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Optimal Control of Coupled Systems of PDE

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Abstract. The Workshop Optimal Control of Coupled Systems of PDE was held from April 17th – April 23rd, 2005 in the Mathematisches Forschungsinstitut Oberwolfach. The scientific program covered various topics such as controllability, feedback control, optimality conditions,analysis and control of Navier-Stokes equations, model reduction of large systems, optimal shape design, and applications in crystal growth, chemical reactions and aviation.

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Introduction by the Organisers

Current research in the control of partial differential equations is driven by a multitude of applications in engineering and science that are modelled by coupled systems of nonlinear differential equations. Associated optimal control problems need efficient numerical methods to deal with the resulting very large problems. There is a fast development of numerical methods and the associated analysis must keep track to justify them and to prepare the basis for further research. It has been the main intention of this Conference to tighten the links between applications, numerics and analysis with some emphasis on the analytic aspects. The meeting was attended by about 50 participants from Europe and the US.

The scientific program consisted of 30 talks that covered various topics such as controllability, feedback control, optimality conditions, analysis and control of Navier-Stokes equations, model reduction of large systems, optimal shape design, and applications in crystal growth, chemical reactions and aviation. It showed that Optimal Control of Partial Differential Equations is a very lively and active mathematical field. Well known experts with long standing experience, Postdocs and PhD students contributed to the program. In particular, 4 PhD students from US took part who received full support from the NSF.

This diversity of topics and mix of participants stimulated an extensive and fruitful discussion and initiated new collaborations, in particular of younger researchers.

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Workshop: Optimal Control of Coupled Systems of PDE

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Abstracts

On an Optimal Obstacle Control Problem

Maïtine Bergounioux
(joint work with Suzanne Lenhart)

We consider an optimal control problem where the state satisfies a bilateral elliptic variational inequality and the control functions are the upper and lower obstacles. We seek a state that is close to a desired profile and for which the H^2 norms of the obstacles are not too large. The motivation of our work is threefold. First, as mentioned above, many shape optimization problems can be modeled as the problem we describe here below. Secondly, as usual, in optimal control theory, we are looking for a first order necessary optimality system that allows us to compute the solution exactly (often not the case) or numerically. Thirdly, from the theoretical point of view, the problem is involved in a wider class of (open) problems, which can be (formally) described as follows:

$$\min\{J(u,\chi),\ u = \mathcal{T}(\chi),\ \chi \in U_{ad} \subset \mathcal{U}\},$$

where $\mathcal{T}$ is an operator which associates u to χ, where u is a (or the only) solution to :

$$\forall v \in K(u,\chi), \quad \langle \mathcal{A}(u,\chi), u - v\rangle \geq 0,$$

where K is a *multiapplication* from $\mathcal{X} \times \mathcal{U}$ to $2^{\mathcal{X}}$, where $\mathcal{X}$ is a Banach space and $\mathcal{U}$ a Hilbert space. Let us give an example: let $\mathcal{Y}$ be a Banach space and A a differential operator (linear or not), parabolic or elliptic from $\mathcal{Y}$ to the dual space $\mathcal{Y}'$, and h an application from $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. The differential equation that relates the control χ to the state function u (i.e., the state "equation") is

$$\langle Au, v - u, \rangle_{\mathcal{Y},\mathcal{Y}'} + h(u,\chi,v) - h(u,\chi,u) \geq (\chi, v - u) \qquad \forall v \in \mathcal{Y},$$

where

(i) $h(u,\chi,v) = h(v)$ gives the classical variational inequalities;
(ii) $h(u,\chi,v) = h(\chi,v)$ gives (for example) obstacle problems (where the obstacle is the control): this is the problem we investigate here;
(iii) $h(u,\chi,v) = h(u,v)$ leads to quasi-variational inequalities whose study is very delicate.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. We consider the bilinear form $a(.,.)$ defined on $H^1_o(\Omega) \times H^1_o(\Omega)$ by

$$a(u,v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\, dx + \sum_{i=1}^{n} \int_\Omega b_i \frac{\partial u}{\partial x_i} v\, dx + \int_\Omega cu\, v\, dx, \tag{1}$$

where a_{ij}, b_i, c belong to $L^\infty(\Omega)$. Moreover, we assume that a_{ij} belongs to $\mathcal{C}^{0,1}(\bar{\Omega})$ (the space of Lipschitz continuous functions in Ω) and that c is nonnegative. The bilinear form $a(.,.)$ is continuous on $H^1_o(\Omega) \times H^1_o(\Omega)$ and coercive on $H^1_o(\Omega)$.

We call $A \in \mathcal{L}(H^1_o(\Omega), H^{-1}(\Omega))$ the linear (elliptic) operator associated with a such that $\langle Au, v\rangle = a(u,v)$. Given $\varphi, \psi \in H^1_o(\Omega)$, we set

$$K(\varphi,\psi) = \{u \in H^1_o(\Omega) \mid \varphi \leq u \leq \psi \text{ a.e. in } \Omega\}, \tag{2}$$

which is a nonempty, closed, convex subset of $H^1_o(\Omega)$. All inequalities as $u \leq \psi$ are understood in the almost everywhere sense. Moreover, $f \in L^2(\Omega)$. For any $\varphi, \psi \in H^2(\Omega)$, the variational inequality

$$\forall v \in K(\varphi,\psi), \qquad a(u, v-u) \geq (f, v-u),\ u \in K(\varphi,\psi), \tag{3}$$

has a unique solution u that belongs to to $H^2(\Omega) \cap H^1_o(\Omega)$([2]. So we may define the operator $\mathcal{T}$ from $(H^2(\Omega) \cap H^1_o(\Omega)) \times (H^2(\Omega) \cap H^1_o(\Omega))$ to $H^2(\Omega) \cap H^1_o(\Omega)$ such that $\mathcal{T}(\varphi,\psi) = u$ is the unique solution to the variational inequality (3). It is known that this operator is not differentiable.

The set of admissible controls is defined as follows:

$$U_{ad} = \{(\varphi,\psi) \in (H^2(\Omega) \cap H^1_o(\Omega)) \times (H^2(\Omega) \cap H^1_o(\Omega)) \ \mid\ \varphi \leq \psi\ \},$$

and we consider the optimal control problem $(\mathcal{P})$:

$$\min\left\{ J(\varphi,\psi) \overset{def}{=} \frac{1}{2}\int_\Omega (\mathcal{T}(\varphi,\psi) - z)^2 dx + \frac{\nu}{2}\left(\int_\Omega \left((\Delta\varphi)^2 + (\Delta\psi)^2\right)\, dx\right), (\varphi,\psi) \in U_{ad}\right\},$$

where $z \in L^2(\Omega)$.

We use a classical technique (see [1,2]) to approximate the variational inequality by a semilinear equation. We define

$$\beta(r) = 0 \text{ if } r \geq 0,\ -r^2 \text{ if } r \in [-\frac{1}{2}, 0],\ r + \frac{1}{4} \text{ if } r < -\frac{1}{2}. \tag{4}$$

and introduce the following semilinear elliptic equation:

$$Au + \frac{1}{\delta}\left(\beta(u-\varphi) - \beta(\psi - u)\right) = f \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega. \tag{5}$$

As $\beta(\cdot - \varphi) - \beta(\psi - \cdot)$ is nondecreasing, it is known that the above equation has a unique solution $u^\delta \in H^2(\Omega) \cap H^1_o(\Omega)$, and we set $u^\delta = \mathcal{T}^\delta(\varphi,\psi)$.

Then we may prove the following results :

Theorem 1. *1. Let $(\varphi^\delta, \psi^\delta) \in U_{ad}$ be a sequence strongly convergent in $H^1_o(\Omega)$ to some (φ,ψ) as δ tends to 0. Then the sequence $u^\delta = \mathcal{T}^\delta(\varphi^\delta,\psi^\delta)$ converges to $u = \mathcal{T}(\varphi,\psi)$ strongly in $H^1_o(\Omega)$.*
2. $\mathcal{T}$ is continuous from U_{ad} endowed with the $H^2(\Omega) \times H^2(\Omega)$ sequential weak topology to $H^1_o(\Omega)$ endowed with the sequential weak topology.
3. Problem $(\mathcal{P})$ has (at least) an optimal solution (φ^*, ψ^*).

Let (φ^*, ψ^*) be an optimal solution to $(\mathcal{P})$ and $u^* = \mathcal{T}(\varphi^*, \psi^*)$.
For any $\delta > 0$, we define $J_\delta(\varphi,\psi) :=$

$$\frac{1}{2}\left[\int_\Omega \left(\mathcal{T}^\delta(\varphi,\psi) - z\right)^2 dx + \nu\int_\Omega \left((\Delta\varphi)^2 + (\Delta\psi)^2\right) dx + \|\varphi - \varphi^*\|_2^2 + \|\psi - \psi^*\|_2^2\right].$$

and define an approximate optimal control problem as follows:

$$(\mathcal{P}_\delta) \qquad \min\{J_\delta(\varphi,\psi),\ (\varphi,\psi) \in U_{ad}\}.$$

Theorem 2. *Problem* $(\mathcal{P}_\delta)$ *has (at least) an optimal solution* $(\varphi^\delta, \psi^\delta)$. *Moreover, the sequence* $(\varphi^\delta, \psi^\delta)$ *weakly converges to* (φ^*, ψ^*) *in* $H^2(\Omega)$, *while* $u^\delta = \mathcal{T}^\delta(\varphi^\delta, \psi^\delta)$ *strongly converges to* $u^* = \mathcal{T}(\varphi^*, \psi^*)$ *in* $H^1_o(\Omega)$.

We now establish a (necessary) optimality system for $(\mathcal{P}_\delta)$.

Theorem 3. *Assume that* $(\varphi^\delta, \psi^\delta)$ *is an optimal solution to* $(\mathcal{P}_\delta)$ *and* $u^\delta = \mathcal{T}^\delta(\varphi^\delta, \psi^\delta)$. *Then there exist* $p^\delta \in H^1_o(\Omega) \cap H^2(\Omega)$ *and* $\mu_1^\delta, \mu_2^\delta \in L^2(\Omega)$ *such that the following optimality system is satisfied:*

$$Au^\delta + \frac{1}{\delta}\left(\beta(u^\delta - \varphi^\delta) - \beta(\psi^\delta - u^\delta)\right) = f \ \textit{in}\ \Omega,\ u^\delta = 0 \ \textit{on}\ \partial\Omega, \tag{6a}$$

$$A^* p^\delta + \mu_1^\delta + \mu_2^\delta = u^\delta - z \ \textit{in}\ \Omega,\ p^\delta = 0 \ \textit{on}\ \partial\Omega, \tag{6b}$$

(6c)
$$\forall(\varphi,\psi) \in U_{ad}, \qquad \left(\mu_1^\delta + \varphi^\delta - \varphi^*, \varphi - \varphi^\delta\right)_2 + \left(\mu_2^\delta + \psi^\delta - \psi^*, \psi - \psi^\delta\right)_2 + \nu\left(\Delta\varphi^\delta, \Delta(\varphi - \varphi^\delta)\right)_2 + \nu\left(\Delta\psi^\delta, \Delta(\psi - \psi^\delta)\right)_2 \geq 0.$$

Finally we prove that we may pass to the limit, thanks to accurate estimates for the different δ-elements and we obtain :

Theorem 4. *Let* (φ^*, ψ^*) *be an optimal solution to* $(\mathcal{P})$. *Then* $\Delta(\varphi^* + \psi^*) \in H^1_o(\Omega)$ *and there exist* $p^* \in H^1_o(\Omega)$ *and* $\lambda^* \geq 0$ *in* $H^{-1}(\Omega)$ *such that the following optimality system is satisfied:*

$$u^* = \mathcal{T}(\varphi^*, \psi^*), \tag{7a}$$

$$A^* p^* = u^* - z^* - \mu^* \ \textit{in}\ \Omega,\ p^* = 0 \ \textit{on}\ \partial\Omega, \tag{7b}$$

$$\langle p^*, \mu^* \rangle \geq 0, \tag{7c}$$

$$\mu^* = \mu_1^* + \mu_2^*,\ \textit{with}\ \mu_1^* = -\lambda^* - \nu\Delta^2\varphi^* \ \textit{and}\ \mu_2^* = \lambda^* - \nu\Delta^2\psi^*, \tag{7d}$$

$$\langle \lambda^*, \varphi^* - u^* \rangle = 0 \ \textit{and}\ \langle \lambda^*, u^* - \psi^* \rangle = 0. \tag{7e}$$

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Thin and Cracked Sets in Image Processing and Related Topics

Michel C. Delfour

As a matter of terminology a subset Ω of the N-dimensional Euclidean space $\mathbb{R}^N$ whose boundary Γ is not empty is said to have a *thin boundary* if the N-dimensional Lebesgue measure of its boundary Γ is zero (cf. [8], pp. 210–225). In this talk, two sets of results are presented in relation to the use of the oriented distance function in shape/geometric analysis and optimization with potential applications in image processing and level sets methods.

The first one [10] is the introduction of the new family of *cracked sets* [10] which forms a compact family of sets in the $W^{1,p}$-topology associated with the oriented distance function [8] with an original application to the celebrated image segmentation problem formulated by Mumford and Shah and some variations of the associated original image functional that do not require a penalization term on the *length of the segmentation*. The sets in this family have thin boundary. It contains non-regular sets and submanifolds of variable dimension. They can have cusps [5–7] and a wide range of singularities.

In the classical formulation of the segmentation problem, there is a penalization term that makes the length of the segmentation finite. Yet, it is easy to construct simple examples of segmentation where that length is infinite. Moreover that term contributes to neglect long slender objects with a large perimeter and a small surface area to the benefit of objects with a large surface area and a small perimeter. Therefore thin crack-like objects will be more difficult to see.

The originality of the approach is that it does not require a penalization term on the length of the segmentation and that, within the set of solutions, there exists one with minimum density perimeter as defined by Bucur and Zolésio [4]. It is different from the approach by SBV functions [2] where the locus of discontinuities of the function (and hence the perimeter of the segmenting interface) is finite. In the process, we revisit and recast in the $W^{1,p}$-framework the earlier existence theorem of Bucur and Zolésio [4] for sets with a uniform bound or a penalization term on the density perimeter.

The second set of results [9] is a new nonlinear evolution equation that describes the time evolution Ω_t of the oriented distance function[1]

$$b_{\Omega_t} \stackrel{\text{def}}{=} d_{\Omega_t} - d_{\complement\Omega_t}$$

of an initial set Ω at time $t = 0$ with only a *thin boundary* under the influence of a velocity field $V(t)$. This equation

$$\frac{\partial}{\partial t} b_{\Omega_t} + V(t) \circ p_{\Gamma_t} \cdot \nabla b_{\Omega_t} = 0 \text{ a.e. in } \mathbb{R}^N$$

makes sense almost everywhere in the space variable and not only on the boundary (or the front) Γ_t of the time-varying set Ω_t. In our analysis the velocity field $V(t)$

[1] d_A is the usual distance function to a set A and $\complement A$ is the complement of the set A in $\mathbb{R}^N$.

is assumed to be Lipschitz, but the projection p_{Γ_t} is at best BV as the gradient of the convex function

$$f_t(x) \stackrel{\text{def}}{=} \frac{1}{2}\left(|x|^2 - |b_{\Omega_t}(x)|^2\right)$$

(cf. [8], p. 214 and Thm 6.3, p. 230). The singularities of p_{Γ_t} occur on the skeleton of Ω_t which is a set of zero measure. The term $V(t) \circ p_{\Gamma_t}$ makes sense as a BV mapping since it is the composition of a Lipschitz mapping $V(t)$ and a BV mapping p_{Γ_t} (cf. [3]).

In [9], we relate our results to equations and constructions used in the context of level set methods [13]. A simple example illustrates that our equation still holds even if the restriction of the equation to the boundary does not make sense. It also turns out that the velocity term $V(t) \circ p_{\Gamma_t}$ in our equation is related to the concept of *extension velocity* introduced by Malladi, Sethian, and Vemuri [11] in 1995 (see also [1]). The natural extension velocity that comes out of our analysis is the original velocity $V(t)$ evaluated at the projection $p_{\Gamma_t}(x)$ of the point x onto the time-varying boundary (or front) Γ_t. This was one of the choices of extension velocity suggested in [11]. We further introduce in [9] a new *moving narrow-band method* that does not theoretically require a reinitialization of the band and can be readily implemented to solve our evolution equation only in the narrow band. It is based on the introdution of a special two-parameter extension/cut-off function that creates two tubular neighborhoods $U_{h'}(\Gamma_t)$ and $U_h(\Gamma_t)$ around Γ_t of fixed thicknesses $0 < h' < h$ (independent of t). In the smaller tubular neighborhood $U_{h'}(\Gamma_t)$ we have exactly b_{Ω_t} while outside the larger tubular neighborhood $U_h(\Gamma_t)$ we have zero.

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Unique Continuation for the Stationary Anisotropic Maxwell System

Matthias M. Eller

The Cauchy problem is one of the classical boundary value problems in partial differential equations. It is well known that the hyperbolic Cauchy problem is well-posed. Problems of boundary control and inverse problem for partial differential equations have led to a more detailed study of non-hyperbolic or ill-posed Cauchy problems. Roughly speaking, these ill-posed Cauchy problems become relevant whenever one measures some some physical quantity on the boundary of a region (or part thereof) and wants to predicts behavior of the same quantity inside of that region. The most interesting question for the ill-posed Cauchy problem is the question about the uniqueness of the solution. In other words, do zero Cauchy data on the boundary of a region force a solution to a partial differential equation to vanish inside that region? This is essentially a local problem and is then often called *unique continuation.*

Unique continuation is very well understood for operators with analytic coefficients. Holmgren's Theorem (1901) gives the exhaustive answer: There is uniqueness of the continuation across all non-characteristic surfaces. As soon as the coefficients are non-analytic the problem becomes much more difficult. By now there is a large class of positive results, in particular for scalar second order operator. On the other hand there are still only few positive results for higher order equations or for systems of equations. For some striking counterexamples even for operators with C^∞-coefficients we refer to [K63] and [P61].

Most uniqueness results rely on Carleman estimates which are weighted energy estimates carrying a large parameter. Given a partial differential operator P of order m and a level surface $S = \{\psi(x) = \psi(x_0)\}$ where $\psi \in C^2$ with $\psi'(x_0) \neq 0$ one looks for an estimate of the form

$$(1) \qquad \sum_{|\alpha|\leq m-1} \tau^{2(m-|\alpha|)-1} \int e^{2\tau\phi}|D^\alpha u|^2 dx \leq C \int e^{2\tau\phi}|P(x,D)u|^2 dx \qquad \tau \geq \tau_0$$

for all $u \in C_0^\infty$ compactly supported in a neighborhood of x_0. Here $\phi = e^{\lambda\psi}$ for some $\lambda \geq \lambda_0$ and τ is a large positive parameter. An inequality of this form implies unique continuation for solutions to $P(x, D)v = 0$ across the surface C^2-surface S, i.e. if v is zero in $S^+ = \{\psi(x) > \psi(x_0)\}$ then v vanishes in a full neighborhood of x_0. This follows from the estimate (1) after a localization and perturbation argument [H83, Chapter XXVII].

T. Carleman pioneered this type of estimate in 1939 for the purpose of unique continuation [C39]. L. Hörmander developed estimates of the from (1) systematically and proved unique continuation across so-called strongly pseudo-convex surfaces for a large class of scalar operators [H63, Chapter 8]. His results are optimal for second order elliptic operators and provide also some results for second order hyperbolic operators. Hörmander's theorem does not apply to systems of partial differential equations; however, a number of systems of partial differential equations can be principally decoupled into a diagonal system of second order operators. Carleman estimates are applied to each component and uniqueness can be proved. This has been done for the isotropic elastic wave equations [W69] [EINT02] as well as the isotropic Maxwell equations [EINT02]. However, this approach breaks down as soon as the system becomes anisotropic, i.e. when the coefficients are not scalars but rather matrices.

A precursor of Hörmander's Theorem is Calderón's Theorem [C58]. He proved unique continuation across some surfaces for a scalar mth order operator. His proof is quite instructive. Using pseudo-differential operators he transforms the mth order equation into a first order system without changing the characteristic set. Then he considers the system as an evolution in direction of the surface function's gradient ψ'. The symbol of the first order system is brought into Jordan form. A Carleman estimate is derived provided the Jordan form is smooth and satisfies certain structural assumptions.

Rather recently, Imanuvilov and Yamamoto have observed that Calderón's approach will lead to a Carleman estimate for the stationary isotropic elastic system [IY04]. In this talk we will use Calderón's method to obtain a Carleman estimate and unique continuation for the anisotropic stationary Maxwell equations.

Let $E(x)$ and $H(x)$ be two vector-valued functions $\Omega \to \mathbb{R}^3$, the electric field intensity and the magnetic field intensity, respectively. Furthermore, the electric permeability $\varepsilon(x)$ and the magnetic permittivity $\mu(x)$ are 3×3 positive definite, symmetric matrices with C^1 entries. The homogeneous Maxwell system consists of the following equations

$$\begin{aligned} \varepsilon(x)E(x) - \mathrm{curl}H(x) &= 0 \\ \mu(x)H(x) + \mathrm{curl}E(x) &= 0 \\ \mathrm{div}(\varepsilon(x)E(x)) &= 0 \\ \mathrm{div}(\mu(x)H(x)) &= 0 \end{aligned} \tag{2}$$

We will discuss the following Carleman estimate.

Theorem 1. *Let Ω be a connected, open set in $\mathbb{R}^3$ and let $\psi \in C^1(\Omega)$ such that $\nabla\psi \neq 0$. Let $E, H \in H^1(\Omega)$ with compact support in Ω. Then there exist positive constants λ_0,s_0 and C such that for $s \geq s_0$ and $\lambda \geq \lambda_0$*

$$\frac{1}{s\lambda}\sum_{j=1}^{3}\int_\Omega \phi^{-1}(|\partial_j E|^2 + |\partial_j H|^2)e^{2s\phi}dx + s\lambda^2\int_\Omega \phi(|E|^2+|H|^2)e^{2s\phi}dx$$

$$\leq C\int e^{2s\phi}\left\{|\varepsilon E - \nabla\times H|^2 + |\mu H + \nabla \times E|^2 + |\nabla\cdot(\varepsilon E)|^2 + |\nabla\cdot(\mu H)|^2\right\}dx$$

Here $\phi = e^{\lambda\psi}$.

This theorem gives the following uniqueness result.

Theorem 2. *Solutions to the homogeneous anisotropic Maxwell equations with positive C^1-coefficients satisfy the unique continuation property across any C^2-surface.*

To our best knowledge this is the first uniqueness result for the fully anisotropic Maxwell system and improves over earlier works by V.Vogelsang [V91] and T. Okaji [O02]. In both papers there were some structural assumptions on the relationship of ε and μ, more specifically, the assumption $\varepsilon(x_0) = \mu(x_0) = I$ was imposed in [V91] and the assumption $\varepsilon(x_0) = \kappa\mu(x_0)$ for a scalar κ in [O02].

This Carleman estimate is proved using Calderón's approach. For that purpose it will suffice to consider the div-curl system, i.e.

$$\begin{aligned}\mathrm{curl} w(x) &= F(x)\\ \mathrm{div}(A(x)w(x)) &= G(x)\end{aligned} \tag{3}$$

where $A(x)$ is a symmetric, positive definite matrix with C^1 entries. Maxwell's system (2) is a weakly coupled system of two div-curl systems.

So far unique continuation for solutions to systems of partial differential equations have been proved on a case by case basis. A more general result is highly desirable. The central question here is whether certain Gårding type inequalities which are known to be valid for scalar operators can be generalized to the case of matrix symbols.

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Aero-Structural Wing Design Optimization Using a Coupled Adjoint Approach

Nicolas R. Gauger
(joint work with Antonio Fazzolari and Joel Brezillon)

The aerospace industry is increasingly relying on advanced numerical flow simulation tools in the early aircraft design phase. Today's flow solvers, which are based on the solution of the compressible Euler and Navier-Stokes equations, are able to predict aerodynamic behaviour of aircraft components under different flow conditions quite well [1]. Within the next few years numerical shape optimization will play a strategic role for future aircraft design. It offers the possibility of designing or improving aircraft components with respect to a pre-specified figure of merit, subject to geometrical and physical constraints.

Here, aero-structural analysis is necessary to reach physically meaningful optimum wing designs. The use of single disciplinary optimizations applied in sequence is not only inefficient but in some cases has been shown to lead to wrong, non-optimal designs [2]. Although multidisciplinary optimizations are possible with classical approaches for sensitivity evaluations by means of finite differences, these methods are extremely expensive in terms of calculation time, requiring the reiterated solution of the coupled problem for every design variable.

However, adjoint approaches are known to allow the evaluation of these sensitivities in an efficient way and to lead to high accuracy. First, we present the development and application of a continuous adjoint approach for single disciplinary aerodynamic shape designs. This approach was previously developed at the German Aerospace Center (DLR) [3] and was the starting point for the extension to aero-structural wing designs. Second, we describe the adjoint approach and its implementation for the evaluation of the sensitivities for coupled aero-structure optimization problems [4] and its application for the drag reduction of the AMP wing by constant lift while taking into account the static deformation of this wing caused by the aerodynamic forces (see figure 1). Finally, we show the application of the coupled aero-structural adjoint approach for the Breguet formula of aircraft range, where next to the lift to drag ratio the weight of the AMP wing is taken into account (see also figure 1).

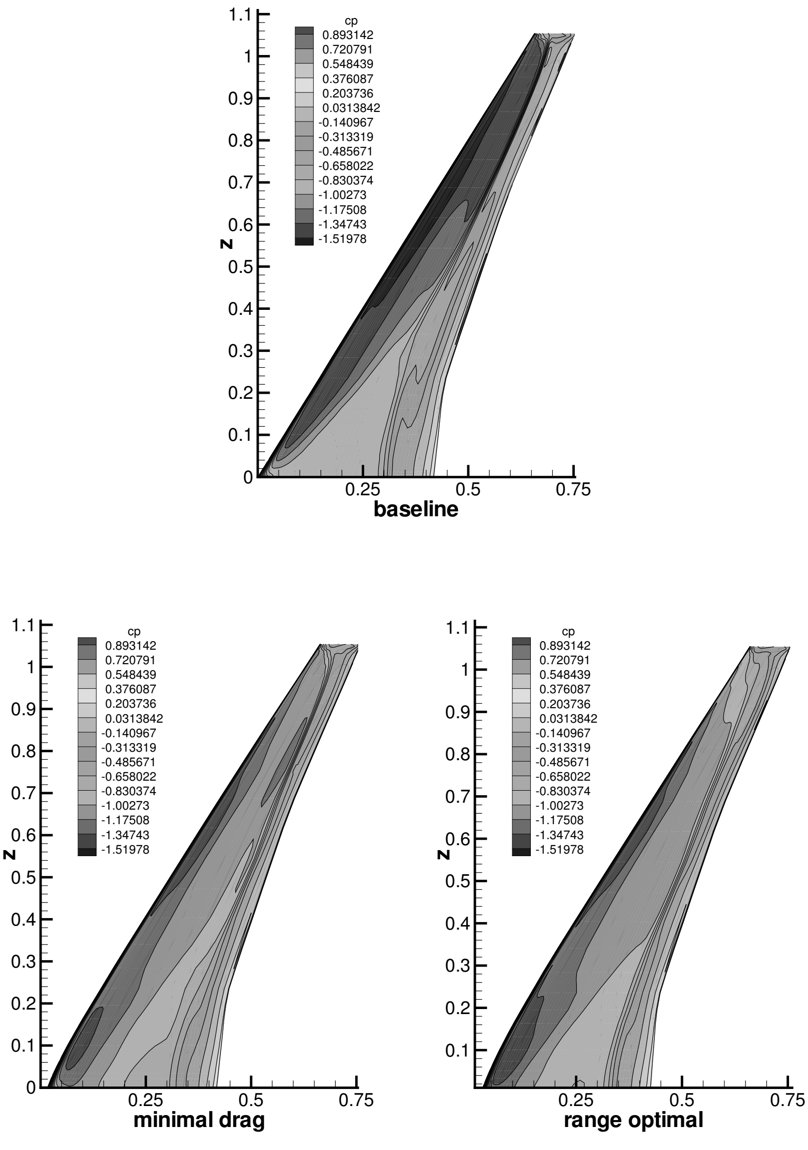

Figure 1. Pressure distribution for the baseline AMP wing shape and for the optimal wing shapes for drag minimization and range maximization ($Mach = 0.78$, $\alpha = 2.83°$).

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Optimal Control in Magnetohydrodynamics

Roland Griesse
(joint work with Karl Kunisch)

Magnetohydrodynamics, or MHD, deals with the mutual interaction of electrically conducting fluids and magnetic fields. The nature of the coupling between fluid motion and the electromagnetic quantities arises from the following three phenomena:

(i) The relative movements of a conducting fluid and a magnetic field induce an electromotive force (Faraday's law) to the effect that an electric current develops in the fluid.
(ii) This current in turn induces a magnetic field (Ampère's law).
(iii) The magnetic field interacts with the current in the fluid and exerts a Lorentz force in the fluid.

It is the third feature in the nature of MHD which renders it so phenomenally attractive for exploitation especially in metallurgical processes. The Lorentz force offers a unique possibility of generating a volume force in the fluid and hence to control its motion in a contactless fashion and without any mechanical interference.

Essentially, the MHD system consists of the Navier-Stokes equation with Lorentz force, yielding the fluid velocity $\boldsymbol{u}$ and its pressure p, plus Maxwell's equations describing the interaction of the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$. In the stationary case, the complete MHD system is given by

(1) $$\nabla \cdot \boldsymbol{J} = 0 \quad \text{(Charge conservation)} \qquad \nabla \times \boldsymbol{E} = 0 \quad \text{(Faraday's law)}$$

(2) $$\nabla \cdot \boldsymbol{B} = 0 \quad \text{(No magnetic monopoles)} \qquad \nabla \times (\mu^{-1}\boldsymbol{B}) = \boldsymbol{J} \quad \text{(Ampère's Law)}$$

(3) $$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) \quad \text{(Ohm's law)}$$

together with the Navier-Stokes system with Lorentz force

$$\varrho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\eta\Delta\boldsymbol{u}+\nabla p=\boldsymbol{J}\times\boldsymbol{B} \tag{4}$$

$$\nabla\cdot\boldsymbol{u}=0. \tag{5}$$

We refer to [1, 4] for more details. Here and throughout, μ denotes the magnetic permeability of the matter occupying a certain point in space, and ϱ, η and σ denote the fluid's density, viscosity and conductivity. All of these numbers are positive. We emphasize that we consider μ a constant throughout space, hence we assume a non-magnetic fluid and no magnetic material present in its relevant vicinity.

It is an outstanding feature in magnetohydrodynamics that from the set of state variables $(\boldsymbol{u}, p, \boldsymbol{E}, \boldsymbol{B}, \boldsymbol{J})$, the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ extend to all of $\mathbb{R}^3$, whereas the velocity $\boldsymbol{u}$ and pressure p are confined to the bounded region $\Omega\subset\mathbb{R}^3$ occupied by the fluid. The current density $\boldsymbol{J}$ is defined within the fluid region and possibly also in external conductors. In what appear to be the practically relevant cases where the outside of the fluid region Ω is finitely conducting or non-conducting, hence permitting control by distant magnetic fields, the proper boundary condition for $\boldsymbol{B}$ is an interface condition requiring $\boldsymbol{B}$ to be continuous across $\partial\Omega$ in both its normal and tangential components, *i.e.*,

$$[\boldsymbol{B}]_{\partial\Omega}=0 \tag{6}$$

where $[\cdot]_{\partial\Omega}$ denotes the jump of any quantity when going from the interior of Ω to its exterior. In the velocity–current formulation in terms of the variables $(\boldsymbol{u}, \boldsymbol{J})$ of the state equation system (1)–(5), see [3], the magnetic field $\boldsymbol{B}$ is eliminated by means of a solution operator $\mathcal{B}(\boldsymbol{J})$ which uniquely solves the div–curl system (2) for divergence-free currents $\boldsymbol{J}$ and respects the interface condition (6). Moreover, the irrotational electric field $\boldsymbol{E}$ is replaced by its potential ϕ (unique only up to a constant). In our case of constant permeability μ, the operator $\mathcal{B}(\boldsymbol{J})$ is given by the Biot-Savart law,

$$\mathcal{B}(\boldsymbol{J})(\boldsymbol{x})=-\frac{\mu}{4\pi}\int_{\mathbb{R}^3}\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3}\times\boldsymbol{J}(\boldsymbol{y})\,d\boldsymbol{y}. \tag{7}$$

Inserting $\boldsymbol{B}=\mathcal{B}(\boldsymbol{J})$ into (1)–(5), we are left with the velocity–current formulation of the stationary MHD system,

$$\varrho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\eta\Delta\boldsymbol{u}+\nabla p-\boldsymbol{J}\times\mathcal{B}(\boldsymbol{J})=0 \qquad \nabla\cdot\boldsymbol{u}=0 \tag{8}$$

$$\sigma^{-1}\boldsymbol{J}+\nabla\phi-\boldsymbol{u}\times\mathcal{B}(\boldsymbol{J})=0 \qquad \nabla\cdot\boldsymbol{J}=0 \tag{9}$$

for the unknowns $(\boldsymbol{u}, p, \boldsymbol{J}, \phi)$. Here $\boldsymbol{u}$ and p and the electric potential ϕ are confined to the region Ω occupied by our conducting fluid, while $\boldsymbol{J}$ may additionally extend to external conductors, see Figure 1.

To complete the specification of the state equation, boundary conditions are required for the current density $\boldsymbol{J}$ and the fluid velocity $\boldsymbol{u}$. For the former, we require

$$\boldsymbol{J}\cdot\boldsymbol{n} = \boldsymbol{J}_{\text{inj}}\cdot\boldsymbol{n} \qquad \text{on } \partial\Omega_{\text{inj}}\cap\partial\Omega \tag{10}$$

$$\boldsymbol{J}\cdot\boldsymbol{n} = 0 \qquad \text{on } \partial\Omega\setminus\partial\Omega_{\text{inj}} \tag{11}$$

where the injected current $\boldsymbol{J}_{\text{inj}}$ can be controlled in magnitude. For the fluid velocity, we impose Dirichlet boundary conditions

$$\boldsymbol{u} = \boldsymbol{h} \qquad \text{on } \partial\Omega. \tag{12}$$

In [2] we consider an optimal control problem of the form

$$(\mathbf{P}) \quad \begin{aligned} &\text{Minimize } \frac{\alpha_u}{2}\|\boldsymbol{u}-\boldsymbol{u}_{\text{d}}\|^2_{L^2(\Omega_{u,\text{obs}})} + \frac{\alpha_B}{2}\|\boldsymbol{B}-\boldsymbol{B}_{\text{d}}\|^2_{L^2(\Omega_{B,\text{obs}})} \\ &\qquad + \frac{\alpha_J}{2}\|\boldsymbol{J}-\boldsymbol{J}_{\text{d}}\|^2_{L^2(\Omega_{J,\text{obs}})} + \frac{\gamma_{\text{ext}}}{2}|I_{\text{ext}}|^2 + \frac{\gamma_{\text{inj}}}{2}|I_{\text{inj}}|^2 + \frac{\gamma_B}{2}|B_{\text{ext}}|^2 \\ &\text{subject to (8)–(12).} \end{aligned}$$

The control variables I_{ext}, I_{inj} and B_{ext} denote the strengths of the currents in external conductors, and of an external magnetic field, respectively, see Figure 1. The total magnetic field $\boldsymbol{B}$ is a superposition of the field $\mathcal{B}(\boldsymbol{J})$ induced by the

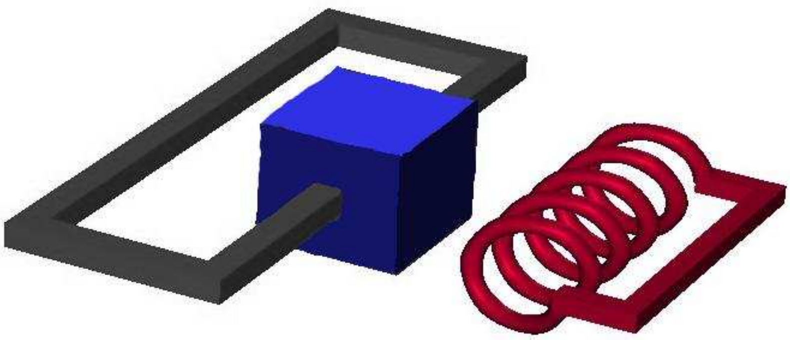

FIGURE 1. General Setup: Fluid region Ω (blue cube), external conductor Ω_{inj} attached to the fluid region (grey), and external conductor Ω_{ext} separate from the fluid region (red).

current $\boldsymbol{J}$ inside the fluid domain, the fields $\mathcal{B}(\boldsymbol{J}_{\text{ext}})$ and $\mathcal{B}(\boldsymbol{J}_{\text{inj}})$ induced by the currents in the external conductors (whether or not attached to the fluid domain), and the magnetic field $\boldsymbol{B}_{\text{ext}}$ associated with the permanent magnet, *i.e.*,

$$\boldsymbol{B} = \mathcal{B}(\boldsymbol{J}) + \mathcal{B}(\boldsymbol{J}_{\text{ext}}) + \mathcal{B}(\boldsymbol{J}_{\text{inj}}) + \boldsymbol{B}_{\text{ext}}. \tag{13}$$

We present a proper function space framework, first order necessary and second order sufficient optimality conditions for $(\mathbf{P})$ and prove a convergence result for an operator splitting scheme concerning the numerical solution of the MHD state equation.

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Optimal Boundary Control of Conservation Laws

Martin Gugat

Many parts of the infrastructure can be modelled as networked systems of conservation laws (water channels, streets, pipeline systems). The solution of optimal control problems for such systems can help to run these systems in an efficient way. For the development of numerical methods that solve these problems, an analysis of systems of this type is essential. As a first example problem, we consider problems of optimal exact boundary control for the wave eqation. For these problems, analytical solutions of the optimal control problems can be given for a large class of spaces (see [3], [2]).

For networked systems of water channels, the de St. Venant equations coupled by algebraic node conditions are used as a model. The de St. Venant equations form a system of two scalar conservation laws, so that information can travel in two directions (downstream and upstream) at the same time. Various *controllability* results have been achieved so far:

- *Frictionless horizontal channels*, *Local Controllability* see [1], [9]: Locally around a subcritical stationary state (also for star–shaped networks) exact boundary control with continuously differentiable states is possible.

For the applications, it is important to have some information about the controllability properties of the system for states that are far away from each other.

- *Frictionless horizontal channels*, *Transcritical Global Controllability* see [5]: Between all stationary states (both sub- and supercritical withour restriction on their distance) exact boundary control with continuously differentiable states is possible.
- *Trees of sloped channels with friction*, *Global Controllability* see [8]: Between supercritical stationary states with the same orientation exact boundary control with continuously differentiable states is possible.
- Recent results (see [10]) state that locally around a subcritical stationary state for *tree shaped networks* of channels *local controllability* is possible.

For the numerical solution of optimal control problems with networked water channels, an adjoint based sensitivity calculus has been developed for continuously differentiable solutions (see [6]). This calculus has been used for the numerical solution of example problems by gradient based optimization methods. Traffic flow through networks of street has also been studied (see [7]). In this case, the flow is modelled by a single scalar conservation law.

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Spatial Domain Decomposition and Model Reduction for Parabolic Optimal Control Problems

Matthias Heinkenschloss
(joint work with M. Herty and D. C. Sorensen)

This work combines an optimization-level spatial domain decomposition method with model reduction for the efficient solution of linear-quadratic parabolic optimal control problems. Such problems arise directly in many applications, but also as subproblems in Newton or sequential quadratic programming methods for the solution of nonlinear parabolic optimal control problems. The motivation for this work is threefold. First, our approach addresses the storage issue that arises in the numerical solution of parabolic optimal control problems due to the strong coupling in space and time of the state, adjoint, and control variables. Secondly, our domain decomposition method introduces parallelism at the optimization level. The third motivation arises from the availability of sensor networks that offer in-network computing capabilities, allow neighbor-to-neighbor communication, but for which communication among distant nodes is prohibitive because of communication bandwidth and battery power limitations. Our combination of domain decomposition and model reduction offers the possibility for in-network computing, in which the global problem is solved using spatially distributed processors that communicate primarily with their immediate neighbors.

To illustrate our ideas, we consider the example problem

$$\text{(1)} \qquad \text{minimize} \quad \frac{1}{2}\int_0^T \int_\Omega (y(x,t)-z(x,t))^2 dxdt + \frac{\alpha}{2}\int_0^T \int_\Omega u^2(x,t)dxdt,$$

subject to

$$\text{(2)} \qquad \begin{array}{c} \partial_t y(x,t) - \mu\Delta y(x,t) + \mathbf{a}(x)\nabla y(x,t) + c(x)y(x,t) = f(x,t) + u(x,t) \\ \text{in } \Omega \times (0,T),\ y(x,t) = 0 \text{ on } \partial\Omega \times (0,T), \quad y(x,0) = y_0(x) \text{ in } \Omega, \end{array}$$

where $\Omega \subset \mathbb{R}^d$, $d = 1, 2$ or 3, is a given domain, $y_0 \in L^2(\Omega)$, $z \in L^2(\Omega \times (0,T))$, $\mathbf{a} \in [W^{1,\infty}(\Omega)]^d$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega \times (0,T))$ are given functions and $\alpha > 0$, $\mu > 0$ are given parameters. We refer to y as the state and to u as the control. Extension of our framework to several other problems are possible.

It is well known [7] that the optimal control problem (1,2) has a unique solution $y \in \{y \ : \ y \in L^2(0,T;H_0^1(\Omega)), y' \in L^2(0,T;H^{-1}(\Omega))\}$ and $u \in L^2(\Omega \times (0,T))$. The system of necessary and sufficient optimality conditions consists of (2) and

$$\text{(3)} \qquad \begin{array}{rl} -\partial_t p(x,t) - \mu\Delta p(x,t) - \mathbf{a}(x)\nabla p(x,t) & \\ +(c(x) - \nabla\mathbf{a}(x))p(x,t) & = -(y(x,t) - z(x,t)) \text{ in } \Omega \times (0,T), \\ p(x,t) = 0 \text{ on } \partial\Omega \times (0,T), & p(x,T) = 0 \text{ in } \Omega, \end{array}$$

$$\text{(4)} \qquad \alpha u(x,t) + p(x,t) = 0 \qquad\qquad \text{in } \Omega \times (0,T).$$

The domain decomposition approach for the solution of (2,3,4) is based on [4] (see also [3,5]). We divide Ω into nonoverlapping subdomains Ω_i, $i = 1, \dots, s$, and we define the subdomain interfaces $\Gamma_i = \partial\Omega_i \setminus \partial\Omega$ and $\Gamma = \cup_{i=1}^s \Gamma_i$. We introduce states and adjoints y_Γ, p_Γ defined on $\Gamma \times (0,T)$. The system (2,3,4) of optimality conditions is imposed onto Ω_i. Let y_i, u_i, p_i be the solution of the restriction of (2,3,4) onto Ω_i with the condition that $y_i = y_\Gamma$ and $p_i = p_\Gamma$ on $\Gamma_i \times (0,T)$. We may view y_i, u_i, p_i as functions of y_Γ and p_Γ. For the example problem, these functions are affine linear. To ensure that the subdomain states, controls and adjoints y_i, u_i, p_i are the restriction of the global states, controls and adjoints y, u, p (the solution of (2,3,4)) onto Ω_i, we require

$$\text{(5)} \qquad \begin{array}{rcl} \Big(\mu\frac{\partial}{\partial\mathbf{n}_i} - (\frac{1}{2}\mathbf{a}(x)\cdot\mathbf{n}_i)\Big)y_i(x,t) & = & -\Big(\mu\frac{\partial}{\partial\mathbf{n}_j} - (\frac{1}{2}\mathbf{a}(x)\cdot\mathbf{n}_j)\Big)y_j(x,t), \\ \Big(\mu\frac{\partial}{\partial\mathbf{n}_i} + (\frac{1}{2}\mathbf{a}(x)\cdot\mathbf{n}_i)\Big)p_i(x,t) & = & -\Big(\mu\frac{\partial}{\partial\mathbf{n}_j} + (\frac{1}{2}\mathbf{a}(x)\cdot\mathbf{n}_j)\Big)p_j(x,t) \end{array}$$

on $\partial\Omega_i \cap \partial\Omega_j \times (0,T)$ for adjacent subdomains Ω_i, Ω_j. Here $\mathbf{n}_i$ denotes the outward unit normal for the ith subdomain. The subdomain solutions y_i, u_i, p_i can be viewed as affine linear functions of the interface variables y_Γ and p_Γ. Hence, satisfying the transmission conditions (5) can be written as an operator equation

$$\text{(6)} \qquad \sum_{i=1}^s S_i(y_\Gamma, p_\Gamma) = \sum_{i=1}^s r_i.$$

The equation (6) is solved using a preconditioned Krylov subspace method.

The evaluation of $S_i(y_\Gamma, p_\Gamma)$ requires the solution of the restriction of the homogeneous problem (2,3,4) onto Ω_i with the condition that $y_i = y_\Gamma$ and $p_i = p_\Gamma$ on $\Gamma_i \times (0,T)$. The right hand sides r_i are determined by similar subdomain optimization problems, but with interface condition $y_i = 0$ and $p_i = 0$ on $\Gamma_i \times (0,T)$. It can be shown (see [3–5]) that the evaluation of $S_i(y_\Gamma, p_\Gamma)$ is equivalent to the solution of an optimization problem, which is essentially a smaller copy of (1,2) restricted onto the subdomain Ω_i.

The operators S_i, restricted to functions defined on $\Gamma_i \times (0,T)$ can shown to be invertible. The application of the inverse of S_i also requires the solution of a subdomain optimization problem. The inverses of S_i can be used to construct preconditioners for (6) that extend to the optimization context Neumann-Neumann preconditioners well known for the solution of single PDEs [8].

The application of S_i and their inverses, which are needed in preconditioners for (6), correspond to subdomain optimal control problems. These problems are significantly smaller than the original one (1,2) and they can be solved in parallel. However, these are still expensive tasks. Moreover, for computations in sensor-networks, sensors that will be available in the near future are not expected to have sufficient computing power to solve the subdomain optimal control problems. This motivates the introduction of model reduction. Model reduction seeks to replace a large-scale system by a system of substantially lower dimension that has nearly the same response characteristics. Specifically, we apply balanced model reduction [1, 2] to the subdomain problems associated with S_i and their inverses. The advantages are that reduced order models for the subdomain problems can be computed in parallel, and that model reduction can be tailored to localized features of the problem more effectively than would be possible if balanced model reduction was applied to the global problem (1,2) directly. This combination of domain decomposition and model reduction also makes it possible to derive distributed solution algorithms applicable in sensor networks.

As we have stated before, the evaluation of $S_i(y_\Gamma, p_\Gamma)$ requires the solution of the restriction of the homogeneous problem (2,3,4) onto Ω_i with the condition that $y_i = y_\Gamma$ and $p_i = p_\Gamma$ on $\Gamma_i \times (0,T)$. We apply model reduction to this system. This leads to operators $\widehat{S}_i$ that are much cheaper to evaluate computationally than S_i. Each $\widehat{S}_i$ is invertible. Similarly, we can derive operators $\widehat{S_i^{-1}}$ that replace S_i^{-1}. We can now use model reduction to design preconditioners for (6). This will lead to a faster algorithm for the solution of (1,2). We can also modify (6) by replacing S_i, r_i with $\widehat{S}_i, \widehat{r}_i$. The solution of the resulting system is no longer a solution of (1,2), but the error in the solution can be bounded by operator norms $\|\widehat{S}_i - S_i\|$ [6]. The advantage of this modification of (6) is that the subdomain optimal control problems that correspond to $\widehat{S}_i$ are small and can be easily solved with small computing resources, which is important for computing in sensor networks.

The crucial task now is to generate reduced models with bounds on $\|\widehat{S}_i - S_i\|$. Such bounds are automatically provided as a result of the balanced reduction. In [6] the subdomain problems associated with S_i, which are essentially restrictions of (2,3,4) onto Ω_i are identified with dynamical systems with inputs u_i, y_Γ (the control restricted to Ω_i and the states on the interface) and with outputs y_i and $\mu\frac{\partial}{\partial \mathbf{n}_i}y_i - (\frac{1}{2}\mathbf{a}\cdot\mathbf{n}_i)y_i$ (the state restricted to Ω_i and the Robin interface data (cf. (5)). The so-called state space representation of these subdomain dynamical systems is precisely in the form that allows the application of balanced truncation [1, 2] and results in bounds for the operator norms $\|\widehat{S}_i - S_i\|$ that can be controlled.

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Boundary Feedback Control for the Stabilization of Unstable Parabolic Systems

Vincent Heuveline
(joint work with T. Carraro and A. Fursikov)

We consider the stabilization problem by means of Dirichlet feedback control for systems modeled by scalar or vector parabolic equations. The main emphasis is put on the treatment of highly nonlinear reaction-advection-diffusion type equations which are unstable if uncontrolled.

We present in that context two approaches:

(i) Formulation by means of an optimal control problem (joint work with T. Carraro)
(ii) Extension operator and invariant manifold (joint work with A. Fursikov)

In the formulation by means of optimal control, the aspects related to online checkpointing and a posteriori error estimation by means of model reduction (SPOD) are addressed.

Applications of stabilization for reactive multicomponent flows are presented. In the second proposed approach we address numerical aspects related to the quadratic approximation of invariant manifolds.

Optimal Control of Variational Inequalities

Michael Hintermüller

(joint work with D. Ralph, S. Scholtes, and K. Kunisch)

Many practical applications result in minimization problems of the form

(1a) minimize $J(y,u)$ over $(y,u) \in H \times U$

(1b) subject to $y \in K$, $\langle A(u)y, v-y\rangle \geq (f(u), v-y) \quad \forall v \in K$,

with $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, a bounded and sufficiently smooth domain, H a suitable Hilbert space,

$$K = \{v \in H | v \geq 0\}$$

and U denoting the Hilbert space of controls. Instances are $H = H_0^1(\Omega)$ and $U \subseteq L^2(\Omega)$. Further, the objective J is assumed to be sufficiently smooth, the operator $A(u)$ maps from H to its dual space H^*, and $f \in \mathcal{L}(U, L^2(\Omega))$.

Typical applications which lead to problems of structure (1) come from engineering sciences (like, e.g., elastohydrodynamic lubrication of rolling element bearings [2]), or, also in a transient regime, mathematical finance in form of volatility estimation in the Black-Scholes model for American options; [1].

It is well known that due to the variational inequality-type constraint, problem (1) does not satisfy any of the classical constraint qualifications which would guarantee the existence of Lagrange multipliers. This fact necessitates an independent approach for the proof of the existence of Lagrange multipliers associated to (1). We also point out that the first order necessary systems derived in the literature might accept spurious stationary points; see, e.g., the recent papers [3,6]. In fact, the critical set in the context of the derivation of first order conditions is the so-called *bi-active* set B, i.e., the set where

$$y^* = 0 \quad \text{and} \quad \lambda^* = 0 \text{ a.e. on } B,$$

simultaneously. If this set would have measure zero, then typically the overall problem (and the existence proof of multipliers) is more amenable. In this case, in finite dimensions one obtains that, under the assumption that the discretization of $A(u)$ is well behaved, the solution $y(u)$ of the discretized lower level problem is Fréchet-differentiable with respect to u [8, Thm. 4.2.28]; otherwise one can at best guarantee directional differentiability. Usually, however, B has positive measure and therefore has to be taken into account.

In this presentation a new first order optimality characterization is discussed which involves dual quantities (multipliers) associated to $y^* = 0$ and $\lambda^* = 0$ on B satisfying certain sign conditions, respectively. For $H = H_0^1(\Omega)$ and $A(u)$ a second order linear elliptic partial differential operator, the multipliers associated to $\lambda^* = 0$ is in $L^2(B)$ while the quantity associated to $y^* = 0$ is only a measure.

The technique of proof hinges on the notion of *pieces*. Given some subset $A \subset \Omega$, a piece problem is a problem associated to (1) with

$$y = 0, \quad \lambda \geq 0 \text{ on } A,$$
$$\lambda = 0, \quad y \geq 0 \text{ on } I = \Omega \setminus A,$$

and the variational inequality in (1) is replaced by

$$A(u)y - \lambda = f(u) \text{ in } H^*.$$

Notice that the resulting problem resembles a state-constrained optimal control problem; [4]. Pieces are called *adjacent* if they differ only on a subset of the bi-active set.

In a second part, an algorithmic framework is considered. It is based on a *feasibility restoration* phase and a *(semismooth) Newton* step: In its core, given u, first a solution of the variational inequality is computed. Then, based on this solution and neglecting the conditions on the bi-active set, a linearization of the first order system is performed. The solution of this system provides a search direction for updating the current primal and dual variables. If the algorithm stops, then the sign conditions of the multipliers on the current bi-active set are checked. As long as these conditions are not satisfied, a new piece is defined and the algorithm is continued; otherwise it stops at a stationary point in the sense of the new first order conditions.

Particular attention is payed to the solver for the variational inequality in the restoration step. Here, the focus will be given on variational inequalities where $A(u)$ is a second order linear elliptic differential operator with smooth coefficients, and $H = H^1_0(\Omega)$. In this case, the variational-inequality in (1) denotes the first order necessary and sufficient condition for

$$\text{minimize } \langle \frac{1}{2}A(u)y - f(u), y\rangle \tag{2a}$$
$$\text{subject to } y \geq 0 \text{ a.e. in } \Omega \tag{2b}$$

and (1) becomes a *bi-level* optimization problem. The proposed solution algorithm for the lower level problem (2) is a semismooth Newton path-following method. In a first step, (2) is regularized by considering

$$\text{minimize } \langle \frac{1}{2}A(u)y - f(u), y\rangle + \frac{1}{2\gamma}\| \max(0, \bar{\lambda} - \gamma y)\|^2_{L^2(\Omega)}. \tag{3}$$

Notice that the inequality constraint in (2) is softened by adding a quadratic relaxation term replacing the hard constraint. The parameter $\gamma \in (0, \infty)$ denotes the so-called path- (or penalty) parameter. It induces a Lipschitz continuous *primal-dual path*

$$\{(y_\gamma, \lambda_\gamma) : \gamma \in (0, \infty)\},$$

where y_γ solves (3), and $\lambda_\gamma = \max(0, \bar{\lambda} - \gamma y_\gamma)$ approximates the Lagrange multiplier λ^* associated to the inequality constraint in (2) at its solution $y^* \geq 0$. It is shown that by solving (3) as $\gamma \to \infty$ a solution of the original problem (2) is approach, and that for every fixed γ the multiplier λ_γ enjoys more regularity than λ^*.

For the solution of the first order system of (3), which is given by

$$A(u)y_\gamma - \lambda_\gamma = f(u),$$
$$\lambda_\gamma = \max(0, \bar{\lambda} - \gamma y_\gamma),$$

a semismooth Newton method is used and, for fixed γ, its locally superlinear convergence in function space is shown. The latter analysis relies on the concept of slant differentiability and the particular problem structure; see [5, 7].

Based on an analysis of the smoothness properties of the primal-dual path, differentiability properties of the value function

$$V(\gamma) = \langle \frac{1}{2}A(u)y_\gamma - f(u), y_\gamma\rangle + \frac{1}{2\gamma}\|\max(0, \bar{\lambda} - \gamma y_\gamma)\|^2_{L^2(\Omega)}$$

are proved. These considerations allow to introduce a model function which closely approximates the behavior of V along the primal-dual path and which serves as a reliable tool for updating the path-parameter γ.

Further, the proposed algorithm relies on an inexact path-following technique. Here, the "inexactness" refers to the fact that early along the major iterations, i.e., for small γ when solving (3), it is sufficient that the iterates stay in a certain neighborhood of the path. This neighborhood, however, becomes smaller as γ is enlarged. Finally, the efficiency of the new path following method and the overall solution procedure is demonstrated for a class of parameter estimation problems.

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Control of Crystal Growth Processes

Michael Hinze
(joint work with Günter Bärwolff, Ulrich Matthes, Axel Voigt, and Stefan Ziegenbalg)

Crystal growth technology is essential for the developments in microelectronics, optical communication, laser technology and numerous other high technologies. An increase in crystal sizes and demands for structural perfection, homogeneity and defect control will even increase the importance of reliable crystal growth technologies in the future. The economic pressure requires increase of yields at optimized performance.

Crystal growth processes involve many different related physical mechanisms which interact on very different spatial and temporal scales. Mathematically, this is expressed by a hierarchy of weakly coupled models of pdes. In model-based simulation this weak coupling of the different components in the models can be used to derive algorithms which relate microscopic crystal properties to macroscopic growth parameters. Model-based optimization is 'dual' to the model-based simulation in the sense that its hierarchy is directed from specific microscopic to more general macroscopic pde models. Output variables of the different models in the simulation loop take now the role of gains, and input variables those of control parameters. In this spirit model-based optimization and design of crystal growth processes are formidable tasks whose solution not only neccesitates interdisciplinary efforts of applied mathematicians and process engineers. It moreover also serves as prototyping test application for the development and specification of new mathematical and numerical approaches in the emerging field of optimization problems with systems of coupled nonlinear pdes.

As model design applications we discuss control of the crystal melt considering as mathematical model the Boussinesq approximation, control of solidification for a sharp interface model, and control of weakly conductive fluids by Lorentz forces.

Control of the crystal melt: The flow in the crystal melt is gouverned by the Boussinesq approximation of the Navier-Stokes system for the velocity $\vec{u} = (u, v, w)$, the pressure p and the temperature θ;

$$(1) \qquad \left\{ \begin{array}{rll} \vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} - \Delta\vec{u} + \nabla p - Gr\,\theta\,\vec{g} & = 0 & \text{on } \Omega_T, \\ -\text{div}\,\vec{u} & = 0 & \text{on } \Omega_T, \\ \theta_t + \vec{u}\cdot\nabla\theta - \frac{1}{Pr}\Delta\theta - f & = 0 & \text{on } \Omega_T. \end{array} \right.$$

Here $\vec{g} = (0, 0, 1)$, and $\Omega_T = \Omega \times (0, T)$ denotes the space-time cylinder with cylindrical melt zone of height H and radius R. Furthermore, Gr denotes the Grashof number, and Pr the Prandtl number. Since here we are interested in control via boundary temperatures the absence of external forces is assumed. System (1) is supplied with temperature boundary conditions of third kind on the crucible walls (which form the control boundary Γ_c), at the solid-liquid interface Γ_d the melting temperature is prescribed, and Dirichlet boundary conditions at the remaining parts of the boundary. For the flow Dirichlet boundary conditions are prescribed on the whole boundary Γ. This leads to

$$(2) \qquad \begin{cases} u = u_d,\ v = v_d,\ w &= w_d \quad \text{on } \Gamma_T, \\ a\frac{\partial\theta}{\partial \mathbf{n}} + b\theta &= \theta_c \quad \text{on } \Gamma_{cT}, \\ \theta &= \theta_d \quad \text{on } \Gamma_{dT}, \end{cases}$$

where $\Gamma_T := \Gamma \times [0,T]$, and a, b denote physical constants which may not vanish simultaneously. System (1) is further supplied with appropriate initial values for the velocity and temperature. We note that it is possible to include via u_d, v_d, w_d certain crystal and crucible rotations, as it is common in the case of Czochralski growth. In the case of zone melting techniques one would require $\vec{u} = \vec{0}$ on Γ_T. The material properties and the dimensionless parameters depend on the specific application and have to be defined appropriately. The optimization problem now is given by

$$\min_{\vec{u},\theta_c} J(\vec{u},\theta_c) \text{ s.t. } (1) - (2),$$

where the cost functional J models the control gain and costs. Examples of control of Czochralski growth and floating zone devices for realistic material parameters can be found in [1], [2]. Model predictive control of the Boussinesq approximation is investigated in [3].

Control of solidification: Let $\Omega := S \times (0, X_{n+1}) \subset \mathbb{R}^{n+1}$ denote the container with the substrate, where $S := (0,X_1) \times \cdots \times (0,X_n)$. For $t \in [0,T]$ we denote by $\Omega_s(t), \Omega_l(t) \subset \Omega$ the parts containing the solid and the liquid phases, where we assume that $\Omega_s(t) \cap \Omega_l(t) = \emptyset$, and $\overline{\Omega} = \overline{\Omega_s(t)} \cup \overline{\Omega_l(t)}$. The free boundary of dimension n then is defined by $\Gamma(t) = \overline{\Omega_s(t)} \cap \overline{\Omega_l(t)}$ and in our approach is modeled as a graph $\Gamma(t) = \{(y, f(t,y)) : y \in S\}$. As mathematical model for the solidification process we take the Stefan problem

$$(3) \qquad \begin{cases} \partial_t u &= \frac{k_s}{c_s\rho}\Delta u \ \text{ in } \Omega_s \text{ and } \partial_t u = \frac{k_l}{c_l\rho}\Delta u \ \text{ in } \Omega_l, \\ V_\Gamma &= k_s\partial_\mu u\big|_{\Omega_s} - k_l\partial_\mu u\big|_{\Omega_l} \text{ on } \Gamma, (\text{ where } V_\Gamma = \frac{f_t}{\sqrt{1+|\nabla f|^2}}), \\ \partial_\eta u &= \frac{\alpha_{s/l}}{k_{s/l}}(u_{b0} + \beta u_{bc} - u) \text{ on } \partial\Omega, \text{ and} \\ u &= u_0 \text{ in } \Omega \text{ and } \Gamma(0) = \Gamma_0. \end{cases}$$

In this model $k_{s/l}$ denote the heat conductivities, $c_{s/l}$ the specific heat in the solid and liquid part, respectively, $\alpha_{s/l}$ the heat transfer coefficients, ρ the density of the substrate, and μ denotes the normal on $\Gamma(t)$ directed from the solid to the liquid phase. Further, u_{bc} denotes the control temperature on the container wall, and u_{b0} some temperature field from experience. For tracking a given evolution

$\bar{f}(t,y)$ the optimization problem is given by

$$\min_{f,u_{bc}} J(f,u_{bc}) := \frac{1}{2}\int_0^T\int_S (f-\overline{f})^2 + \frac{\lambda_1}{2}\int_0^T\int_{\partial\Omega}\beta^2 u_{bc}^2 + \frac{\lambda_2}{2}\int_S (f-\overline{f})^2\big|_{t=T} + \frac{\lambda_3}{2}\int_0^T\int_\Gamma \frac{1}{\left(\partial_\mu u\big|_{\Omega_s}+\partial_\mu u\big|_{\Omega_l}\right)^2} \text{ s.t. (3).}$$

We have not included the Gibbs-Thomson law in our mathematical model, since it describes effects on the meso- and micro-scale, whereas the boundary control acts on the macro scale. However, the λ_3-addend in the cost functional prevents the regions along the free boundary where dendritic growth may occur from getting large. For a discussion and more details we refer to [4].

Control of weakly conductive fluids: Weakly conductive fluids like sea water and other electrolytes can be controlled by means of Lorentz forces

$$F_L = J \times B,$$

which exponentially decay into the fluid. Here J is the current density and B denotes the magnetic induction. Practically Lorentz forces can be generated by certain electrode-magnet arrangements. Since the magnetic Reynolds number Re_m (of order 10^{-12} for electrolytes) and the conductivity of weakly conductive fluids are very small F_L in this case can be modeled as an external force. As mathematical model for the controlled flow in the domain Ω over the time horizon $[0,T]$ we therefore take the incompressible Navier-Stokes system with volume force F_L;

$$\text{(NS)}\begin{cases} y_t - \nu\Delta y + (y\nabla)y + \nabla p &= F_L \quad \text{in } Q := (0,T)\times\Omega,\\ -\nabla\cdot y &= 0 \quad \text{in } Q,\\ y(0) &= y^0 \quad \text{in } \Omega.\end{cases}$$

For F_L we make the Ansatz

$$F_L(t,x) = \sum_{i=1}^m u_i(t)F_i(x)e^{-\mathrm{dist}[x,\partial\Omega]},$$

with F_i denoting spatial vector fields modeling the direction of the Lorentz force, and control variables u_i modeling time dependent amplitudes related to the electric current. The optimization problem then is given by

$$\min_{y,u} J(y,u) \text{ s.t. } (NS), \text{ and } -a \le u_i(t) \le a \text{ for all } t.$$

Here the cost functional J again models the control gain and costs, and a denotes the maximal absolute value of admissible amplitudes. For details we refer to [5].

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Optimal Control of Solid-Solid Phase Transitions Including Mechanical Effects

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(joint work with Wolf Weiss)

During the last 15 years, the thermomechanical modeling of phase transitions in steel became an active research topic of physical metallurgy (cf., eg., [1–3] and the references therein). It seems that there is no unified thermomechanical model at hand so far that is well accepted and that allows to reproduce all experiments. However, it is quite clear what the principle effects are that a macroscopic model should account for:

- The metallurgical phases have material parameters with different thermal characteristics, hence their effective values have to be computed by a mixing rule.
- The different densities of the metallurgical phases result in a different thermal expansion. This thermal and transformation strain is the major contribution to the evolution of internal stresses during heat treatments.
- Experiments with phase transformations under applied loading show an additional irreversible deformation even when the equivalent stress corresponding to the load is far below the normal yield stress. This effect is called *transformation-induced plasticity.*
- The irreversible deformation leads to a mechanical dissipation that acts as a source term in the energy balance.
- The internal stresses influence the transformation kinetics.

Assuming that the density only depends on the different volume fractions via a mixture ansatz as well as disregarding the mechanical contribution to the phase transition kinetics leads to the folowing model which has been investigated in [4]: Find volume fractions $z = (z_1, \ldots, z_5)$, a stress tensor σ, a displacement field u and a temperature field θ, such that the following system is satisfied:

$$z(t) = \mathcal{P}[\theta](t), \tag{1}$$

$$\operatorname{div} \sigma = f, \tag{2}$$

$$\varepsilon(u) = C(z)\sigma + \varepsilon^{th} + \int_0^t \Lambda(z)_\xi S(\xi)\, d\xi, \tag{3}$$

$$(4) \quad \rho(z)c_\varepsilon\theta_t - \operatorname{div}\Big(k(\theta)\operatorname{grad}\theta\Big) = -\rho L(z)z_{1,t} + \sigma : \varepsilon_t^{th} + \Lambda(z)_t\,|S|^2 + d\,w.$$

The operator equation (1) describes the evolution of the volume fractions $z = (z_1, \ldots, z_5)$ depending on the temperature θ. Typically, the operator $\mathcal{P}$ is given as the solution operator to an ordinary differential equation. Equation (2) is the usual quasistatic momentum balance with stress tensor σ and an external force f. The linearized strain tensor $\varepsilon(u)$ is defined by

$$\varepsilon(u)_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad 1 \le i, j \le 3.$$

The constitutive equation (3) is derived from Hooke's law under the assumption that the overall strain $\varepsilon(u)$ can be additively decomposed into an elastic part, $C(z)\sigma$, a thermal one, ε^{th} and one that stems from the transformation induced plasticity, which is given by $\int_0^t \Lambda(z)_\xi S(\xi)\,d\xi$. Here, $C(z)$ is the inverse of the stiffness tensor, $\Lambda(z)$ is a coefficient allowed to depend on the phase volume fractions, and S is the trace-free part of σ. The last equation is the energy balance, with the density ρ, heat capacity at constant strain c_ε, and the latent heat L. The heat source w, multiplied by a coefficient $d = d(x,t)$, may serve as a distributed control. The main analytical difficulty of the coupled system stems from the quadratic nonlinearity in S on the right-hand side of (4). Adding appropriate initial and boundary conditions it has been shown in [4] that (1)–(4) admit a weak solution, while uniqueness is still an open problem.

In the heat treatment of steel the goal is to obtain a desired distribution of phases at end-time T. This corresponds to minimizing the cost functional

$$J(d) = \frac{\alpha_1}{2}\int_\Omega (z(T) - z_d)^2$$

subject to (1)–(4). In [7] this control problem has been investigated disregarding the anisotropic strain component in (3).

An important heat treatment technology is the laser hardening of steel. Neglecting mechanical effects this problem has been investigated in [5] using model reduction. An even more efficient approach based on PID- control has been described in [6] including a strategy for the interplay between simulation and machine based control which has also been verified by experiments.

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Convergence Analysis of an Adaptive Finite Element Method for Distributed Control Problems with Control Constraints

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(joint work with Michael Hintermüller)

Although adaptive finite element methods are widely and successfully used for the efficient numerical solution of PDE related problems [1–3, 9, 12], very little is known so far with regard to conditions that guarantee an error reduction and thus lead to a convergent scheme. A convergence analysis of adaptive finite element methods for standard Lagrangian type finite element discretizations of second order elliptic boundary value problems has been performed in [8, 11]. Very recent results, obtained in cooperation with C. Carstensen [5–7], deal with adaptive edge element discretizations, adaptive mixed and adaptive nonconforming finite element approximations. As far as the a posteriori error analysis of adaptive finite element schemes for optimal control problems is concerned, the unconstrained case has been addressed in [4] (cf. also [3]). In the control constrained case, residual-type a posteriori error estimators have been derived and analyzed in [10].
In this contribution, we are concerned with the convergence analysis of an adaptive finite element method for distributed optimal control problems with control constraints of the form

$$\text{(1a)}\qquad \text{minimize } J(y,u) \;:=\; \frac{1}{2}\,\|y-y^d\|^2_{0,\Omega} \;+\; \frac{\alpha}{2}\,\|u-u^d\|^2_{0,\Omega}$$

$$\text{over } (y,u)\in H^1_0(\Omega)\times L^2(\Omega),$$

$$\text{(1b)}\qquad \text{subject to } -\Delta\, y \;=\; f \;+\; u\;,$$

$$\text{(1c)}\qquad u\in K \;:=\; \{v\in L^2(\Omega)\mid v\le\psi \text{ a.e. in } \Omega\}\;.$$

Here, $\Omega\subset\mathbb{R}^2$ is a bounded, polygonal domain with boundary $\Gamma := \partial\Omega$. Moreover, we suppose that

$$\text{(2)}\qquad u^d, y^d\in L^2(\Omega)\;,\; f\in L^2(\Omega)\;,\; \psi\in L^2(\Omega)\;,\; \alpha\in\mathbb{R}_+\;.$$

It is well-known that under the assumption (2) the distributed optimal control problem (1a)-(1c) admits a unique solution $(y,u)\in H^1_0(\Omega)\times L^2(\Omega)$ which is characterized by the existence of an adjoint state (co-state) $p\in H^1_0(\Omega)$ and a Lagrange multiplier for the inequality constraint (co-control) $\sigma \in L^2(\Omega)$ such that

(3a) $$a(y,v) \ = \ (f+u,v)_{0,\Omega} \quad , \quad v \in H_0^1(\Omega) \ ,$$

(3b) $$a(p,v) \ = \ -\ (y-y^d,v)_{0,\Omega} \quad , \quad v \in H_0^1(\Omega) \ ,$$

(3c) $$u \ = \ u^d \ + \ \frac{1}{\alpha}\ (p-\sigma) \ ,$$

(3d) $$\sigma \in \partial I_K(u) \ .$$

Here, $a(\cdot,\cdot)$ stands for the bilinear form

$$a(w,z) \ := \ \int\limits_\Omega \nabla w \cdot \nabla z \ dx \quad , \quad w,z \in H_0^1(\Omega) \ ,$$

and $\partial I_K : L^2(\Omega) \to 2^{L^2(\Omega)}$ denotes the subdifferential of the indicator function I_K of the constraint set K.

The control problem is discretized with respect to a shape regular simplicial triangulation of the computational domain using continuous, piecewise linear finite elements for the state and the adjoint state and elementwise constant approximations of the control and the adjoint control.

The methods provide an error reduction and thus guarantee convergence of the adaptive loop which consists of the essential steps 'SOLVE', 'ESTIMATE', 'MARK', and 'REFINE'. Here, 'SOLVE' stands for the efficient solution of the finite element discretized problems. The following step 'ESTIMATE' is devoted to a residual-type a posteriori error estimation of the global discretization errors in the state, the adjoint state, the control and the adjoint control. A bulk criterion is the core of the step 'MARK' to indicate selected edges and elements for refinement, whereas the final step 'REFINE' deals with the technical realization of the refinement process itself.

The residual-type a posteriori error estimator consists of edge and element residuals, a complementarity consistency error term and lower and higher order data oscillations.

The main result states conditions that guarantee an error reduction of the error in the state, the adjoint state, the control, and the adjoint control and thus establishes convergence of the adaptive scheme. The proof of this result is based on three significant properties: the first one is the reliability of the error estimator, that is, it provides an upper bound for the global discretization errors. The second one is what is called strict discrete local efficiency. Here, it has to be shown that the components of the error estimator can be bounded locally by the respective norms of the differences of the coarse and fine mesh finite element approximations of the state, the adjoint state, the control, and the adjoint control. Finally, the third important property is a perturbed Galerkin orthogonality of the finite element approximations. We will show that an error reduction can be achieved by a subtle interaction of these three basic properties. The proof does not require any regularity of the solution.

Numerical results illustrate the performance of the error estimator.

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Modeling and Controllability of Linked Elastic Structures of Differing Dimensions

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(joint work with Günter Leugering)

Controllability and stability properties of linked structures composed of multiple elastic elements give rise to an abundance of mathematical challenges. When a structure is composed of a number of interconnected elastic elements, the behavior becomes much harder to both predict and to control. It cannot simply be considered as a single large flexible structure, because the effects of each element upon the next are critical in determining its motion. While it may be known that a single element is exactly controllable with an appropriate choice of boundary feedback, a connected system composed of the same types of elements may not even be approximately controllable due to issues arising as energy is transmitted across the joints. Yet flexible structures consisting of a combination of strings, beams, plates and shells arise in many applications, including but not limited to trusses, robot arms, solar panels and suspension bridges [3].

To construct feasible models, it is typically necessary to return to the basic kinematic hypotheses and apply Hamilton's Principle to derive the equations of motion and the associated junction conditions at the interfaces [2]. Once appropriate models are developed, the use of boundary control becomes even more important in linked dynamic systems. Within a multi-link structure, the natural locations to implement control are at the joints or edges of the structure. Initially, two plate-beam configurations are considered, the first consisting of a beam orthogonally attached to the edge of the plate and the second composed of a beam whose centerline is orthogonal to one of the faces of the plate. Nonlinear plate-beam systems are motivated by issues arising from large-amplitude periodic oscillations, but such systems have not been seen in the literature.

A model comprised of a nonlinear von Kármán plate coupled with a nonlinear beam equation is developed from first principles [1]. Dynamic junction conditions are imposed at the interface. Wellposedness is established by first considering a corresponding linear problem, then applying a perturbation theorem for nonlinear semigroups. Proof of regularity takes advantage of elliptic theory, as well as the regularity of the Airy's stress function. Stabilization through the use of velocity feedback along the boundary of the plate (and, possibly, at the free end of the beam) is based on energy methods and critically relies on unique continuation properties for the system. Without unique continuation of the solution, the traditional alternative has been to impose strict restrictions on the domain, e.g., assuming the domain is either convex or star-shaped. However, no unified theory exists for unique continuation and results are highly dependent on the model under consideration.

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Applications of Semi–Smooth Newton Method to Variational Inequalities

Kazufumi Ito

This talk discusses semismooth Newton methods for solving nonlinear non-smooth equations in Banach spaces. These investigations are motivated by complementarity problems, variational inequalities and optimal control problems with control or state constraints, for example. The function $F(x)$ for which we desire to find a root is typically Lipschitz continuous but not C^1 regular. A generalized Newton iteration for solving the nonlinear equation $F(x) = 0$ with is defined by

$$x^{k+1} = x^k - V_k^{-1} F(x^k), \text{ where } V_k \in \partial_B F(x^k). \tag{1}$$

where a generalized Jacobian $V_k \in \partial_B F(x^k)$ in the finite dimensional case. Local convergence of $\{x^k\}$ to x^*, a solution of $F(x) = 0$, is based on the following concepts;

$$|F(x^* + h) - F(x^*) - V\,h| = o(|h|), \tag{2}$$

where $V = V(x^* + h) \in \partial_B F(x^* + h)$, for h in a neighborhood of x^*. Thus, letting $h = x^k - x^*$ and $V^k = V(x^k)$ we have

$$|x^{k+1} - x^*| = |V_k^{-1}(F(x^k) - F(x^*) - V_k(x^k - x^*))| = o(|x^k - x^*|).$$

The condition (2) is equivalent to the semismoothness of F at x^* under appropriate assumptions in the finite dimensional case. This leads to the following definition in Banach spaces X, Z.

Definition 1. *(a) Let $D \subset X$ be an open set. $F\colon D \subset X \to Z$ is called Newton differentiable at x, if there exists an open neighborhood $N(x) \subset D$ and mappings $G\colon N(x) \to \mathcal{L}(X, Z)$ such that*

$$\lim_{|h| \to 0} \frac{|F(x+h) - F(x) - G(x+h)h|_Z}{|h|_X} = 0.$$

The family $\{G(x) : x \in N(x)\}$ is called a N-derivative of F at x.
(b) F is called semismooth at x, if it is Newton differentiable at x and

$$\lim_{t \to 0^+} G(x + t\,h)h \quad \textit{exists uniformly in } |h| = 1.$$

Examples which motivate our study include nonlinear variational inequalities of the form: find $x \in C$ such that

$$(f(x), y - x) \geq 0 \quad \text{for all} \quad y \in C,$$

where C is a closed convex set in a Hilbert X. It can equivalently be written as

$$F(x) = x - Proj_C(x - f(x)) = 0, \tag{3}$$

where $Proj_C$ is the projection of X onto C. In particular, if C is a hypercube $\{x|\, \phi \leq x \leq \psi\}$ in $X = L^2(\Omega)$, with $\phi \leq \psi$ and the inequalities are defined pointwise, then (3) can be expressed as

$$F(x) = f(x) + \max(0, -f(x) + x - \psi) + \min(0, -f(x) + x - \phi) = 0. \tag{4}$$

Our particular interest in (1) is due to the fact that the primal-dual active set strategy for (4) is a specific semi-smooth Newton method in case f is linear. The primal-dual active set strategy is known to be extremely efficient for solving discretized variational inequalities and constrained optimal control problems.

The globalization of the iteration (1) in R^m on the basis of the merit functional $\theta(x) = |F(x)|^2$ is achieved by

Algorithm 1. *Let* $\beta,\ \gamma \in (0,1)$ *and* $\sigma \in (0,\bar{\sigma})$. *Choose* $x^0 \in R^m$ *and set* $k = 0$. *Given* x^k *with* $F(x^k) \neq 0$. *Then:*
(i) If there exits a solution h^k *to*

$$V_k\, h^k = -F(x^k)$$

with $|h^k| \leq b|F(x^k)|$, *and if further*

$$|F(x^k + h^k)| < \gamma\, |F(x^k)|\,,$$

set $d^k = h^k$, $x^{k+1} = x^k + d^k$, $\alpha_k = 1$, *and* $m_k = 0$.

(ii) Otherwise choose $d^k = d(x^k)$ *according to (A.2) and let* $\alpha_k = \beta^{m_k}$, *where* m_k *is the first positive integer* m *for which*

$$\theta(x^k + \beta^m\, d^k) - \theta(x^k) \leq -\sigma\beta^m\, \theta(x^k).$$

Set $x^{k+1} = x^k + \alpha_k\, d^k$,

under the following assumptions:

- (A.1) $S = \{x \in R^m : |F(x)| \leq |F(x^0)|\}$ is bounded.
- (A.2) There exist $\bar{\sigma}$ and $b > 0$ such that for each $x \in S$ there exists $d = d(x) \in R^m$ satisfying

(1) $$\theta'(x; d) \leq -\bar{\sigma}\theta(x) \quad \text{and} \quad |d| \leq b\,|F(x)|.$$

- (A.3) The following closure property holds: if $x_k \to \bar{x}$ and $d(x_k) \to \bar{d}$ with $x_k \in S$, then $\theta'(\bar{x}; \bar{d}) \leq -\bar{\sigma}\theta(\bar{x})$.
- (A.4) θ is subdifferentiably regular for all $x \in S$, i.e., $\theta^o(x; d) = \theta'(x; d)$ for all $d \in R^m$.

Concerning conditions (A.2) and (A.3) we introduce the notions of quasi-directional derivative. This will allow us to construct descent directions which satisfy these two conditions. Combining conditions (A.1)-(A.4) and the notion of quasi-directional derivative provides us with a rather axiomatic approach to globalization of the semi-smooth Newton method based on the norm functional as merit functional.

Some Inverse Problems in Piezoelectricity

Barbara Kaltenbacher
(joint work with Manfred Kaltenbacher, Tom Lahmer, and Marcus Mohr)

Piezoelectric transducers that transform mechanical into electric energy and vice versa, play an important role in many technical applications ranging from ultrasound generation in medical imaging and therapy to injection valves in automotive industry. For the development of piezoelectric sensors and actuators by means of numerical simulation, precise knowledge of the elastic stiffness coefficients, the dielectric coefficients and the piezoelectric coupling coefficients is necessary.

The topic of this talk is identification of these material parameters, that appear as coefficients in a system of partial differential equations modelling piezoelectric behaviour. Especially, we focus on the nonlinear situation, where due to large excitations, some of the material parameters are not constants any more but but depend on the electric field strength.

Starting with the constant coefficient case, we show different possible formulations of the forward problem as a system of coupled PDEs and discuss well-posedness as well as fast solution approaches. Concerning the inverse problem, identifiability of the material parameters from non-eigenfrequencies is shown, which leads to a PDE based approach for determining these material parameters, as opposed to the conventional purely experimental scheme that is based on measurements of resonance frequencies.

In the nonlinear case one is faced with the infinite dimensional problem of determining the functional dependence of the material parameters on the states from the given measurements. This amounts to a parameter identification problem for a nonlinear system of PDEs. We discuss the question of identifiability by means of a one-dimensional model problem and point out the inherent instability of the identification problem. Moreover, we propose a reconstruction method based on a frequency domain formulation of the PDEs using a multiharmonic ansatz for the states. Here, regularization is introduced by an appropriate kind of time discretization as well as early stopping of the Newton type method that is employed for inverting the nonlinear parameter-to measurement-map.

The last part of the talk is devoted to a discussion of first ideas on an important future research issue in this context, namely the characterization of hysteresis in piezoelectric transducers.

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Boundary Feedback Stabilization of a Navier Stokes Flow

Irena Lasiecka

(joint work with Viorel Barbu and Roberto Triggiani)

We consider a Navier Stokes flow defined on a simply connected bounded domain $\Omega \subset R^d$, $d = 2, 3$ with a smooth boundary Γ and boundary control u in the no-slip (Dirichlet) boundary conditions:

$$
\begin{aligned}
y_t - \nu_0 \Delta y + (y \cdot \nabla) y = \nabla p_1 + f, &\quad in\ \Omega \times (0, \infty) \\
\nabla \cdot y = 0, &\quad in\ \ \Omega \times (0, \infty) \\
y = u, &\quad on\ \ \Gamma \times (0, \infty) \\
y(0) = y_0, &\ \ in\ \Omega
\end{aligned} \tag{1}
$$

Our aim is to locally stabilize the flow in a neighborhood of an unstable (steady-state) equilibrium y_e. Thus let $y_e \in H^2(\Omega) \cap V$ and $p_e \in H^1(\Omega)$ satisfy

$$(2) \qquad -\nu_0 \Delta y_e + (y_e \cdot \nabla) y_e = \nabla p_e + f, \quad \nabla \cdot y_e = 0 \;\; in\ \Omega, \quad y_e = 0 \; on\ \Gamma$$

Here and below, for simplicity, we omit the notation $(.)^d$ to denote function spaces of d-vectors. We seek a feedback operator $F : Y \to U \subset L_2(\Gamma)$ such that the control law $u = F(y - y_e)$ stabilizes the flow exponentially in the neighborhood of the unsteady equilibrium y_e. The choices of the state space W and control spoace U are important. In what follows we shall take:

$$W \equiv H^{1/2-\epsilon}(\Omega) \cap H, \quad when\ d = 2, \quad and \quad W \equiv H^{1/2+\epsilon}(\Omega) \cap H, \quad when\ d = 3$$

where $0 < \epsilon < 1/2$, and

$$H \equiv \{y \in L_2(\Omega);\ \nabla \cdot y = 0\ in\ \Omega, \quad y \cdot \nu = 0 \quad on \quad \Gamma\}, \quad V = H^1_0(\Omega) \cap H$$

The control space is given by: $\quad U = \{u \in L_2(\Gamma); u \cdot \nu = 0\}$

Thus, we restrict the class of boundary controls to have purely tangential action. It is natural to consider the linearization around the equilibrium $v \equiv y - y_e$ with the resulting pressure $p = p_1 - p_e$. After taking the Leray's projection $P : L_2(\Omega) \to H$ and noting that $Pv = v$ we are led to consider

$$\begin{aligned} v_t - \nu_0 P \Delta v + P[(y_e \cdot \nabla) v + (v \cdot \nabla) y_e] = 0, \quad in\ \Omega \times (0, \infty) \\ \nabla \cdot v = 0, \quad in \quad \Omega \times (0, \infty) \\ v = u, \quad on \quad \Gamma \times (0, \infty) \end{aligned} \qquad (3)$$

It is known that the Stokes operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset H \to H$ given by

$$\mathcal{A} v \equiv \nu_0 P[\Delta v - P(y_e \cdot \nabla) v + (v \cdot \nabla) y_e], \quad \mathcal{D}(\mathcal{A}) \equiv H^2(\Omega) \cap V$$

generates an analytic semigroup on H. Hence, the fractional powers $(cI - \mathcal{A})^\theta$ are well defined on H for a suitable translation c. Moreover, we have $\mathcal{D}((cI - \mathcal{A})^\theta) = H^{2\theta}(\Omega) \cap H; \quad \theta < 1/4$.

Let the operator $D : U \to H$ be the Dirichlet map, ie the extension of the boundary data into the interior via the steady state problem. With the above notation the standard boundary control model corresponding to (3) [2] takes the form;

$$(4) \qquad v_t = \mathcal{A}(v - Du), \quad v(0) = v_0$$

We are ready to state our main result which asserts the existence of a feedback operator stabilizing the nonlinear model, provided that the initial condition is sufficiently close to the unstable equilibrium point y_e. We shall consider separately the three and two dimensional cases.

1. Three dimensional case

Let $R : W = H^{1/2+\epsilon}(\Omega) \cap H \to W'$ be the Riccati operator (positive self-adjoint on H) which defines the value function corresponding to the following minimization problem

$$MinJ(u,v) \equiv \int_0^\infty |v(t)|^2_{H^{3/2+\epsilon}(\Omega)} + |u(t)|^2_U dt$$

subject to the condition that $v(u)$ satisfies (3) with $u \in L_2(0,\infty;U)$.

It is shown in [2] (a highly non-trivial fact, which in turn rests on the interior stabilization results of [3]) that the functional $J(u,v(u))$ is proper in the sense of optimization theory; that is, the Finite Cost Condition of the above optimal control problem is satisfied [4]. As a consequence, the Riccati operator R defining the value function is well defined $R : W \to W'$ and in fact is an isomorphism between these two spaces. This then allows one to define an equivalent norm on the state space W by $|y|_W \sim |Ry|_H$. The main result is the statement that the operator

$$u = F(y-y_e) \equiv \nu_0 \partial_\nu R(y-y_e), \quad on\ \Gamma$$

stabilizes exponentially the flow (1). This is to say that the feedback control given by $u = \nu_0\partial_\nu R(y-y_e)$ when inserted into the dynamics (1) gives the following estimate for the nonlinear solutions

Theorem 1. *[2] Under the above setting, the (unbounded, densely defined) feedback $F = \nu_0\partial_\nu R$ leads to a strongly continuous semigroup generated by $\mathcal{A}(I-DF)$ on the space $W = H^{1/2+\epsilon}(\Omega)\cap H$. Moreover, $u = F(y-y_e)$ stabilizes exponentially problem (1) in a sufficiently small neighborhood of the equilibrium state y_e;ie for $y_0 \in \mathcal{V}(\rho, y_e) \equiv \{y \in W : \quad |y-y_e|_W \leq \rho\}$, for sufficiently small ρ. In this case we then have*

$$|y(t)-y_e|_{H^{1/2+\epsilon}(\Omega)} \leq Ce^{-\omega t}|y_0-y_e|_{H^{1/2+\epsilon}(\Omega)}$$

where the constant ω is independent of ρ.

In addition to the above pointwise stability result in W, the following additional regularity of the flow holds true: $y \in L_2(0,\infty;H^{3/2+\epsilon}(\Omega))$, and $u = F(y-y_e) \in L_2(0,\infty;H^{1+\epsilon}(\Omega))$.

2. Two dimensional case

The two-dimensional case is, as expected, more regular. In fact, we can work within a lower topology and we select the state space $W \equiv H^{1/2-\epsilon}(\Omega)\cap H$, where $0 < \epsilon < 1/2$. The control space is the same as in the three-dimensional case. Since in the two dimensional case the state space W is below compatibility conditions, we may have several choices for the Riccati feedback opertors.

In analogy with the 3-d case, we may take $R : W = H^{1/2-\epsilon}(\Omega) \cap H \to W'$ to be the Riccati operator (positive self-adjoint on H) defining the value function corresponding to the following minimization problem

$$MinJ(u,v) \equiv \int_0^\infty |v(t)|^2_{H^{3/2-\epsilon}(\Omega)} + |u(t)|^2_U dt$$

subject to the condition that $v(u)$ satisfies (3) with $u \in L_2(0,\infty;U)$. As before, it is shown in [2] that the functional $J(u,v(u))$ is proper (the Finite Cost Condition is satisfied). In addition, in the two-dimensional case, one can select the feedback operators to be active only on an arbitrary small portion of the boundary. Furthermore - under a finite dimensional spectral assumption (FDSA) in [2] that $\mathcal{A}$ restricted on its finite-dimensional unstable sub-space be diagononalizable, the feedback operator may also be finite dimensional) Thus the Riccati operator R (positive self-adjoint on H and defining the value function), is a well defined operator $R : W = H^{1/2-\epsilon}(\Omega) \cap H \to W'$. Moreover, it is also an isomorphism between these two spaces. As in the 3-d case the operator

$$F(y - y_e) \equiv \nu_0 \partial_\nu R(y - y_e)$$

stabilizes exponentially the dynamics.

Theorem 2. *[[2]] The control feedback $u = F(y - y_e)$ stabilizes exponentially problem (1) in a sufficiently small neighborhood of the equilibrium y_e ie for $y_0 \in \mathcal{V}(\rho, y_e) \equiv \{y \in W = H^{1/2-\epsilon}(\Omega) \cap H : \ |y - y_e|_W \leq \rho\}$ for sufficiently small ρ. In this case we have:*

$$|y(t) - y_e|_{H^{1/2-\epsilon}(\Omega)} \leq Ce^{-\omega t}|y_0 - y_e|_{H^{1/2-\epsilon}(\Omega)}$$

where the constant ω is independent of ρ. In addition, the feedback operator F can be taken to be supported on an arbitrary small portion of the boundary Γ. Under the Finite Dimensional Spectral Assumption, it can be taken to be, in addition, finite dimensional.

In the two-dimensional case, other selections of the gain operators in the optimization problem are possible. For instance, one may minimize

$$MinJ(u, v) \equiv \int_0^\infty |v(t)|_H^2 + |u(t)|_U^2 dt$$

subject to the condition that $v(u)$ satisfies (3) with $u \in L_2(0, \infty; U)$. In that case, the Riccati operator not only is bounded $R \in \mathcal{L}(H)$, but it has additional regularity properties in line with the Riccati theory for analytic semigroups and bounded observation [4]. More specifically, while the formally same feedback operator $u = F(y) = \nu_0 \partial_\nu R(y - y_e)$ stabilizes exponentially the nonlinear dynamics (in line with Theorem 2) , we also have that

- The linearized feedback generator $A_F \equiv \mathcal{A}(I - DF)$ generates an analytic semigroup on H and on $W = H^{1/2-\epsilon}(\Omega) \cap H$, which is exponentially stable and satisfies $\int_0^\infty |e^{A_F t}x|^2_{H^{3/2-\epsilon}(\Omega)} dt \leq C|x|_W^2$
- The feedback $F = \nu_0 \partial_\nu R$ is bounded $H \to U$
- The corresponding Riccati equation takes the classical form $(\mathcal{A}^* Rx, y)_H + (R\mathcal{A}x, y)_H + (x, y)_H = \nu_0^2 < \partial_\nu Ry, \partial_\nu Rx >_U, \ for \ x, y \in \mathcal{D}(\mathcal{A}^{*\epsilon})$

We also have the regularity: $y \in L_2(0, \infty; H^{3/2-\epsilon}(\Omega)), \quad u \in L_2(0, \infty; H^{1-\epsilon}(\Gamma))$

Remark 1. *We note that the key point to obtaining a stabilization result of the nonlinear problem is the ability to achieve an "improved" regularity of the feedback semigroup $e^{A_F t}$ which is sufficiently high to control the nonlinear terms. In the three dimensional case we obtain $\int_0^\infty |e^{A_F t}x|^2_{H^{3/2+\epsilon}(\Omega)} dt \leq C|x|^2_{H^{1/2+\epsilon}(\Omega)}$ by using* *a "high gain" functional cost. In the two dimensional case, the corresponding inequality takes the form* $\int_0^\infty |e^{A_F t}x|^2_{H^{3/2-\epsilon}(\Omega)}dt \leq C|x|^2_{H^{1/2-\epsilon}(\Omega)}$ *and is obtained with both* $L_2(\Omega)$ *and* $H^{3/2-\epsilon}(\Omega)$ *gains. While the* L_2 *case is much simpler, since it relies on classical Riccati theory [4], the* $H^{3/2-\epsilon}$ *penalization requires a construction of a "new" class of Riccati equations (as in the 3-dimensional case). However, it appears that this latter option provides for more robust algorithms and gives a larger base of attraction. In all these cases (regardless of the penalization), the fundamental issue to be resolved is showing the Finite Cost Condition for the corresponding functional cost [2]. This issue is particularly subtle due to the constraints imposed on the control*

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Feedback Controller Design for PDE Systems in *COMPl$_e$ib*

Friedemann Leibfritz

We consider static output feedback (SOF) control design problems, e. g. SOF–$\mathcal{H}_\infty$ synthesis, and focus the discussion on the numerical solution of SOF problems if the control system is described by partial differential equations (PDEs). The discretization of those problems leads to large–scale non–convex and nonlinear semidefinite programs (NSDPs). We discuss some practical difficulties which arise in the solution of such problems and consider some algorithmic strategies for solving the non–convex NSDPs. Moreover we state some PDE–based models which are currently implemented in *COMPl$_e$ib* 1.1: the *CO*nstrained *M*atrix-optimization *P*roblem *li*brary [4] which contains actually 171 test examples drawn from a variety of control systems engineering applications. For example, *COMPl$_e$ib* contains variants of the following PDE models: 2D unstable convection diffusion equations with distributed as well as boundary control input, 2D nonlinear perturbed heat equation models with boundary control, 1D modified Burgers as well as Korteweg–de Vries–Burgers models, nonlinear damped wave/mass spring systems and a 1D coupled diffusion radiation model in a thin circular disc (e. g. see [4], [5], [6]).

COMPl$_e$ib may serve as a useful benchmark tool for NSDP , BMI and other matrix optimization problem (including linear SDP) solvers (e. g. IPCTR [3], [5], SSDP [1] for NSDPs arising in feedback control design, or, PENBMI [2] for bilinear matrix inequality problems, or, SeDuMi (Sturm), SDPT3 (Todd), DSDP (Ye) and so forth for linear SDPs). As a byproduct, *COMPl$_e$ib* can be used as a test environment for parts of control design procedures, e. g. model reduction algorithms. A numerical strategy for computing a linear SOF control law for discretized PDE control systems can be found in [5], [6]. In example, a finite difference or finite element discretization of a PDE control problem yields a large–scale finite dimensional control system of the following form (e. g. see [4], [5], [6])

$$\begin{array}{rcl} E\dot{x}(t) & = & (A+\delta A)x(t) + B_1 w(t) + Bu(t), \\ z(t) & = & C_1 x(t) + D_{11} w(t) + D_{12} u(t), \quad y(t) \; = \; Cx(t) + D_{21} w(t), \end{array} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, $y \in \mathbb{R}^{n_y}$, $z \in \mathbb{R}^{n_z}$, $w \in \mathbb{R}^{n_w}$ denote the state, control input, measured output, regulated output, and noise input, respectively. $E \in \mathbb{R}^{n_x \times n_x}$ is a regular diagonal matrix (very often we have that E is equal to the identity matrix) and all other data matrices are given. If $\delta A \equiv 0$ the system matrix A is not affected by a perturbation, and, if $G(x(t)) \equiv 0$, the system is linear. Depending on the corresponding PDE model, we get linear or nonlinear control systems. If the PDE model is nonlinear, we linearize or neglect the nonlinear term $G(x(t)$ for computing a SOF control. The goal is to determine the matrix $F \in \mathbb{R}^{n_u \times n_y}$ of the SOF control law $u(t) = Fy(t)$ such that the closed loop system

$$\dot{x}(t) \; = \; A(F)x(t) + B(F)w(t), \quad z(t) \; = \; C(F)x(t) + D(F)w(t), \tag{2}$$

fulfills some specific control design requirements, where $A(F) = E^{-1}(A + \delta A + BFC)$, $B(F) = E^{-1}(B_1 + BFD_{21})$, $C(F) = C_1 + D_{12}FC$, $D(F) = D_{11} + D_{12}FD_{21}$. In particular, the optimal SOF–$\mathcal{H}_\infty$ problem can be formally stated in the following term: *Find F such that $A(F)$ is Hurwitz and the $\mathcal{H}_\infty$–norm of (2) is minimal.* If $D_{11} = 0, D_{21} = 0$, the simplified $\mathcal{H}_\infty$–NSDP is given by (e. g. see [4], [5])

(3)

$$\begin{array}{ll} \min \ \beta^{-1}, & A(F)^T L + LA(F) + C(F)^T C(F) + \beta^2 L B_1 B_1^T L = 0, L \succeq 0, \beta > 0, \\ & (A(F) + \beta^2 B_1 B_1^T L)^T V + V(A(F) + \beta^2 B_1 B_1^T L) + I = 0, V \succ 0, \end{array}$$

where $X = (\beta, F, L, V) \in \mathbb{R} \times \mathbb{R}^{n_u \times n_y} \times \mathbb{S}^{n_x} \times \mathbb{S}^{n_x}$ are matrix variables, $\mathbb{S}^m$ denotes the space of all real symmetric $m \times m$ matrices and $Z \succ 0$ ($Z \succeq 0$) is used to indicate that $Z \in \mathbb{S}^m$ is positive (semi–) definite. For solving the non–convex $\mathcal{H}_\infty$–NSDP we use the interior point constrained trust region (IPCTR) solver as described in [3], [5]. IPCTR is specialized to NSDPs of the form (3) which arise in SOF/ROC (reduced order control) design. It is a combination of a primal–dual interior point method, a modified conjugate gradient approach and a constrained reduced SQP–type trust region method which exploits the inherent structure of the SOF/ROC–NSDP problems. IPCTR is a fully iterative method and, thus, no explicit evaluation of the Hessian of the Lagrangian is needed in IPCTR which can be a very time consuming process, in particular for large NSDPs. Solving a NSDP of the form (3) can be very difficult since, in general, it is nonlinear and non–convex in the matrix variables $X = (\beta, F, L, V)$ and the constraint set contains nonlinear and non–convex matrix equalities/inequalities. Moreover, the number of NSDP variables can be very huge if n_x is big, e. g. let $n_x = 4000$, $n_u = n_y = 2 \ll n_x$ then the $\mathcal{H}_\infty$–NSDP has approximately sixteen million variables (e. g. $2\frac{1}{2}n_x(n_x+1) + n_u n_y + 1 \approx O(n_x^2) = 16 \cdot 10^6$). To my knowledge, it is impossible to solve a NSDP of that size. Furthermore, due to the matrix product BFC, the closed loop matrix $A(F)$ can be a dense matrix even if A is sparse. Thus, it is not clear in which way a sparsity pattern in A can be exploited in an algorithm for solving very large NSDPs. One way out of this dilemma is to perform a model reduction procedure to the discretized PDE model prior we solve the $\mathcal{H}_\infty$–NSDP as proposed by [5], [6]. For example the computational design of a SOF control for a linear (unstable) convection–diffusion model leads to large–scale NSDPs of the form (3). In $\Omega = [0,1] \times [0,1]$ the infinite dimensional control problem of the convection–diffusion model is given by (e. g. see [6])

$$\begin{array}{rcl} v_t &=& \kappa\Delta v - \varepsilon_1(v_\xi + v_\eta) + \varepsilon_2 v + \sum_{i=1}^{n_u} u_i(t) b_i, \ \text{ in } \Omega, t > 0, \\ v(\xi,\eta;t) &=& 0, \text{ on } \partial\Omega, t > 0 \quad \text{and} \quad v(\xi,\eta,0) \ = \ v_0(\xi,\eta), \text{ in } \Omega, \end{array} \tag{4}$$

where $v := v(\xi,\eta;t)$, $(\xi,\eta) \in \Omega$, $t > 0$, Δ is the Laplace operator, $\varepsilon_1, \varepsilon_2 \geq 0$ are given constants, $\partial\Omega$ denotes the boundary of Ω, $\kappa > 0$ is the diffusion coefficient, $b_i, i = 1,\ldots,n_u$ are given shape functions for the control inputs $u_1,\ldots,u_{n_u}$ and $v_0(\cdot)$ is the initial state in Ω at $t = 0$. After a spatial finite difference discretization we end up with a linear control system of the form (1) with $n_x = 3600$ states. We choose $n_u = 2$ and $b_i = \chi_{\Omega_i^u}$, $i = 1, 2$, where $\chi_{\Omega_i^u}$ denotes the characteristic function on the control input domain $\Omega_i^u \subset \Omega$ of u_i and $\Omega_1^u = [0.1, 0.4] \times [0.1, 0.4]$, $\Omega_2^u = [0.6, 0.9] \times [0.7, 0.9]$. Moreover, we set $n_y = 2$ and measure the state on the observation domains $\Omega_i^y \subset \Omega, i = 1, 2$ of $y(t) = (y_1(t), y_2(t))^T$, where $\Omega_1^y = [0.1, 0.4] \times [0.5, 0.7]$, $\Omega_2^y = [0.6, 0.9] \times [0.1, 0.4]$. For computing the SOF–$\mathcal{H}_\infty$ control law for the large–scale discretized (unstable) *COMPl$_e$ib* PDE model, we use the approach of [5], [6]. This method is a combination of a POD approximation of the large–scale dynamical system (1) with IPCTR for solving a low dimensional $\mathcal{H}_\infty$–NSDP. In particular, first we compute a POD approximation of the large–scale discrete model (1) to derive a control system of the form (1), but with dimension $n_{pod} \ll n_x$, e. g. $n_{pod} = 5$. Then, neglecting or linearizing the nonlinear part $G(x(t))$ in (1), we solve the n_{pod} dimensional $\mathcal{H}_\infty$–NSDP (3) to obtain the linear SOF control law $u(t) = Fy(t)$. Finally, we fit this SOF control law into the high dimensional (nonlinear) PDE control system. For more details and several numerical experiments we refer to [5] and [6]. Another PDE control system in *COMPl$_e$ib* is a 2D model of a nonlinear instable diffusion equation. In the domain $\Omega = [0,1] \times [0,1]$ we consider a initial boundary value problem of this nonlinear reaction–diffusion model for the unknown function $v(\xi,\eta;t)$, $(\xi,\eta) \in \Omega$, $t > 0$:

$$\begin{array}{rcl} v_t &=& \kappa\Delta v + \varepsilon_2 v - \varepsilon_1 v^3 + \sum_{i=1}^{n_u} u_i(t) b_i, \ \text{ in } \Omega, t > 0, \\ v(\xi,\eta;t) &=& 0, \ \text{ on } \partial\Omega, t > 0 \quad \text{and} \quad v(\xi,\eta,0) \ = \ v_0(\xi,\eta), \ \text{ in } \Omega, \end{array} \tag{5}$$

where $\varepsilon_1, \varepsilon_2 \geq 0$ are positive constants and the other quantities are defined as in the previous example. For $\varepsilon_1 = 0$ and $\varepsilon_2 = 0$, the open loop system of (5) is the heat equation, which is asymptotically stable. However, it is unstable if $\varepsilon_2 > 0$ is large enough even if $\varepsilon_1 = 0$. The unstable viscous modified Burgers equation with boundary control input u and given $\kappa = 0.5$, $\varepsilon = 0.25$ is a 1D PDE instance in *COMPl$_e$ib* (e. g. see [6]):

$$\begin{gathered} v_t - \kappa v_{\xi\xi} + v v_\xi - \varepsilon v = 0, \qquad\qquad \xi \in (0, 2\pi),\ t > 0 \\ v_\xi(t,0) = 0, \quad v_\xi(t, 2\pi) = u, \quad v(0,\xi) = sin(\xi). \end{gathered}$$

Finally, we state an unstable 1D Korteweg–de Vries–Burgers model. For given $\kappa = 0.5$, $\lambda = 0.1$, $\varepsilon = 0.345$ the following boundary control problem can be found in *COMPl$_e$ib* :

$$\begin{gathered} v_t - \kappa v_{\xi\xi} + \lambda v_{\xi\xi\xi} + v v_\xi - \varepsilon v = 0, \qquad\qquad \xi \in (0, 2\pi),\ t > 0 \\ v(t,0) = 0, \quad v_\xi(t, 2\pi) = u, \quad v_{\xi\xi}(t, 2\pi) = 0, \quad v(0,\xi) = \xi sin(\xi). \end{gathered}$$

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Optimal Control of an Elliptic PDE with Nonlocal Radiation Interface Conditions

Christian Meyer

(joint work with O. Klein, P. Philip, J. Sprekels, and F. Tröltzsch)

This work deals with the optimal control of the production of silicon carbide (SiC) single crystals that represent an important semiconductor material. The state-of-the-art technique to produce SiC single crystals is the physical vapor transport (PVT) method. A corresponding growth apparatus mainly consists of a graphite crucible Ω_s with a cavity inside that is denoted by Ω_g. This cavity is filled with argon and, at the bottom, with polycrystalline SiC powder. The crucible is heated up to 2000–3000 K, usally by induction heating. Due to the high temperature, the SiC powder sublimates and crystallizes at the cooled top of the cavity. In this way, the desired single crystal grows into the reaction chamber (see e.g. [3] for details). A simplified optimal control problem that arises from this application but still covers some main difficulties, is given by the following semilinear elliptic problem with pointwise control constraints

$$
\text{(P)} \quad \begin{cases}
\text{minimize} \quad J(y,u) := \dfrac{1}{2}\displaystyle\int_{\Omega_g} |\nabla y - z|^2\,dx + \dfrac{\nu}{2}\displaystyle\int_{\Omega_s} u^2\,dx \\
\text{subject to} \qquad -\text{div}(\kappa_s\,\nabla y) = u \qquad \text{in } \Omega_s \\
\qquad\qquad\qquad -\text{div}(\kappa_g\,\nabla y) = 0 \qquad \text{in } \Omega_g \\
\qquad \kappa_g \left(\dfrac{\partial y}{\partial n_r}\right)_g - \kappa_s \left(\dfrac{\partial y}{\partial n_r}\right)_s = G(\sigma|y|^3 y) \quad \text{on } \Gamma_r = \overline{\Omega}_s \cap \overline{\Omega}_g \\
\qquad \kappa_s \dfrac{\partial y}{\partial n_0} + \varepsilon\sigma\,|y|^3 y = \varepsilon\sigma\,y_0^4 \qquad \text{on } \Gamma_0 = \partial\Omega_s \\
\text{and} \quad u_a(x) \le u(x) \le u_b(x) \quad \text{a.e. in } \Omega_s,
\end{cases}
$$

with a desired state gradient $z \in L^2(\Omega)^2$ and thermal conductivities $\kappa_s \in L^\infty(\Omega_s)$, $\kappa_g \in L^\infty(\Omega_g)$, $\kappa_s, \kappa_g > 0$ a.e. in Ω_s and Ω_g, respectively. Furthermore, $y_0 \in L^{16}(\Gamma_0)$ is the external temperature with $y_0 \ge \vartheta > 0$ a.e. on Γ_0. The bounds u_a and u_b are functions in $L^\infty(\Omega_s)$ with $0 < u_a(x) < u_b(x)$ a.e. in Ω_s. We assume that the outer boundary Γ_0 and the interface Γ_r are Lipschitz and Γ_r is additionally piecewise $C^{1,\delta}$. Moreover, $\varepsilon \in [0,1]$ denotes the emissivity, $\sigma > 0$ is the Stefan-Boltzmann constant, and G represents a nonlocal radiation operator that is defined by

$$G(\sigma|y|^3 y) := (I-K)(I-(1-\varepsilon)K)^{-1}\varepsilon\,\sigma|y|^3 y.$$

Here, K denotes an integral operator with symmetic kernel ω, i.e. $(Kv)(x) := \int_{\Gamma_r} \omega(x,z)v(z)\,ds_z$, for further details see [8], [9].

The operator G was investigated in detail by Laitinen and Tiihonen who proved that G is a bounded linear operator from $L^p(\Gamma_r)$ to itself for $1 \le p \le \infty$ ([4, Lemma 8]). In general, G is not positive, i.e. $v(x) \ge 0$ a.e. on Γ_r does not imply $(Gv)(x) \ge 0$ a.e. on Γ_r. This property causes that the nonlinearity in the state equation in (P) is not monotone. However, Laitinen and Tiihonen showed in [4] that it is pseudomonotone. Hence, Brezis' theorem for pseudomonotone operators (cf. [11]) implies the existence of solutions of the state equation in the state space $V = \{v \in H^1(\Omega)\,|\,\tau_r v \in L^5(\Gamma_r), \tau_0 v \in L^5(\Gamma_0)\}$. Using a technique introduced by Stampacchia in the linear case (see [2], [7]), we prove the boundedness of the solutions in Ω and on $\Gamma_r \cup \Gamma_0$ and introduce a corresponding state space by $V^\infty = H^1(\Omega) \cap L^\infty(\Omega)$ (see [5]). By the theory of Fredholm operators, the existence of a unique solution to the linearized state equation is also shown in [5]. Based on this result, the implicit function theorem gives the differentiability of the control-to-state operator $S : L^2(\Omega_s) \to V^\infty$. In a standard way, a pointwise discussion of the variational inequality yields the well-known projection formula for control constrained problems:

$$\bar{u}(x) = P_{ad}\left\{-\frac{1}{\nu}p(x)\right\},$$

where $P_{ad} : \mathbb{R} \to \mathbb{R}$ is the pointwise projection operator on $[u_a(x), u_b(x)]$. Here, $p \in H^1(\Omega)$ denotes the adjoint state that solves the following adjoint equation

$$\begin{aligned}
\operatorname{div}(\kappa_g \nabla p) &= \Delta \bar{y} - \operatorname{div} z && \text{in } \Omega_g \\
\operatorname{div}(\kappa_s \nabla p) &= 0 && \text{in } \Omega_s \\
\kappa_s \left(\frac{\partial p}{\partial n_r}\right)_s - \kappa_g \left(\frac{\partial p}{\partial n_r}\right)_g + 4\sigma |\bar{y}|^3 G^* p &= \frac{\partial \bar{y}}{\partial n_r} - z \cdot n_r && \text{on } \Gamma_r \\
\kappa_s \frac{\partial p}{\partial n_0} + 4\varepsilon\sigma |\bar{y}|^3 p &= 0 && \text{on } \Gamma_0.
\end{aligned}$$

Second-order sufficient optimality conditions are established in a very recent work [6]. The corresponding analysis is based on the techniques introduced in [1] and [10]. These conditions account for strongly active sets and give local optimality in a L^s-neighborhood of a reference function, where s can be chosen smaller than ∞.

Finally some numerical results are presented that were computed using a Gauß–Newton method. The arising linear-quadratic subproblems were solved with an active set strategy. The PDEs were discretized with linear finite elements.

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Recent Results on Exact Controllability of the Navier-Stokes System

Jean-Pierre Puel

1. Introduction and main results

Let $\Omega \subset \mathbf{R}^N$ be a bounded regular domain, with $N = 2$ or $N = 3$. Assume that $\omega \subset \Omega$ is a nonempty (small) open subset and $T > 0$ is given. In the sequel, we will use the following notation: $Q = \Omega \times (0, T)$, $\Sigma = \partial\Omega \times (0, T)$; C will stand for a generic positive constant that may depend on Ω and ω.

We will be concerned here with some controllability properties of the Navier-Stokes system

$$(1) \qquad \begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = v1_\omega & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y^0 & \text{in } \Omega \end{cases}$$

and the similar linear Stokes-like problem

$$(2) \qquad \begin{cases} y_t - \Delta y + \nabla \cdot (\overline{y} \otimes y + y \otimes \overline{y}) + \nabla p = f + v1_\omega & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y^0(x) & \text{in } \Omega, \end{cases}$$

where $\overline{y} = \overline{y}(x,t)$ is given and satisfies adequate regularity assumptions. In (2), the symbol $\otimes$ stands for the usual tensor product in $\mathbf{R}^N$. As usual, it will be convenient to analyze the observability properties of the following system, which can be viewed as the adjoint of (2):

$$(3) \qquad \begin{cases} -\varphi_t - \Delta\varphi - (D\varphi)\overline{y} + \nabla\pi = g & \text{in } Q, \\ \nabla \cdot \varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(x,T) = \varphi^0(x) & \text{in } \Omega. \end{cases}$$

Here, $D\varphi = \nabla\varphi + \nabla\varphi^T$. We will need some function spaces:

$$H = \{\, z \in L^2(\Omega)^N : \nabla \cdot z = 0 \text{ in } \Omega,\ z \cdot n = 0 \text{ on } \partial\Omega \,\},$$

$$V = \{\, z \in H_0^1(\Omega)^N : \nabla \cdot z = 0 \text{ in } \Omega \,\}.$$

Furthermore, the following hypotheses over $\overline{y}$ will be needed in order to have a suitable Carleman inequality for the solutions to (3):

$$(4) \qquad \overline{y} \in L^\infty(Q)^N, \quad \overline{y}_t \in L^2(0,T; L^\sigma(\Omega))^N \quad \begin{pmatrix} \sigma > 6/5 & \text{if } N = 3 \\ \sigma > 1 & \text{if } N = 2 \end{pmatrix}.$$

Our first main result is a new global Carleman estimate for the solutions to (3). Several weight functions will be needed:

$$\alpha(x,t) = \frac{e^{5/4\,\lambda\, m\,\|\eta^0\|_\infty} - e^{\lambda(m\,\|\eta^0\|_\infty + \eta^0(x))}}{t^4(T-t)^4}, \quad \xi(x,t) = \frac{e^{\lambda(m\,\|\eta^0\|_\infty + \eta^0(x))}}{t^4(T-t)^4}, \tag{5}$$
$$\widehat{\alpha}(t) = \min_{x\in\overline{\Omega}} \alpha(x,t), \quad \alpha^*(t) = \max_{x\in\overline{\Omega}} \alpha(x,t), \quad \widehat{\xi}(t) = \max_{x\in\overline{\Omega}} \xi(x,t).$$

Here, $m > 4$ is a fixed real number and $\eta^0 \in C^2(\overline{\Omega})$ is a function satisfying

$$\eta^0 > 0 \text{ in } \Omega, \quad \eta^0 = 0 \text{ on } \partial\Omega, \quad |\nabla\eta^0| > 0 \text{ in } \overline{\Omega\setminus\omega'},$$

where $\omega' \subset\subset \omega$ is a nonempty open set. The existence of such a function η^0 is proved in [4].

Theorem 1. *Let us assume that* (4) *holds. There exist positive constants $\overline{s}$, $\overline{\lambda}$ and C, only depending on Ω and ω such that, for every $g \in L^2(Q)^N$ and $\varphi^0 \in H$, the associated solution to* (3) *satisfies*

$$\begin{aligned} &\iint_Q e^{-2s\alpha}\left((s\xi)^{-1}(|\varphi_t|^2 + |\Delta\varphi|^2) + s\lambda^2\xi|\nabla\varphi|^2 + s^3\lambda^4\xi^3|\varphi|^2\right) dx\,dt \\ &\quad\le C(1+T^2)\left(s^{15/2}\lambda^{20}\iint_Q e^{-4s\widehat{\alpha}+2s\alpha^*}\,\widehat{\xi}^{15/2}\,|g|^2\,dx\,dt \right.\\ &\quad\left. + s^{16}\lambda^{40}\iint_{\omega\times(0,T)} e^{-8s\widehat{\alpha}+6s\alpha^*}\,\widehat{\xi}^{16}\,|\varphi|^2\,dx\,dt\right) \end{aligned} \tag{6}$$

for any $\lambda \ge \overline{\lambda}\,(1 + \|\overline{y}\|_\infty + e^{\overline{\lambda}\,T\,\|\overline{y}\|_\infty^2} + \|\overline{y}_t\|^2_{L^2(L^\sigma)})$ and any $s \ge \overline{s}\,(T^7 + T^8)$.

This Carleman inequality provides, in a classical way, an observability inequality for the solutions to (3) i.e.,

$$\|\varphi(0)\|_{L^2(\Omega)} \le C\iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt \tag{7}$$

for a positive constant C. It is now classical to prove that the inequality (7) implies the null controllability of (2) for $f = 0$.

The second main result concerns the local exact controllability to the trajectories of (1). It is the following:

Theorem 2. *Let $(\overline{y}, \overline{p})$ be a solution to the Navier-Stokes problem*

$$\begin{cases} \overline{y}_t - \Delta\overline{y} + (\overline{y}\cdot\nabla)\overline{y} + \nabla\overline{p} = 0 & \text{in } Q, \\ \nabla\cdot\overline{y} = 0 & \text{in } Q, \\ \overline{y} = 0 & \text{on } \Sigma, \\ \overline{y}(x,0) = \overline{y}^0(x) & \text{in } \Omega, \end{cases} \tag{8}$$

satisfying (4) *and $\overline{y}^0 \in L^{2N-2}(\Omega)^N \cap H$. Then there exists $\delta > 0$ such that, for any $y^0 \in L^{2N-2}(\Omega)^N \cap H$ satisfying $\|y^0 - \overline{y}^0\|_{L^{2N-2}(\Omega)^N} \le \delta$, we can find controls $v \in L^2(\omega\times(0,T))^N$ and associated states (y,p) such that one has* (1) *and*

$$y(x,T) = \overline{y}(x,T) \quad \text{in} \quad \Omega.$$

In the following sections, we will indicate the main ideas of the proofs of theorems 1 and 2. The detailed proofs are given in [3].

2. A new Carleman inequality

We will use the notation $I(s,\lambda;\varphi)$ to denote the left hand side of (6). Let $g \in L^2(Q)^N$ and $\varphi^0 \in H$ be given and let (φ, π) be the associated solution to (3). We can first apply to each component of φ the usual Carleman inequality for the heat equation with right hand side in $L^2(Q)$. After some arrangements, we get

$$(9)\quad I(s,\lambda;\varphi) \le C \left(\iint_Q e^{-2s\alpha} \left(|g|^2 + |\nabla\pi|^2\right) dx\, dt + s^3\lambda^4 \iint_{\omega'\times(0,T)} e^{-2s\alpha}\, \xi^3\, |\varphi|^2\, dx\, dt \right),$$

for all $\lambda \ge C(1+\|\overline{y}\|_\infty)$ and $s \ge C(T^7+T^8)$. For the proof of (9), see [6] and [4]; for the explicit values of λ and s, see [2].

In view of the main result in [8] and following the ideas of [7], we can estimate the pressure gradient in (9) and deduce that

$$(10)\quad \begin{aligned} I(s,\lambda;\varphi) \le C \Bigg(& s^3\lambda^4 \iint_{\omega'\times(0,T)} e^{-2s\alpha}\, \xi^3\, |\varphi|^2\, dx\, dt \\ & + s^2\,\lambda^2 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\, \xi^2\, |\pi|^2\, dx\, dt + s \iint_Q e^{-2s\alpha}\, \xi\, |g|^2\, dx\, dt \Bigg), \end{aligned}$$

for any $\lambda \ge C(1+\|\overline{y}\|_\infty)$ and any $s \ge C(T^7+T^8)$, where ω_1 is an open set such that $\omega' \subset\subset \omega_1 \subset\subset \omega$. The rest of the proof is oriented towards the absorption of the local pressure term in (10). Let us remark that we have only used the assumption $\overline{y} \in L^\infty(Q)^N$ until this moment, while more regularity on $\overline{y}$ will be needed to perform a local estimate of the pressure.

We can assume that the pressure has been chosen with zero mean in ω_1. Then,

$$\iint_{\omega_1\times(0,T)} e^{-2s\alpha}\, \xi^2\, |\pi|^2\, dx\, dt \le C \iint_{\omega_1\times(0,T)} e^{-2s\widehat{\alpha}}\, \widehat{\xi}^2\, |\nabla\pi|^2\, dx\, dt$$

and using the equation satisfied by φ and π we see that the task is to obtain local estimates of $\Delta\varphi$ and φ_t.

For the estimate of $\Delta\varphi$, we can use classical arguments for the heat equation; observe that $u = \Delta\varphi$ fulfills a heat equation where the pressure is absent. On the other hand, integrating by parts in time and using well known *a priori* estimates for the Stokes system (see [5]), we can find a local estimate of φ_t in terms of local integrals of φ and $\nabla\varphi$ and $I(s,\lambda;\varphi)$. More precisely, with $q = s^{15/2} e^{-2s\widehat{\alpha}+s\alpha^*}\,\widehat{\xi}^{15/2}$

and ω_2 an open set satisfying $\omega_1 \subset\subset \omega_2 \subset\subset \omega$, for any small $\varepsilon > 0$ we obtain

$$\begin{aligned} & s^2\lambda^2 \iint_{\omega_1\times(0,T)} e^{-2s\widehat{\alpha}}\widehat{\xi}^2\, |\varphi_t|^2\, dx\, dt \\ & \leq \varepsilon\, I(s,\lambda;\varphi) \\ & + C_\varepsilon \lambda^{20}(1+T)\left(\|q\, g\|^2_{L^2(L^2)} + \|q\,\varphi\|^2_{L^2(L^2(\omega_2))} + \|q\,\nabla\varphi\|^2_{L^2(L^2(\omega_2))}\right) \end{aligned}$$

for $\lambda \geq C(1 + \|\overline{y}\|_\infty + e^{C\,T\,\|\overline{y}\|^2_\infty} + \|\overline{y}_t\|^2_{L^2(L^\sigma)})$. Let us remark that proving such a local estimate requires many technical computations and led us to assume $\overline{y}_t \in L^2(L^\sigma)$.

The local estimates of $\Delta\varphi$ and φ_t lead to the desired Carleman inequality (6).

3. The local null controllability of the Navier-Stokes system

The proof of theorem 2 follows the ideas in [7]. Thus, we deduce in a first step a null controllability result for (2) with suitable right hand side f.

More precisely, let us set $Ly = y_t - \Delta y + \nabla\cdot(\overline{y}\otimes y + y\otimes\overline{y})$ and let us introduce in dimension $N = 3$ the space E_3, with

$$(11)\qquad \begin{aligned} E_3 = \{(y,v) \in E_0 : e^{s\beta^*/2}(\gamma^*)^{-1/4} y \in L^4(0,T;L^{12}(\Omega)^3), \\ \exists p : e^{s\beta^*}(\gamma^*)^{-1/2}(Ly + \nabla p - v1_\omega) \in L^2(0,T;W^{-1,6}(\Omega)^3)\}, \end{aligned}$$

where

$$\begin{aligned} E_0 = \{(y,v) : e^{2s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{-15/4} y,\, e^{4s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^{-8} v1_\omega \in L^2(Q)^N, \\ e^{s\beta^*/2}(\gamma^*)^{-1/4} y \in L^2(0,T;V)\cap L^\infty(0,T;H)\}. \end{aligned}$$

and where the new weight functions β, β^*, etc. are given by

$$\beta(x,t) = \frac{e^{5/4\lambda m\|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{\ell(t)^4},$$

$$\widehat{\beta}(t) = \min_{x\in\overline{\Omega}} \beta(x,t), \quad \beta^*(t) = \max_{x\in\overline{\Omega}} \beta(x,t),$$

$$\gamma(x,t) = \frac{e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{\ell(t)^4}, \quad \widehat{\gamma}(t) = \max_{x\in\overline{\Omega}} \gamma(x,t), \quad \gamma^*(t) = \min_{x\in\overline{\Omega}} \gamma(x,t).$$

Here, we have introduced

$$\ell(t) = \begin{cases} T^2/4 & \text{in } (0,T/2) \\ t(T-t) & \text{in } (T/2,T). \end{cases}$$

We then have:

Proposition 3. *Let us assume that $\overline{y}$ satisfies* (4) *and the following hypotheses on the initial condition and the right hand side hold:*

$$y^0 \in H\cap L^4(\Omega)^3, \quad e^{s\beta^*}(\gamma^*)^{-1/2} f \in L^2(0,T;W^{-1,6}(\Omega)^3).$$

Then there exists a control v such that the associated solution (y,p) to (2) *satisfies $(y,v) \in E_3$.*

Notice that this is actually a null controllability result for (2). Indeed, if $(y, v) \in E_3$, we have in particular that $y(x, T) = 0$ in Ω.

The rest of the proof of theorem 2 relies on an appropriate *inverse mapping theorem*. More precisely, we use the following result (see [1]):

Proposition 4. *Let E, F be two Banach spaces and let $\mathcal{A} : E \mapsto F$ satisfy $\mathcal{A} \in C^1(E; F)$. Assume that $e_0 \in E$, $\mathcal{A}(e_0) = h_0$ and $\mathcal{A}'(e_0) : E \mapsto F$ is an epimorphism. Then, there exists $\delta > 0$ such that, for every $h \in F$ satisfying $\|h - h_0\|_F < \delta$, there exists a solution of the equation*

$$\mathcal{A}(e) = h, \quad e \in E.$$

We can apply this result to the mapping $\mathcal{A} : E \mapsto F$ given by

$$\mathcal{A}(y, v) = (Ly + (y \cdot \nabla)y + \nabla p - v1_\omega, y(\cdot, 0)) \quad \forall (y, v) \in E_3,$$

where E_3 is as in (11) and $F = L^2(e^{s\beta^*}(\gamma^*)^{-1/2}; W^{-1,6}(\Omega)^3) \times (L^4(\Omega)^3 \cap H)$, with $e_0 = (0, 0, 0)$ and $h_0 = (0, 0)$.

From the definition of E_3, one can easily check that $\mathcal{A}$ is well defined and satisfies $\mathcal{A} \in C^1(E; F)$. Finally, the identity

$$\mathrm{Im}(\mathcal{A}'(0, 0, 0)) = F$$

is equivalent to the result stated in proposition 3. This completes the proof of theorem 2.

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Feedback Boundary Stabilization of the Two and the Three Dimensional Navier-Stokes Equations

Jean-Pierre Raymond

Let Ω be a bounded and connected domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with a regular boundary Γ, $\nu > 0$, and consider a couple $(\mathbf{w}, \chi)$ – a velocity field and a pressure – solution to the stationary Navier-Stokes equations in Ω:

$$-\nu\Delta\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{w} + \nabla\chi = \mathbf{f} \quad \text{and } \operatorname{div}\mathbf{w} = 0 \text{ in } \Omega, \qquad \mathbf{w} = \mathbf{u}_s^\infty \quad \text{on } \Gamma.$$

We assume that $\mathbf{w}$ is regular and is an unstable solution of the instationary Navier-Stokes equations. We want to determine a Dirichlet boundary control $\mathbf{u}$, in feedback form, localized in a part of the boundary Γ, so that the corresponding controlled system:

$$\text{(1)} \qquad \begin{aligned} &\frac{\partial \mathbf{y}}{\partial t} - \nu\Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = 0, \\ &\operatorname{div}\mathbf{y} = 0 \quad \text{in } Q_\infty, \quad \mathbf{y} = M\mathbf{u} \;\text{ on } \Sigma_\infty, \quad \mathbf{y}(0) = \mathbf{y}_0 \text{ in } \Omega, \end{aligned}$$

be stable for initial values $\mathbf{y}_0$ small enough in an appropriate space $\mathbf{X}(\Omega)$. In this setting, $Q_\infty = \Omega \times (0,\infty)$, $\Sigma_\infty = \Gamma \times (0,\infty)$, $\mathbf{X}(\Omega)$ is a subspace of $\mathbf{V}_n^0(\Omega) = \Big\{\mathbf{y} \in \mathbf{L}^2(\Omega) \mid \operatorname{div}\mathbf{y} = 0 \text{ in } \Omega,\ \mathbf{y}\cdot\mathbf{n} = 0 \text{ on } \Gamma\Big\}$, $\mathbf{w} \in \mathbf{X}(\Omega)$, and the operator M is a restriction operator ensuring that the control is localized on a part of the boundary Γ (see [10]). If we set $(\mathbf{z}, q) = (\mathbf{w} + \mathbf{y}, \chi + p)$ and if $\mathbf{u} = 0$, we see that $(\mathbf{z}, q)$ is the solution to the Navier-Stokes equations

$$\begin{aligned} &\frac{\partial \mathbf{z}}{\partial t} - \nu\Delta\mathbf{z} + (\mathbf{z}\cdot\nabla)\mathbf{z} + \nabla q = \mathbf{f}, \quad \operatorname{div}\mathbf{z} = 0 \quad \text{in } Q_\infty, \\ &\mathbf{z} = \mathbf{u}_s^\infty \;\text{ on } \Sigma_\infty, \quad \mathbf{z}(0) = \mathbf{w} + \mathbf{y}_0 \text{ in } \Omega. \end{aligned}$$

Thus $\mathbf{y}_0$ is a perturbation of the stationary solution $\mathbf{w}$. When $\mathbf{w} \in \mathbf{L}^\infty(\Omega)$ and $\mathbf{y}_0 \in \mathbf{V}_n^0(\Omega) \cap \mathbf{L}^4(\Omega)$ with $|\mathbf{y}_0|_{\mathbf{L}^4(\Omega)}$ small enough, the existence of a boundary control $\mathbf{u}$ such that the solution to equation (1) exponentially decreases in the norm of the space $\mathbf{X}(\Omega) = \mathbf{V}_n^0(\Omega) \cap \mathbf{L}^4(\Omega)$, follows from a local exact controllability result stated in [5, Theorem 2]. But the proof in [5] does not give any way to define such a control in feedback form. In the three-dimensional case, and when $\mathbf{X}(\Omega) = \Big\{\mathbf{y} \in \mathbf{H}^1(\Omega) \mid \operatorname{div}\mathbf{y} = 0 \text{ in } \Omega,\ \langle \mathbf{y}\cdot\mathbf{n}, 1\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = 0\Big\}$, the existence of a control exponentially stabilizing (1) is proved in [6]. One way to construct robust feedback laws consists in using the methods of the optimal control theory. This approach has been studied in the case of an internal control [1–3], and has been numerically tested with a boundary control in the very specific geometry of the rectangular driven cavity [7] and when the normal component of the control is equal to zero. As it is the situation corresponding to many engineering applications [8, 9], here we do not assume that the normal component of the control variable is zero.

The Linear-Quadratic theory for the Dirichlet control of the linearized Navier-Stokes equations has been studied in a very recent work [4], in the case when the normal component of the boundary control is zero, and when the control is applied everywhere on the boundary. To the best of our knowledge the case when the normal component is not equal to zero has not yet been studied in the literature. The main objectives of this talk are:

- first to develop the Linear-Quadratic theory over an infinite time horizon of the Dirichlet boundary control of the Oseen equations when the control is localized on a part of the boundary, and when the normal component of the control is not zero,

- next to show that the linear feedback law, calculated with the linearized model, and applied to the nonlinear equation (1), provides a local exponential stabilization of the state in some appropriate space $\mathbf{X}(\Omega)$.

In the two dimensional case the feedback control law is obtained by studying the control problem

$$(\mathcal{P}) \qquad \inf\Big\{J(\mathbf{y},\mathbf{u}) \ \mid\ (\mathbf{y},\mathbf{u}) \text{ satisfies (2)},\ \mathbf{u}\in L^2(0,\infty;\mathbf{V}^0(\Gamma))\Big\},$$

where

$$J(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty\int_\Omega |\mathbf{y}|^2\,dxdt + \frac{1}{2}\int_0^\infty\int_\Gamma |\mathbf{u}|^2\,dxdt,$$

and

$$(2) \qquad \begin{aligned} &\frac{\partial \mathbf{y}}{\partial t} - \nu\Delta\mathbf{y} + (\mathbf{w}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{w} - \omega P\mathbf{y} + \nabla p = 0, \quad \text{in } Q_\infty,\\ &\operatorname{div}\mathbf{y} = 0 \quad \text{in } Q_\infty, \quad \mathbf{y} = M\mathbf{u} \text{ on } \Sigma_\infty, \quad \mathbf{y}(0) = \mathbf{y}_0 \quad \text{in } \Omega, \end{aligned}$$

where $\mathbf{V}^0(\Gamma) = \Big\{\mathbf{y}\in\mathbf{L}^2(\Gamma)\ \mid\ \langle\mathbf{y}\cdot\mathbf{n},1\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = 0\Big\}$. The coefficient $\omega > 0$, which is not present in (1), is added in equation (2) in order to guarantee the exponential decay in the norm $\mathbf{H}^{1/2-\varepsilon}(\Omega)$, $0<\varepsilon<1/4$, of the solution of the nonlinear closed loop system defined below. We show that the control problem $(\mathcal{P})$ can be rewritten in the form of another control problem in which the state variable is $P\mathbf{y}$ – where P is the so-called Helmholtz projection operator – and not $\mathbf{y}$. This transformation is essential in our approach. It leads to a Riccati equation which is the natural one for the new control problem, but which is not the expected one if we only consider problem $(\mathcal{P})$. This transformation of $(\mathcal{P})$ into a new control problem is a direct consequence of rewriting equation (2) in the form:

$$(3) \qquad \begin{aligned} &P\mathbf{y}' = AP\mathbf{y} - \omega P\mathbf{y} + BM\mathbf{u}, \qquad \mathbf{y}(0) = \mathbf{y}_0,\\ &(I-P)\mathbf{y} = (I-P)D_A\gamma_n M\mathbf{u}. \end{aligned}$$

The operator A is the Oseen operator, the control operator is defined by $B = (\lambda_0 I - A)D_A$ for some $\lambda_0 > 0$, and D_A is the Dirichlet operator associated with $\lambda_0 I - A$. We refer to [10] for the transformation of equation (2) into (3), and for regularity results for equation (3). Denoting by Π_ω the solution to the Riccati equation of the control problem $(\mathcal{P})$, and setting $R_A = MD_A^*(I-P)D_AM + I$, we show that the closed loop system

$$\begin{aligned} &\frac{\partial \mathbf{y}}{\partial t} - \nu\Delta\mathbf{y} + (\mathbf{w}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{w} + \nabla p = 0, \quad \text{in } Q_\infty,\\ &\operatorname{div}\mathbf{y} = 0 \quad \text{in } Q_\infty, \quad \mathbf{y} = -MR_A^{-1}MB^*\Pi_\omega P\mathbf{y} \quad \text{on } \Sigma_\infty, \quad \mathbf{y}(0) = \mathbf{y}_0 \quad \text{in } \Omega, \end{aligned}$$

is exponentially stable if $|\mathbf{y}_0|_{\mathbf{H}^{1/2-\varepsilon}(\Omega)\cap\mathbf{V}^0_n(\Omega)}$ is small enough for some $0 < \varepsilon < 1/4$.

In the three dimensional case we obtain a similar result by studying the control problem

$$(\mathcal{Q}) \qquad \inf\Big\{I(\mathbf{y},\mathbf{u}) \ \mid\ (\mathbf{y},\mathbf{u}) \text{ satisfies (4)},\ \mathbf{u}\in L^2(0,\infty;\mathbf{V}^0(\Gamma))\Big\},$$

where

$$I(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_s^T\int_\Omega |(-P\Delta)^{-1/2}P\mathbf{y}|^2 + \frac{1}{2}\int_s^T\int_\Gamma |R_A^{1/2}\mathbf{u}|^2,$$

and

$$(4) \qquad \begin{aligned} &\frac{\partial \mathbf{y}}{\partial t} - \nu\Delta\mathbf{y} + (\mathbf{w}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{w} - \omega\mathbf{y} + \nabla p = 0, \quad \text{in } Q_\infty,\\ &\text{div } \mathbf{y} = 0 \quad \text{in } Q_\infty, \quad \mathbf{y} = \theta(t)M\mathbf{u} \text{ on } \Sigma_\infty, \quad \mathbf{y}(0) = \mathbf{y}_0 \quad \text{in } \Omega. \end{aligned}$$

The weight function θ is a C^2 function from $\mathbb{R}^+$ into $[0,1]$, satisfying $\theta(0) = 0$, $\theta(t) = 1$ for $t \geq T$ for some $T > 0$. In that case the solution Π_ω of the Riccati equation of the control problem $(\mathcal{Q})$ depends on t in the intervall $[0,T]$. The exponential decay is obtained in the norm $\mathbf{H}^{1/2+\varepsilon}(\Omega)$ if $|\mathbf{y}_0|_{\mathbf{H}_0^{1/2+\varepsilon}(\Omega)\cap\mathbf{V}^0_n(\Omega)}$ is small enough.

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Sufficient Second-Order Optimality Conditions for Mixed Constrained Optimal Control Problems

Arnd Rösch
(joint work with Fredi Tröltzsch)

In this talk we consider the optimal control problem to minimize

$$F(y,u) = \int_\Omega f(x,y(x))\,dx + \int_\Gamma g(x,y(x),u(x))\,ds(x) \tag{1}$$

subject to the state equations

$$\begin{aligned} Ay + y &= 0 && \text{in } \Omega \\ \partial_{n_A} y &= b(x,y,u) && \text{on } \Gamma, \end{aligned} \tag{2}$$

the control constraints

$$0 \le u(x) \quad \text{for } x \in \Gamma, \tag{3}$$

and to the mixed control-state constraints

$$c(x) \le u(x) + \gamma(x)y(x) \quad \text{for } x \in \Gamma. \tag{4}$$

The main task of our talk is to establish second-order sufficient optimality conditions that are close to the associated necessary ones. For control-constrained problems, this issue was discussed quite completely in literature for semilinear elliptic and parabolic equations.

A suitable Lagrange functional can be defined by

$$\begin{aligned} L(y,u,p,\mu_1,\mu_2) &= F(y,u) - \int_\Omega (\sum_{i,j=1}^m a_{ij} D_j y D_i p + yp)\,dx - \int_\Gamma bp\,ds(x) \\ &\quad - \int_\Gamma \mu_1 u\,ds(x) - \int_\Gamma (u + \gamma y - c)\mu_2\,ds(x) \end{aligned}$$

since the Lagrange multipliers can be expressed by regular functions, i.e. $p \in Y = C(\bar\Omega) \cap H^1(\Omega)$ and $\mu_i \in L^\infty(\Gamma)$: The existence of such regular Lagrange multipliers has been proved in Tröltzsch [6], Bergounioux, Tröltzsch [2], and Arada, Raymond [1] for the parabolic case and Tröltzsch [7], Rösch, Tröltzsch [4] for the elliptic case. The existence of regular multipliers can be shown under a Slater type condition and the assumption $\gamma \ge 0$.

Moreover any local solution $\bar u$, the associated state $\bar y$, the corresponding adjoint state $\bar p \in Y$, and regular Lagrange multipliers $\bar\mu_i \in L^\infty(\Gamma)$, have to satisfy together the following first order necessary optimality system (FON),

$$(FON)\begin{cases} D_yL(\bar{y},\bar{u},\bar{p},\bar{\mu}_1,\bar{\mu}_2) & = & 0\\ D_uL(\bar{y},\bar{u},\bar{p},\bar{\mu}_1,\bar{\mu}_2) & = & 0\\ \text{and for a.a. } x\in\Gamma & & \\ \bar{\mu}_1(x) & \geq & 0\\ \bar{\mu}_2(x) & \geq & 0\\ \bar{u}(x)\bar{\mu}_1(x) & = & 0\\ (\bar{u}(x)+\gamma(x)\bar{y}(x)-c(x))\bar{\mu}_2(x) & = & 0. \end{cases}$$

We define strongly active sets by

$$\begin{aligned} A_1(\delta_1) &:= \{x\in\Gamma : \bar{\mu}_1(x)\geq\delta_1\},\\ A_2(\delta_2) &:= \{x\in\Gamma : \bar{\mu}_2(x)\geq\delta_2\}. \end{aligned}$$

Moreover, we say that $(y,u)\in C(\bar{\Omega})\times L^\infty(\Gamma)$ belongs to the critical subspace, if

$$\begin{aligned} u &= 0 && \text{on } A_1,\\ u+\gamma y|_\Gamma &= 0 && \text{on } A_2, \end{aligned}$$

and

$$\begin{aligned} Ay+y &= 0 && \text{in } \Omega\\ \partial_{n_A}y-\bar{b}_y\, y &= \bar{b}_u\, u && \text{in } \Gamma \end{aligned}$$

Now, we are able to state sufficient second-order optimality conditions:

SSC: There exist positive numbers $\delta,\ \delta_1,\ \delta_2$ such that the definiteness condition

$$L''_{(y,u)}(\bar{y},\bar{u},\bar{p},\bar{\mu}_1,\bar{\mu}_2)[y,u]^2\geq\delta\|u\|^2_{L^2(\Gamma)}$$

is satisfied for all (y,u) belonging to the critical subspace.

Theorem: Let (SSC) and a regularity condition be satisfied. Then there exist $\delta_s>0$ and $\varepsilon>0$ such that the quadratic growth condition

$$F(y,u)-F(\bar{y},\bar{u})\geq\delta_s\|u-\bar{u}\|^2_{L^2(\Gamma)}$$

holds for all admissible pairs (y,u) with $\|u-\bar{u}\|_{L^\infty(\Gamma)}<\varepsilon$. Therefore, $\bar{u}$ is a locally optimal control in the sense of $L^\infty(\Gamma)$.

The regularity condition is a solvability condition of an elliptic partial differential equation (see [5]). This condition is similar to the linear independence constraint qualification in finite dimensional optimization.

The presented theory is more general than a former result of the authors [3]: We abstain from any nonnegativity property of involved operators. Moreover, the definition of the strongly active sets seems to be more natural than in [3].

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On compactness, Domain Dependence and Existence of Steady State Solutions to Compressible Isothermal Navier-Stokes Equations.

Jan Sokolowski
(joint work with P. I. Plotnikov)

We prove the existence of stationary solutions to the Navier-Stokes equations of compressible isentropic flows

$$\alpha\varrho\mathbf{u} + \text{div } (\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla\varrho^\gamma = \varrho\mathbf{F} + \Delta\mathbf{u} + (1+\nu)\nabla\text{div}\mathbf{u} \text{ in } \mathcal{D}'(\Omega), \tag{1a}$$

$$\alpha\varrho + \text{div } (\varrho\mathbf{u}) = f \text{ in } \mathcal{D}'(\Omega), \quad u = 0 \text{ on } \partial\Omega \tag{1b}$$

in a bounded domain $\Omega \subset R^3$ on the condition that the adiabatic constant $\gamma \geq 1$. The main result is the following

Theorem. If $\gamma > 1$, then for every $\mathbf{F} \in C(\Omega)$ problem (1) has a weak solution $\varrho \in L^\gamma(\Omega)$, $\mathbf{u} \in H_0^{1,2}(\Omega)$. If $\gamma = 1$, then there are $\varrho \in L^1(\Omega)$ and $\mathbf{u} \in H_0^{1,2}(\Omega)$ satisfying (1b) such that

$$\alpha\varrho\mathbf{u} + \text{div } (\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla\varrho + \text{div } \mathcal{S} = \varrho\mathbf{F} + \Delta\mathbf{u} + (1+\nu)\nabla\text{div}\mathbf{u}.$$

Here the weak star defect measure $\mathcal{S}$ is concentrated on the one-dimensional rectifiable set Ω_{sing} and has the representation

$$\int_\Omega \varphi(x) : d\mathcal{S}(x) = \int_{\Omega_{sing}} \mathbf{s}(x)\otimes\mathbf{s}(x) : \varphi(x)m(x)d\mathcal{H}^1 \text{ for all } \varphi \in C_0^1(\Omega)^9,$$

Theorem yields the alternative: Either the concentration set is empty or its Hausdorff dimension is equal to one. Whether concentrations are cancelled or a non-trivial singular set really exists is a question which we cannot decide with certainty. Note only that if approximate solutions and a flow region are also invariant under the action of some group $x \to x'$, then a concentration set and a measure density θ also are invariant under the action of the this group. The precise definition of the concentration set and the measure density are given in [2]. In particular, the velocity field and the pressure are invariant with respect to the shift $x_3 \to x_3 +$ const, in the case of a two-dimensional flow in the plane (x_1, x_2). Therefore, in this case Ω_{sing} is the union of a countable set of straight lines and θ is a constant along each of those. From this we conclude that $\div\mathcal{S} = 0$ and concentrations are cancelled in agreement with results of P.L. Lions and [1]. The same results hold true for axially symmetric flows. On the other hand, the simple examples show that singularities definitely exist for solutions of the pressureless Navier-Stokes equations, which are used in astrophysics. Finally, let us point out that the results can be used in three dimensional case, in the same way as in [1] in two dimensional case, to establish the existence of solutions for the associated shape optimization problems of the drag minimization.

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An Optimization Approach for Frictional Contact Problems

Georg Stadler

We are concerned with the development and convergence analysis of *second-order algorithms* for the solution of frictional contact problems in function space. The main difficulty of these problems lies in the contact and friction conditions, which are inherently nonlinear thus making both theoretical analysis as well as an efficient numerical realization challenging (we refer to the selected contributions [1,2,5] and the references given therein). Here, we mainly consider the contact problem with *Tresca friction* (also known as *given friction*) and remark that the more realistic contact problem with *Coulomb friction* can be approached by solving a sequence of Tresca friction problems and using a fixed point idea.

The contact problem with Tresca friction can be stated as the following constrained and non-differentiable optimization problem:

$$(\mathcal{P}) \qquad \min_{\substack{\boldsymbol{\tau y}=0 \text{ on } \Gamma_d \\ \tau_N \boldsymbol{y} \le d \text{ a.e. on } \Gamma_c}} J(\boldsymbol{y}) := \frac{1}{2} a(\boldsymbol{y},\boldsymbol{y}) - L(\boldsymbol{y}) + \int_{\Gamma_c} \mathfrak{F} g \|\boldsymbol{\tau}_T \boldsymbol{y}\|\, dx.$$

Here, $\Omega \subset \mathbb{R}^n$, $n = 2,3$ is the region occupied by the elastic body whose boundary is split into three disjoint parts Γ_d, Γ_n and Γ_c. By $\boldsymbol{\varepsilon}(\boldsymbol{y}) = \frac{1}{2}\big(\nabla \boldsymbol{y} + (\nabla \boldsymbol{y})^\top\big)$ and $\boldsymbol{\sigma}(\boldsymbol{y}) = \lambda \operatorname{tr}(\boldsymbol{\varepsilon}(\boldsymbol{y}))\mathrm{Id} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{y})$ we denote the linear strain and stress tensors, respectively, where λ and μ are the Lamé parameters given by $\lambda = (E\nu)/\big((1+\nu)(1-2\nu)\big)$ and $\mu = E/\big(2(1+\nu)\big)$ with Young's modulus $E > 0$ and the Poisson ration $\nu \in (0, 0.5)$. The symmetric bilinear form $a(\cdot\,,\cdot)$ and the linear form $L(\cdot)$ are defined by $a(\boldsymbol{y},\boldsymbol{z}) := \int_\Omega \boldsymbol{\sigma}(\boldsymbol{y}) : \boldsymbol{\varepsilon}(\boldsymbol{z})\, dx$ and $L(\boldsymbol{y}) = \int_\Omega \boldsymbol{f}\boldsymbol{y}\, dx + \int_{\Gamma_n} \boldsymbol{t}\,\boldsymbol{\tau}\boldsymbol{y}\, dx$, where ":" denotes the sum of the componentwise products and $\boldsymbol{f} \in L^2(\Omega)$ and $\boldsymbol{t} \in L^2(\Gamma_n)$ denote inner force and surface tractions, respectively. The function $g \in L^2(\Gamma_c)$

denotes the given friction and $\mathfrak{F} : \Gamma_c \to \mathbb{R}$ is such that $\mathfrak{F}g \in L^2(\Gamma_c)$. Moreover, $d \geq 0$ models a possible gap between elastic body and rigid foundation. Finally, τ_N and $\boldsymbol{\tau}_T$ denote the normal and tangential component of the trace operator, respectively.

While $(\mathcal{P})$ is a constrained and non-differentiable optimization problem, its *Fenchel dual* is a constrained maximization problem with a differentiable functional [7]. The variables appearing in this dual problem can be interpreted as components of the stress tensor $\boldsymbol{\sigma}(\boldsymbol{y})$ and the first-order optimality conditions for $(\mathcal{P})$ (in the context of duality theory often called extremality conditions) involve both primal and dual variables.

Due to a lack of regularity of the dual variables we introduce a Tichonov-type regularization in the dual problem, depending on parameters $\gamma_1, \gamma_2 > 0$. In the corresponding primal problem this regularization leads to a quadratic penalization of the pointwise inequality constraint corresponding to the contact condition, and to a local C^1-smoothing of the non-differentiable friction term. As the regularization parameters tend to infinity, it can be shown that both the solution of the primal and of the dual problem converge to the solution of the original (primal and dual) problem in the corresponding function spaces.

For simplicity, in what follows we restrict ourselves to the case of planar elasticity, *i.e.*, $n = 2$ (see also [6]). In this case the primal solution $\boldsymbol{y}_\gamma \in (H^1(\Omega))^n$ of the smoothed problem is characterized by the existence of dual variables $\lambda_\gamma, \mu_\gamma \in L^2(\Gamma_c)$ such that

$$
\begin{aligned}
&a(\boldsymbol{y}_\gamma, \boldsymbol{z}) - L(\boldsymbol{z}) + (\mu_\gamma, \tau_T \boldsymbol{z})_{\Gamma_c} + (\lambda_\gamma, \tau_N \boldsymbol{z})_{\Gamma_c} = 0 \text{ for all } \boldsymbol{z} \in (H^1(\Omega))^n,\\
&\lambda_\gamma - \max(0, \hat{\lambda} + \gamma_1(\tau_N \boldsymbol{y}_\gamma - d)) = 0 \quad \text{on } \Gamma_c,\\
&\left\{\begin{array}{l} \gamma_2(\xi_\gamma - \tau_T \boldsymbol{y}_\gamma) + \mu_\gamma - \hat{\mu} = 0,\\ \xi_\gamma - \max(0, \xi_\gamma + \sigma(\mu_\gamma - \mathfrak{F}g)) - \min(0, \xi_\gamma + \sigma(\mu_\gamma + \mathfrak{F}g)) = 0 \end{array}\right. \quad \text{on } \Gamma_c,
\end{aligned}
$$

where $\sigma > 0$ is arbitrary and $\hat{\lambda}, \hat{\mu} \in L^2(\Gamma_c)$ denote fixed shifting functions whose introduction is motivated by augmented Lagrangians.

The above equations are *semismooth in function space* in the sense of [3, 8]. Thus, we can apply a generalized Newton method for their solution. Due to the appearance of the pointwise max- and min- functions, this results in an algorithm having the form of an *active set method.* It is related to the strategy in [1] and can be shown to converge locally superlinear in infinite dimensions. In our numerical experiments we observe that the method performs very reliably and that it converges from any initialization after very few (usually 5–12) iterations. Moreover, the number of iterations depends only weakly on the mesh-size and on the value of the regularization parameters, *i.e.*, we can solve the regularized problem very efficiently.

Naturally, the question arises how a family of regularized solutions can be used to obtain the solution of the original problem. Here, we propose two possibilities: Firstly, a *first-order augmented Lagrangian method,* which is an update strategy for the shift parameters $\hat{\lambda}$ and $\hat{\mu}$. While having desirable properties such as convergence from arbitrary initialization, the convergence of this method is quite slow if the regularization parameters γ_1, γ_2 are kept fixed. The second method we propose is a continuation strategy with respect to the parameters γ_1, γ_2. These parameters can either be increased heuristically or, more conveniently, using an *infinite-dimensional path-following strategy* as proposed in [4] for an obstacle problem. Similarly as for interior point methods the overall number of iterations of the path-following method can be decreased if the auxiliary problems are only solved approximately. The performance of our methods for solving contact problems with Tresca friction carries over to problems with Coulomb friction using a fixed point algorithm.

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Dirichlet Boundary Stabilization of the Plate Equation

Marius Tucsnak

(joint work with Kaïs Ammari and Gérald Tennenbaum)

The aim of this work is to give an exponential stability result for a Bernoulli-Euler plate equation in a square, damped by a feedback bending moment acting on a part of the boundary. The main novelty brought in by this work is that it gives a complete characterization of the control regions for which the exponential stability property holds. The main result of this work is the following.

Theorem 1. *Consider the square* $\Omega = (0,\pi)\times(0,\pi)$ *and let* Γ *be an open subset of* $\partial\Omega$. *Consider the following control problem*

$$\ddot{w} + \Delta^2 w = 0, \qquad x \in \Omega,\ t > 0, \tag{1}$$

$$w(x,t) = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{2}$$

$$\Delta w = 0, \qquad x \in \partial\Omega \setminus \Gamma,\ t > 0 \tag{3}$$

$$\Delta w(x,t) = -\frac{\partial}{\partial\nu}(G\dot{w}), \qquad x \in \Gamma,\ t > 0 \tag{4}$$

$$w(x,0) = w_0(x), \quad \dot{w}(x,0) = w_1(x), \qquad x \in \Omega, \tag{5}$$

where G *denotes the inverse of the Laplace operator with homogenous Dirichlet boundary conditions in* Ω *and* ν *denotes the outer unit normal field to* $\partial\Omega$. *Then the following assertions are equivalent:*
1) The system (1)-(5) *is exponentially stable in* $H_0^1(\Omega)\times H^{-1}(\Omega)$.
2) The control region Γ *contains both a horizontal and a vertical segment of non zero length.*

Sketch of the Proof.
First step. We show that equations (1)-(5) are equivalent, in a precise sense, to an initial value problem of the form

$$\ddot{w}(t) + A_0 w(t) + B_0 B_0^* \dot{w}(t) = 0,$$

$$w(0) = w_0, \quad \dot{w}(0) = w_1,$$

where H is an appropriate Hilbert space, $A_0 : \mathcal{D}(A_0) \to H$ is a self-adjoint and strictly positive operator, and B_0 is an unbounded admissible control operator defined in Hilbert space U with values in $\left[\mathcal{D}(A_0^{\frac{1}{2}})\right]^*$.
Second step. We show that the operators A_0 and B_0 defined at the first step satisfy the condition

If $\beta > 0$ is fixed and $C_\beta = \left\{\lambda \in \mathbb{C} \mid Re\lambda = \beta\right\}$, the function

$$\lambda \in \mathbb{C}_+ = \left\{\lambda \in \mathbb{C} \mid Re\lambda > 0\right\} \to H(\lambda) = \lambda\, B_0^*(\lambda^2 I + A_0)^{-1} B_0 \in \mathcal{L}(U)$$

is bounded on C_β

The basic technical tool in the proof of the above estimate is the following elementary lemma.

Lemma 2. *Assume that* $\alpha > 0$. *Then there exists a constant* $C > 0$, *depending only on* α, *such that for all* $m \in \mathbb{N}$ *and for all* $\lambda \in \mathbb{C}$ *with real part equal to* α *we have*

$$\sum_{n\geq 0} \frac{|\lambda|}{|\lambda^2 + (m^2+n^2)^2|} \leq C.$$

Third step.
By using the result obtained at step 2, combined with the general result from [1] it follows that that the exponential stability of (1)-(5) is equivalent to an exact observability estimate for the corresponding undamped problem. The proof is concluded by using the recent result in [2] which asserts that this exact observability result property is equivalent to property *2)* in the statement of the theorem. ■

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Quantum Control: from Theory to Experimental Practice

Gabriel Turinici

This summary will present activities related to the control of quantum phenomena. More specifically, the focus of the research presented here is on the interaction of the laser with matter. The applications include not only tailored construction of chemical compounds (through selective dissociation of chemical bounds in molecules or through creation of new ones), but also preparation of specific quantum states (that can for instance be later used in logic gates for quantum computers) and fast switches in semiconductors. Yet other applications are related to the *High Harmonic Generation* techniques, where a laser of given frequency is input to a system (typically a chrystal) and lasers of integer multiples of this frequency are obtained as output; this is a very promising technique to build e.g. high frequency X-rays lasers. The mathematical description of the laser-matter interaction is formulated within the framework of the quantum theory through a time dependent Schrödinger equation containing on the one part the internal Hamiltonian of the system H_0 and on the other part terms describing the its interaction with the laser. If the interaction is considered up to the first order, a bi-linear system results where the control term (the laser intensity) $\epsilon(t)$ multiplies the quantum state wavefunction:

$$(1) \qquad \begin{cases} i\frac{\partial}{\partial t}\Psi(x,t) = (H_0 - \epsilon(t)\mu)\Psi(x,t) \\ \Psi(x, t=0) = \Psi_0(x). \end{cases}$$

Here x is the set of internal variables, Ψ is the state of the system and $\mu(x)$ is the coupling dipole operator.

1. Quantum controllability

Successful control of chemical phenomena has been demonstrated on a variety of experimental settings. A natural question is concerned to the controllability of the systems, which requires to assess the set of all attainable final states. Results have already been obtained in finite-dimensional settings related to the computation of the Lie algebra spanned by $-iH_0$ and $-i\mu$ [1, 2] or directly with tangent space results [3], and a fairly complete description of phenomena at work is present. However, fundamental questions remains still unanswered in this field such as the good notions of controllability of the infinite-dimensional equations complemented with easily implementable criterions to assess this controllability property.

2. Monotonic algorithms for quantum optimal control

At the level of the numerical simulations, much work is still required in order to bring the size of the systems that can be treated accurately to practical dimensions. Of course, using efficient algorithms to solve the quantum control critical point equations is crucial to the overall cost reduction; some of the most used schemes nowadays fall within the class of *monotonically convergent algorithms* that are guaranteed to improve the performance (measured through a cost functional $J(\epsilon)$) at each iteration. Joint work with several co-authors lead to the introduction of new classes of monotonic algorithms [4, 5] and to the study of their convergence properties. Further works concerned the Lyapunov functional approaches [6]. Finally, similarities between the two classes of algorithms have been demonstrated.

3. Algorithms for experimental realization of quantum control

Experimental realization of quantum control is performed in practice through the minimization of the cost functional $J(\epsilon)$, realized on a computer, but which calls an experimental cycle each time when the value $J(\epsilon)$, is to be computed, and measures the result; the minimization algorithm used is most often a derivative of Genetic Algorithms or Evolutionary Strategy paradigm. The understanding of how this algorithm manages to find the good solutions and its practical implications are the goal of this part of my research. During our studies, we realized that other algorithms can be used in the experimental setting [7], and one of those is actually under implementation at Princeton University.

4. Parallel in time discretization schemes.

Whereas massively parallel computers enable simulations on larger and larger space scales, very few methods are available to achieve similar results in the time domain. Naturally, contrary to space, time is sequential and this precludes a priori the straightforward implementation of a parallel approach. The parareal discretization scheme (designed initially with JL Lions and Y Maday) [8] is a possible solution to this endeavor. It combines very precise simulations run in parallel on disjoint time segments with a coarse (approximate) simulation over the entire time span. This scheme has also been extended to the control of quantum evolution equations [9].

5. Dynamical discrimination of molecules

Similar molecules often may be characterized as sharing common chemical structures and as such, they are expected to have related Hamiltonians and similar chemical and physical properties. Examples range from simple isotopic variants of diatomics to highly complex molecules including those of biological relevance. A common need is to analyze or separate one molecular species in the presence of possibly many other similar agents. To enhance the ability to distinguish molecules, we advocate the use of the optimal dynamic discrimination (ODD) approach: all similar molecules are excited by a common laser pulse optimized to maximize signals from only one species, while suppressing signals from all the others. A controllability analysis and implementations [10, 11] shown that this techniques gives good results in practice.

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Second-Order Approaches to Constrained Large-Scale Optimization Problems with Partial Differential Equations

Michael Ulbrich

We consider two modern second order approaches to constrained large-scale optimization problems with PDEs: Interior-point methods and semismooth Newton methods. It is shown that the resulting linear systems for the step computation have very similar structure. In order to implement these algorithms efficiently, fast solvers for the Newton systems have to be developed. We describe how the semismooth Newton system arising from 3D two-body elastic contact problems can be solved very efficiently by multigrid methods [3]. As a second application, we consider free material optimization (see, e.g., [1, 4]) and develop a preconditioner for the primal-dual/semismooth Newton system. Finally, free material optimization with contact is considered. This problem results in an infinite-dimensional mathematical program with equilibrium constraints (MPEC). Currently, MPECs [2] are under intensive investigation in the field of finite-dimensional nonlinear optimization. We show that the approaches developed there are successfully applicable to the problem at hand. The investigations of this study are illustrated and supported by large-scale numerical tests.

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Generalized SQP-Methods with "Parareal" Time-Domain Decomposition for Time-Dependent PDE-Constrained Optimization

Stefan Ulbrich

We present recent results from [6] on a generalized SQP-framework with iterative solvers based on the parallel "Parareal" time-decomposition algorithm for the parallel solution of time-dependent PDE-constrained optimization problems.

The Parareal algorithm was recently proposed by J.-L. Lions, Y. Maday et al. [1–4] as a parallel solver for time-dependent PDEs. The Parareal algorithm is a time-domain-decomposition method that combines the parallel solution of the PDE on the subdomains by a high resolution scheme with an error propagation step by a sequential low resolution scheme on a coarse grid. The Parareal algorithm allows the fast solution of time-dependent PDEs on parallel computers and is capable of handling nonlinear PDEs efficiently without linearization. The Parareal algorithm can be seen as a preconditioned parallel iterative solver for a multiple shooting formulation of the PDE.

In this talk we propose a generalized SQP-framework for time-dependent PDE-constrained optimization that allows the flexible use of external iterative Parareal solvers for the state equation and adjoint equation. The generalized SQP-method uses a novel nonmonotone SQP-concept without penalty function inspired by [5] that enforces convergence and controls the inexactness of the Parareal state and adjoint solvers efficiently. The algorithm is capable of using arbitrary, also nonlinear, user-provided state and adjoint solvers, in particular Parareal solvers. This leads to a modular parallel SQP-type algorithm based on time-decomposition techniques for time-dependent PDE-constrained optimization.

The efficiency of the approach is demonstrated by numerical results for the optimal control of semilinear parabolic PDEs in 2-D.

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Adaptive Finite Element Methods for Optimization Problems

Boris Vexler

We present a systematic approach to error control and mesh adaptation in the numerical solution of optimization problems governed by partial differential equations.

The infinite dimensional optimization problems are discretized by finite element methods leading to discrete problems. This procedure may be interpreted as *model reduction*. It is desirable to carry out the optimization process on a cheap discrete model which still captures the "essential" features of the physical problem under consideration. The three main questions which arise are:

- What are the "essential" features?
- How can they be measured?
- How such a cheap discrete model can be designed?

For measuring a quality of a given discretization (model), it is crucial to introduce a quantity describing the goal of the computation, called *quantity of interest*. Due to different types of the quantity of interest we distinguish between the following types of problems:

(A) *Functional minimization problem*, if the quantity of interest coincides with the cost functional;

(B) *Parameter identification problem*, if the quantity of interest is given as a functional on the control (parameter) space;
(C) *Parameter (model) calibration problem*, if the quantity of interest depends on both the state and the control variable.

For these three types of optimization problems, we derive a posteriori error estimates accessing the discretization error with respect the quantity of interest, see [2, 3].

Since we consider finite element (mesh-based) discretization, a choice of a discrete model is equivalent to a choice of an appropriate finite element mesh. Therefore, we develop algorithms for finding efficient (cheap) discretizations by automatically constructing locally refined meshes with a "minimal" number of mesh points. These algorithms are based on the error estimators for the discretization error with the quantity of interest. Our approach extends the concepts for error estimation from [1].

The developed methods are applied to optimal control problems in fluid dynamics as well as to estimation of chemical models in multidimensional reactive flow problems [4]. We demonstrate the behavior of our method on the problem of calibration of the diffusion coefficients for a hydrogen flame with detailed chemistry. The underlying model includes the compressible Navier-Stokes equations and nine (nonlinear) convection-diffusion-reaction equations for chemical species.

In addition, we present recent results concerning space time finite element discretization, see [5], and a posteriori error estimation for time-dependent (parabolic) optimization problems.

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Optimal Control Problems with Pointwise Convex Control Constraints

Daniel Wachsmuth

Introduction. In fluid dynamics the control can be brought into the system by blowing or suction on the boundary. Then the control is a velocity, which is a directed quantity, hence it is a vector in $\mathbb{R}^2$ respectively $\mathbb{R}^3$. That is, the optimal control problem is to find a vector-valued function $u \in L^p((0,T) \times \Omega)^n$.

In the literature, optimal control problems with control constraints in the form of box constraints $u_a(\xi) \le u(\xi) \le u_b(\xi)$ are mostly investigated. This is the most suitable choice in cases where the control is a scalar quantity such as heating, cooling and so on. But as already mentioned, in some applications the control $u(\xi)$ is a vector. In this case, it is more adequate to have control constraints of the form $g(\xi; u(\xi)) = g(\xi; u_1(\xi), \ldots, u_n(\xi)) = 0$ or $u(\xi) \in U(\xi) \subset \mathbb{R}^n$.

There are a few articles about optimal control problems with such constraints. Second-order necessary conditions for problems with the control constraint $u(\xi) \in U(\xi)$ were proven by Páles and Zeidan [4] involving second-order admissible variations. Second-order necessary as well as sufficient conditions were established in Bonnans [1], Bonnans and Shapiro [2], and Dunn [3]. However, the set of admissible controls has to be polygonal and independent of ξ, i.e. $U(\xi) \equiv U$.

In contrast, we will follow another approach. We treat the control constraint as an inclusion $u(\xi) \in U(\xi)$. The advantage of this approach is that the analysis is based on rather elementary say geometrical arguments, hence there is no need of any constraint qualification. For the details, we refer to the forth-coming article [5].

The optimization problem. We will investigate optimality conditions for optimal control problems with a general set-valued control constraint. For the sake of brevity, we deal with the abstract problem

$$\min f(u) \text{ subject to } u \in U_{ad} \subset L^2(Q)^n. \tag{1}$$

The function f is required to be twice Fréchet differentiable from $L^2(Q)^n$ to $\mathbb{R}$. Here, Q is a measurable subset of $\mathbb{R}^n$. It represents the set where the control acts. The set of admissible controls U_{ad} is defined by

$$U_{ad} = \{u \in L^2(Q)^n : \; u(\xi) \in U(\xi) \text{ a.e. on } Q\}.$$

The admissible set is built by a set-valued function $U : Q \rightsquigarrow \mathbb{R}^n$. We will impose the following assumptions on this mapping:

(i) U is a measurable set-valued function, whose images $U(\xi)$ are closed and convex with non-empty interior a.e. on Q.
(ii) There exists a function $f_U \in L^2(Q)^n$ with $f_U(x,t) \in U(x,t)$ a.e. on Q.

Please note, we did not impose any conditions on the sets $U(x,t)$ that are beyond convexity such as boundedness or regularity of the boundaries $\partial U(x,t)$. Assumption (i) guarantees the existence of a measurable selection of U, i.e. a measurable single-valued function f_M with $f_M(x,t) \in U(x,t)$ a.e. on Q. However, no measurable selection needs to be square-integrable, therefore, the second assumption guarantees the existence of an admissible control.

Optimality conditions. The first order necessary condition for our abstract problem can be derived from known results: Let $\bar{u}$ be a local minimizer of (1). Then it holds

$$\nabla f(\bar{u})(u - \bar{u}) \ge 0 \qquad \forall u \in U_{ad}, \tag{2}$$

which is equivalent to $-\nabla f(\bar{u}) \in \mathcal{N}_{U_{ad}}(\bar{u})$. Here $\mathcal{N}_C(u)$ denotes the normal cone on a convex set C at a point u.

Now, we will present a second-order sufficient condition for local optimality of a reference control $\bar{u}$. Let us first introduce for $\epsilon > 0$ the set of points, where the control constraint is strongly active by

$$Q_\epsilon(\bar{u}) = \left\{\xi \in Q : \operatorname{dist}\left(-\nabla f(\bar{u})(\xi),\, \mathrm{rb}\mathcal{N}_{U(\xi)}(\bar{u}(\xi))\right) > \epsilon\right\}.$$

Here $\mathrm{rb}\mathcal{N}$ is the relative boundary of the normal cone. This definition means that $-\nabla f(\bar{u})(\xi)$ lies not only in the normal cone but has also some positive distance to its relative boundary.

We assume the following coercivity condition to be satisfied for some $\delta > 0$

$$\text{(3)} \qquad \nabla^2 f(\bar{u})[h,h] \geq \delta \|h\|_2^2 \qquad \forall h \in L^2(Q)^n : h_N(\xi) = 0 \text{ a.e. on } Q_\epsilon(\bar{u}).$$

Thus, we need coercivity of $\nabla^2 f$ only in directions h whose normal component h_N vanishes on the strongly active set. The function h_N is defined as the pointwise projection of h on the space of normal directions, i.e. $h_N(\xi) = \mathrm{proj}_{\mathrm{span}\,\mathcal{N}_{U(\xi)}(\bar{u}(\xi))}(h(\xi))$. Assuming that (2) and (3) are satisfied, we can prove local optimality of $\bar{u}$.

Theorem. Let $\bar{u}$ satisfy the conditions (2) *and* (3) *for some $\epsilon, \delta > 0$. Then $\bar{u}$ is locally optimal in L^∞, and there are constants $\rho, \alpha > 0$ such that the quadratic growth*

$$f(\bar{u}) \leq f(u) + \alpha \|u - \bar{u}\|_2^2$$

holds for all $u \in U_{ad}$ with $\|u - \bar{u}\|_\infty \leq \rho$.

For a proof in the context of optimal control of non-stationary Navier-Stokes equations we refer to [5]. The crucial point in the proof is to show that the function h_N, which is the projection on the normal directions, is measurable.

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Shape Control for Wave Equations

Jean-Paul Zolesio

We consider classical wave and heat equations in a moving domain. The moving domain is contained in a fixed bounded universe D and the evolution is in finite time $T < \infty$. The non-cylindrical evolution domain Q is derived from the convection of the caracteristic function χ_{Ω_0} for a given speed vector field V, given in

$\mathcal{H} := L^1(0, T, BV(D, \mathbb{R}^N))$ with div$V \in L^2(0, T, L^2(D)))$ and $V \cdot n = 0$ at ∂D. That speed field V turns out to be the control parameter of some "classical like" functional. Actually, the wave equation solution is not known to exist in such a nonsmooth tube Q. In order to close the analysis we consider in fact the weak closure in $\mathcal{H}$ of some family f of smooth vector fields $V_n \in L^1(0, T, W^{1,\infty}(D, \mathbb{R}^N)) = E$ for which the classical flow mapping $T_t \subset V$ is classical one to one defined in D.

From Cooper, Cooper and Strauss, and P. Acquistapace etc. it is known that non cylindrical wave and heat equations do have unique solutions y_n in Q_n, associated to the control $V_n \in E$. With D. Bucur, we introduce the density perimeter or fractal perimeter as

$$P_\gamma(A) = \sup_{0<\varepsilon<\gamma} \text{meas}_{\mathbb{R}^N} \frac{A^\varepsilon}{2\varepsilon}$$

for any closed set A and A^ε, the ε-delation, $A^\varepsilon = \bigcup_{n \in A} B(n, \varepsilon)$.

Among all good properties it is known that $P_\gamma(\partial\Omega) < \infty$ implies that $\Omega \backslash \partial\Omega$ is an open set and $\text{meas}_{\mathbb{R}^N}(\partial\Omega) = 0$ so that, using the weak formulation of the wave problem, we get existence for the limiting domain Q. This turns out to the clean – using the transverse field Z – solution of the evolution problem $Z_t + [Z, V] = W$ where $[\cdot, \cdot]$ is the Lie-bracket, and of its adjoint backward problem. We derive the gradient with respect to the control V for the cost functional.

We also present the cubic derivative of the energy.

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