rceil$ in the upper left resp. lower right corner. For the orthogonal Lie algebras, they are often both of size $\lceil d_i/2 \rceil$.

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Tensor product structure of affine Demazure modules and limit constructions

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(joint work with Peter Littelmann)

Let $\mathfrak{g}$ be a simple complex Lie algebra, we denote by $\widehat{\mathfrak{g}}$ the untwisted affine Kac–Moody algebra associated to the extended Dynkin diagram of $\mathfrak{g}$. Let Λ_0 be the fundamental weight of $\widehat{\mathfrak{g}}$ corresponding to the additional node of the extended Dynkin diagram. Let $P^\vee$ be the coweight lattice of $\mathfrak{g}$. An element $\lambda^\vee$ in the coroot lattice, can be viewed as an element of the affine Weyl group W^{aff}, and one can associate to $\lambda^\vee$ the Demazure submodule $V_{\lambda^\vee}(\Lambda)$ of $V(\Lambda)$, Λ dominant, integral for Lhg. Actually, this construction generalizes to arbitrary $\lambda^\vee \in P^\vee$ in the following way: one can write $\lambda^\vee$ as $w\sigma \in \widetilde{W}^{\text{aff}}$ in the extended affine Weyl group, where $w \in W^{\text{aff}}$ and σ corresponds to an automorphism of the Dynkin diagram of $\widehat{\mathfrak{g}}$. Denote by $V_{\lambda^\vee}(\Lambda)$ the Demazure submodule $V_w(\sigma\Lambda)$.

If $\lambda^\vee$ is a dominant coweight, then the Demazure module $V_{-\lambda^\vee}(m\Lambda_0)$ is in fact a $\mathfrak{g}$–module, and it is interesting to study its structure as $\mathfrak{g}$–module. We write $\overline{V}_{-\lambda^\vee}(m\Lambda_0)$ for the Demazure module viewed as a $\mathfrak{g}$–module.

A first reduction step is the following theorem describing the Demazure module as a tensor product. Such a decomposition formula for Demazure modules was first observed by Sanderson in the affine rank two case. We provide a description of the Demazure module as a tensor product of modules of the same type, but for "smaller coweights". More precisely, let $\lambda^\vee$ be a dominant coweight and suppose we are given a decomposition $\lambda^\vee = \lambda_1^\vee + \lambda_2^\vee + \ldots + \lambda_r^\vee$ of $\lambda^\vee$ as a sum of dominant coweights. The theorem is a generalization of a result by Magyar, who proved this statement in the case $m = 1$ and under the additional assumption that all the $\lambda_i^\vee$ are minuscule fundamental weights, and the decomposition formulas by the Kyoto school where in the framework of perfect crystals for classical groups many cases have been discussed.

Theorem 1. *For all $m \geq 1$, we have an isomorphism of $\mathfrak{g}$–representations between the Demazure module $\overline{V}_{-\lambda^\vee}(m\Lambda_0)$ and the tensor product of Demazure modules:*

$$\overline{V}_{-\lambda^\vee}(m\Lambda_0) \simeq \overline{V}_{-\lambda_1^\vee}(m\Lambda_0) \otimes \overline{V}_{-\lambda_2^\vee}(m\Lambda_0) \otimes \cdots \otimes \overline{V}_{-\lambda_r^\vee}(m\Lambda_0).$$

Of course, to analyse the structure of $\overline{V}_{-\lambda^\vee}(m\Lambda_0)$ as a $\mathfrak{g}$–module, the simplest way is to take a decomposition of $\lambda^\vee$ as a sum of fundamental coweights $\lambda^\vee = \sum b_i\omega_i^\vee$. So by Theorem 1, it remains to describe the structure of the $\overline{V}_{-\omega_i^\vee}(m\Lambda_0)$ as a $\mathfrak{g}$–module. We give such a description in our article for all fundamental coweights for the classical groups. For the exceptional groups we give the decomposition in the cases interesting for the limit constructions considered later.

There is a very interesting conjectural connection with certain $U_q'(\widehat{\mathfrak{g}})$–modules, called Kirillov-Reshetikhin modules. Here $U_q'(\widehat{\mathfrak{g}})$ denotes the quantized affine algebra without derivation.

Let $c_k^\vee = \frac{a_k}{a_k^\vee}$ and $l \in \mathbb{N}$. Let $W(lc_k^\vee\omega_k)$ be the Kirillov-Reshitikhin–module associated to the weight $lc_k^\vee\omega_k$. It is conjectured that the $W(lc_k^\vee\omega_k)$ admits a crystal bases (moreover it is a perfect crystal) and that the crystal is isomorphic to the crystal of a Demazure module, after omitting the 0–arrows in both crystals.

Chari, Kleber and the Kyoto school have calculated for classical Lie-algebras and some fundamental coweights for non-classical Lie-algebras the decomposition of the Kirillov–Reshetikhin module $W(lc_k^\vee\omega_k))$ into irreducible $U_q(\mathfrak{g})$–modules. By comparing the $U_q(\mathfrak{g})$–structure of the Kirillov–Reshetikhin module $W(lc_k^\vee\omega_k))$ with our list of Demazure modules, we conclude:

Corollary 2. *In classical case (and for the non-classical case calculated by ourselves) the Demazure module $(\overline{V}_{-\omega^\vee,q}(m\Lambda_0))$ and the Kirillov–Reshetikhin module $W(lc_k^\vee\omega_k{}^*)$ are, as $U_q(\mathfrak{g})$–modules, isomorphic.*

Let Λ be an arbitrary dominant integral weight for $\widehat{\mathfrak{g}}$. The $\widehat{\mathfrak{g}}$–module $V(\Lambda)$ is the direct limit of the Demazure-modules $V_{-N\lambda^\vee}(\Lambda)$ for some dominant, integral, nonzero coweight of $\mathfrak{g}$. We give a construction of the $\mathfrak{g}$–module $\overline{V}(\Lambda)$ as a direct limit of tensor products of Demazure modules. This has been done before in the case of classical Lie-algebras by the Kyoto school via the theory of perfect crystals. In addition they have also considered some special weights in the case of non-classical groups. For $\mathtt{G}_2$, such a construction has been given by Yamane. For the Lie algebras of type $\mathtt{E}_6$ and $\mathtt{E}_7$ a construction (only for the case $\Lambda = \Lambda_0$) was given by Magyar using the path model.

We provide such a direct limit construction for arbitrary simple Lie algebras $\mathfrak{g}$. Let Λ be a dominant, integral weight for $\widehat{\mathfrak{g}}$, then we can write $\Lambda = r\Lambda_0 + \lambda$ with λ dominant, integral for $\mathfrak{g}$.

Let W be the $\mathfrak{g}$-module $W := \overline{V}_{-\theta^\vee}(r\Lambda_0)$, where θ is the highest root of $\mathfrak{g}$, we showed that W contains a unique one-dimensional submodule. Fix $w \neq 0$ a $\mathfrak{g}$-invariant vector in W. Let $V(\lambda)$ be the irreducible $\mathfrak{g}$-module with highest weight λ and define the $\mathfrak{g}$-module $V^\infty_{\lambda,r}$ to be the direct limit of:

$$V^\infty_{\lambda,r}: \quad V(\lambda) \hookrightarrow W \otimes V(\lambda) \hookrightarrow W \otimes W \otimes V(\lambda) \hookrightarrow W \otimes W \otimes W \otimes V(\lambda) \hookrightarrow \dots$$

where the inclusions are always given by taking a vector u to its tensor product $u \mapsto w \otimes u$ with the fixed $\mathfrak{g}$-invariant vector in W.

Theorem 3. *For any integral dominant weight Λ of $\widehat{\mathfrak{g}}$, $\Lambda = r\Lambda_0 + \lambda$, the $\mathfrak{g}$-modules $V^\infty_{\lambda,r}$ and $\overline{V}(\Lambda)$ are isomorphic.*

Remark 1. The choice of W is convenient because it avoids case by case considerations. But, in fact, one could choose any other module $W = V_{-\mu^\vee}(r\Lambda_0)^{\otimes m}$, where $V_{-\mu^\vee}(r\Lambda_0)$ is the Demazure module for a dominant integral coweight $\mu^\vee$ and m is such that $V_{-\mu^\vee}(r\Lambda_0)^{\otimes m}$ contains a one-dimensional submodule.

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A note on characters of tilting modules

Oleksandr Khomenko

Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$ and let G be a semi-simple split connected simply connected algebraic group over $\mathbb{K}$. Fix a maximal torus T in G and a Borel subgroup $B \subset G$ containing T. Let $\mathcal{R} \subset \mathcal{R}^+ = -\mathcal{R}(B,T)$ be the root system of G and the subset of positive roots, and let h be the coxeter number of G.

An important topic in the representation theory of algebraic groups is the study of simple modules $L(\lambda)$ over split semi-simple reductive algebraic groups over fields of prime characteristic. A formula (or rather an algorithm) for computing the above multiplicities was conjectured by Lusztig. In [1] Lusztig's conjecture is proved for $p \gg 0$. However at the moment, it is not known whether Lusztig conjecture holds in a given characteristic or not.

An important special case of the discussed problem was considered by Soergel in [2]. He introduced a category $\mathcal{O}$ which is a subquotient category of G-mod, so called "regular subquotient around the Steinberg point". This category carries the information about the multiplicities $[H^0(\mathrm{st} + x\rho) : L(\mathrm{st} + y\rho)]$ for x, $y \in W$ where W is the Weyl group of G, ρ is half the sum of positive roots of $\mathcal{R}$, and $\mathrm{st} = (p-1)\rho$ is the Steinberg weight. If $p > h$ then the validity of Lusztig conjecture over $\mathbb{K}$ in this case follows from the decomposition theorem for certain intersection cohomology sheaves on flag varieties with coefficients in $\mathbb{K}$ (at the moment this theorem is available only for $\mathrm{char}\,\mathbb{K} \gg 0$).

Soergel's construction can not be directly applied to study simple and tilting modules which are "near" the wall of the dominant chamber. Here we report on the further investigation of Soergel's "regular subquotient" $\mathcal{O}$ in a slightly more general context. First, we enable taking subquotients around arbitrary point $\lambda \in X(T)$ (in particular on the walls of the dominant chamber). Second, we study not only rational representations of G but also representations of Frobenius kernels G_r. Following Soergel we define subquotient categories $\mathcal{O}^{\mathfrak{A},\lambda}$ for $\lambda \in X(T)$ and $\mathfrak{A}$ is either the algebra of distributions $\mathrm{Dist}(G)$ on G or on $\mathrm{Dist}(G_r)$ supported in the neutral element. First we present the following

Theorem 1. *Assume that $p > 2h$ is such that the decomposition theorem holds for intersection cohomology sheaves with coefficients in $\mathbb{K}$ on complex Schubert varieties associated to Langland dual group of G. Let $\lambda \in X(T)$ be such that $\lambda + \rho$ is dominant, $W_s = \mathrm{Stab}_{W_p} \lambda$, and $W_\mathfrak{p} = \{w \in W_s \mid w(\lambda + \rho)$ is non-dominant$\}$. Then the Grothendieck group of the category $\mathcal{O}^{\mathrm{Dist}(G),\lambda}$ is equal to the Grothendieck group of the regular block in the parabolic subcategory $\mathcal{O}_\mathfrak{p}$ of Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for a semi-simple complex finite dimensional Lie algebra $\mathfrak{f}$ with Weyl group W_s and parabolic subalgebra $\mathfrak{p} \subset \mathfrak{f}$ corresponding to $W_\mathfrak{p} \subset W_s$.*

The above theorem provides some information about characters of tilting modules for G. Let $q \in \mathbb{C}$ be a primitive p-th root of 1 and U_q be the quantum group associated to $\mathcal{L}ie(G)$.

Theorem 2. *Let p be as in Theorem 1, and $\lambda \in X(T)$ be dominant. The character of the tilting G-module with highest weight λ coincides with the character of the tilting U_q-module with highest weight λ modulo the characters of tilting U_q-modules with highest weights in facette whose closure do not intersect the closure of the facette containing λ.*

Note that the characters of tilting U_q-modules are known by [1, 3].

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An application of free Lie algebras to current algebras

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(joint work with Vyjayanthi Chari)

Let $\mathfrak{a}$ be a Lie algebra satisfying $\mathfrak{a} = [\mathfrak{a}, \mathfrak{a}]$. Then $\mathfrak{a}$ acts naturally on its free Lie algebra $F(\mathfrak{a})$. Let $\mathfrak{F} = \mathfrak{a} \oplus F(\mathfrak{a})$ be the semi-direct product of Lie algebras corresponding to that action. We use a "graded version" of the canonical Lie algebra homomorphism $\tau : F(\mathfrak{a}) \to \mathfrak{a}$ to construct a surjective homomorphism $\tau[t]$ of graded Lie algebras of $\mathfrak{F}$ onto the current algebra $\mathfrak{a}[t] = \mathfrak{a} \otimes \mathbb{C}[t]$, thus obtaining a new realization of the current algebra.

We use this realization to describe the set of isomorphism classes of $\mathfrak{a}[t]$-module structures extending a given $\mathfrak{a}$-module structure. On the other hand, this realization allows one to associate to an $\mathfrak{a}[t]$-module V a two-sided ideal $\mathbf{I}(V)$ in the tensor algebra $T(\mathfrak{a})$, which is invariant under the natural action of $\mathfrak{a}$ on $T(\mathfrak{a})$ and contains the kernel of $\tau[t]$. Conversely, such an ideal $\mathbf{I}$ corresponds to a canonical $\mathfrak{a}[t]$-module structure on $T(\mathfrak{a})/\mathbf{I}$ arising from the left multiplication.

Similarly, any $\mathfrak{a}[t]$-module M satisfying $(x \otimes t^2)M = 0$ for all $x \in \mathfrak{a}$ gives rise to an $\mathfrak{a}$-invariant ideal $I(M)$ in the symmetric algebra $S(\mathfrak{a})$ and to any $\mathfrak{a}$-invariant ideal I in $S(\mathfrak{a})$ corresponds a canonical $\mathfrak{a} \otimes \mathbb{C}[t]/(t^2)$-module structure on $S(\mathfrak{a})/I$ arising from multiplication. In particular, for $\mathfrak{a} = \mathfrak{g}$ simple and finite dimensional, this shows that the ring of regular functions on a G-invariant variety in $\mathfrak{g}^*$ always admits a $\mathfrak{g}[t]$-module structure. Since classical limits of Kirillov-Reshetikhin $\mathfrak{g}[t]$-modules ([2]) are actually $\mathfrak{g} \otimes \mathbb{C}[t]/(t^2)$-modules for $\mathfrak{g}$ of classical types ([1]), they can be obtained using our construction. Also, for any self-dual $\mathfrak{g}$-module V, a construction of Kostant ([3]) gives a homomorphism of $\mathfrak{g}$-modules $\mathfrak{g} \to \bigwedge^2 V$, which allows us to define a $\mathfrak{g}[t]$-module structure on $\bigwedge^* V$. Classical limits of fundamental Kirillov-Reshetikhin modules for $\mathfrak{g}$ of type D_n can be obtained this way.

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Equivariant sheaves and Lie superalgebras

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(joint work with M. Duflo)

We suggest a notion of associated variety for a module over a Lie superalgebra. This is a superanalogue of an associated variety for Harish-Chandra modules. Associated varieties have many interesting applications in classical representation theory (see, for example, [2, 3, 4]).

An associated variety for a Lie superalgebra is a subvariety of a cone $X \subset \mathfrak{g}_1$ of self-commuting odd elements. This cone X was studied by Caroline Gruson, see [5, 1, 6]. She used geometric properties of X to obtain important results about cohomology of Lie superalgebras.

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite-dimensional Lie superalgebra, G_0 denote a simply-connected connected Lie group with Lie algebra $\mathfrak{g}_0$. Let

$$X = \{x \in \mathfrak{g}_1 \mid [x, x] = 0\}.$$

It is clear that X is a G_0-invariant Zariski closed cone in $\mathfrak{g}_1$. Let M be a $\mathfrak{g}$-module. For each $x \in X$ put $M_x = \operatorname{Ker} x / xM$ and define

$$X_M = \{x \in X \mid M_x \neq 0\}.$$

We call X_M the *associated variety* of M. If M is a finite-dimensional $\mathfrak{g}$-module, then X_M is a Zariski closed G_0-invariant subvariety. The following properties of X_M are easy to check

(1) If $M = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M_0$ for some $\mathfrak{g}_0$-module M_0, then $X_M = \{0\}$;
(2) If M is trivial, then $X_M = X$;
(3) For any $\mathfrak{g}$-modules M and N, one has $X_{M \oplus N} = X_M \cup X_N$;
(4) For any $\mathfrak{g}$-modules M and N, one has $X_{M \otimes N} \subset X_M$;
(5) For any finite-dimensional $\mathfrak{g}$-module M, $X_{M^*} = X_M$;
(6) For any $x \in X$, sdim M =sdim M_x.

Let $\mathcal{O}_X$ denote the structure sheaf of X. Then $\mathcal{O}_X \otimes M$ is the sheaf of sections of a trivial vector bundle with fiber isomorphic to M. Let $\partial : \mathcal{O}_X \otimes M \to \mathcal{O}_X \otimes M$ be the map defined by

$$\partial\varphi(x) = x\varphi(x)$$

for any $x \in X$, $\varphi \in \mathcal{O}_X \otimes M$. Clearly $\partial^2 = 0$ and the cohomology $\mathcal{M}$ of ∂ is a quasi-coherent sheaf on X. If M is finite-dimensional, then $\mathcal{M}$ is coherent.

For any $x \in X$ denote by $\mathcal{O}_x$ the local ring of x, by $\mathcal{I}_x$ the maximal ideal. Then the fiber $\mathcal{M}_x$ is the the cohomology of $\partial : \mathcal{O}_x \otimes M \to \mathcal{O}_x \otimes M$. The evaluation map $j_x : \mathcal{O}_x \otimes M \to M$ satisfies $j_x \circ \partial = x \circ j_x$. Hence we have the maps

$$j_x : \operatorname{Ker} \partial \to \operatorname{Ker} x,\ j_x : \operatorname{Im} \partial \to xM.$$

One can easily check that the latter map is surjective. Therefore j_x induces the map $\bar{j}_x : \mathcal{M}_x \to M_x$, and $\operatorname{Im} \bar{j}_x \cong \mathcal{M}_x / \mathcal{I}_x \mathcal{M}_x$.

Lemma 1. *Let M be a finite-dimensional $\mathfrak{g}$-module. The support of $\mathcal{M}$ is contained in X_M. The map $\bar{j}_x$ is surjective for a generic point $x \in X$. In particular, if $X_M = X$, then* $\operatorname{supp} \mathcal{M} = X$.

Corollary 1. *Let $x \in X$ be a generic point, then in some neighborhood U of x, the sheaf $\mathcal{M}_U$ coincides with the sheaf of sections of a vector bundle with fiber M_x.*

Let $\mathcal{F}$ be the category of finite-dimensional $\mathfrak{g}$-modules semisimple over $\mathfrak{g}_0$. The latter condition is automatic if $\mathfrak{g}_0$ is semisimple.

Theorem 1. *Assume that $\mathfrak{g}_0$ is a reductive Lie algebra and elements of X span $\mathfrak{g}_1$. Then $M \in \mathcal{F}$ is projective iff $X_M = \{0\}$.*

Let $\mathfrak{g}$ be a contragredient finite-dimensional Lie superalgebra with indecomposable Cartan matrix, i.e. $\mathfrak{g}$ is isomorphic to one from the following list: $\mathfrak{sl}(m|n)$ if $m \neq n$, $\mathfrak{gl}(n|n)$, $\mathfrak{osp}(m|2n)$, $D(\alpha)$, F_4 or G_3. Let S denote the family of sets of linearly independent mutually orthogonal isotropic roots of $\mathfrak{g}$ and S_k denote the family of such k-element subsets. The Weyl group W acts on S and S_k in the natural way.

Theorem 2. *There are finitely many G_0-orbits on X. These orbits are in one-to one correspondence with W-orbits in S.*

An element $x \in X$ *has rank* k if its G_0-orbit corresponds to a W-orbit in S_k.

The maximal number of isotropic mutually orthogonal linearly independent roots is called the *defect* of $\mathfrak{g}$. This notion was introduced in [7]. One can see that the defect of $\mathfrak{g}$ is equal to the dimension of maximal isotropic subspace in $\mathfrak{h}^*$. All exceptional Lie superalgebras has defect 1. The defect of $\mathfrak{sl}(m|n)$ is $\min(m,n)$, the defect $\mathfrak{osp}(2l+1|2n)$ and $\mathfrak{osp}(2l|2n)$ is $\min(l,n)$. One can check that the rank of any element $x \in X$ is not greater than the defect of $\mathfrak{g}$.

Theorem 3. *Let d be the defect of $\mathfrak{g}$. Then the irreducible components of X are in bijection with W-orbits on S_d. If all odd roots of $\mathfrak{g}$ are isotropic, then the dimension of each component equals $\frac{\dim \mathfrak{g}_1}{2} = \frac{|\Delta_1|}{2}$.*

Let Z denote the center of the universal enveloping algebra $U(\mathfrak{g})$. Define the Harish-Chandra homomorphism as for Lie algebras. Then every weight $\lambda \in \mathfrak{h}^*$ defines a central character $\chi_\lambda : Z \to \mathbb{C}$. *The degree of atypicality* of $\chi = \chi_\lambda$ is k if the maximal set of isotropic linearly independent mutually orthogonal roots orthogonal to $\lambda + \rho$ has k elements. (Here ρ is the half sum of positive even roots minus the half-sum of positive odd roots.)

Theorem 4. *Let $\mathfrak{g}$ be a contragredient simple Lie superalgebra, M be a $\mathfrak{g}$-module which admits a central character χ, the degree of atypicality of χ be equal to k. Let X_k be the set of all elements of rank k. Then $X_M \subset \bar{X}_k$.*

Theorem 5. *Let $\mathfrak{g} = \mathfrak{gl}(m|n)$ or $\mathfrak{sl}(m|n)$, M be an irreducible finite-dimensional $\mathfrak{g}$-module which has a central character with degree of atypicality k. Then $X_M = \bar{X}_k$.*

Now we discuss properties of the fiber M_x over a point $x \in X_M$. Let $C_{\mathfrak{g}}(x)$ be the centralizer of $x \in X$, then by definition $\mathfrak{g}_x = C_{\mathfrak{g}}(x) / [x, \mathfrak{g}]$.

Lemma 2. *The subspace $[x, \mathfrak{g}]$ is an ideal in $C_{\mathfrak{g}}(x)$. Let $\mathfrak{m}^{\perp}$ denote the orthogonal complement to $\mathfrak{m}$ with respect to the invariant form on $\mathfrak{g}$. Then $[x, \mathfrak{g}]^{\perp} = C_{\mathfrak{g}}(x)$.*

Lemma 3. *M_x is a $C_{\mathfrak{g}}(x)$-module trivial over $[x, \mathfrak{g}]$.*

Theorem 6. *Let M admit a central character with degree of atypicality k, and $x \in X_k$. Then $\mathfrak{g}_x$-module M_x admits a typical central character. In particular, if M_x is finite dimensional, it is semi-simple over $\mathfrak{g}_x$, and therefore over $C_{\mathfrak{g}}(x)$.*

The properties of M_x allow one to say something about the superdimension and supercharacter of M. First, observe that sdim M_x =sdim M. Therefore

Lemma 4. *If $X_M \neq X$, then sdim $M = 0$. In particular, if a finite-dimensional module M admits a central character whose degree of atypicality is less than the defect of $\mathfrak{g}$, then sdim $M = 0$.*

Now let M be a finite-dimensional $\mathfrak{g}$-module and $h \in \mathfrak{h}$. Write

$$\mathrm{ch}_M(h) = \mathrm{str}_M\left(e^h\right).$$

Obviously, ch_M is a W-invariant analytic function on $\mathfrak{h}$. We can write the Taylor series for ch_M at $h = 0$

$$\mathrm{ch}_M(h) = \sum_{i=0}^{\infty} p_i(h),$$

where $p_i(h)$ is a homogeneous polynomial of degree i on $\mathfrak{h}$. The order of ch_M at zero is by definition the minimal i such that $p_i \not\equiv 0$.

Theorem 7. *Assume that all odd roots of $\mathfrak{g}$ are isotropic. Let M be a finite-dimensional $\mathfrak{g}$-module, s be the codimension of X_M in X. The order of ch_M at zero is greater or equal than s. Moreover, the polynomial $p_s(h)$ in the Taylor series for ch_M is determined uniquely up to proportionality.*

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Elliptic curves and canonical bases

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(joint work with I. Burban)

One of the oldest and still most fundamental objects in representation theory and combinatorics is the ring of symmetric (Laurent) polynomials

$$\Lambda = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots]^{\mathfrak{S}_\infty} := \underset{\longleftarrow}{\mathrm{Lim}}\, \mathbb{C}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]^{\mathfrak{S}_r}.$$

This ring is well-known to admit numerous algebraic and geometric realizations, but one of the historically first constructions, which dates back to the work of Steinitz in 1900, later completed by P. Hall, is in terms of what is now called the classical Hall algebra $\mathbf{H}$ (see [Mac] , Chapter III). This algebra has a defining basis consisting of isomorphism classes of abelian q-groups (for a fixed prime power q), and the structure constants are defined by counting extensions between such abelian groups. In fact, these structure constants are polynomials in q, and we may thus consider $\mathbf{H}$ as a $\mathbb{C}[q^{\pm 1}]$-algebra. The main theorem of Steinitz and Hall provides an isomorphism $\mathbf{H} \simeq \Lambda_q^+ = \mathbb{C}[q^{\pm 1}][x_1, x_2, \ldots]^{\mathfrak{S}_\infty}$. Under this isomorphism, the natural basis of $\mathbf{H}$ (resp. the natural scalar product) is mapped to the basis of Hall-Littlewood polynomials (resp. the Hall-Littlewood scalar product). In addition, Zelevinsky [Z] endowed Λ_q^+ with a structure of a (cocommutative) Hopf algebra, and the whole algebra $\Lambda_q = \Lambda \otimes \mathbb{C}[q^{\pm 1}]$ may be recovered from Λ_q^+ by the Drinfeld double procedure. The corresponding Hopf algebra structure is also naturally intrinsically defined in the Hall algebra.

The definition of the Hall algebra admits a geometric version, in which one considers an algebra $\mathbf{U}$ of $GL(r)$-equivariant perverse sheaves on the nilpotent cones $\mathcal{N}_r \subset \mathfrak{gl}(r)$, for all r. The algebras $\mathbf{U}$ and $\mathbf{H}$ are then related by Grothendieck's faisceaux-fonctions correspondence. This construction has the advantage of yielding, by application of the above correspondence to simple perverse sheaves, a "canonical basis" of $\mathbf{H}$ satisfying some strong positivity and integrality properties. This "canonical basis" is in fact none other than the basis of Schur functions.

The aim of the present work is to initiate a similar approach for the ring of *diagonal* symmetric polynomials

$$R = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, y_1^{\pm 1} y_2^{\pm 1}, \dots]^{\mathfrak{S}_\infty} := \varprojlim \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, y_1^{\pm 1}, y_2^{\pm 1} \dots]^{\mathfrak{S}_r},$$

(where $\mathfrak{S}_\infty$ acts simultaneously on the variables x_i and y_i) based on the category of coherent sheaves on an elliptic curve.

The category of abelian q-groups is equivalent to the category of nilpotent representations over the finite field $\mathbb{F}_q$ of the Jordan quiver, i.e the quiver with one vertex and one loop. More generally, Ringel [R] considered the Hall algebra of an arbitrary quiver $\vec{Q}$ and showed that it contained the positive part $U_q^+(\mathfrak{g})$ of the quantized enveloping algebra of the Kac-Moody algebra associated to $\vec{Q}$. In a similar direction, Kapranov [Ka1] studied a natural subalgebra $\mathbf{U}_{\mathcal{E}}^+$ of the Hall algebra $\mathbf{H}_{\mathcal{E}}$ of the category $Coh(\mathcal{E})$, for a smooth projective curve $\mathcal{E}$ defined over a finite field. When $\mathcal{E} = \mathbb{P}^1$, the algebra $\mathbf{U}_{\mathcal{E}}^+$ is isomorphic to the positive part of the quantum loop algebra $U_q(L\mathfrak{sl}_2)$, and in higher genus, Kapranov defined a surjective map from a certain algebra $\mathfrak{U}_{\mathcal{E}}^+$ (defined by generators and relations) to $\mathbf{U}_{\mathcal{E}}^+$. Unfortunately, this map has a large kernel, and it is not known how to describe it explicitly.

We study the Hall algebra $\mathbf{U}_{\mathcal{E}}^+$ in details when $\mathcal{E}$ is an elliptic curve, defined over $\mathbb{F}_q$. We show that the structure constants for this algebra are polynomials in $q^{\pm 1}$ and $\tau^{\pm 1}$ where τ, τ^{-1} are the Frobenius eigenvalues on $H^1(\overline{\mathcal{E}}, \overline{\mathbb{Q}_l})$, and hence it may be considered as a $\mathbb{C}[q^{\pm 1}, \tau^{\pm 1}]$-algebra. It turns out that $\mathbf{U}_{\mathcal{E}}^+$ is a (flat) deformation of the ring of *diagonal* symmetric functions

$$R^+ := \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, y_1, y_2, \dots]^{\mathfrak{S}_\infty}.$$

This ring has recently attracted a lot of attention, due in particular to its intimate relation to Macdonald polynomials and double affine Hecke algebras. We describe explictly this deformed (Hopf) algebra by generators and relations. To obtain a more symmetric and canonical object, we consider the Drinfeld double $\mathbf{U}_{\mathcal{E}}$ of $\mathbf{U}_{\mathcal{E}}^+$, which is now a deformation of $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, y_1^{\pm 1} y_2^{\pm 1}, \dots]^{\mathfrak{S}_\infty}$. This deformation is neither commutative, nor cocommutative. However, we prove that the group of autoequivalences of the derived category $D^b(Coh(\mathcal{E}))$ naturally acts by *algebra* automorphisms of $\mathbf{U}_{\mathcal{E}}$, yielding an action of the universal cover $\widetilde{SL}(2, \mathbb{Z})$ of $SL(2, \mathbb{Z})$ on $\mathbf{U}_{\mathcal{E}}$. In addition, there is a natural "monomial" basis of $\mathbf{U}_{\mathcal{E}}^+$ (resp. of $\mathbf{U}_{\mathcal{E}}$) indexed by the collection of (finite) *convex* paths in the region $(\mathbb{Z}^2)^+ = \{(p, q) \in \mathbb{Z}^2 \mid p \geq 1 \text{ or } p = 0, q \geq 1\}$ (resp. in $\mathbb{Z}^2$). In fact, there is one such basis of $\mathbf{U}_{\mathcal{E}}$ for any line L in $\mathbb{Z}^2$, and these bases are interchanged by the $\widetilde{SL}(2, \mathbb{Z})$-action.

In addition to determining the algebra structure of the Hall algebra $\mathbf{U}_{\mathcal{E}}^+$, it is natural to seek a geometric version of $\mathbf{U}_{\mathcal{E}}^+$. We are thus led to consider an algebra $\mathbf{A}_{\mathcal{E}}^+$ of perverse sheaves on the moduli space (stack) $Coh(\mathcal{E})$ parametrizing all coherent sheaves on $\mathcal{E}$. We describe in details all simple perverse sheaves occuring in this construction (which is reminiscent of Lusztig's construction in the context of quiver representations [L]), and prove that the faisceaux-fonction correspondence relates the algebras $\mathbf{A}^+_{\mathcal{E}}$ and $\mathbf{U}^+_{\mathcal{E}}$. In particular, this provides us with a canonical basis for the algebra $\mathbf{U}^+_{\mathcal{E}}$.

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Periods, Subconvexity of L-functions and Representation Theory

Joseph Bernstein

(joint work with Andre Reznikov)

Let $\mathbb{H}$ denote the upper half plane equipped with the standard Riemannian metric of constant curvature -1. We denote by dv the associated volume element and by Δ the corresponding Laplace-Beltrami operator on $\mathbb{H}$.

Fix a discrete group Γ of motions of $\mathbb{H}$ and consider the Riemann surface $Y = \Gamma\backslash\mathbb{H}$. For simplicity we assume that Y is compact.

Consider the spectral decomposition of the operator Δ in the space $L^2(Y, dv)$ of functions on Y. It is known that the operator Δ is non-negative and has purely discrete spectrum; we will denote by $0 = \mu_0 < \mu_1 \leq \mu_2 \leq ...$ the eigenvalues of Δ. For these eigenvalues we always use a natural from representation-theoretic point of view parameterization $\mu_i = \frac{1-\lambda_i^2}{4}$, where $\lambda_i \in \mathbb{C}$. We denote by $\phi_i = \phi_{\lambda_i}$ the corresponding eigenfunctions (normalized to have L^2-norm one).

In the theory of automorphic forms, the functions ϕ_{λ_i} are called automorphic functions or *Maass forms* (after H. Maass). The study of Maass forms plays an important role in analytic number theory.

Triple products. In my talk I consider the following problem. We fix two Maass forms $\phi = \phi_\tau$ and $\phi' = \phi_{\tau'}$ as above and consider the coefficients defined by the triple period:

$$c_i = \int_Y \phi\phi'\phi_i dv,$$

as the ϕ_i run over an orthonormal basis of Maass forms. We would like to estimate the coefficients c_i in terms of parameters $|\lambda_i|$ when i tends to ∞.

Let me explain why this problem is interesting. The bounds on the coefficient c_i are related to bounds on automorphic L-functions. Namely, in many cases it is proven that

$$|c_i|^2 = G(\lambda_i) \cdot L(\lambda_i),$$

where G is some rational expression in ordinary Euler Γ-functions and L is essentially the value at the critical point 1/2 of the triple automorphic L-function $L(s; \phi_\tau \otimes \phi_{\tau'} \otimes \phi_i)$ (see details in [1]). Thus the estimates of periods c_i lead to estimates of L-functions (this idea comes from Selberg's work [2]). Since these L-functions play absolutely central role in number theory any new method of analyzing them is very interesting.

The first non-trivial observation is that the coefficients c_i have exponential decay in $|\lambda_i|$ as $i \to \infty$. Namely, as we have shown in [1], it is natural to introduce normalized coefficients

$$d_i = \gamma(\lambda_i)|c_i|^2$$

Here $\gamma(\lambda)$ is given by an explicit rational expression in terms of the standard Euler Γ-function (see [1]) and, for purely imaginary λ, it has an asymptotic $\gamma(\lambda) \sim \beta|\lambda|^2 \exp(\frac{\pi}{2}|\lambda|)$ when $|\lambda| \to \infty$ with some explicit $\beta > 0$. It turns out that the normalized coefficients d_i have at most *polynomial growth* in $|\lambda_i|$, and hence the coefficients c_i decay exponentially. This is consistent with the general experience from the analytic theory of automorphic L-functions.

In fact, in [1] we proved the following mean value bound

$$\sum_{|\lambda_i| \le T} d_i \le AT^2$$

According to Weyl's law the number of terms in this sum is of order CT^2. So this formula says that on average the coefficients d_i are bounded by some constant.

More precisely, let us fix an interval $I \subset \mathbb{R}$ around point T and consider the finite set of all Maass forms ϕ_i with parameter $|\lambda_i|$ inside this interval. Then the average value of coefficients d_i in this set is bounded by a constant *provided* the interval I is long enough (i.e., of the size $\approx T$).

Note that the best individual bound which we can get from this formula is $d_i \le A|\lambda_i|^2$. For Hecke-Maass forms this bound corresponds to the convexity bound for the corresponding L-function.

In my talk I outline the proof of the following bound.

Theorem 1. *There exist effectively computable constants $B,\ b > 0$ such that, for an arbitrary $T > 1$ we have the following bound*

$$\sum_{|\lambda_i| \in I_T} d_i \le BT^{5/3} ,$$

where I_T is the interval of size $bT^{1/3}$ centered at T.

Note that this theorem gives an individual bound $d_i \le B|\lambda_i|^{5/3}$ (for $|\lambda_i| > 1$). This leads to the following *subconvexity* bound for the triple L-function

Corollary 2. *Let ϕ and ϕ' be fixed Hecke-Maass cusp forms. For any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that the bound*

$$L(\tfrac{1}{2}\ , \phi \otimes \phi' \otimes \phi_{\lambda_i}) \le C_\epsilon |\lambda_i|^{5/3+\epsilon}$$

holds for any Hecke-Maass form ϕ_{λ_i}.

The convexity bound for the triple L-function corresponds to similar bound with the exponent 5/3 replaced by 2.

We note that the above bound is the first subconvexity bound for an L-function of degree 8. All previous subconvexity results were obtained for L-functions of degree at most 4.

We can formulate the following natural *conjecture*: For any $\epsilon > 0$ we have $d_i \ll |\lambda_i|^\epsilon$.

Our proof follows the method described in [1]. It is based on the detailed analysis of representation spaces of representations of the principal series of the group $G = PGL(2, \mathbf{Z})$. We can make very explicit computations for these spaces since they can be realized as a *classical* space of smooth functions on the unit circle. As a result the computation of some quantities we need can be performed using stationary phase method. It is difficult to generalize these computations to groups of higher rank. In fact it is quite difficult to make similar computations even for representations of discrete series of the group $G = PGL(2, \mathbf{Z})$ since we do not have *classical* model for these representations.

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