Mathematisches Forschungsinstitut Oberwolfach

Report No. 19/2005

Kommutative Algebra

Organised by
Winfried Bruns (Osnabrück)
Hubert Flenner (Bochum)
Craig Huneke (Lawrence)

April 24th – April 30th, 2005

Abstract. The lectures of the conference whose participants included a significant group of researchers from affine geometry were devoted to the following areas: characteristic p-methods, combinatorial commutative algebra and effective methods, homological methods and invariants, multiplier ideals, affine geometry.

Mathematics Subject Classification (2000): 13xx, 14Rxx.

Introduction by the Organisers

The workshop on Commutative Algebra was very well attended by the important senior researchers in the field and many promising young mathematicians. A major subgroup of the participants was formed by researchers in affine geometry, a neighboring field that has strong interactions with commutative algebra.

The NSF Oberwolfach program made it possible to increase the number of young participants considerably, and the organizers are very grateful to the NSF for its support.

The conference took place in a very lively atmosphere, made possible by the excellent facilities of the institute. There were 53 participants and 19 talks with a considerable number of lectures given by young researchers. The program left plenty of time for cooperation and discussion among the participants. We highlight the areas in which new results were presented by the lecturers:

(a) *Characteristic p-methods* Starting with the landmark results of Peskine and Szpiro, characteristic p methods have had a extremely strong influence on the development of commutative algebra. They were crystalized by Hochster and

Huneke in the notion of tight closure, and have led to remarkable results in ideal and module theory.

(b) *Combinatorial commutative algebra and effective methods* The spectacular applications of commutative algebra to enumerative combinatorics two decades ago have developed into a subfield of commutative algebra that is very active now. The main objects are algebraic structures defined by monomials, in particular face rings of simplicial complexes and affine monoid algebras. Another driving force of this development are the powerful effective methods based on Gröbner bases since they ultimately rely on monomial computations.

(c) *Homological methods and invariants* This area started from the fundamental theorems of Auslander, Buchsbaum and Serre on regular rings. Homological properties are used in the major classification of commutative rings and their modules. In the last decade considerable progress has been made through the use of differential graded algebras, derived categories and the duality between the polynomial ring and the exterior algebra.

(d) *Multiplier ideals* The multiplier ideals of an ideal $I \subset \mathbb{C}[X_1, \dots, X_n]$ constitute subtle invariants of the singularity defined by I. Originally defined in analytic terms, they have now been applied to algebraic problems and have led to surprising results in ideal theory, especially on uniform bounds for the asymptotic behavior of various kinds of powers of an ideal.

(e) *Affine geometry* Affine geometry deals with algebro-geometric questions of affine varieties. The talks concerned actions of the additive and multiplicative groups $\mathbf{G_a}$ and $\mathbf{G_m}$ on affine varieties and, in particular, the affine space. Moreover there were lectures on the famous Jacobian problem and possible generalizations to other varieties. In many of the contributions the close ties to commutative algebra became apparent and led to fruitful discussions.

Workshop: Kommutative Algebra

Table of Contents

Abstracts

Recent Progress on the Jacobian Conjecture

Shreeram S. Abhyankar

Section 1: Introduction. Let $S^\times$ = set of all nonzero elements of a ring S. Call a pair of integers (N, M) principal if either M divides N, or N divides M.

Let X, Y be indeterminates over an algebraically closed field k of characteristic zero, and let there be given any two elements $f = f(X, Y)$ and $g = g(X, Y)$ in $k[X, Y]$. Call (f, g) an automorphic pair if we have $k[f, g] = k[X, Y]$. Let $J(f, g)$ be the jacobian of f, g relative to X, Y, i.e., $J(f, g) = f_X g_Y - f_Y g_X$ where subscripts denote partial derivatives. Call (f, g) a jacobian pair if we have $J(f, g) \in k^\times$. The jacobian problem conjectures that every jacobian pair is an automorphic pair.

To do the jacobian problem, without loss of generality, we may and we shall assume that f and g are monic polynomials of positive degrees N and M in Y with coefficients in $k[X]$. By Theorem (20.4) on page 149 of [Ab1] we see that

(1.1) if (f, g) is a jacobian pair then $k(X, f, g) = k(X, Y)$.

Let $m_0, \ldots, m_{h+1} = \infty$ be the characteristic sequence when we expand g in terms of f as polynomials in Y, and let $d_1, \ldots, d_{h+1} = 1$ be the corresponding GCD sequence. Note that $(m_0, m_1) = (-N, -M)$, for $1 \le i \le h+1$ we have $m_{i-1} \in \mathbb{N}$ and $d_i = \mathrm{GCD}(m_0, \ldots, m_{i-1})$, and h is called the number of characteristic pairs of the pair (f, g). Detailed definitions of h, m_i, d_i in Section 2. Observe that,

$\mathrm{GCD}(N, M) = 1 \Leftrightarrow h = 1$,

whereas

$\mathrm{GCD}(N, M) =$ a prime number $\Rightarrow h = 2$.

As reported on page 181 of the Engineering Book [Ab3], in Abhyankar's Purdue lectures of 1971, it was shown that if (f, g) is a jacobian pair and

either: $h \le 2$ (two characteristic pair case),

or: $h = 3$ with $d_3 \le 4$ (plus epsilon case),

then (f, g) is an automorphic pair.

Using approximate roots, we propose to write down the modification of the proof of the two characteristic pair case (recalled in [Ab4]) required to get a proof of the plus epsilon case. Actually, we propose to prove the SHARPER RESULT which says that if $h = 3$ and d_3 is even then (f, g) is an automorphic pair.

Let us note that the following two results (1.2) and (1.3) were proved in Theorem (18.13) and (19.4) on pages 138 and 143 of [Ab1] respectively. In these results, and in the Remarks following them, we do not assume f, g to be monic in Y but we let positive integers N, M stand for their total degrees. Recall that f has r points at infinity means the degree form of f, i.e., the homogeneous polynomial consisting of the highest degree terms in f, is a product of powers of r pairwise coprime homogeneous linear polynomials.

(1.2) if (f, g) is a jacobian pair then f has at most two points at infinity.

$$(1.3)\quad \begin{cases} \text{The following four implications are equivalent.} \\ \text{(i) } (f,g) \text{ is a jacobian pair} \Rightarrow (f,g) \text{ is an automorphic pair.} \\ \text{(ii) } (f,g) \text{ is a jacobian pair} \Rightarrow \text{the pair } (N,M) \text{ is principal.} \\ \text{(iii) } (f,g) \text{ is a jacobian pair} \Rightarrow f \text{ has only one point at infinity.} \\ \text{(iv) } (f,g) \text{ is a jacobian pair} \Rightarrow \text{the Newton polygon of } f \text{ is a triangle.} \end{cases}$$

Remarks. Theorem (18.13) of [Ab1] proves (1.2) also for "weighted degree forms" of f. On page 181 of [Ab3] it was observed that the jacobian conjecture was settled also for $\min(N, M) \leq 52$.

Now we revert to assuming f, g to be monic in Y.

Section 2: Characteristic Sequences and Approximate Roots. Let

$$\Phi = \Phi(X,Y) = Y^N + \sum_{1\leq i\leq N} A_i(X)Y^{N-i} \in R = L((X))[Y]$$

with $N \in \mathbb{N}_+$ and $A_i(X) \in L((X))$ be irreducible in R where L is an algebraically closed field of characteristic zero. By Newton's Theorem

$$\Phi(T^N, Y) = \prod_{\omega^N=1} (Y - y(\omega T))$$

where $y(T) = \sum_{i\in\mathbb{Z}} y_i T^i \in L((T))$ with $y_i \in L$. Recall that the support $\mathrm{Supp}(y(T))$ of $y(T)$ is defined to be the set of all $i \in \mathbb{Z}$ for which $y_i \neq 0$, and note that it is independent of which root of Φ is called $y(T)$. The said support gives rise to certain finite sequences of integers. Let us define them for any $J \subset \mathbb{Z}$ rather than just for $J = \mathrm{Supp}(y(T))$. First, following the Kyoto Paper [Ab2] faithfully, let us define them abstractly.

A GCD-sequence is a system d consisting of its length $h(d) \in \mathbb{N}$ and its sequence $(d_i)_{0\leq i\leq h(d)+2}$ where $d_0 = 0$, $d_i \in \mathbb{N}_+$ for $1 \leq i \leq h(d)+1$, $d_i \in d_{i+1}\mathbb{Z}$ for $0 \leq i \leq h(d)$, and $d_{h(d)+2} \in \mathbb{R}^*$. A charseq (= characteristic sequence) is a system m consisting of its length $h(m) \in \mathbb{N}$ and its sequence $(m_i)_{0\leq i\leq h(d)+1}$ where $m_0 \in \mathbb{Z}^\times$, $m_i \in \mathbb{Z}$ for $1 \leq i \leq h(d)$, and $m_{h(d)+1} \in \mathbb{R} \cup \{-\infty, \infty\}$. The GCD-sequence of a charseq m is the GCD-sequence $d(m)$ obtained by putting $h(d(m)) = h(m)$ and $d_i = \mathrm{GCD}(m_0, \ldots, m_{i-1})$ for $0 \leq i \leq h(m)+2$, with $\mathrm{GCD}(P) = \infty$ if $P \not\subset \mathbb{Z}$. Given any charseq m, by the difference sequence of m we mean the charseq $q(m)$ defined by putting $h(q(m)) = h(m)$, $q_i(m) = m_i$ for $0 \leq i \leq 1$, and $q_i(m) = m_i - m_{i-1}$ for $2 \leq i \leq h(m)+1$. Given any charseq q, by the inner product sequence of q we mean the charseq $s(q)$ defined by putting $h(s(m)) = h(q)$, $s_0(q) = q_0$, and $s_i(m) = \sum_{1\leq j\leq i} q_j d_j(q)$ for $1 \leq i \leq h(m)+1$. Given any charseq q, by the normalized inner product sequence of q we mean the charseq $r(q)$ defined by putting $h(r(m)) = h(q)$, $s_0(q) = s_0(q)$, and $r_i(m) = r_i(q)/d_i(q)$ for $1 \leq i \leq h(m)+1$. A charseq m is upper-unbounded means $m_{h(m)+1} = \infty$.

Given any $J \subset \mathbb{Z}$ which is bounded from below and any $l \in \mathbb{Z}^\times$, we define the GCD-dropping sequence of J relative to l to be the unique upper-unbounded charseq $m(J,l)$ such that

$$m_i(J,l) = \begin{cases} l & \text{if } i = 0 \\ \min(J) & \text{if } i = 1 \\ \min(J \setminus \sum_{0\le j\le i-1} m_j(J,l)\mathbb{Z}) & \text{if } 2 \le i \le h(m(J,l)) + 1. \end{cases}$$

Given any $l \in \mathbb{Z}^\times$, we define the newtonian charseq $m(\Phi, l)$ of Φ relative to l by putting $m(\Phi, l) = m(\mathrm{Supp}(y(T)), l)$. Let $m = m(\Phi, l)$, $h = h(m)$, $d = d(m)$, $q = q(m)$, $s = s(q)$, $r = r(q)$. According to the classical theory of Halphen and Smith [Zar], the pairs of coprime integers $(m_i/d_{i+1}, d_i/d_{i+1})_{1\le i\le h}$ are called the characteristic pairs of (Φ, l). Let

$$R^\natural = \text{the set of all monic nonunit irreducibles in } R$$

and note that the Y-degrees of members of $R^\natural$ belong to $\mathbb{N}_+$. Recall that for a monic polynomial $U(Y)$ of positive degree E in Y with coefficients in a domain S of characteristic zero, and a positive integer D which divides E, the approximate D-th root $\mathrm{App}_D(U)$ of U is the unique monic polynomial $V = V(Y)$ of degree E/D in Y with coefficients in S such that $\deg_Y(V - U^D) < E - (E/D)$. We define the approximate root sequence of Φ relative to l to be the sequence $(\Phi_{[l,j]})_{1\le j\le h+1}$ where $\Phi_{[l,j]} = \Phi_{l,j}(X,Y) \in R$ is obtained by putting $\Phi_{[l,j]} = Y$ or $\mathrm{App}_{d_j}(\Phi)$ according as $j = 1$ or $2 \le j \le h+1$. The main result of [Ab2] (which is the Theorem on Orders of Approximate Roots on page 368 of that paper) says that

$$(2.1) \qquad \begin{cases} \text{if } l = N \text{ or } -N \text{ then for } 1 \le j \le h+1 \text{ we have} \\ \Phi_{[l,j]} \in R^\natural \text{ with } \mathrm{ord}_X(\mathrm{Res}_Y(\Phi, \Phi_{[l,j]})) = r_j. \end{cases}$$

Consider a polynomially parameterized plane curve $X = u(Z)$ and $Y = v(Z)$ where $u(Z), v(Z)$ are univariate monic polynomials of positive degrees N, M with coefficients in L. Assume that $L(u(Z), v(Z)) = L(Z)$, and let $\phi(X,Y) \in L[X,Y]^\times$ be irreducible such that $\phi(u(Z), v(Z)) = 0$ and, regarding ϕ as a member of $(L[X])[Y]$, the highest Y-degree term of ϕ is a monic polynomial in X. Then upon letting $\Phi(X,Y) = \phi(X^{-1}, Y)$ we have $\Phi(X,Y) \in R^\natural$ with $\deg_Y \Phi = N$.

In the situation of Section 1, let W, Z be indeterminates over $k(X,Y)$, and let L be an algebraic closure of $k(W)$ inside an algebraic closure of $k(W,X,Y,Z)$. Assuming $k(X,f,g) = k(X,Y)$ and taking $u(Z) = f(W,Z)$ and $v(Z) = f(W,Z)$ let us use the above set-up. Define the jacobian charseq $m(f,g)$ of the pair (f,g) by putting $m(f,g) = m(\Phi, -N)$. Let $m = m(f,g)$, $h = h(m)$, $d = d(m)$, $q = q(m)$, $s = s(q)$, $r = r(q)$. Now $(m_i/d_{i+1}, d_i/d_{i+1})_{1\le i\le h}$ may be called the characteristic pairs of the pair (f,g).

References

[Ab1] S. S. Abhyankar, *Expansion Techniques in Algebraic Geometry,* Tata Institute of Fundamental Research, Bombay, 1977.

[Ab2] S. S. Abhyankar, *On the semigroup of a Meromorphic Curve (Part I),* International Symposium on Algebraic Geometry (Kyoto), Kinokuniya, Tokyo, 1977, pages 249-414. Kinokuniya, Tokyo, 1977, pages 249-414.

[Ab3] S. S. Abhyankar, *Algebraic Geometry for Scientists and Engineers,* American Mathematical Society, 1990.
[Ab4] S. S. Abhyankar, *Some Remarks on the Jacobian Question,* Proceedings of the Indian Academy of Sciences, vol. 104 (1994), pages 515-542.
[Zar] O. Zariski, *Algebraic Surfaces,* Springer-Verlag, 1935.

Local cohomology multiplicities via local étale cohomology

Manuel Blickle
(joint work with Raphael Bondu)

Let $A = R/I$ for I an ideal in a regular (local) ring (R, m) of dimension n and containing a field k. The main results of [Lyu93, HS93] state that the local cohomology module $H^a_m(H^{n-i}_I(R))$ is injective and supported at m. Therefore it is a finite direct sum of $e = e(H^a_m(H^{n-i}_I(R)))$ many copies of the injective hull $E_{R/m}$ of the residue field of R. Lyubeznik shows in [Lyu93] that this number

$$\lambda_{a,i}(A) \stackrel{\text{def}}{=} e(H^a_m(H^{n-i}_I(R)))$$

does not depend on the auxiliary choice of R and I. The aim of this talk is to completely describe these invariants for a large class of rings; namely the ones that behave cohomologically like an isolated singularity. This includes all local rings which are complete intersections on the punctured spectrum and, in positive characteristic, even all rings which are set theoretically Cohen–Macaulay rings on the punctured spectrum.

Theorem ([Bli05]). *Let $A = \mathcal{O}_{Y,x}$ for Y a closed k–subvariety of dimension $d \geq 2$ of a smooth variety X. If for $i \neq d$ the modules $H^{n-i}_{[Y]}(\mathcal{O}_X)$ are supported in the point x then*

(1) for $2 \leq a \leq d$ one has

$$\lambda_{a,d}(A) - \delta_{a,d} = \lambda_{0,d-a+1}(A)$$

and all other $\lambda_{a,i}(A)$ vanish.

(2) Furthermore,

$$\lambda_{a,d}(A) - \delta_{a,d} = \begin{cases} \dim_{\mathbb{F}_p} H^{d-a+1}_{\{x\}}(Y_{\acute{e}t}, \mathbb{F}_p) & \text{if } \operatorname{char} k = p \\ \dim_{\mathbb{C}} H^{d-a+1}_{\{x\}}(Y_{\text{an}}, \mathbb{C}) & \text{if } k = \mathbb{C} \end{cases}$$

where $\delta_{a,d}$ is the Kroneker delta function.

In the case that A has only an isolated singularity and $k = \mathbb{C}$, this was shown by Garcia Lopez and Sabbah in [GLS98], using the Riemann–Hilbert correspondence and duality for holonomic D–modules. In our proof of the positive characteristic case we replace the Riemann–Hilbert correspondence by a correspondence of Emerton–Kisin [EK04], but have to work somewhat harder to make up for the lack of duality in this setting. The benefit is that the proof presented here applies to a more general situation and works, up to the plugin of the appropriate correspondence, in all characteristics.

Instead of going into too many details I will try to point out the key features of the proof (taking the risk that they might not become apparent to be "key" without all the details...).

Part (1) is a consequence of the spectral sequence for the composition of the functors $\Gamma_m \circ \Gamma_I = \Gamma_m$ and can be found in [BB04] in all detail.

In order to ease notation in the proof of part (2) I focus on the positive characteristic case. In [EK04] Emerton and Kisin relate (on the level of derived categories) the category of finitely generated unit $R[F]$–modules and the category of constructible sheaves on the étale site of $X = \operatorname{Spec} R$. The functor that yields the correspondence is denoted $\operatorname{Sol}(\_\_)$. It can be thought of as (dual to higher derived) fixed points of the Frobenius action such that we have (up to a shift) that $\operatorname{Sol} \mathcal{O}_X = \mathbb{F}_p$ by the Artin–Schreyer sequence. This implies that Sol of $E_{R/m} = H^d_x(\mathcal{O}_X)$ is equal to the constant sheaf $\mathbb{F}_p$ supported at the point x, hence

$$\lambda_{a,i}(A) = e(H^a_{[x]}(H^{n-i}_{[Y]}(\mathcal{O}_X))) = \dim_{\mathbb{F}_p} \operatorname{Sol}(H^a_{[x]}(H^{n-i}_{[Y]}(\mathcal{O}_X)).$$

Hence the problem becomes now to compute this solution functor. The trick in the proof is to replace $H^{n-i}_{[Y]}(\mathcal{O}_X)$ by something more accessible – that is by something whose solutions $\operatorname{Sol}(\_\_)$ can readily be computed. The assumption that $H^{n-i}_{[Y]}(\mathcal{O}_X)$ is supported at x for $i \neq d$ we rephrase by saying that one has a quasi-isomorphism of complexes

$$H^{n-d}_{[Y-x]}(\mathcal{O}_{X-x}) \cong \mathbf{R}\Gamma_{[Y-x]}(\mathcal{O}_{X-x})[n-d],$$

and, in fact, the solutions of the latter can be computed easily, namely they are equal to $i'_!(\mathbb{F}_p)_{Y-x}[d]$. To get this favorable situation one has to shift attention away from the point x. For this I use the intermediate extension $j_{!*}H^{n-d}_{[Y-x]}(\mathcal{O}_{X-x})$ which is close enough to $H^{n-i}_{[Y]}(\mathcal{O}_X)$ so that for the matters concerned with here it can actually replace the latter. This substitution via middle extensions was the key point in unlocking the theorem in positive characteristic since it is the appropriate substitute of $j_!$ which one would like to use but which unfortunately is not available in the positive characteristic context. The argument is then finished with a fairly straightforward computation.

Despite the apparent similarity of the positive and zero characteristic case which is suggested by the theorem, one should not expect that the invariants $\lambda_{a,i}(A)$ do behave well under reduction mod p. In fact, some of the $\mathbb{Q}$–algebras considered in Anurag Sings talk should yield an example of a $\mathbb{Q}$–algebra for which some $\lambda_{a,i}(A)$ is nonzero but $\lambda_{a,i}(A_p) = 0$ for almost all reductions mod p of an integral model of A. Also an example of a $\mathbb{Q}$–algebra such that the property $\lambda_{a,i}(A_p) = 0$ for reductions mod p of A varies in an arithmetic progression can probably be obtained from these.

References

[BB04] Manuel Blickle and Raphael Bondu, *Local cohomology multiplicities in terms of étale cohomology*, 2004, to appear in Ann. Inst. Fourier.

[Bli05] Manuel Blickle, *Lyubezniks invariants for cohomologically isolated singularities*, 2005, preprint.

[EK04] Mathew Emerton and Mark Kisin, *Riemann–Hilbert correspondence for unit $\mathcal{F}$-crystals*, Astérisque **293** (2004), vi+257 pp.

[GLS98] R. García López and C. Sabbah, *Topological computation of local cohomology multiplicities*, Collect. Math. **49** (1998), no. 2-3, 317–324, Dedicated to the memory of Fernando Serrano.

[HS93] Craig L. Huneke and Rodney Y. Sharp, *Bass numbers of local cohomology modules*, Trans. Amer. Math. Soc. **339** (1993), no. 2, 765–779.

[Lyu93] Gennady Lyubeznik, *Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra)*, Invent. Math. **113** (1993), no. 1, 41–55.

On the arithmetic of tight closure and of strong semistability

Holger Brenner
(joint work with Moty Katzman)

This talk reports on joint work [1] with M. Katzman (university of Sheffield). For a noetherian domain R containing a field K of positive characteristic, the tight closure of an ideal is defined to be

$$I^* = \{f \in R : \exists c \neq 0,\, cf^{p^e} \in I^{[p^e]} \text{ for all } e \geq 0\}.$$

This notion is due to M. Hochster and C. Huneke (see [2], [3]). How does the containment $f \in I^*$ depend on the prime characteristic? To make sense of this question suppose that $R_{\mathbb{Z}}$ is a finitely generated ring extension of $\mathbb{Z}$ and that $I \subseteq R_{\mathbb{Z}}$ is an ideal, $f \in R_{\mathbb{Z}}$. Then we may consider for every prime number p the specialization $R_{\mathbb{Z}/(p)} = R_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ of characteristic p together with the extended ideal $I_p \subseteq R_{\mathbb{Z}/(p)}$, and one may ask whether $f_p \in I_p^*$ holds or not. In this setting f is said to be in the tight closure of the ideal $IR_{\mathbb{Q}}$ in $R_{\mathbb{Q}}$ (in characteristic zero) if $f_p \in I_p^*$ holds for almost all prime reductions. Our main question is then:

(Q) Does the containment $f_p \in I_p^*$ for infinitely many prime numbers (dense property) implies that $f_p \in I_p^*$ holds true for almost all prime numbers.

This question is as old as the definition of tight closure in characteristic zero.

We deal with this question in the two-dimensional graded situation and relate it via the geometric interpretation of tight closure to questions about the arithmetical behavior of strongly semistable bundles. Recall that a vector bundle $\mathcal{S}$ on a smooth projective curve C is called semistable if for every subbundle $\mathcal{T} \subseteq \mathcal{S}$ the inequality $\deg(\mathcal{T})/\operatorname{rk}(\mathcal{T}) \leq \deg(\mathcal{S})/\operatorname{rk}(\mathcal{S})$ holds true. In positive characteristic, $\mathcal{S}$ is called strongly semistable if $F^{e*}(\mathcal{S})$ is semistable for all pull-backs of the absolute Frobenius morphism. If now $C \to \operatorname{Spec}\mathbb{Z}$ is a smooth projective relative curve and $\mathcal{S}$ is a vector bundle on C which is semistable in the generic fiber $C_{\mathbb{Q}}$ of characteristic zero, then one may ask the following questions:

(Q 1) Is the bundle $\mathcal{S}_p$ on C_p strongly semistable for almost all prime numbers p (Shepherd-Barron, [5])?

(Q 2) Is $\mathcal{S}_p$ strongly semistable for infinitely many prime numbers p (Miyaoka, [4])?

(Q 3) Is $\mathcal{S}_p$ strongly semistable for at least one prime numbers p?

(Q 4) What is the density of the set p such that $\mathcal{S}_p$ is strongly semistable?

It is known that $\mathcal{S}_p$ is semistable for almost all prime reductions. A positive answer to (Q 1) would imply a positive answer to our main question (Q) for homogeneous ideals in two-dimensional graded domains. Since semistable bundles on an elliptic curve are strongly semistable, the geometric interpretation gives that (Q) has a positive answer for cones over elliptic curves, like e.g. $R = K[X,Y,Z]/\text{smooth cubic}$.

The main goal of this talk is however to present counterexamples to (Q 1) and to (Q). We show that on the Fermat septic $C = \operatorname{Proj} K[X,Y,Z]/(Z^7 - X^7 - Y^7)$ the syzygy bundle $\operatorname{Syz}(X^4, Y^4, Z^4)$, which is given by

$$0 \longrightarrow \operatorname{Syz}(X^4,Y^4,Z^4) \longrightarrow \bigoplus_{i=1}^{3} \mathcal{O}_C(-4) \stackrel{X^4,Y^4,Z^4}{\longrightarrow} \mathcal{O}_C \longrightarrow 0$$

is not strongly semistable under the characteristic condition $p = 2 \bmod 7$. For that we show that the first Frobenius pull-back of the syzygy bundle has global non-trivial sections in a degree which contradicts semistability.

Using a destabilizing sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow F^*(\operatorname{Syz}(X^4,Y^4,Z^4)(6)) = \operatorname{Syz}(X^{4p},Y^{4p},Z^{4p})(6p) \longrightarrow \mathcal{M} \longrightarrow 0\,,$$

where $\mathcal{L}$ is an invertible sheaf of positive degree and $\mathcal{M}$ is an invertible sheaf of negative degree, and its Frobenius pull-back, we get an isomorphism

$$H^1(C, \operatorname{Syz}(X^{4p^2}, Y^{4p^2}, Z^{4p^2})(6p^2)) \longrightarrow H^1(C, \mathcal{M}^p)\,.$$

From this we deduce for a homogeneous element f of degree 6 the equivalence $f \in (X^4,Y^4,Z^4)^*$ if and only if $f^{p^2} \in (X^{4p^2}, Y^{4p^2}, Z^{4p^2})$, which reduces the tight closure computation to one single ideal membership test. For $f = X^3Y^3$ this ideal membership reduces after eliminating Z to the question whether the $(\ell+1)$-th elementary column vector is in the span of the $(2\ell+1, 2\ell)$ matrix given by the entries

$$\binom{4\ell+1}{2\ell+i-j}_{1 \leq i \leq 2\ell+1, 1 \leq j \leq 2\ell}\,.$$

Some extended matrix operations show at the end that this is not possible, and therefore $X^3Y^3 \notin (X^4,Y^4,Z^4)^*$ in characteristic $p = 2 \bmod 7$.

On the other hand we show that $X^3Y^3 \in (X^4,Y^4,Z^4)^*$ holds in characteristic $p = 3 \bmod 7$, which establishes the following theorem (using Dirichlet theorem on primes in an arithmetic progression):

Theorem. *In $R = K[X,Y,Z]/(Z^7 - X^7 - Y^7)$ the containment*

$$X^3Y^3 \in (X^4,Y^4,Z^4)^*$$

holds for infinitely many prime characteristics and holds not for infinitely many prime characteristics.

From this it follows also that the syzygy bundle $\operatorname{Syz}(X^4, Y^4, Z^4)$ is semistable in characteristic zero. Our example provides also an example of a $\mathbb{Z}$-algebra

$$A = \mathbb{Z}[X, Y, Z, U, V, W]/(UX^4 + VY^4 + WZ^4 + X^3Y^3)$$

(the forcing algebra of these data) such that the open subset $D(X, Y) \subset \operatorname{Spec} A$ is an affine scheme for infinitely many prime reductions and not affine for infinitely many prime reductions. Therefore the cohomological dimension varies arithmetically between 0 and 1 with the characteristic.

References

[1] H. Brenner and M. Katzman. On the arithmetic of tight closure. *ArXiv*, 2004.

[2] M. Hochster and C. Huneke. Tight closure, invariant theory, and the Briançon-Skoda theorem. *J. Amer. Math. Soc.*, 3:31–116, 1990.

[3] C. Huneke. *Tight Closure and Its Applications*, volume 88 of *CBMS Lecture Notes in Mathematics*. AMS, Providence, 1996.

[4] Y. Miyaoka. The Chern class and Kodaira dimension of a minimal variety. In *Algebraic Geometry, Sendai 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 449–476, 1987.

[5] N. I. Shepherd-Barron. Semi-stability and reduction mod *p*. *Topology*, 37(3):659–664, 1997.

Noncommutative Desingularization of the Generic Determinant

Ragnar-Olaf Buchweitz

(joint work with Graham Leuschke and Michel van den Bergh)

1. Let K be a field, $S = K[x_{ij}]$ the polynomial ring over K on the entries of the generic $(n \times n)$–matrix $X = (x_{ij})$, and $R = S/(\det X)$ the hypersurface given by the generic determinant. With $I_t(X)$ the ideal in S generated by all t–minors of X, we set as well $R_i := S/I_{n-i}(X)$, so that, in particular, $R = R_0$, and R_1 is the ring of the singular locus of $\operatorname{Spec} R$.

We view the x_{ij} as the coordinate functions on the K–affine space $\operatorname{Spec} S \cong \operatorname{Hom}_K(G, F)$, for a pair of free K–modules F, G of rank n. Let $\mathcal{G} := G \otimes_K S \cong \oplus_{j=1}^n g_j S$ and $\mathcal{F} := F \otimes_K S \cong \oplus_{i=1}^n f_i S$ be the corresponding free S–modules, with respect to ordered bases $(g_j)_j$ of G and $(f_i)_i$ of F. The *generic S–linear map* is then given by

$$\varphi : \mathcal{G} \longrightarrow \mathcal{F} \quad , \quad \varphi(g_j) = \sum_{i=1}^n f_i x_{ij} \, .$$

2. Writing exterior powers over S as $\Lambda^\bullet = \Lambda^\bullet_S$, for each integer $a = 0, ..., n$, the S–module $M_a := \operatorname{cok}(\Lambda^a \phi : \Lambda^a \mathcal{G} \to \Lambda^a \mathcal{F})$ is annihilated by $\det \varphi$ and so naturally a *maximal Cohen-Macaulay* (MCM) R–module, of rank $\binom{n-1}{a-1}$. In particular, $M_0 = \operatorname{cok}(\mathrm{id}_S) = 0, M_1 \cong \operatorname{cok} \varphi$, and $M_n \cong \operatorname{cok}(\det \varphi) \cong R$. Each of the two equations

$$(\Lambda^a \varphi)\, (\Lambda^a \varphi)^{adj} = (\det \varphi)\, \mathrm{id}_{\Lambda^a \mathcal{F}} \quad , \quad (\Lambda^a \varphi)^{adj}\, (\Lambda^a \varphi) = (\det \varphi)\, \mathrm{id}_{\Lambda^a \mathcal{G}}$$

defines uniquely the *adjoint* or *adjugate* $(\Lambda^a\varphi)^{adj} : \Lambda^a\mathcal{F} \to \Lambda^a\mathcal{G}$ to $\Lambda^a\varphi$. Then $M'_a = \operatorname{cok}(\Lambda^a\phi)^{adj}$ is again a MCM R–module, of rank $\binom{n-1}{a}$, and it fits into exact sequences of MCM R–modules

$$0 \longrightarrow M'_a \longrightarrow \Lambda^a\mathcal{F} \otimes_S R \longrightarrow M_a \longrightarrow 0$$
$$0 \longrightarrow M_a \longrightarrow \Lambda^a\mathcal{G} \otimes_S R \longrightarrow M'_a \longrightarrow 0 \ .$$

With $\underline{\operatorname{Hom}}_R(?, ??)$ the bifunctor of *stable* R–homomorphisms, one obtains thus isomorphisms of R–modules

$$\operatorname{Ext}^{2i+1}_R(M_a, M_b) \cong \underline{\operatorname{Hom}}_R(M'_a, M_b) \qquad \text{for each } i \geq 0, \text{ and}$$
$$\operatorname{Ext}^{2j}_R(M_a, M_b) \cong \underline{\operatorname{Hom}}_R(M_a, M_b) \qquad \text{for each } j > 0.$$

3. A key theme that motivated this work was to determine the size and structure of these extension modules and their associated Yoneda algebras; see the work Graham Leuschke presents elsewhere in this report.

To formulate the results succinctly, set further $M := \oplus_a M_a \cong \operatorname{cok}(\Lambda^\bullet\varphi)$ and $M' := \oplus_a M'_a \cong \operatorname{cok}(\Lambda^\bullet\varphi)^{adj}$, and let $E := \operatorname{End}_R(M)$ denote the R–endomorphism algebra of M, with $\underline{E} := \underline{\operatorname{End}}_R(M)$ its stable version.

Theorem 4. *Using the notations introduced above, one has*

(1) $E = \operatorname{End}_R(M)$ *is MCM as R–module, whereas $\underline{E}$ is MCM as R_1–module.*
(2) $\operatorname{Ext}^{2i+1}_R(M, M) \cong \underline{\operatorname{Hom}}_R(M', M) = 0$ *for each* $i \geq 0$.
(3) As a ring, E is left and right noetherian and of finite global dimension.
(4) As an R–algebra, E is Frobenius *with respect to the natural trace map* $E = \operatorname{End}_R(M) \to R$.

5. Denoting $\mathcal{F}^*$ the S–dual of $\mathcal{F}$, the map $q_\varphi : \mathcal{F}^* \oplus \mathcal{G} \to S$ determined through $q_\varphi(\lambda, g) := \lambda\varphi(g) \in S$ is S*–quadratic* with associated symmetric matrix $\begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix}$. Assigning degree -1 to $\mathcal{F}^*$ and degree $+1$ to $\mathcal{G}$, the Clifford algebra $\mathcal{C}(q_\varphi)$ becomes naturally $\mathbb{Z}$–graded over S, and $\operatorname{Mod} E$, resp. $\operatorname{Mod} \underline{E}$, becomes equivalent to the abelian category of graded $\mathcal{C}(q_\varphi)$–modules that are concentrated in degrees $a = 1, ..., n$, respectively, $a = 1, ..., n-1$. Explicitly, E and $\underline{E}$ can be presented as "*quiverized Clifford algebras*":

Theorem 6. *As an S–algebra, $E \cong S\langle e_1, ..., e_n; u_1, ..., u_n; v_1, ..., v_n\rangle / J$, where J is the (two-sided) ideal generated by the relations*

(1) $\qquad e_ie_j = \delta_{ij}e_i\,;$

(2) $\qquad u_iu_j + u_ju_i = 0 = u_i^2 \quad$ *and* $\quad v_iv_j + v_jv_i = 0 = v_i^2\,;$

(3) $\qquad e_au_j = u_je_{a+1} \quad$ *and* $\quad e_av_j = v_je_{a-1} \quad$ *(with* $e_0 = 0 = e_{n+1}$*)*;

(4) $\qquad u_iv_j + v_ju_i = x_{ij}\,.$

The stable endomorphism algebra has the presentation $\underline{E} \cong E/Ee_nE$.

7. In this description, the occurring idempotents correspond to the canonical projectors $e_a\colon M \twoheadrightarrow M_a \hookrightarrow M$, the u_i arise from the contractions $f_i^* : \Lambda^\bullet\mathcal{F} \to \Lambda^{\bullet-1}\mathcal{F}$, whereas the v_j are induced by the multiplication maps $g_j\wedge ? : \Lambda^\bullet\mathcal{G} \to \Lambda^{\bullet+1}\mathcal{G}$.

These purely algebraic results are a consequence of the fact that E provides a *noncommutative desingularization* of $\operatorname{Spec} R$, as we now explain.

8. Let $\mathbb{P} = \mathbb{P}(F^*)$ be the K–projective space on the K–dual of F, set $Y = \mathbb{P}\times \operatorname{Hom}_K(G,F)$, with the canonical projections $p : Y \to \mathbb{P}, q : Y \to \operatorname{Hom}_K(G,F)$, and set, as usual, $?\boxtimes ?? := q^*(?)\otimes_{\mathcal{O}_Y} p^*(??)$ for sheaves or morphisms ? on $\operatorname{Spec} S$ and ?? on $\mathbb{P}$, respectively. The *incidence variety*

$$Z := \{([\lambda],\psi) \mid \lambda\in F^*, \psi\in \operatorname{Hom}_K(G,F), \lambda\psi = 0\} \subseteq Y$$

serves as a *desingularization* of the generic determinant hypersurface. Indeed, it is classical; see, for example, [2]; that

- The inclusion $j : Z \hookrightarrow Y$ is a *regular immersion* of codimension n, zero-locus of the cosection $\Phi : q^*\mathcal{G} \to p^*(\mathcal{O}_\mathbb{P}(1)) = \mathcal{O}_Y(1)$, determined by

  $$\Phi(q^*g_j) = \sum_i f_i\otimes x_{ij} \in H^0(Y,\mathcal{O}_Y(1)) \cong F\otimes_K \operatorname{Hom}_K(G,F)^*\,.$$

  Accordingly, $j_*\mathcal{O}_Z$ is resolved through $(\Lambda^\bullet_Y(\mathcal{G}\boxtimes\mathcal{O}_\mathbb{P}(-1)),\partial_{\Phi(-1)})$, the Koszul complex constructed over the $\mathcal{O}_Y$–linear form $\Phi(-1)$.
- The composition $p' = pj : Z \to \mathbb{P}$ is an *affine vector bundle*,

  $$Z \cong \mathbb{V}_\mathbb{P}(G\otimes_K \mathcal{T}_\mathbb{P}(-1)) \xrightarrow{p'} \mathbb{P}\,,$$

  where $\mathcal{T}_\mathbb{P}$ denotes the tangent sheaf on $\mathbb{P}$. In particular, Z is smooth, and the inclusion $j : Z \hookrightarrow Y$ corresponds, as a morphism over $\mathbb{P}$, to the canonical epimorphism of $\mathcal{O}_\mathbb{P}$–bundles $G\otimes_K (F^*\otimes_K \mathcal{O}_\mathbb{P} \twoheadrightarrow \mathcal{T}_\mathbb{P}(-1))$.
- The composition $q' = qj : Z \to \operatorname{Spec} S \cong \operatorname{Hom}_K(G,F)$ is a *resolution of singularities* of its image $q'(Z) = \operatorname{Spec} R$, the generic determinant.

9. Let $D^b(?) := D^b(\mathfrak{Coh}(?))$ denote the bounded derived category of coherent $\mathcal{O}_?$–modules, and, if A is a (right) noetherian ring, $D^b(A) := D^b(\operatorname{mod} A)$ that of the finite (right) A–modules. Write, as usual, $\mathbb{R}q'_*\colon D^b(Z) \to D^b(R)$ for the right derived functor of q'_*, and $p'^*\colon D^b(\mathbb{P}) \to D^b(Z)$ for the (exact) pullback along the projection $p' : Z \to \mathbb{P}$.

The crucial result is then as follows:

Theorem 10. *The R–algebra E provides a* noncommutative desingularization *of* $\operatorname{Spec} R$. *In detail, the $\mathcal{O}_Z$–module $\mathbb{T} := p'^*\left(\bigoplus_{a=1}^{n-1}\Omega^{a-1}_{\mathbb{P}/K}(a)\right)$ is a* tilting bundle *in $\mathfrak{Coh}(Z)$, with endomorphism algebra isomorphic to E, that is,*

(1) $E \cong \operatorname{End}_{D^b(Z)}(\mathbb{T})$ *as algebras,*
(2) $\operatorname{Ext}^i_Z(\mathbb{T},\mathbb{T}) := \operatorname{Hom}_{D^b(Z)}(\mathbb{T},\mathbb{T}[i]) = 0$ *for* $i\neq 0$,
(3) $\mathbb{R}\operatorname{Hom}_{D^b(Z)}(\mathbb{T},?) : D^b(Z) \to D^b(\operatorname{mod} E)$ *is an exact equivalence of triangulated categories with* $??\otimes^{\mathbb{L}}_E \mathbb{T}$ *as an inverse.*

In particular, E is of finite global dimension.

The reader may recognize these claims as a relative version of Beilinson's original "tilting description" of $D^b(\mathbb{P})$ in [1]. The relation of this geometric result to the algebraic facts asserted above is then established along the following lines.

Proposition 11. *With notation as before,*

(1) $\mathbb{R}q'_*\mathbb{T} \cong M[0]$ *and* $\mathbb{R}q'_*(\mathbb{T}(-1)) \cong M'[0]$*, equivalently,* $q'_*\mathbb{T} \cong M$*, whereas* $\mathbb{R}^i q'_*\mathbb{T} = 0$ *for* $i \neq 0$*, and analogously for* $\mathbb{T}(-1)$*.*

(2) *The object* $\mathbb{R}q'_*\mathcal{E}nd_Z(\mathbb{T})$ *is isomorphic in* $D^b(S)$ *to the cokernel of a single morphism between free* S*–modules situated in (cohomological) degrees* -1 *and* 0*. This implies*

  (a) $E \cong q'_*\mathcal{E}nd_Z(\mathbb{T})$ *is MCM over* R*,*

  (b) *the higher direct images vanish, and so, in particular,*

$$\mathbb{R}^1 q'_*\mathcal{E}nd_Z(\mathbb{T}) \cong \operatorname{Ext}^1_R(M, M) \cong \underline{\operatorname{Hom}}_R(M', M) = 0\,.$$

References

[1] A. A. Beĭlinson, *Coherent sheaves on* $\mathbf{P}^n$ *and problems in linear algebra*, Funktsional. Anal. i Prilozhen. **12**, (1978), 68–69.

[2] J. Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Mathematics **149**, Cambridge University Press, Cambridge 2003, xiv+371,

Products of linear spaces, polymatroids and integral posets

Aldo Conca

In this talk we report on the recent preprint [C]. The goal is to discuss the properties of a family of algebras associated with linear spaces and the relations with two problems in algebraic combinatorics: White's and related conjectures on matroids and polymatroids and the study of integral posets. We start be recalling White's conjecture and the notion of integral poset.

Matroids and Polymatroids: Let n be a positive integer. Denote by $[n]$ the set $\{1, 2, \ldots, n\}$ and by $2^{[n]}$ the set of the subsets $[n]$. Let R be the polynomial ring $K[x_1, \ldots, x_n]$ over the field K. For a subset $F \in 2^{[n]}$ we denote by $|F|$ its cardinality and by x_F the product of the x_i with $i \in F$. A subset M of $2^{[n]}$ is a matroid if it satisfies the following conditions:

a) if $F \in M$ and $G \subset F$ then $G \in M$.

b) If $F, G \in M$ and $|G| < |F|$ then there exists $i \in F \setminus G$ such that $G \cup \{i\} \in M$.

A maximal element of a matroid M is called a base of M. The bases of a matroid M have all the same cardinality. The set B of the bases of M identifies completely the matroid M and satisfies the following property, called the symmetric exchange property:

(S) For every $F, G \in B$ and for every $i \in F \setminus G$ there exists $j \in G \setminus F$ such that $(F \setminus \{i\}) \cup \{j\}$ and $(G \setminus \{j\}) \cup \{i\}$ belongs to B.

Associated with any matroid M with base set B we have a ring $K[B]$, called the base ring of M, defined as the K-subalgebra of R generated by the monomials

x_F with $F \in B$. Consider new variables t_F with $F \in B$ and the presentation $\phi : K[t_F : F \in B] \to K[B]$ defined by $t_F \to x_F$. The symmetric exchange property gives raise to quadrics in $\operatorname{Ker}\phi$: with the notation of (S) and setting $A = (F \setminus \{i\}) \cup \{j\}$ and $B = (G \setminus \{j\}) \cup \{i\}$ we have $t_F t_G - t_A t_B \in \operatorname{Ker}\phi$. White's conjecture [W] asserts that $\operatorname{Ker}\phi$ is generated by the quadrics $t_F t_G - t_A t_B$ arising from (S). The notion of matroid can be generalized in a natural way by considering subsets with multiplicities. This gives raise to the notion of (discrete) polymatroid. Herzog and Hibi in [HH] asked whether White's conjecture holds also for polymatroids. They also asked whether the base ring $K[B]$ associated with a polymatroid is Koszul or defined by a Gröbner basis of quadrics.

A polymatroid P is called transversal if its bases arise as follows: there are non-empty subsets $C_1, C_2, \ldots, C_m$ of the set $[n]$ such that the bases of P are exactly the multi-subsets of the form $\{i_1, i_2, \ldots, i_m\}$ with $i_j \in C_j$.

ASL and integral posets: Algebras with straightening laws (ASL for short) on posets were introduced by De Concini, Eisenbud and Procesi [DEP, Ei]. The abstract definition of ASL was inspired by earlier work of Hochster, Hodge, Laksov, Musili, Rota, and Seshadri among others. It was motivated by the existence of many families of classical algebras, such as coordinate rings of Grassmannians and their Schubert subvarieties and various kinds of determinantal rings, which could be treated within that frame. The homogeneous ASL can be defined in terms of revlex Gröbner bases. Given a poset H denote by $K[x : x \in H]$ the polynomial ring over the field K whose indeterminates are the elements of H. A standard graded K-algebra A is a homogeneous ASL on H if A has a presentation $A = K[x : x \in H]/I$ and I is contains polynomials (called straightening relations) of the form

$$xy - \sum \lambda zt$$

with $x, y, z, t \in H$ and $\lambda \in K$ such that

a) x, y are incomparable in H, $z \leq t, z < x, z < y$ in H whenever $\lambda \in K \setminus \{0\}$.

b) the set of all straightening relations form a Gröbner basis of I for a (equivalently for all) revlex order associated with a linear extension of the partial order on H.

A finite poset H is integral (with respect to a field K) if there exists a homogeneous ASL integral domain on H. The problem of identifying integral posets is quite open. The main results are a beautiful theorem, due to Hibi [H1], that says that any distributive lattice L is integral and a classification, due to Hibi and Watanabe [HW1, HW2], of the 3-dimensional Gorenstein integral posets.

Algebras associated with linear spaces: Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring graded in the standard way by setting $\deg x_i = 1$ over an infinite field K. Let $V = V_1, \ldots, V_m$ be a collection of vector spaces of linear forms. Denote by $A(V)$ the K-subalgebra of R generated by the elements of the product $V_1 \cdots V_m$. Results from [CH] show immediately that for every choice of the V_i the algebra $A(V)$ is normal.

If the V_i are generated by variables, say $V_i = \langle x_j : j \in C_i \rangle$ then $A(V)$ is exactly the base ring associated to the transversal polymatroid identified by the C_i's. In this case we show that $A(V)$ is Koszul. The ingredients of the proof are:

(1) an elimination and degree selection process,

(2) a theorem of Sturmfels and Villarreal which identifies the universal Gröbner basis of the ideal of 2-minors of a matrix of variables in terms of cycles in the complete bipartite graph.

(3) a result of Herzog, Hibi and Restuccia showing that certain quotients of multiple Segre products of polynomial rings are Koszul.

One can try to adapt this proof to the general case (i.e. when the V_i need not be generated by variables). While (1) and (3) extend immediately to the general situation, one needs a replacement for (2). This boils down to the following conjecture:

Conjecture: Let t_{ij} be distinct variables over a field K with $1 \leq i \leq m$ and $1 \leq j \leq n$. Let $L = (L_{ij})$ be a $m \times n$ matrix with $L_{ij} = \sum_{k=1}^{n} a_{ijk} t_{ik}$ and $a_{ijk} \in K$ for all i, j, k. Denote by $I_2(L)$ the ideal of the 2-minors of L. We conjecture that for every choice of a_{ijk}'s, for every term order $<$ on $K[t_{ij}]$ the initial ideal of $I_2(L)$ is square-free in the $\mathbf{Z}^m$-graded sense, i.e. it is generated by elements the form $t_{i_1 j_1} \cdots t_{i_k j_k}$ with $i_1 < i_2 < \cdots < i_k$.

We prove the conjecture for generic L. From this we deduce that $A(V)$ is Cohen-Macaulay and Koszul for generic V_i. More precisely, we show that in this case $A(V)$ is a homogeneous ASL on the poset H_n defined as the subposet of the hypercube $\Pi_{i=1}^{m}[d_i]$ of the elements of rank $< n$. Here $d_i = \dim V_i$. This shows that the any rank truncation of any hypercube is an integral poset. Whether the same statement holds for any distributive lattice is an open question. If we take $V_i = R_1$ for all i we obtain that the m-th Veronese subring of R is a homogeneous ASL on the poset $\{\alpha \in [n]^m : \operatorname{rank} \alpha < n\}$ where $\operatorname{rank} \alpha$ is the poset rank which, in this case, is nothing but $\alpha_1 + \alpha_2 + \cdots + \alpha_m - m$.

References

[C] A.Conca, Linear spaces, transversal polymatroids and ASL domains, preprint 2005, see math.AC/0504111.

[CH] A.Conca, J.Herzog, Castelnuovo-Mumford regularity of products of ideals. Collect. Math. 54 (2003), no. 2, 137–152.

[DEP] C.De Concini, D.Eisenbud, C.Procesi, Hodge algebras , Astérisque, 91. Société Mathématique de France, Paris, 1982. 87 pp.

[Ei] D.Eisenbud, Introduction to algebras with straightening laws. Ring theory and algebra, III (Proc. Third Conf., Univ. Oklahoma, Norman, Okla., 1979), pp. 243–268, Lecture Notes in Pure and Appl. Math., 55, Dekker, New York, 1980.

[HH] J.Herzog, T.Hibi, Discrete polymatroids. J. Algebraic Combin. 16 (2002), no. 3, 239–268.

[H1] T.Hibi, Distributive lattices, affine semigroup rings and algebras with straightening laws. Commutative algebra and combinatorics (Kyoto, 1985), 93–109, Adv. Stud. Pure Math., 11, North-Holland, Amsterdam, 1987.

[HW1] T.Hibi, K.Watanabe, Study of three-dimensional algebras with straightening laws which are Gorenstein domains. I. Hiroshima Math. J. 15 (1985), no. 1, 27–54.

[HW2] T.Hibi, K.Watanabe, Study of three-dimensional algebras with straightening laws which are Gorenstein domains. II. Hiroshima Math. J. 15 (1985), no. 2, 321–340.
[W] N.White, A unique exchange property for bases, Linear Algebra Appl. 31 (1980), 81–91.

Singularities of pairs

Lawrence Ein

Let X be a smooth complex variety and V be a closed subscheme of X. We study the singularities of the pair (X, V) using embedded resolution of singularities. See the excellent paper of Kollár for a general survey of this area. In particular, we study the log-canonical threshold of the pair (X, V), which we'll denote by $\mathrm{lc}(X, V)$. Interest in bounds for log canonical thresholds is motivated by techniques that have recently been developed in higher dimensional birational geometry. In this work, we study this invariant using intersection theory, degeneration techniques and jet schemes.

Suppose that Z is an irreducible component of V. We denote by n the codimension of Z in X, and by $\mathfrak{a} \subset \mathcal{O}_{X,Z}$ the image of the ideal defining V. The following theorem in [4] gives a bound on the log-canonical threshold of $\mathfrak{a}$ in term of the Samuel multiplicity of $\mathfrak{a}$.

Theorem 1.1. *With the above notations, we have*

$$e(\mathfrak{a}) \geq \frac{n^n}{\mathrm{lc}(\mathfrak{a})^n}. \tag{1}$$

Remark 1.2. For $n = 2$, inequality (1.1) gives a result of Corti from [3]. Suppose that $\mathfrak{a}$ is the monomial ideal $\langle x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n} \rangle$. Then $e(\mathfrak{a}) = \prod_{i=1}^n a_i$ and $\mathrm{lc}(\mathfrak{a}) = \sum_{i=1}^n 1/a_i$. The above inequality between $e(\mathfrak{a})$ and $\mathrm{lc}(\mathfrak{a})$ is equivalent to the well known inequality which says that the arithmetic mean of the set $\{1/a_1, \ldots, 1/a_n\}$ is larger than its geometric mean.

The boundary of the above theorem is characterized in the following result in [4].

Theorem 1.3.

$$e(\mathfrak{a}) = \frac{n^n}{\mathrm{lc}(\mathfrak{a})^n} \tag{2}$$

if and only if there is a positive integer q, such that the integral closure $\overline{\mathfrak{a}}$ of $\mathfrak{a}$ is equal to $\mathcal{M}^q$, where $\mathcal{M}$ is the maximal ideal of $\mathcal{O}_{X,Z}$. Moreover, in this case $q = \frac{n}{\mathrm{lc}(\mathfrak{a})}$.

Similarly we obtain an inequality between the colength of $\mathfrak{a}$ and its log-canonical threshold.

Theorem 1.4. *With the above notations, if $n \geq 2$, then we have*

$$l(\mathcal{O}_{X,Z}/\mathfrak{a}) > \frac{n^n}{n!\,\mathrm{lc}(\mathfrak{a})^n}. \tag{3}$$

As an application of the above theorems, we can study the behavior of log-canonical threshold under generic projections [5].

Theorem 1.5. *Let X be a smooth complex variety. Suppose that V is a Cohen-Macaulay pure codimension k closed subscheme of X. and let $f : X \to Y$ be a proper, dominant, smooth morphism of relative dimension $k-1$, with Y smooth. If $f|_V$ is finite, then*

$$\mathrm{lc}(Y, f_*[V]) \leq \frac{k! \cdot \mathrm{lc}(X,V)^k}{k^k},$$

and the inequality is strict if $k \geq 2$. Moreover, if V is locally complete intersection, then

$$\mathrm{lc}(Y, f_*[V]) \leq \frac{\mathrm{lc}(X,V)^k}{k^k}.$$

In their influential paper, Iskovskikh and Manin [6] proved that a smooth quartic threefold is what is called nowadays birationally superrigid; in particular, every birational automorphism of such a hypersurface is an isomorphism. This shows that the variety is not rational. There has been a lot of work to extend this result to other Fano varieties of index one. In particular to smooth hypersurfaces of degree N in $\mathbb{P}^N$, for $N > 4$. The case $N = 5$ was done by Pukhlikov [8], and the cases $N = 6, 7, 8$ were proven by Cheltsov. Moreover, Pukhlikov [9] showed that a general hypersurface as above is birationally superrigid, for every $N > 4$. We use our results and the methods of Pukhlikov to give an easy and uniform proof of birational superrigidity for arbitrary smooth hypersurfaces of degree N in $\mathbb{P}^N$ when N is small [5].

Theorem 1.6. *If $X \subset \mathbb{P}^N$ is a smooth hypersurface of degree N, and if $4 \leq N \leq 12$, then X is birationally superrigid.*

References

[1] A. Cheltsov, On a smooth four-dimensional quintic, (Russian) Mat. Sb. **191** (2000), 139–160; translation in Sb. Math. **191** (2000), 1399–1419.

[2] A. Corti, Factoring birational maps of threefolds after Sarkisov, J. Algebraic Geom. **4** (1995), 223–254.

[3] A. Corti, Singularities of linear systems and 3-fold birational geometry, in *Explicit birational geometry of 3-folds*, 259–312, Cambridge Univ. Press, Cambridge, 2000.

[4] T. de Fernex, L. Ein and M. Mustaţă, Multiplicities and log canonical threshold, J. Algebraic Geom. **13** (2004), 603-615.

[5] T. de Fernex, L. Ein and M. Mustaţă, Bounds fir log canonical thresholds with applications to birational rigidity Math. Research Letters **10**, (2003), 219-236.

[6] V. A. Iskovskikh and Yu. I. Manin, Three-dimensional quartics and counterexamples to the Lüroth problem, Mat. Sb. **86** (1971), 140–166; English transl., Math. Sb. **15** (1972), 141–166.

[7] J. Kollár, Singularities of pairs, in *Algebraic Geometry, Santa Cruz 1995*, volume **62** of Proc. Symp. Pure Math. American Math. Society, 1997, 221–286.

[8] A. V. Pukhlikov, Birational automorphisms of a four-dimensional quintic, Invent. Math. **87** (1987), 303–329.

[9] A. V. Pukhlikov, Birational automorphisms of Fano hypersurfaces, Invent. Math. **134** (1998), 401–426.

[10] A. V. Pukhlikov, Birationally rigid Fano hypersurfaces, preprint 2002, arXiv: math.AG/0201302.

Computing Direct Images with Free Resolutions

David Eisenbud
(joint work with Frank-Olaf Schreyer)

The direct image of a coherent sheaf under a projective (or proper) morphism is a right-bounded complex of free modules, well-defined up to quasi-isomorphism. There are several known ways to describe a representative of this class of complexes: one can push forward an injective resolution, or push forward the Čech complex. In the case where the base is affine, one can take a suitable dual of a graded piece of a free resolution of an appropriately conditioned graded module. In the most basic situation $\mathcal{F}$ is a sheaf on $\mathbf{P}^n_A$, where A is a local ring, and the morphism is the projection π to Spec(A). In this case there is a unique minimal complex of free A-modules representing the direct image $\pi_*\mathcal{F}$, and one would like to compute that complex directly. In this abstract and the primary source [1] we give a new method that works for any Noetherican affine base, and which:

- starts from any finitely generated graded A-module representing $\mathcal{F}$;
- leads directly to this minimal free complex representing $\pi_*\mathcal{F}$; and
- uses using only a regularity computation and the computation of a minimal free resolution of a module over an exterior algebra.

Our method can easily be implemented on computer algebra systems: at the website [4] we provide an implementation for Macaulay2 ([3]). It can be used for theoretical purposes too, for example to show that every free complex appears as a direct image, even of something flat and nice:

Theorem 1. *Let A be a Noetherian ring. If*

$$\mathbf{F}: \qquad 0 \to F_n \to \cdots \to F_0 \to 0$$

is a complex of finitely generated free modules over A, then there is a vector bundle $\mathcal{F}$ on $\mathbf{P}^n_A$ such that $\mathbf{F} = \pi_*\mathcal{F}$, where π is the projection to Spec(A).

The proof is given in [1].

To describe the construction, let A be a Noetherian ring, let $S = A[x_0, \dots, x_n]$ be the polynomial ring over A in $n+1$ variables, graded with A in degree 0 and each x_i in degree 1, and let $\mathbf{P} = \mathbf{P}^n_A = \mathrm{Proj}(S)$ be the corresponding projective space. Let $W = \bigoplus Ax_i \cong A^{n+1}$ be the free A-module of elements of S of degree 1, and let $V = \bigoplus Ae_i$ be its dual, where the e_i are a dual basis to the x_i, and are regarded as elements of degree -1. Let E be the exterior algebra generated over A by the e_i, so that $E \cong \wedge V$.

If T is a finitely generated graded A-module we define reg(T) to be the supremum of the degrees of nonzero elements of T. Since elements of A have degree 0, this supremum is finite unless $T = 0$, when it is $-\infty$. If M is a finitely generated graded S-module, we set

$$\operatorname{reg}(M) = \sup_i \big(\operatorname{reg}(\operatorname{Tor}_i^S(A, M)) - i\big).$$

It is not hard to show that these definitions agree in the case where they both apply.

Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}$ represented by the finitely generated graded S-module M. Let $s = \max(0, \operatorname{reg}(M))$, and let P be the graded E-module that is the kernel of the map $E \otimes M_s \to E \otimes M_{s+1}$ sending $1 \otimes m \mapsto \sum_i e_i \otimes x_i m$. Let $\mathbf{T}$ be an E-free resolution of P, which we choose to be a minimal resolution in the case where A, and thus also E, is a local ring.

Theorem 2. *With notation as above the degree 0 part, $(A \otimes_E \mathbf{T})_0$, of the complex $A \otimes \mathbf{T}$ of free A-modules, represents $\pi_* \mathcal{F}$. If A is local and $\mathbf{T}$ is chosen minimal, as above, then it is the unique minimal representative.*

The proof depends on the construction, also using the exterior algebra, of a relative Beilinson monad for $\mathcal{F}$, following ideas in [2].

References

[1] D. Eisenbud and F.-O. Schreyer, *Computing the higher direct image complex of a coherent sheaf*, Preprint.

[2] D. Eisenbud, G. Fløystad and F.-O. Schreyer, *Sheaf cohomology and free resolutions over exterior algebras.* Trans. Amer. Math. Soc. **355** (2003), no. 11, 4397–4426 (electronic); MR1990756 (2004f:14031)

[3] D. Grayson and M. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, Available at http://www.math.uiuc.edu/Macaulay2/

[4] http://www.math.uni-sb.de/ ag-schreyer/

Effective factorization of invertible matrices over monoid rings

Joseph Gubeladze

1. Motivation

Our monoids are submonoids of rational vector spaces, i. e. additive submonoids of vector spaces over $\mathbb{Q}$. Unless specified otherwise, for a monoid M we assume neither the finite generation, nor the absence of nontrivial invertible elements, nor the normality condition.

For a natural number c and a monoid M we let $M^{1/c}$ denote the corresponding monoid of *cth roots.* (We use multiplicative terminology for the monoid operation in M).

For a ring Λ its *special linear group of order* n is the group of $n \times n$-matrices over Λ having determinant 1. It is denoted by $\mathrm{SL}_n(\Lambda)$. *The subgroup of elementary matrices* is denoted by $\mathrm{E}_n(\Lambda)$, i. e. $\mathrm{E}_n(\Lambda)$ is generated by the matrices $e_{ij}(\lambda)$,

$i \neq j$, with the diagonal entries 1 and with at most one nonzero off-diagonal entry λ on the ij-position.

For the rest of the text we fix: a coefficient ring R, a monoid M, natural numbers $n \geq 3$ and $c \geq 2$ and a matrix $\mathcal{A} \in \mathrm{SL}_n(R[M])$. An old result of ours says:

Theorem 1.1. [G90] *If R is Dedekind domain (like $\mathbb{Z}$, or a field) then there exists a natural number t such that $\mathcal{A} \in \mathrm{E}_n\left(R\left[M^{1/c^t}\right]\right)$.*

It can be shown that in the special case $M = \mathbb{Z}_+^n$ this result is *equivalent* to Suslin's structural description of $\mathrm{SL}_n(R[X_1, \ldots, X_n])$ [Su77]. Moreover, we conjecture that $t = 1$ is enough in Theorem 1.1.

On the other hand the Cohn's well known matrix

$$\begin{pmatrix} 1 + X_1X_2 & X_1^2 \\ -X_2^2 & 1 - X_1X_2 \end{pmatrix} \in \mathrm{SL}_2(R[X_1, X_2]) \setminus \mathrm{E}_2(R[X_1, X_2])$$

and the results of [G95, Sr87] show that the inequalities $n \geq 3$ and $c \geq 2$ are both sharp.

An algorithm for factoring invertible matrices over a polynomial ring (when R is a field) into elementary ones was developed in [PW95], basing on Suslin's work. This type of algorithms have application in *signal processing*.

It is very likely that the original proof, given in [G90], can serve as a basis for an algorithm that finds $t \in \mathbb{N}$ and an explicit factorization of $\mathcal{A}$ into elementary matrices over $R\left[M^{1/c^t}\right]$. This would be in the lines of the algorithmic treatment of the work [G88], developed in [LW97]. However, such an approach has a serious drawback: it uses *Milnor patching of invertible matrices* for certain *Karoubi squares*. Even worse, the number of such Karoubi squares grows rapidly as the pattern of support monomials in $\mathcal{A}$ becomes complicated. In practice this means that such an algorithm will be substantially slower than the Parker-Woodburn algorithm for polynomial algebras.

Is there a way to speed up the effective factorization into elementary matrices in the monoid ring setting?

2. Almost separated factorization

It turns out that Mushkudiani's forgotten work [M95] has a key to a substantially faster algorithm. Actually, [M95] contains an aesthetic nontrivial mathematical fact (Theorem 2.1 below). This work was published in the mid 90s in a local journal hardly available to a broader mathematical audience, the exposition is poor, and it contains a number of small inaccuracies. Currently we are preparing for publication a revised version of the result, together with its algorithmic consequences [G].

The general outline of the argument in [M95] follows the framework set up in [G90] with one principal technical difference – instead of using Karoubi squares, associated to certain localizations of monoid rings, Mushkudiani uses the following *almost separated factorization.*

Consider the special case when the monoid M is finitely generated and without nontrivial units. Then $C = \mathbb{R}_+ M$ is a rational, finite and pointed cone in some ambient Euclidean space $\mathbb{R}^r$, $r \in \mathbb{N}$. Using an appropriate rational transformation if needed, we can always achieve that $\dim C = r$.

Let $\mathcal{H} \subset \mathbb{R}^r$ be a rational codimension 1 subspace, cutting C into two full dimensional rational subcones $C = C_1 \cup C_2$. For a real number $\epsilon > 0$ we let $C_2(\epsilon)$ denote the cone of the radial rays within the ϵ-vicinity of those in C_2:

$$C_2(\epsilon) = \mathbb{R}_+ \left\{ z \in \mathbb{R}^n \setminus \{0\} \mid \exists x \in C_2 \setminus \{0\} \qquad \frac{z}{\|z\|} \in B_\varepsilon\left(\frac{x}{\|x\|}\right) \right\} \subset \mathbb{R}^r.$$

Let $M_1 = C_1 \cap M$ and $M_2(\epsilon) = C \cap C_2(\epsilon) \cap M$.

Theorem 2.1. [Almost separated factorization] *Let R be a Dedekind domain and $\mathcal{A} \in \mathrm{E}_n(R[M])$. Then there exist $t \in \mathbb{N}$ and matrices $\mathcal{B} \in \mathrm{SL}_n(R[M_1^{1/c^t}])$ and $\mathcal{C} \in \mathrm{SL}_n(R[M_2(\epsilon)^{1/c^t}])$ such that $\mathcal{A} = \mathcal{B} \cdot \mathcal{C}$.*

We remark that the deduction of Theorem 1.1 from 2.1 requires an additional work, and that the proof of Theorem 2.1 is algorithmic in nature.

3. Higher nilpotence

Another motivation for us for returning to [M95] has been the recent proof of the nilpotence conjecture for all higher K-groups of toric varieties over a characteristic 0 field [G05]. The 'K_1-slice' of this result is Theorem 1.1. More precisely, the main (stable) results of [G90] and [M95] are proved for arbitrary *regular* coefficient rings and the aforementioned almost separated factorization is valid even on the level of $\mathrm{St}(R[M])$, the *Steinberg group* of $R[M]$. In particular, algebraic K-theory of monoid rings over a regular coefficient ring is nilpotent up to K_2. This is a strong evidence that the main result of [G05] extends from fields to all regular coefficient rings. Although, from technical point of view, it is clear that Mushkudiani's method does not extend to higher K-groups ($i \geq 3$). A rescue should be coming from a different direction – the two landmark results in commutative algebra: Lindel's proof of the Bass-Quillen conjecture in the geometric case, based on his techniques of *étale neighborhoods* [Li81], and Popescu's desingularization process [Po85, Sw98].

References

[G88] J. Gubeladze, Anderson's conjecture and the maximal class of monoids over which projective modules are free, *Math. USSR Sbornik* **135** (1988), 169–181. (Russian.)

[G90] J. Gubeladze, Classical algebraic K-theory of monoid algebras, *Lect. Notes Math.* **1437**, Springer, Berlin, 1990, 36–94.

[G95] J. Gubeladze, Nontriviality of $SK_1(R[M])$, *J. Pure Appl. Alg.* **104** (1995), 169–190.

[G05] J. Gubeladze, The nilpotence conjecture in K-theory of toric varieties, Invent. math. **160** (2005), 173–216.

[G] J. Gubeladze, Computations in the Steinberg group of a monoid ring (after Z. Mushkudiani), in preparation.

[Li81] H. Lindel, *On the Bass-Quillen conjecture concerning projective modules over polynomial rings*, Invent. Math. **65** (1981), 319–323.

[M95] Z. Mushkudiani, *K_2-groups of monoid algebras over regular rings*, Proc. A. Razmadze Math. Inst., Georg. Ac. Sci. **113** (1995), 120–137.

[LW97] R. Laubenbacher and C. Woodburn, An algorithm for the Quillen-Suslin theorem for monoid rings, *J. Pure Appl. Algebra* **117** (1997), 395–429.

[PW95] H. Park and C. Woodburn, *An algorithmic proof of Suslin's stability theorem for polynomial rings*, J. Algebra **178** (1995), 277–298.

[Po85] D. Popescu, *General Nérone desingularizations,* Nagoya J. Math. **100** (1985), 97–126.

[Sr87] V. Srinivas, *K_1 of the cone over a curve*, J. Reine Angew. Math. **381** (1987), 37–50.

[Su77] A. A. Suslin, *On the structure of the special linear group over polynomial rings*, Math. USSR Izvestija **11** (1977), 221–238.

[Sw98] R. G. Swan, *Néron-Popescu desingularizations*, in *Algebra and Geometry*, 135–192, Lect. Algebra Geom. **2**, Internat. Press, Cambridge, MA, 1998.

A geometric proof of Boutot's result on singularities of quotients

R.V. Gurjar

Abstract. We will give a new geometric proof of J.F. Boutot's result on rationality of singularities of a quotient of $\mathbf{C}^n$ modulo a reductive algebraic group. The proof uses flattening of the quotient morphism.

Theorem. *Let G be a reductive algebraic group over an algebraically closed field of characteristic 0 acting algebraically on a smooth affine variety V. Then $V//G$ has rational singularities.*

We believe that our proof will also give different proofs of some results of R. Elkik on rational singularities [2, Theorem 2].

In our proof we assume that the ground field is the field of complex numbers $\mathbf{C}$. At one point in the proof we will use integration of differential forms (Proposition 2) hence we will work with complex varietes. By usual arguments the result is then true for any algebraically closed field of characteristic 0.
The Theorem was also partly proved by the author in [4]. In [4] we used the Hochster-Roberts theorem that $V//G$ is Cohen-Macaulay [6]. Here we give a different proof of Boutot's result which appears to be more geometric. In the case when the connected component of identity of G is semisimple we give a very short proof of Boutot's result by making use of a result of Flenner (Proposition 1). A proof for this special case was given by G. Kempf [7] using the Hochster-Roberts result. Our proof of Boutot's result is inspired by a result of M. Miyanishi which says that an affine normal surface over $\mathbf{C}$ which has an $\mathbf{A}^1$-fibration has atmost cyclic quotient singularities ([9], Chapter I, §6). In our proof we use flattening of the quotient map using results of Hironaka and Raynaud, induction on $dimG$ and study of C^*-actions such that the quotient is smooth and the quotient map is flat. This is the novel part of our proof. The flattening is used only to make the fibers of the quotient map equidimensional. Our proof can be considered as an "upside" down proof. First we reduce to the case when the reductive group

is a 1-dimensional torus $\mathbf{C}^*$ acting on an affine variety with rational singularities with the same quotient. By flattening and resolution of singularities we make the quotient smooth and prove that the (normalized) fiber product has rational singularities. The proof of this part is purely algebraic geometric. Our proof for the semisimple case appears to be new.

Proposition 1. *Let k be an algebraically closed field of char. 0 and let X be a normal algebraic variety/k of dimension d. If the canonical divisor K_X of X is Cartier and any regular d-form on the smooth locus of X extends to a regular form on a resolution of singularities of X then X has rational singularities.*

We will use the following result due to H. Laufer in [8].

Proposition 2. *Let V be a normal algebraic variety of dimension $n > 1$ and $p \in V$. Let U be a small neighborhood of p in V with compact closure. Let ω be a rational n-form on V which is regular in U and let $\overline{\omega}$ be its complex conjugate. If $\int_{U-SingU} \omega \wedge \overline{\omega} < \infty$ then ω extends to a regular form on any resolution of singularities of U. Conversely, if ω extends to a regular form on a resolution of singularities of V then the above integral is finite.*

References

[1] J.F. Boutot, Singularités rationelles et quotients par les groupes rèductifs, Invent. Math., **88** (1987), 65-68.

[2] R. Elkik, Singularites rationnelles et deformations, Invent. Math. **47** (1978), 139-147.

[3] F. Flenner, Rationale quasihomogene Singularitäten, Arch. Math. (Basel), **36** (1981), no. 1, 35-44.

[4] R.V. Gurjar, On a conjecture of C.T.C. Wall, Journal of Math. Kyoto Univ., **31**, no. 4 (1991), 1121-1124.

[5] H. Hironaka, Flattening of analytic maps, Manifolds-Tokyo, 1973. Univ. of Tokyo Press.

[6] M. Hochster and J. Roberts, Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay, Adv. in Math., **13** (1974), 115-175.

[7] G. Kempf, Some quotient varieties have rational singularities, Michigan Math. Journal, **24** (1977), 347-352.

[8] H. Laufer, On rational singularities, Amer. J. Math. **94** (1972), 597-608.

[9] M. Miyanishi, Non-complete algebraic surfaces, Lectures Notes in Mathematics, 857, Springer-Verlag 1981.

[10] M. Raynaud, Flat modules in algebraic geometry, Algebraic Geometry, Oslo 1970, F. Oort, Editor.

Gorenstein liaison of algebraic varieties

Robin Hartshorne

1. The classical case

Max Noether in his study of space curves [8] described a curve in 3-space by giving two surfaces containing the curve, and then describing the residual intersection (Restcurve). The idea was to describe a complicated curve by linking it to a simpler curve. An easy example is the twisted cubic curve, which is contained in two quadric surfaces, with residual intersection a line.

This idea was brought up to date by Peskine and Szpiro [9], who defined liaison of subschemes of $\mathbb{P}^n$ as follows. Two equidimensional subschemes X, Y of dimension r are *linked* if there is a complete intersection scheme Z of dimension r such that $X \cup Y = Z$ and $\mathcal{I}_{X,Z} \cong \mathcal{H}om(\mathcal{O}_Y, \mathcal{O}_Z)$ and conversely $\mathcal{I}_{Y,Z} \cong \mathcal{H}om(\mathcal{O}_X, \mathcal{O}_Z)$. A chain of linkages gives the equivalence relation of *liaison*.

Peskine and Szpiro [loc. cit.] showed that a curve Y in $\mathbb{P}^3$ is arithmetically Cohen–Macaulay (ACM) if and only if it is in the liaison class of a complete intersection (licci). Rao [10] generalized this result by showing that two curves X and Y in $\mathbb{P}^3$ are in the same even liaison equivalence class if and only if they have the same *Rao module* ($M_X = H^1_*(\mathcal{I}_X)$) up to shift of degree.

This whole theory generalizes nicely to codimension 2 subschemes of $\mathbb{P}^n$.

2. Gorenstein liaison

For subschemes of codimension > 2 in $\mathbb{P}^n$, the above notion of linking by complete intersections is too restrictive. Hence one defines *Gorenstein liaison* in the same way as above, but taking Z to be *arithmetically Gorenstein* (AG) (i.e., $k[x_0, \ldots, x_n]/I_Z$ is a Gorenstein ring) instead of complete intersection.

The big open question now is whether an ACM subscheme of $\mathbb{P}^n$ is in the Gorenstein liaison class of a complete intersection (glicci). Many special cases are known (e.g., good determinental schemes) but the problem remains open in general. See [7] and [6] for background and study of this question.

3. Codimension 2 on an AG variety

Since the case of codimension ≥ 3 in $\mathbb{P}^n$ is difficult, we change the problem slightly and consider codimension 2 subschemes of a normal arithmetically Gorenstein scheme X in $\mathbb{P}^n$ (for example, X could be a hypersurface). We consider Gorenstein liaison in X, defined as above, and also Gorenstein biliaison: If $Y \subseteq X$ has codimension 2 and S is an ACM codimension 1 subscheme of X containing Y, then we say any Y' linearly equivalent to $Y + mH$, where H is the hyperplane class on S, is obtained by an *elementary Gorenstein biliaison* from Y. Chains of these generate the equivalence relation of Gorenstein biliaison. Easy examples show that even complete intersection liaison $\Rightarrow$ Gorenstein biliaison $\Rightarrow$ even Gorenstein liaison, and these three notions are distinct in general.

4. ACM sheaves

In joint work with Marta Casanellas and Elena Drozd [2], [3], we show that the relations of Gorenstein liaison and biliaison can be interpreted in terms of the category of ACM sheaves on X. We say a coherent sheaf $\mathcal{E}$ on X is ACM if it is locally Cohen–Macaulay on X, and $H^i(\mathcal{E}(n)) = 0$ for $0 < i < \dim X$ and for all $n \in \mathbb{Z}$. This is equivalent to saying that the associated graded module $E = H^0_*(\mathcal{E})$ is a maximal Cohen–Macaulay module (MCM) on the homogeneous coordinate ring $S(X) = k[x_0, \ldots, x_n]/I_X$. These modules have been extensively studied (see for example the book of Yoshino [11]), and we are able to make use of the known structure theorems to determine the Gorenstein liaison and biliaison classes on certain AG schemes X. Two typical results are these:

Theorem 1 [3]. *On a nonsingular quadric three-fold in $\mathbb{P}^4$, two curves are in the same even Gorenstein liaison class if and only if their Rao modules are isomorphic (up to shift). In particular,* ACM $\Rightarrow$ *glicci.*

Theorem 2 [2]. *On a singular quadric threefold with one double point in $\mathbb{P}^4$, two curves are in the same Gorenstein biliaison class if and only if their Rao modules are isomorphic (up to shift). In particular,* ACM $\Rightarrow$ *in the Gorenstein biliaison class of a complete intersection (gobilicci).*

5. Open problems

The first case of ACM schemes of codimension 3 in $\mathbb{P}^n$ is points in $\mathbb{P}^3$. Here we know that any zero-scheme in a nonsingular quadric surface is glicci [2]. On a nonsingular cubic surface, we know that any set of n points in general position is glicci [5]. To determine whether every 0-scheme on the cubic surface X is glicci, it would be sufficient to show that every ACM sheaf $\mathcal{E}$ on X has a resolution $0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{E} \to 0$ where $\mathcal{F}_1, \mathcal{F}_2$ are ACM sheaves that have filtrations whose successive quotients are all rank 1 ACM sheaves. Up to now, rank 2 ACM sheaves on the cubic surface have been studied by Faenzi [4] and by Popescu et al. [1], but no general structure theorem is known.

In view of the above, a good test case for the question ACM $\Rightarrow$ glicci seems to be: is a set of 20 points in general position in $\mathbb{P}^3$ glicci?

References

[1] C. Baciu, V. Ene, G. Pfister and D. Popescu, *Rank 2 Cohen–Macaulay modules over singularities of type $x_1^3 + x_2^3 + x_3^3 + x_4^3$*, Preprint math.AC/0410625.

[2] M. Casanellas and R. Hartshorne, *Gorenstein biliaison and ACM sheaves*, J. Algebra **278** (2004), 314–341.

[3] M. Casanellas, E. Drozd and R. Hartshorne, *Gorenstein liaison and ACM sheaves*, Crelle J. (to appear).

[4] D. Faenzi, *Rank 2 arithmetically Cohen–Macaulay bundles on a nonsingular cubic surface*, Preprint math.AG/0504492.

[5] R. Hartshorne, *Some examples of Gorenstein liaison in codimension three*, Collect. Math. **53** (2002), 21–48.

[6] J. Kleppe, J. C. Migliore, R. M. Miró–Roig, U. Nagel and C. Peterson, *Gorenstein liaison, complete intersection liaison invariants, and unobstructedness*, Memoirs AMS **154**(732) (2001).

[7] J. C. Migliore, *Introduction to liaison theory and deficiency modules*, Birkhäuser, Boston (1998).

[8] M. Noether, *Zur Grundlegung der Theorie der algebraischen Raumkurven*, Berlin (1883).

[9] C. Peskine and L. Szpiro, *Liaison des variétés algébriques*, Invent. Math. **26** (1974), 271–302.

[10] A. P. Rao, *Liaison among curves in $\mathbb{P}^3$*, Invent. Math. **50** (1979), 205–217.

[11] Y. Yoshino, *Cohen–Macaulay modules over Cohen–Macaulay rings*, London Math. Soc. Lec. Notes **146** (1990).

Levels in triangulated categories and perfect complexes over commutative rings

Srikanth Iyengar

(joint work with Luchezar Avramov, Ragnar-Olaf Buchweitz, Claudia Miller)

Let $(R, \mathfrak{m}, k)$ be a local ring and M an R-module. The results reported here may be considered against the background of the following heuristic observation: If M has finite projective dimension, then it cannot be "too small", unless the ring R is regular. Here we measure the "size" of M by means of its *Loewy length*, defined to be the number:

$$\ell\ell_R M = \inf\{i \geq 0 \mid \mathfrak{m}^i M = 0\}$$

The theorem below is one of the principal results in this work. It reveals an unexpected relationship between (free ranks of) conormal modules and the homology of finite free complexes. It may be compared to the New Intersection Theorem, which imposes constraints on lengths of finite free complexes.

Theorem 1. *Let R be a local ring and let $\widehat{R} = Q/I$ where $(Q, \mathfrak{q}, k)$ is a local ring with $I \subseteq \mathfrak{q}^2$. If F is a finite free complex of R-modules with $\mathrm{H}(F) \neq 0$, then*

$$\sum_{n \in \mathbb{Z}} \ell\ell_R \, \mathrm{H}_n(F) \geq \text{f-rank}_R(I/I^2) + 1$$

where $\text{f-rank}_R(I/I^2)$ denotes the maximal rank of R-free summands of I/I^2.

The special case $Q = \mathbb{F}_2[x_1, \ldots, x_c]$ and $I = (x_1^2, \ldots, x_c^2)$ is a theorem of Carlsson [3], who used it to study the action of elementary abelian 2-groups on CW complexes. Our interest here is in a bound which it yields on the Loewy lengths of homology modules:

Corollary 2. *Let $c = \text{f-rank}_R(I/I^2)$ and $d = \mathrm{card}\{n \mid \mathrm{H}_n(F) \neq 0\}$. There exists an integer i such that $\mathfrak{m}^b \, \mathrm{H}_i(F) \neq 0$ for $b = \lfloor c/d \rfloor$.*

The examples below identify special cases where the corollary applies.

Example 3. Let M be a non-zero R-module of finite projective dimension.

(1) If $\widehat{R} = A/\boldsymbol{f}$ where A is a local ring and $\boldsymbol{f} = f_1, \ldots, f_h$ is a regular sequence in A, in particular if R is complete intersection of codimension h, then

$$\mathfrak{m}^b M \neq 0 \quad \text{for} \quad b = \operatorname{edim} R - \operatorname{edim} A + h$$

(2) If $\widehat{R}$ is the closed fibre of a flat local homomorphism $A \to B$, then

$$\mathfrak{m}^b M \neq 0 \quad \text{for} \quad b = \operatorname{edim} R - \operatorname{edim} B + \operatorname{edim} A$$

The following result is a key step in the proof of Theorem 1:

Theorem 4. *Let A be a commutative noetherian ring containing a field. Let G be a differential A-module admitting a filtration*

$$0 \subseteq G^{(0)} \subseteq \cdots \subseteq G^{(l-1)} \subseteq G^{(l)} = G$$

such that $\delta(G^{(i)}) \subseteq G^{(i-1)}$ and the S-module $G^{(i)}/G^{(i-1)}$ is a finitely generated projective module for each i.

If a differential module D is a direct summand of G, then

$$l \geq \operatorname{height}(\operatorname{Ann}_A \mathrm{H}(D))$$

This result improves upon the New Intersection Theorem for local rings containing a field: Indeed, any finite free complex $0 \to G_l \xrightarrow{\partial_l} \cdots \xrightarrow{\partial_1} G_0 \to 0$ gives rise to a differential module $G = \bigoplus_n G_n$, with differential $\bigoplus_n \partial_n$, and admits a filtration $G^{(i)} = \bigoplus_{n=0}^{i} G_n$ satisfying the hypothesis of the theorem above. Thus, applying it with $D = G$, we conclude $l \geq \operatorname{height}(\operatorname{Ann}_A \mathrm{H}(G))$.

The proof of Theorem 4 builds on an idea of Carlsson [3], by using big Cohen-Macaulay modules constructed by Hochster [5]; this remark explains the hypothesis that A contains a field.

A crucial new ingredient in the proof of Theorem 1 is the consideration of numerical invariants of objects in arbitrary triangulated categories, called *levels*. Their introduction is motivated in part by work of Dwyer, Greenlees, and Iyengar [4], where a notion of *building* objects (modules, complexes, etc.) from a given one was transported into commutative algebra from algebraic topology; levels provide a way to *quantify* the complexity of the "building process".

Levels can be defined with respect to an arbitrary class of objects: given a non-empty class $\mathcal{C}$ in a triangulated category $\mathcal{T}$, an object $T \in \mathcal{T}$ has $\operatorname{level}_{\mathcal{T}}^{\mathcal{C}}(T) \leq n$ if it is isomorphic to a direct summand of an n-fold extension of finite direct sums of shifts of objects in $\mathcal{C}$. The utility of this notion was suggested to us by work of Bondal and Van den Bergh [2], and Rouquier [7], relating to dimensions of triangulated categories.

Two levels in the derived category $\mathcal{D}R$ of R-modules play a special role in this work: one, with respect to the class of simple modules, tracks Loewy length; the other, with respect to the class of projective modules, tracks projective dimension. It is remarkable that homological invariants, such as projective dimension, as well as ring theoretic invariants, such as Loewy length, are captured by the same formalism. This attests to the flexibility afforded by the notion of levels.

To elaborate on this point, we consider levels with respect to k, the residue field of R. If C is a complex of R-modules, then

$$(\dagger) \qquad \sum_{n\in\mathbb{Z}} \ell\ell_R \mathrm{H}_n(C) \geq \operatorname{level}_{\mathcal{D}R}^{k}(C) + 1 \geq \max_{n\in\mathbb{Z}}\{\ell\ell_R \mathrm{H}_n(C)\}$$

In particular, for an R-module M one has

$$\operatorname{level}_{\mathcal{D}R}^{k}(M) = \ell\ell_R M - 1$$

On the other hand, levels with respect to R have the property that every finite free complex $F : 0 \to F_l \to \cdots \to F_0 \to 0$ with $\mathrm{H}(F) \neq 0$ satisfies an inequality:

$$l \geq \operatorname{level}^R_{\mathcal{D}R}(F) \tag{‡}$$

It should be noted that the inequality can be strict even when $\partial(F) \subseteq \mathfrak{m}F$. However, if F is the minimal free resolution of an R-module M, then

$$\operatorname{level}^R_{\mathcal{D}R}(M) = \operatorname{level}^R_{\mathcal{D}R}(F) = l = \operatorname{proj\,dim}_R M$$

Now, if A is a DG algebra, the function $\operatorname{level}^A_{\mathcal{D}A}(-)$, defined on the derived category $\mathcal{D}A$ of DG modules over A, provides an analogue of projective dimension of modules. Indeed, many of the formal properties of projective dimension for modules over rings extend to this setting.

An outline of the proof of Theorem 1. Theorem 1 is proved by reduction to a statement that is essentially a corollary of Theorem 4. It involves a passage from the derived category $\mathcal{R}$ of R-modules, to the derived category $\mathcal{S}$ of DG modules over a polynomial ring $S = k[x_1, \cdots, x_c]$, where k is the residue field of R, the degree of x_i is -2 for each i, and $c = \text{f-rank}_R(I/I^2)$. The transition from $\mathcal{R}$ to $\mathcal{S}$ is via a chain of exact functors of triangulated categories:

$$\mathcal{R} \xrightarrow{\mathsf{K}=K\otimes_R -} \mathcal{K} \xrightarrow[\sim]{\mathsf{B}} \mathcal{B} \xrightarrow{\mathsf{L}} \mathcal{L} \xrightarrow{\mathsf{E}=\operatorname{Hom}_\Lambda(X,-)} \mathcal{E} \xrightarrow{\mathsf{S}} \mathcal{S}$$

In this diagram $\mathcal{K}$ is the derived category of DG modules over the Koszul complex K on a minimal generating set for the maximal ideal of R. The presence of a free summand of rank c in I/I^2 entails that K is quasi-isomorphic to a DG algebra B of the form $C \otimes_k \Lambda$, where Λ is an exterior algebra on the k-vectorspace k^c in degree 1; this was proved in [6]. This quasi-isomorphism induces an equivalence B between $\mathcal{K}$ and $\mathcal{B}$, the derived category of DG modules over B. The inclusion of DG algebras $\Lambda \to C \otimes_k \Lambda$ yields the functor L from $\mathcal{B}$ to $\mathcal{L}$, the derived category of DG modules over Λ. Let X be a DG-projective resolution of k over Λ and let E denote the endomorphism DG algebra $\operatorname{Hom}_\Lambda(X, X)$. The category $\mathcal{E}$ is the derived category of DG modules over E. Finally, $S = \mathrm{H}(E) = \operatorname{Ext}_\Lambda(k, k)$, and $\mathcal{S}$ is the derived category of DG modules over S. The equivalence S is induced by a quasi-isomorphism $S \to E$, which is part of the Bernstein-Gelfand-Gelfand [1] correspondence between Λ and S.

Let F be a finite free complex of R-modules, as in the statement of Theorem 1, and let D be its image under the composite functor $\mathcal{R} \to \mathcal{S}$. We prove:

(a) F finite free implies $\mathrm{H}(D)$ has finite length;
(b) $\operatorname{level}^k_{\mathcal{D}R}(F) \geq \operatorname{level}^S_{\mathcal{D}S}(D) + 1$.

The proof is based on the fact that, in contrast to many homological invariants, levels behave predictably under changes of categories. To complete the proof of Theorem 1 it remains to recall (†), and apply the next result:

Theorem 5. *Let S be a graded noetherian commutative ring with S_0 a local ring containing a field, and either $S_{2i+1} = 0$ for each i or* $\operatorname{char}(S) = 2$.

If N is a DG S-module with $\operatorname{length}_S \mathrm{H}(N)$ *finite, then* $\operatorname{level}^S_{\mathcal{D}S}(N) \geq \dim S$.

The proof of this theorem requires basic results concerning levels in the derived category of DG modules over DG algebras, and Theorem 4.

References

[1] I. N. Bernstein, I. M. Gelfand, S. M. Gelfand, *Algebraic vector bundles on P^n and problems of linear algebra* Funct. Anal. Appl. **12** (1978), 212–214.

[2] A. Bondal, M. Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Moscow Math. J. **3** (2003), 1–36.

[3] G. Carlsson, *On the homology of finite free $(Z/2)^n$-complexes*, Invent. Math. **74** (1983), 139–147.

[4] W. G. Dwyer, J. P. C. Greenlees, S. Iyengar, *Finiteness in derived categories of local rings*, preprint, 2004; arXiv:math.AC/0404034

[5] M. Hochster, *Topics in the homological theory of modules over commutative rings*, Conf. Board Math. Sci. **24**, Amer. Math. Soc. 1975.

[6] S. Iyengar, *Free summands of conormal modules and central elements in homotopy Lie algebras of local rings*, Proc. Amer. Math. Soc., **129** (2001) 1563–1572.

[7] R. Rouquier, *Dimensions for triangulated categories*, preprint, 2005; arXiv:math.CT/0310134

$\mathbf{C}_+$–actions on contractible threefolds

Shulim Kaliman

(joint work with Nikolai Saveliev)

In 1968 Rentschler [Re] proved that every algebraic action of the additive group $\mathbf{C}_+$ of complex numbers on $\mathbf{C}^2$ is triangular in a suitable polynomial coordinate system. This implies that a free $\mathbf{C}_+$-action on $\mathbf{C}^2$ (i.e. an action without fixed points) can be viewed as a translation. In 1984 Bass [Ba] found a $\mathbf{C}_+$-action on $\mathbf{C}^3$ which is not triangular in any coordinate system, and in 1990 Winkelmann [Wi] constructed a free $\mathbf{C}_+$-action on $\mathbf{C}^4$ which is not a translation. But the question about free $\mathbf{C}_+$-actions on $\mathbf{C}^3$ remained open (e.g., see [Sn]). While working on this problem in [Ka] we consider a more general situation when there is a nontrivial algebraic $\mathbf{C}_+$-action on a complex three-dimensional smooth affine algebraic variety X such that its ring of regular functions is factorial. By a theorem of Zariski [Za] the algebraic quotient $X//\mathbf{C}_+$ is isomorphic to an affine surface S. Let $\pi : X \to S$ be the natural projection. Then there is a curve $\Gamma \subset S$ such that for $E = \pi^{-1}(\Gamma)$ the variety $X \setminus E$ is isomorphic to $(S \setminus \Gamma) \times \mathbf{C}$ over $S \setminus \Gamma$. The study of morphism $\pi|_E : E \to \Gamma$ is crucial. As an easy consequence of the Stein factorization one can show that $\pi|_E = \theta \circ \kappa$ where $\kappa : E \to Z$ is a surjective morphism into a curve Z with general fibers isomorphic to $\mathbf{C}$, and $\theta : Z \to \Gamma$ is a quasi-finite morphism. A more delicate fact is that in the case of a smooth contractible X morphism θ is, actually, finite and, furthermore, for each irreducible component Z^1 of Z, such that $\theta|_{Z^1}$ is not injective, $\theta(Z^1)$ is a polynomial curve (i.e. each component of the"smallest" Γ is a polynomial curve). Using finiteness one can show that in the case of a free $\mathbf{C}_+$-action and a smooth S such a component Z^1 cannot exist. If the restriction of θ to any irreducible component of Z is injective then Γ can be chosen empty, i.e. X is isomorphic to $S \times \mathbf{C}$ over S. In combination with Miyanishi's theorem [Miy], which says that $\mathbf{C}^3//\mathbf{C}_+ \simeq \mathbf{C}^2$, this yields the long-expected result [Ka].

Every free algebraic $\mathbf{C}_+$-action on $\mathbf{C}^3$ is a translation in a suitable polynomial coordinate system.

We prove also in [KaSa] the following generalization of Miyanishi's theorem:

If X is a smooth contractible affine algebraic threefold with a non-trivial algebraic $\mathbf{C}_+$-action on it then the algebraic quotient $X//\mathbf{C}_+$ is a smooth contractible surface S.

Since all such surfaces are rational [GuSh], we deduce that X is rational as well. In combination with a previous result this implies also the following.

If X is a smooth contractible affine algebraic threefold with a free algebraic $\mathbf{C}_+$-action on it then X is isomorphic to $S \times \mathbf{C}$ and the action is induced by translation on the second factor.

References

[Ba] H. Bass, *A nontriangular action of G_a on A^3*, J. Pure Appl. Algebra **33** (1984), no. 1, 1–5.

[GuSh] R. V. Gurjar, A. R. Shastri, *On the rationality of complex homology 2-cells. II*, J. Math. Soc. Japan **41** (1989), no. 2, 175–212.

[Ka] Sh. Kaliman, *Free $\mathbf{C}_+$-actions on $\mathbf{C}^3$ are translations*, Invent. Math. 156 (2004), no. 1, 163–173.

[KaSa] Sh. Kaliman, N. Saveliev *$\mathbf{C}_+$-actions on contractible threefolds*, Michigan Math. J., 52 (2004), 619-625.

[Miy] M. Miyanishi, *Regular subrings of a polynomial ring*, Osaka J. Math. **17** (1980), no. 2, 329–338.

[Re] R. Rentschler, *Opérations du groupe additif sur le plane affine*, C.R. Acad. Sci. Paris, **267** (1968), 384–387.

[Sn] D. Snow, *Unipotent actions on affine space*, in "Topological methods in algebraic transformation groups", Birkhäuser, Boston, MA, 1989, 165–176.

[Wi] J. Winkelmann, *On free holomorphic $\mathbf{C}$-actions on $\mathbf{C}^n$ and homogeneous Stein manifolds*, Math. Ann. **286** (1990), no. 1-3, 593–612.

[Za] O. Zariski, *Interpretations algébrico-géométriques du quatorzième problème de Hilbert*, in "O. Zariski: Collected Papers", **2**, The MIT Press, Cambridge (Massachusetts), 1973, 261–275.

The singular Riemann-Roch formula and Hilbert-Kunz functions

Kazuhiko Kurano

Recently Huneke, McDermott and Monsky proved the following theorem:

Theorem (Huneke, McDermott and Monsky [3]**)** Let $(A, \mathfrak{m}, k)$ be a Noetherian d-dimensional normal local ring of characteristic p, where p is a prime integer. Assume that the residue class field k is perfect. Let I be an $\mathfrak{m}$-primary ideal of A and M be a finitely generated A-module.

(1) There exist real numbers $e_{HK}(I, M)$ and $\beta(I, M)$ that satisfy the following equation:

$$\ell_A(M/I^{[p^e]}M) = e_{HK}(I, M) \cdot p^{de} + \beta(I, M) \cdot p^{(d-1)e} + O(p^{(d-2)e})$$

(2) Assume that A is F-finite. Then, there exists a $\mathbb{Q}$-homomorphism $\tau_I : \mathrm{Cl}(A)_{\mathbb{Q}} \longrightarrow \mathbb{R}$ that satisfies
$$\beta(I,M) = \tau_I\left(\mathrm{cl}(M) - \frac{\mathrm{rank}_A M}{p^d - p^{d-1}}\mathrm{cl}({}^1A)\right),$$
for any finitely generated torsion-free A-module M. In particular,
$$\beta(I,A) = -\frac{1}{p^d - p^{d-1}}\tau_I\left(\mathrm{cl}({}^1A)\right)$$
is satisfied.

Here, we explain notation which are used in the above theorem.

We say that A is F-finite if the Frobenius map $F : A \to A = {}^1A$ is module-finite. We sometimes denote the e-th iteration of F by $F^e : A \to A = {}^eA$. We set $I^{[p^e]} = (x^{p^e} \mid x \in I)$.

Let $f(e)$ and $g(e)$ be functions on e. We denote $f(e) = O(g(e))$ if there exists a real number K that satisfies $|f(e)| < Kg(e)$ for any e.

For an abelian group N, $N_{\mathbb{Q}}$ stands for $N \otimes_{\mathbb{Z}} \mathbb{Q}$.

The map $\mathrm{cl} : G_0(A) \to \mathrm{Cl}(A)$ is defined in Bourbaki [1] and usually called "determinant".

It is natural to ask when $\mathrm{cl}({}^1A)$ vanish. Here, we state the main theorem.

Theorem 1 Let $(A, \mathfrak{m}, k)$ be a Noetherian d-dimensional normal local ring of characteristic p, Assume that A is a homomorphic image of a regular local ring, k is a perfect field, and A is F-finite.

Then, for each integer $e > 0$, we have
$$\mathrm{cl}({}^eA) = \frac{p^{de} - p^{(d-1)e}}{2}\mathrm{cl}(\omega_A)$$
in $\mathrm{Cl}(A)_{\mathbb{Q}}$.

The following corollary is an immediate consequence of Theorem 1:

Corollary 2 Under the same assumption as in the above theorem, if $\mathrm{cl}(\omega_A)$ is a torsion element in $\mathrm{Cl}(A)$, then $\beta(I,A) = 0$ for any maximal primary ideal I.

The following is an analogue of Theorem 1 for normal algebraic varieties.

Theorem 3 Let k be a perfect field of characteristic p, where p is a prime integer. Let X be a normal algebraic variety over k of dimension d. Let $F : X \to X$ be the absolute Frobenius map.

Then, we have
$$c_1(F^e_* O_X) = \frac{p^{de} - p^{(d-1)e}}{2}[K_X]$$
in $A_{d-1}(X)_{\mathbb{Q}}$, where $c_1(\ )$ is the first Chern class.

Remark that, under this assumption, F is a finite morphism.

Set $U = X \setminus \mathrm{Sing}(A)$. Since $\mathrm{codim}_X \mathrm{Sing}(A) \geq 2$, the restriction $A_{d-1}(X) \to A_{d-1}(U)$ is an isomorphism. In this case, $(F^e_* O_X)|_U = (F|_U)^e_* O_U$ is a vector bundle on U. Here, $c_1(F^e_* O_X)$ is defined to be the first Chern class $c_1((F^e_* O_X)|_U) \in A_{d-1}(U) = A_{d-1}(X)$.

Now we start to prove Theorem 1.

Let $(A, \mathfrak{m})$ be a Noetherian local ring that satisfies the assumption in Theorem 1. Since $(A, \mathfrak{m})$ is a homomorphic image of a regular local ring, we have an isomorphism

$$\tau_A : \mathrm{G}_0(A)_{\mathbb{Q}} \longrightarrow \mathrm{A}_*(A)_{\mathbb{Q}}$$

of $\mathbb{Q}$-vector spaces by the singular Riemann-Roch theorem (Chapter 18 and 20 in [2]), where $\mathrm{A}_*(A) = \oplus_{i=0}^{d} \mathrm{A}_i(A)$ is the Chow group of $\mathrm{Spec}(A)$. Let

$$p : \mathrm{A}_*(A)_{\mathbb{Q}} \longrightarrow \mathrm{A}_{d-1}(A)_{\mathbb{Q}} = \mathrm{Cl}(A)_{\mathbb{Q}}$$

be the projection. We set $\tau_{d-1} = p \circ \tau_A : \mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{Cl}(A)_{\mathbb{Q}}$.

Here, we summarize basic facts on the map τ_{d-1}.

(i) Let $\mathfrak{p}$ be a prime ideal of height 1. Then, by the top-term property (Theorem 18.3 (5) in [2]), we have

$$\tau_{d-1}(A/\mathfrak{p}) = [\mathrm{Spec}(A/\mathfrak{p})] = \mathrm{cl}(\mathfrak{p}) = -\mathrm{cl}(A/\mathfrak{p}).$$

Let $\mathfrak{q}$ be a prime ideal of height at least 2. By the top-term property, we have $\tau_{d-1}(A/\mathfrak{q}) = 0$.

(ii) By the covariance with proper maps (Theorem 18.3 (1) in [2]), we have

$$\tau_{d-1}({}^e A) = p^{(d-1)e} \tau_{d-1}(A)$$

for each $e > 0$.

(iii) We have

$$\tau_{d-1}(A) = \frac{1}{2} \mathrm{cl}(\omega_A)$$

in $\mathrm{Cl}(A)_{\mathbb{Q}}$ by Lemma 3.5 of [4].

Next we prove the following lemma:

Lemma 4 Let $(A, \mathfrak{m})$ be a local ring that satisfies the assumption in Theorem 1. Then, for a finitely generated A-module M, we have

$$\tau_{d-1}(M) = -\mathrm{cl}(M) + \frac{\mathrm{rank}_A M}{2} \mathrm{cl}(\omega_A)$$

in $\mathrm{Cl}(A)_{\mathbb{Q}}$.

Proof. Set $r = \mathrm{rank}_A M$. Then we have an exact sequence $0 \to A^r \to M \to T \to 0$, where T is a torsion module. By this exact sequence, we obtain

$$\mathrm{cl}(M) = r \cdot \mathrm{cl}(A) + \mathrm{cl}(T) = \mathrm{cl}(T).$$

On the other hand, by the basic fact (iii) as above, we obtain

$$\tau_{d-1}(M) = r \cdot \tau_{d-1}(A) + \tau_{d-1}(T) = \frac{r}{2} \mathrm{cl}(\omega_A) + \tau_{d-1}(T).$$

We have only to prove $\tau_{d-1}(T) = -\mathrm{cl}(T)$. We may assume that $T = A/\mathfrak{p}$, where $\mathfrak{p} \neq 0$ is a prime ideal of A. Since $\mathrm{ht}\, \mathfrak{p} \geq 1$, we have $\tau_{d-1}(A/\mathfrak{p}) = -\mathrm{cl}(A/\mathfrak{p})$ by (i) as above. $\square$

Now we start to prove Theorem 1.

By the basic facts (ii) and (iii), we obtain

$$\tau_{d-1}({}^{e}A) = p^{(d-1)e}\tau_{d-1}(A) = \frac{p^{(d-1)e}}{2}\mathrm{cl}(\omega_A).$$

By Lemma 4 as above, we have

$$\tau_{d-1}({}^{e}A) = -\mathrm{cl}({}^{e}A) + \frac{\mathrm{rank}_A\, {}^{e}A}{2}\mathrm{cl}(\omega_A)$$

in $\mathrm{Cl}(A)_{\mathbb{Q}}$. It is easy to see that $\mathrm{rank}_A\, {}^{e}A = p^{de}$. Thus, we have obtained

$$\mathrm{cl}({}^{e}A) = \frac{p^{de} - p^{(d-1)e}}{2}\mathrm{cl}(\omega_A)$$

in $\mathrm{Cl}(A)_{\mathbb{Q}}$. □

Remark 5 By Theorem 1 and Lemma 4, we have

$$\tau_{d-1}(M) = -\mathrm{cl}(M) + \frac{\mathrm{rank}_A\, M}{2}\mathrm{cl}(\omega_A) = -\mathrm{cl}(M) + \frac{\mathrm{rank}_A\, M}{p^d - p^{d-1}}\mathrm{cl}({}^{1}A).$$

Therefore, we have

$$\beta(I, M) = -\tau_I(\tau_{d-1}(M)) \quad \text{and} \quad \beta(I, A) = -\frac{1}{2}\tau_I(\mathrm{cl}(\omega_A)).$$

References

[1] N. Bourbaki, *Algèbre commutative, Éléments de Mathématique, Chap. 1–7*, Hermann Paris, 1961–1965.

[2] W. Fulton, *Intersection Theory, 2nd Edition*, Springer-Verlag, Berlin, New York, 1997.

[3] C. Huneke, M. A. McDermott and P. Monsky, *Hilbert-Kunz functions for normal graded rings*, Math. Res. Letters **11** (2004), 539–546.

[4] K. Kurano, *An approach to the characteristic free Dutta multiplicities*, J. Math. Soc. Japan, **45** (1993), 369–390.

Factoring the Adjoint and Maximal CM Modules

Graham J. Leuschke

(joint work with Ragnar-Olaf Buchweitz)

Let k be a field and $X = (x_{ij})$ the generic $(n \times n)$–matrix over k. Put $S = k[x_{ij}]$. Let $\mathrm{adj}(X)$ denote the "classical adjoint" of X, whose entries are the appropriately signed submaximal minors (or cofactors) of X, and which is characterized by the matrix equation

$$X\ \mathrm{adj}(X) = \det(X) \cdot \mathrm{id}_n = \mathrm{adj}(X)\ X\,. \tag{1}$$

Our motivating question is due to G.M. Bergman [1], who asked whether the equation (1), viewed as a factorization of the diagonal matrix $\det(X) \cdot \mathrm{id}_n$, can be refined by writing $\mathrm{adj}(X) = YZ$ for a pair of noninvertible $(n \times n)$–matrices Y and Z over S. He gave a partial answer to the question:

Theorem (Bergman). *Let k be an algebraically closed field of characteristic zero.*

(a) For n odd, there are no nontrivial factorizations of $\operatorname{adj}(X)$.

(b) For n even, any factorization $\operatorname{adj}(X) = YZ$ *must have either* $\det Y = \det X$ *or* $\det Z = \det X$*, up to units of S.*

We translate Bergman's question into commutative algebraic terms, as follows: The pair $(X, \operatorname{adj}(X))$ forms a *matrix factorization* of $\det X \in S$, in the sense of Eisenbud [3]. In particular, $M := \operatorname{cok}\operatorname{adj}(X)$ and $L := \operatorname{cok} X$ are *maximal Cohen–Macaulay (MCM) modules* over the hypersurface $R := S/(\det X)$. The existence of a nontrivial factorization $\operatorname{adj}(X) = YZ$ is equivalent to a nonsplit short exact sequence

$$0 \to \operatorname{cok} Z \to M \to \operatorname{cok} Y \to 0\,,$$

of maximal Cohen–Macaulay modules over R. The MCM R-modules are not particularly well understood, but it follows from Bruns' calculation of the divisor class group [2] that the only MCM R-modules of rank one, up to isomorphism, are $L = \operatorname{cok} X$ and the dual $L^\vee := \operatorname{cok} X^T$. This translation already allows us to recover Bergman's result for $n = 3$ and any UFD coefficient ring k:

Proposition. *Let $X = (x_{ij})$ be the generic (3×3)–matrix over a unique factorization domain k. Then there are no nontrivial factorizations of* $\operatorname{adj}(X)$ *over* $k[x_{ij}]$.

For $n \geq 4$, we consider the case $\det Y = u \det X$, u a unit in S. This corresponds precisely to assuming that either $\operatorname{cok} Y \cong L$ or $\operatorname{cok} Y \cong L^\vee$. A pushout construction reduces the open case of Bergman's result to the problem of classifying all extensions

$$0 \to \operatorname{cok} Y \to Q \to L \to 0\,,$$

where either $\operatorname{cok} Y \cong L$ or $\operatorname{cok} Y \cong L^\vee$. In other words, we must compute $\operatorname{Ext}^1_R(L, L)$ and $\operatorname{Ext}^1_R(L, L^\vee)$. The first case follows from a recent result of R. Ile [5]:

Theorem (Ile). $\operatorname{Ext}^1_R(L, L) = 0$.

On the other hand, computer calculations [4] reveal that $\operatorname{Ext}^1_R(L, L^\vee) \neq 0$. To better understand the structure of $\operatorname{Ext}^1_R(L, L^\vee)$, we consider first $\operatorname{Hom}_R(M, L^\vee)$.

Theorem. *The R-module* $\operatorname{Hom}_R(M, L^\vee)$ *is maximal Cohen–Macaulay of rank $n-1$, generated by $\binom{n}{2}$ elements. Indeed,* $\operatorname{Hom}_R(M, L^\vee)$ *is generated by the* alternating *matrices over S. More precisely, for any alternating $(n \times n)$–matrix with entries in S, there exists a unique alternating matrix B_A of the same size such that*

$$A \operatorname{adj}(X) = X^T B_A\,,$$

and $\operatorname{Hom}_R(M, L^\vee)$ *consists of all homomorphisms induced by such pairs (A, B_A). The entries of $B_A = (b_{ij})$ are given in terms of those of $A = (a_{kl})$:*

$$b_{ij} = \sum_{k<l} (-1)^{i+j+k+l} a_{kl} [ij \widehat{\mid} kl]\,,$$

where $[\widehat{ij \mid kl}]$ *denotes the* $(n-2) \times (n-2)$ *minor of* X *obtained by removing the* i, j *rows and* k, l *columns.*

In particular, we obtain an answer to the open case of Bergman's question:

Theorem. *When* n *is even, there exist* invertible *alternating matrices* A *over* S*; for such* A*, we have* $\operatorname{adj}(X) = (A^{-1}X^T)B_A$*, a nontrivial factorization of the adjoint.*

Returning to $\operatorname{Ext}^1_R(L, L^\vee)$, we compute the minimal graded S-free resolution and obtain

Theorem. $\operatorname{Ext}^1_R(L, L^\vee)$ *is a MCM module of rank one over* $S/I_{n-1}(X)$*, the ring defined by the submaximal minors of* X*. For each nonzero alternating matrix* A *with polynomial entries, there is an extension of* $L^\vee$ *by* L

$$0 \to L^\vee \to Q \to L \to 0\,,$$

with Q *an orientable MCM* R*-module of rank 2, given by the matrix factorization*

$$\left(\begin{pmatrix} X^T & A \\ 0 & X \end{pmatrix}, \begin{pmatrix} \operatorname{adj}(X)^T & -B \\ 0 & \operatorname{adj}(X) \end{pmatrix} \right).$$

Considering the middle terms Q of extensions in $\operatorname{Ext}^1_R(L, L^\vee)$, we observe that for $n \geq 3$, the MCM-representation theory of the generic determinantal hypersurface is quite "wild", even restricted to orientable MCM modules of rank 2.

Theorem. *Assume* $n \geq 3$*. Then there is a surjection from the isomorphism classes of extensions* $L^\vee$ *by* L *to the principal ideals of a polynomial ring over* k *in* $(n-2)^2$ *variables. In particular, the MCM* R*-modules of rank* 2 *cannot be parametrized by the points of any finite-dimensional variety over* k*.*

This last result stands in stark contrast to the situation when $n = 2$, wherein there are only three indecomposable MCM modules up to isomorphism: R, L, and $L^\vee$.

References

[1] G.M. Bergman, *Can one factor the classical adjoint of a generic matrix?*, Transformation Groups (2004), 1, to appear; see both the preliminary version of 08 June 2003 and a revised version of 08 August 2004 at `http://front.math.ucdavis.edu/math.AC/0306126`.

[2] W. Bruns, *Die Divisorenklassengruppe der Restklassenringe von Polynomenringen nach Determinantenidealen*, Revue Roumaine Math. Pures Appl. **20** (1985), 1109–1111.

[3] D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. **260** (1980), 35–64.

[4] D. Grayson and M. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, Available at `http://www.math.uiuc.edu/Macaulay2/`.

[5] R. Ile, *Deformation theory of rank one maximal Cohen–Macaulay modules on hypersurface singularities and the Scandinavian complex*, Compositio Math. **140** (2004), 435–446.

Affine pseudo-planes and affine pseudo-coverings

Masayoshi Miyanishi

1. Introduction

Let $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ be an étale endomorphism of the affine plane $\mathbb{A}^2$. The Jacobian conjecture asserts that φ is an isomorphism. Suppose on the contrary that there is a splitting of φ by a smooth affine surface X. Namely, suppose that φ splits as $\varphi = \varphi_2 \cdot \varphi_1$, where $\varphi_1 : \mathbb{A}^2 \to X$ and $\varphi_2 : X \to \mathbb{A}^2$ are étale morphisms. What kind of properties does X (or φ_1, φ_2) have ?

This consideration motivates us to define affine pseudo-planes and affine pseudo-coverings. We take the complex field $\mathbb{C}$ as the ground field. We refer to [4, 3] for the details.

2. Affine pseudo-planes and affine pseudo-coverings

Let X be a smooth affine variety and let $f : Y \to X$ be a morphism of algebraic varieties. We say that f is *almost surjective* if $\operatorname{codim}_X(X - f(Y)) \geq 2$ and that Y is an *affine pseudo-covering* of X if Y is affine, f is étale and almost surjective. Note that an affine pseudo-covering is not necessarily an ordinary finite covering. If the field extension $\mathbb{C}(Y)/\mathbb{C}(X)$ is a Galois extension, the affine pseudo-covering is called *Galois*.

Lemma 2.1. *Let $f : Y \to X$ be a quasi-finite morphism of smooth affine varieties. Suppose that* $\mathrm{Pic}(X)$ *is a torsion group and* $\Gamma(Y, \mathcal{O}_Y)^* = \mathbb{C}^*$*. Then f is almost surjective. In particular, if f is étale, then Y is an affine pseudo-covering of X.*

This lemma implies that if X is a smooth affine variety with $\mathrm{Pic}(X)_{\mathbb{Q}} = 0$ and $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$, then a quasi-finite endomorphism $\varphi : X \to X$ is almost surjective. This is, in particular, the case for $X = \mathbb{A}^n$.

Lemma 2.2. *Let $f : Y \to X$ be an affine pseudo-covering of smooth affine varieties. Then the following assertions hold.*

(1) *The image of the natural group homomorphism $\pi_1(f) : \pi_1(Y) \to \pi_1(X)$ is a subgroup of finite index in $\pi_1(X)$. Hence $\pi_1(X)$ is a finite group provided so is $\pi_1(Y)$.*
(2) *Suppose that $f : Y \to X$ is a Galois affine pseudo-covering with Galois group $G :=$ $\mathrm{Gal}(\mathbb{C}(Y)/\mathbb{C}(X))$. Let $\widetilde{X}$ be the normalization of X in $\mathbb{C}(Y)$ and let $\widetilde{f} : \widetilde{X} \to X$ be the normalization morphism. Then $\widetilde{X}$ is smooth and Y is a Zariski open set of $\widetilde{X}$. The group G acts freely on $\widetilde{X}$ and X is the algebraic quotient $\widetilde{X}/G$.*
(3) *If Y is a rational surface with $\overline{\kappa}(Y) = -\infty$ so is X.*

We shall consider mostly affine pseudo-coverings in the case where Y is the affine plane. Let X be a smooth affine rational surface with $\mathbb{A}^1$-fibration $\rho : X \to B$, where B is a rational curve. Let $\Gamma_0 = \sum_{i=1}^{n} \mu_i C_i$ be its fiber, where the C_i are irreducible components and the μ_i are the multiplicities of the C_i in Γ_0. Let $m = \gcd(\mu_1, \dots, \mu_n)$, which we call the multiplicity of Γ_0. If $m > 1$, we say that Γ_0 is a multiple fiber of ρ and write $\Gamma_0 = mF_0$ with $F_0 = \sum_{i=1}^{n} (\mu_i/m)C_i$. We recall the following result (cf. [5]).

Lemma 2.3. *With the above notations, let F be a general fiber of f, let $m_1F_1, \dots, m_sF_s$ exhaust all multiple fibers of f and let $P_i = f(F_i)$. Set $B' = B - \{P_1, \dots, P_s\}$. Then there exists a short exact sequence*

$$\pi_1(F) \to \pi_1(X) \to \Gamma \to (1),$$

where Γ is the quotient of $\pi_1(B')$ by the normal subgroup generated by $e_1^{m_1}, \dots, e_s^{m_s}$ with the e_i corresponding to a small loop in B around the point P_i.

As a corollary of this lemma, we obtain

Lemma 2.4. *Let X be a smooth affine surface with an $\mathbb{A}^1$-fibration $\rho : X \to B$, where B is isomorphic to either $\mathbb{A}^1$ or $\mathbb{P}^1$. Suppose that $\pi_1(X)$ is a finite group. Then we have:*

(1) *When $B \cong \mathbb{A}^1$, there is at most one multiple fiber mF and $\pi_1(X)$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$.*
(2) *When $B \cong \mathbb{P}^1$, there are at most three multiple fibers. If there are three of them, say m_1F_1, m_2F_2, m_3F_3, then $\{m_1, m_2, m_3\}$ is, up to permutations, one of the Platonic triplets $\{2, 2, m\}$ $(m \geq 2), \{2, 3, 3\}, \{2, 3, 4\}, \{2, 3, 5\}$.*

We are interested in the case where the $\mathbb{A}^1$-fibration $\rho : X \to B$ has only irreducible fibers and introduce three kinds of smooth affine surfaces.

Definition **2.5.** Let X be a smooth affine surface with an $\mathbb{A}^1$-fibration $\rho : X \to B$ such that B is isomorphic to $\mathbb{A}^1$ or $\mathbb{P}^1$ and that all fibers of ρ are irreducible.

(1) X is an *affine pseudo-plane* (more precisely, *of type d*) if $B \cong \mathbb{A}^1$ and there is at most one multiple fiber dF_0. An affine pseudo-plane is a $\mathbb{Q}$-homology plane with $\mathrm{Pic}(X) \cong H_1(X; \mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$.
(2) X is a *cyclic $\mathbb{A}^1$-fiber space* if $B \cong \mathbb{P}^1$ and there are at most two multiple fibers. Clearly, the Picard number of X is one, and $\mathrm{Pic}(X)_{\mathrm{tor}} \cong \pi_1(X) \cong H_1(X, \mathbb{Z})$ which is a finite cyclic group and $\mathrm{Pic}(X)/\mathrm{Pic}_{\mathrm{tor}}(X) \cong H_2(X, \mathbb{Z})$ *(see* [1, Lemma 1.4]*)*. If there is at most one multiple fiber, X contains $\mathbb{A}^2$ as an open set and hence is simply-connected. If there are two multiple fibers m_1F_1, m_2F_2, $\pi_1(X)$ has order equal to $\gcd(m_1, m_2)$.
(3) X is a *Platonic $\mathbb{A}^1$-fiber space* if $B \cong \mathbb{P}^1$ and there are three multiple fibers m_1F_1, m_2F_2, m_3F_3 such that $\{m_1, m_2, m_3\}$ is a Platonic triplet. The Picard number of X is one, $\mathrm{Pic}_{\mathrm{tor}}(X) \cong H_1(X, \mathbb{Z})$ and $\mathrm{Pic}(X)/\mathrm{Pic}_{\mathrm{tor}}(X) \cong H_2(X, \mathbb{Z})$, while $\pi_1(X) \cong \langle e_1, e_2, e_3 \rangle / (e_1^{m_1} = e_2^{m_2} = e_3^{m_3} = e_1e_2e_3 = 1)$.

When X is an affine pseudo-plane, it is shown in [2, Lemma 2.6] that the universal covering $\widetilde{X}$ of X has a Galois group $\mathbb{Z}/d\mathbb{Z} = \{\omega; \omega^d = 1\}$ and contains an open set U_ω which is isomorphic to $\mathbb{A}^2$ and mapped surjectively onto X by the covering mapping. Hence $\mathbb{A}^2$ is a Galois affine pseudo-covering of X. Slightly generalizing this result, we can prove the following result.

Lemma 2.6. *Let X be an affine pseudo-plane, cyclic $\mathbb{A}^1$-fiber space or a Platonic $\mathbb{A}^1$-fiber space. Then the following assertions hold.*

(1) *$\mathbb{A}^2$ is a Galois affine pseudo-covering of X if X belongs to one of the classes.*
   (i) *an affine pseudo-plane,*
   (ii) *a cyclic $\mathbb{A}^1$-fiber space with two multiple fibers of equal multiplicity larger than one,*
   (iii) *a Platonic $\mathbb{A}^1$-fiber space.*

(2) *$\mathbb{A}^2$ is not a Galois affine pseudo-covering of X if X is a cyclic $\mathbb{A}^1$-fiber space which is simply-connected.*

In treating affine algebraic varieties, affine pseudo-coverings are more general than ordinary finite étale coverings. As an example, we have the following result.

Lemma 2.7. *Let X be a cyclic $\mathbb{A}^1$-fiber space with two multiple fibers m_1F_1, m_2F_2. Suppose that $m_1 \mid m_2$ and $m_2/m_1 > 1$. Then there exists an affine pseudo-covering $f : \mathbb{A}^2 \to X$ of degree m_2 which is not Galois.*

It is almost clear that a Galois affine pseudo-covering of a topologically simply-connected smooth algebraic variety is trivial, that is to say, the covering morphism is an isomorphism. There are, nonetheless, non-trivial affine pseudo-coverings of the affine space. We just give few examples in the case of $\mathbb{A}^1$. It is obvious that we can produce various examples of affine pseudo-coverings of $\mathbb{A}^n$ by taking direct products.

Example **2.8.** Let C be a smooth cubic curve in $\mathbb{P}^2$. Let P be a flex of C and let ℓ_P be the tangent line of C at P. Choose a point Q on ℓ_P such that $Q \neq P$ and that the other tangent lines of eight other flexes do not meet ℓ_P on Q. Consider the projection of $\mathbb{P}^2$ to $\mathbb{P}^1$ with center Q. Since every line ℓ through Q other than ℓ_P meets the curve C either in three distinct points or in two distinct points with intersection multiplicities 1 and 2, we obtain an open set Y of C by omitting the point P and the four points where the line through Q meets C with intersection multiplicity 2 and a surjective étale morphism $f : Y \to \mathbb{A}^1$ by restricting the projection onto Y. This morphism has degree 3 and is non-Galois.

The following result shows that some of the surfaces appearing in Definition 2.5 cannot be affine pseudo-coverings of $\mathbb{A}^2$.

Theorem 2.9. *Let X be either an affine pseudo-plane which is an ML_0 surface or a Platonic $\mathbb{A}^1$-fiber space. Then there are no étale morphisms from X to $\mathbb{A}^2$.*

References

[1] R.V. Gurjar, K. Masuda, M. Miyanishi and P. Russell, Affine lines on affine surfaces, preprint.
[2] K. Masuda and M. Miyanishi, Affine pseudo-planes and Cancellation Problem, to appear in Trans.AMS.
[3] M. Miyanishi, Lectures on geometry and topology of polynomials - surrounding the Jacobian Conjecture, Lecture Notes in Mathematics **8** (2004), Department of Mathematics, Osaka University.

[4] Affine pseudo-coverings of algebraic surfaces, to appear in J. Algebra.
[5] G. Xiao, π_1 of elliptic and hyperelliptic surfaces, Internat. J. Math. **2** (1991), 599–615.

Cohomology of partially ordered sets

Tim Römer
(joint work with Morten Brun, Winfried Bruns)

Let P be a finite poset. We consider P as a topological space with the *Alexandrov topology*, that is, the topology where the open sets are the lower subsets (also called order ideals) of P. Let K be a field and $\mathcal{T} = (\mathcal{T}_x)_{x \in P}$ be a system of $\mathbb{Z}^d$-graded K-algebras together with homogeneous ring homomorphisms $\mathcal{T}_{xy} : \mathcal{T}_y \to \mathcal{T}_x$ of degree 0 for $x, y \in P$, $x < y$, such that $\mathcal{T}_{xx} = \mathrm{id}_{\mathcal{T}_x}$ and $\mathcal{T}_{xz} = \mathcal{T}_{xy} \circ \mathcal{T}_{yz}$ for $x, y, z \in P$, $x < y < z$. Thus $\mathcal{T}$ determines a sheaf of $\mathbb{Z}^d$-graded K-algebras on P. We are interested in the inverse limit $\lim \mathcal{T}$ which is isomorphic to the zeroth cohomology group $H^0(P, \mathcal{T})$ of P with coefficients in $\mathcal{T}$. This ring is also called the ring of global sections of $\mathcal{T}$. Such rings appear naturally in Algebraic Combinatorics as was observed by Baclawski [1] and Yuzvinsky [8]. See also Brun and Römer [3], Bruns and Gubeladze [4, 5] and Caijun [6] for related results.

For example, let Σ be a rational pointed fan in $\mathbb{R}^d$, i.e. Σ is a finite collection of rational pointed cones in $\mathbb{R}^d$ such that for $C' \subseteq C$ with $C \in \Sigma$ we have that C' is a face of C if and only if $C' \in \Sigma$, and such that if $C, C' \in \Sigma$, then $C \cap C'$ is a common face of C and C'. The face poset $P(\Sigma)$ of Σ is the partially ordered set of faces of Σ ordered by inclusion. Stanley [7] defined the *face ring* $K[\Sigma]$ of Σ follows: As a K-vectors space $K[\Sigma]$ has one basis element x^a for each a in the intersection of $\mathbb{Z}^d$ and the union of the faces of Σ. Multiplication in $K[\Sigma]$ is defined by

$$x^a \cdot x^b = \begin{cases} x^{a+b} & \text{if } a \text{ and } b \text{ are elements of a common face of } \Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{T}(\Sigma)_C$ be the monoid ring $K[C \cap \mathbb{Z}^d]$ for $C \in \Sigma$ and $\mathcal{T}(\Sigma)_{CC'} \colon \mathcal{T}(\Sigma)_{C'} \to \mathcal{T}(\Sigma)_C$ be the natural face projection for $C, C' \in \Sigma$, $C \subseteq C'$. It is easy to see that $\mathcal{T}(\Sigma)$ is a sheaf of $\mathbb{Z}^d$-graded K-algebras on $P(\Sigma)$ and that $\lim \mathcal{T}(\Sigma) \cong K[\Sigma]$.

Given an ideal I in a commutative ring R and an R-module M we denote the local cohomology groups of M with support at I by $H_I^i(M)$ for $i \geq 0$. A version of our main result in [2] is the following

Theorem 1. *Let $\mathcal{T}$ be a sheaf of $\mathbb{Z}^d$-graded K-algebras on a finite poset P and let I be an ideal of $\lim \mathcal{T}$. For $x \in P$ we let d_x denote the Krull dimension of $\mathcal{T}_x$ at $x \in P$ and we assume that:*

(a) $\lim \mathcal{T}$ is a Noetherian ring,
(b) $\mathcal{T}$ is flasque, i.e. $\lim \mathcal{T}|U \to \lim \mathcal{T}|V$ is surjective for $V \subseteq U \subseteq P$ open,
(c) $H_I^i(\mathcal{T}_x) = 0$ for every $x \in P$ and every $i \neq d_x$,
(d) if $x < y$ in P then $d_x < d_y$.

Then there is an isomorphism

$$H^i_I(\lim \mathcal{T}) \cong \bigoplus_{x\in P} \widetilde{H}^{i-d_x-1}((x,1_{\widehat{P}});K) \otimes_K H^{d_x}_I(\mathcal{T}_x)$$

of $\mathbb{Z}^d$*-graded* K*-modules, where* $\widetilde{H}^{i-d_x-1}((x,1_{\widehat{P}});K)$ *denotes the reduced cohomology of the partially ordered set* $(x,1_{\widehat{P}}) = \{y \in P\colon x < y\}$ *with coefficients in* K*.*

The following immediate corollary of Theorem 1 generalizes the results [6, Theorem 2.4] and [8, Theorem 6.4] of Cajun and Yuzvinsky.

Theorem 2. *Suppose in the situation of Theorem 1 that there exists a unique* $\mathbb{Z}^d$*-graded and maximal ideal* $\mathfrak{m}$ *in* $\lim \mathcal{T}$*. If the assumptions of Theorem 1 are satisfied by the ideal* $I = \mathfrak{m}$*, then the ring* $\lim \mathcal{T}$ *is a Cohen-Macaulay ring if and only if there exists a number* n *such that the reduced cohomology* $\widetilde{H}^*(\Delta((x,1_{\widehat{P}})),K)$ *of the simplicial complex* $\Delta((x,1_{\widehat{P}}))$ *associated to the poset* $(x,1_{\widehat{P}})$ *is concentrated in degree* $(n-d_x-1)$ *for every* $x \in P$ *with* $\mathcal{T}_x \neq 0$*.*

Let us return to the situation where $P = P(\Sigma)$ is the face poset of a rational pointed fan Σ in $\mathbb{R}^d$ and $\mathcal{T} = \mathcal{T}(\Sigma)$ is the sheaf of $\mathbb{Z}^d$-graded K-algebras associated to Σ as constructed above. Note that the face ring $K[\Sigma]$ is Noetherian, $\mathbb{Z}^d$-graded and has a unique maximal $\mathbb{Z}^d$-graded ideal $\mathfrak{m}$. The K-algebras $\mathcal{T}(\Sigma)_C$ are normal and thus Cohen-Macaulay of Krull dimension $d_C = \dim(C)$. Hence the following result is a direct consequence of Theorem 1.

Theorem 3. *Let* Σ *be a rational pointed fan in* $\mathbb{R}^d$ *with face poset* $P(\Sigma)$ *and* $\mathfrak{m}$ *be the* $\mathbb{Z}^d$*-graded maximal ideal of the face ring* $K[\Sigma]$*. Then there is an isomorphism*

$$H^i_{\mathfrak{m}}(K[\Sigma]) \cong \bigoplus_{C\in P} \widetilde{H}^{i-\dim(C)-1}((C,1_{\widehat{P}});K) \otimes_K H^{\dim(C)}_{\mathfrak{m}}(K[C\cap \mathbb{Z}^d])$$

of $\mathbb{Z}^d$*-graded* K*-modules.*

Applications of Theorems 2 and 3 include Hochster's decomposition of local cohomology of Stanley-Reisner rings and Reisner's topological characterization of the Cohen-Macaulay property of Stanley-Reisner rings. Also Stanley's observation in [7, Lemma 4.6] that the face ring $K[\Sigma]$ is a Cohen-Macaulay ring if the Stanley-Reisner ring $K[\Delta(P(\Sigma))]$ of the order complex of the face poset of Σ is a Cohen-Macaulay ring can be easily shown using our results.

References

[1] K. Baclawski, *Cohen-Macaulay ordered sets.* J. Algebra **63**, 226–258 (1980).

[2] M. Brun, W. Bruns and Tim Römer, *Cohomology of partially ordered sets.* `math.AC/0502517`, Preprint (2005).

[3] M. Brun and T. Römer, *Subdivisions of toric complexes.* To appear in Journal of Algebraic Combinatorics.

[4] W. Bruns and J. Gubeladze, *Polytopal linear groups.* J. Algebra **218**, 715–737 (1999).

[5] W. Bruns and J. Gubeladze, *Polyhedral algebras, arrangements of toric varieties, and their groups.* Adv. Stud. Pure Math. **33**, 1–51 (2001).

[6] Z. Caijun, Cohen-Macaulay Section Rings. Trans. Am. Math. Soc. **349**, No.11, 4659–4667 (1997).
[7] R. P. Stanley, *Generalized h-vectors, intersection cohomology of toric varieties, and related results.* Commutative algebra and combinatorics, 1985, Adv. Stud. Pure Math. **11**, 187–213 (1987).
[8] S. Yuzvinsky, *Cohen-Macaulay rings of sections.* Adv. Math. **63**, 172–195 (1987).
[9] S. Yuzvinsky, *Flasque sheaves on posets and Cohen-Macaulay unions of regular varieties.* Adv. Math. **73**, 24–42 (1989).

Remarks on F-pure rings

Anurag K. Singh
(joint work with Uli Walther)

A ring homomorphism $\varphi : R \longrightarrow S$ is *pure* if the map

$$\varphi \otimes 1 : R \otimes_R M \longrightarrow S \otimes_R M$$

is injective for every R-module M. If R is a ring containing a field of prime characteristic $p > 0$, then R is F*-pure* if the Frobenius homomorphism $F : R \longrightarrow R$ is pure. This notion was introduced by Hochster and Roberts [3, 4] in the course of their study of rings of invariants. Some examples of F-pure rings are:

- regular rings,
- determinantal rings,
- Plücker embeddings of Grassmannians,
- homogeneous coordinate rings of non-supersingular elliptic curves,
- polynomial rings modulo square-free monomial ideals,
- normal affine semigroup rings.

Also, direct summands of F-pure rings are F-pure.

Let R be a polynomial ring over a field. The *local cohomology* modules of R with support at an ideal $\mathfrak{a}$ are the modules

$$H^i_{\mathfrak{a}}(R) = \varinjlim_t \operatorname{Ext}^i_R(R/\mathfrak{a}^t, R), \qquad i \geq 0,$$

where the maps in the direct limit system are induced by the natural surjections

$$R/\mathfrak{a}^{t+1} \longrightarrow R/\mathfrak{a}^t.$$

Any chain of ideals which is cofinal with the chain $\{\mathfrak{a}^t\}_{t\in\mathbb{N}}$ yields the same direct limit. In [1] Eisenbud, Mustaţă and Stillman raised the following:

Question. *For which ideals $\mathfrak{a}$ does there exist a descending chain of ideals $\{\mathfrak{a}_t\}_{t\in\mathbb{N}}$, cofinal with $\{\mathfrak{a}^t\}_{t\in\mathbb{N}}$, such that the natural maps*

$$\operatorname{Ext}^i_R(R/\mathfrak{a}_t, R) \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{a}_{t+1}, R)$$

are injective for all i and t ?

In [6] Mustaţă proved that if $\mathfrak{a}$ is generated by square-free monomials $\mu_1, \dots, \mu_m$, then the ideals $\mathfrak{a}_t = (\mu_1^t, \dots, \mu_m^t)$ form a descending chain such that the maps between the Ext modules as above are indeed injective.

In a ring of prime characteristic $p > 0$, the Frobenius powers of an ideal $\mathfrak{a}$ are the ideals

$$\mathfrak{a}^{[p^t]} = (x^{p^t} : x \in \mathfrak{a}), \qquad t \geq 0.$$

The main result of the talk is the following:

Theorem. *Let R be a regular ring of prime characteristic $p > 0$, and $\mathfrak{a}$ be an ideal such that $R/\mathfrak{a}$ is an F-pure ring. Then the natural maps*

$$\operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^t]}, R) \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^{t+1}]}, R)$$

are injective for all i, t.

The proof is inspired by Lyubeznik's paper [5].

We conclude with an open question. Let K be a field, and consider the K-linear ring homomorphism

$$\varphi : R = K[w, x, y, z] \longrightarrow K[s^4, s^3t, st^3, t^4]$$

where φ sends w, x, y, z to the elements s^4, s^3t, st^3, t^4 respectively. Let $\mathfrak{a}$ be the kernel of φ. It may be verified that $H^i_{\mathfrak{a}}(R) = 0$ for $i \geq 3$.

If K has characteristic $p > 0$, Hartshorne [2] showed that $\mathfrak{a}$ is a set-theoretic complete intersection, i.e., that there exist $f, g \in R$ such that $\mathfrak{a} = \operatorname{rad}(f, g)$. In this case, the ideals $\mathfrak{a}_t = (f^t, g^t)$ form a chain such that the maps between the appropriate Ext modules are injective.

Next, let K be a field of characteristic 0. If $\mathfrak{b}$ is an ideal with $\operatorname{rad}\mathfrak{b} = \mathfrak{a}$ such that

$$\operatorname{Ext}^i_R(R/\mathfrak{b}, R) \hookrightarrow H^i_{\mathfrak{a}}(R), \qquad i \geq 0,$$

then

$$\operatorname{Ext}^3_R(R/\mathfrak{b}, R) = \operatorname{Ext}^4_R(R/\mathfrak{b}, R) = 0$$

implies that $R/\mathfrak{b}$ is Cohen-Macaulay.

Question. *If K is a field of characteristic 0 and $\mathfrak{a} \subset R$ as above, is there an ideal $\mathfrak{b}$ with $\operatorname{rad}\mathfrak{b} = \mathfrak{a}$ such that the ring $R/\mathfrak{b}$ is Cohen-Macaulay?*

References

[1] D. Eisenbud, M. Mustaţă, and M. Stillman, *Cohomology on toric varieties and local cohomology with monomial supports*, J. Symbolic Comput. **29** (2000), 583–600.

[2] R. Hartshorne, *Complete intersections in characteristic $p > 0$*, Amer. J. Math. **101** (1979), no. 2, 380–383. 45–79.

[3] M. Hochster and J. Roberts, *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Advances in Math. **13** (1974), 115–175.

[4] M. Hochster and J. Roberts, *The purity of the Frobenius and local cohomology*, Advances in Math. **21** (1976), 117–172.

[5] G. Lyubeznik, *On the vanishing of local cohomology in characteristic $p > 0$*, Preprint (2004).

[6] M. Mustaţă, *Local cohomology at monomial ideals*, J. Symbolic Comput. **29** (2000), 709–720.

Combinatorial Secant Varieties

Bernd Sturmfels
(joint work with Seth Sullivant)

Given two varieties $X, Y \subset \mathbb{P}^{n-1}$, their join is the new variety $X * Y$ which is the Zariski closure of all lines spanned by a point in X and a point in Y. The join of a variety X with itself $X * X = X^{\{2\}}$ is the secant variety of X and r-fold join of X with itself $X * X * \cdots * X = X^{\{r\}}$ is the r-th secant variety of X. Joins and secant varieties have been much studied in algebraic geometry. They also occur in algebraic statistics as mixture models. Typically, we would like to study invariants of these varieties, such as the dimension and degree, or, in the best case, compute the ideal of functions $I(X^{\{r\}})$. The dimensions of the secant varieties of classical varieties was studied in [1, 2, 4, 5].

In this ongoing project, we develop a combinatorial framework for approaching these problems. Our strategy is described by the following steps. First, we take secants and joins of arbitrary projective schemes, and, hence, of arbitrary homogeneous ideals in a polynomial ring. Second, we develop the combinatorial study of secants and joins of monomial ideals, relating secants and joins to Alexander duality, coloring properties of graphs, antichains in posets, and regular triangulations of polytopes. Third, we use Gröbner degeneration as a tool to reduce questions about secants and joins of arbitrary projective schemes to secants and joins of monomial schemes.

During the conference at Oberwolfach, we learned that this approach was already suggested in the work of Simis and Ulrich [8], and some of our basic results appear there. However, we develop the combinatorial study of the secants of monomial ideals considerably further, and we offer a range of new applications of these techniques.

On the combinatorial side, we show that the Strong Perfect Graph Theorem of Chudnovsky, Robertson, and Seymour [6] leads to the following result about monomial ideals.

Theorem 0.1. *Let X be a subscheme of $\mathbb{P}^{n-1}$ defined by quadratic monomials. Then either $I(X^{\{r\}})$ is generated in degree $r+1$ for all r, or $I(X^{\{2\}})$ has a minimal generator of odd degree bigger than three, or there exists $r > 2$ such that $I(X^{\{r\}})$ has a minimal generator in degree $2r+1$ and $I(X^{\{s\}})$ is generated in degree $s+1$ for $s < r$.*

References

[1] J. Alexander and A. Hirschowitz. *Polynomial interpolation in several variables.* J. Algebraic Geom. **4** (1995), no. 2, 201–222.

[2] M. Catalano-Johnson. The possible dimensions of the higher secant varieties. *Amer. J. Math.* **118** (1996), no. 2, 355–361.

[3] M. Catalano-Johnson. The homogeneous ideals of higher secant varieties. *Journal of Pure and Applied Algebra* **158** (2001), 123-129.

[4] M. V. Catalisano, A. V. Geramita, and A. Gimigliano. Ranks of tensors, secant varieties of Segre varieties and fat points. *Journal of Linear Algebra and Its Applications* **355** (2002), 263-285.
[5] M. V. Catalisano, A. V. Geramita, and A. Gimigliano. Secant varieties of Grassmann varieties. *Proc. Amer. Math. Soc.* **133** (2005), no. 3, 633–642.
[6] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. To appear in *Annals of Mathematics*, 2002, math.CO/0212070.
[7] J. M. Landsberg and L. Manivel. On the ideals of secant varieties to Segre varieties. *Foundations of Computational Mathematics.* **4** (2004), no. 4, 397 - 422.
[8] A. Simis and B. Ulrich. On the ideal of an embedded join. *Journal of Algebra* **226** (2000), 1-14.

A subadditivity formula for multiplier ideals on singular varieties

Shunsuke Takagi

Ein, Lazarsfeld and Smith discovered a surprising fact about the behavior of symbolic powers of ideals in affine regular rings of equal characteristic zero.

Theorem 1 ([2, Theorem 2.2]). *Let R be an affine regular ring of equal characteristic zero and $P \subset R$ be a prime ideal of height $h > 0$. Then $P^{(hn)} \subseteq P^n$ for all integer $n > 0$.*

To show this result, they introduced the notion of asymptotic multiplier ideals $\mathcal{J}(t \cdot \mathfrak{a}_\bullet)$, which is a variant of multiplier ideals defined for a graded family $\mathfrak{a}_\bullet$ of ideals and a real number $t \geq 0$. In this abstract, a graded family $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m \geq 1}$ of ideals on a Noetherian ring R means a collection of ideals $\mathfrak{a}_m \subseteq R$, satisfying $\mathfrak{a}_1 \cap R^\circ \neq \emptyset$ and $\mathfrak{a}_k \cdot \mathfrak{a}_l \subseteq \mathfrak{a}_{k+l}$ for all $k, l \geq 1$. Just for convenience, we decree that $\mathfrak{a}_0 = R$. One of the most important examples of graded families of ideals is a collection of symbolic powers $\mathfrak{a}^{(\bullet)} = \{\mathfrak{a}^{(m)}\}_{m \geq 1}$. The reader is referred to [2] and [10] for the definition and fundamental properties of asymptotic multiplier ideals. For the proof of Theorem 1 the essential property of these multiplier ideals is the subadditivity formula given by Demailly, Ein and Lazarsfeld [1] which holds only on nonsingular varieties.

After a short while, Hochster and Huneke generalized Theorem 1 for arbitrary regular rings using the theory of tight closure. The notion of tight closure is a powerful tool in commutative algebra introduced in 1980's by Hochster and Huneke [7] using the Frobenius map. The test ideal $\tau(R)$ of a Noetherian ring R of characteristic $p > 0$ is defined to be the anihilator ideal of all tight closure relations in R and plays a central role in the theory of tight closure. Recently Hara introduced a generalization of the test ideal corresponding asymptotic multiplier ideals $\mathcal{J}(t \cdot \mathfrak{a}_\bullet)$.

Definition 2. Let $\mathfrak{a}_\bullet = \{a_m\}$ be a graded family of ideals on a Noetherian reduced ring R of characteristic $p > 0$ and let $t \geq 0$ be a real number. we denote by R° the set of elements of R which are not in any minimal prime ideal.

(1) ([4, Definition 2.7]) Let $N \subseteq M$ be R-modules. The $t \cdot \mathfrak{a}_\bullet$-tight closure of N in M, denoted by $N_M^{*t \cdot \mathfrak{a}_\bullet}$, is defined to be the submodule of M consisting of all

elements $z \in M$ for which there exists $c \in R^\circ$ such that
$$c\mathfrak{a}_{\lceil tq \rceil} z^q \subseteq N_M^{[q]}$$
for all large $q = p^e$.

(2)(cf. [4, Definition 2.9]) Let $E = \bigoplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$ be the direct sum, taken over all maximal ideals $\mathfrak{m}$ of R, of the injective hulls of the residue fields $R/\mathfrak{m}$. We define
$$\widetilde{\tau}(t \cdot \mathfrak{a}_\bullet) = \widetilde{\tau}(\mathfrak{a}_\bullet^t) = \mathrm{Ann}_R(0_E^{*t\cdot\mathfrak{a}_\bullet}).$$
When $\mathfrak{a}_\bullet = \{\mathfrak{a}^m\}$ is the trivial family consisting of powers of a fixed ideal $\mathfrak{a} \subseteq R$, we denote $\widetilde{\tau}(t \cdot \mathfrak{a}_\bullet) = \widetilde{\tau}(\mathfrak{a}^t)$.

Hara and Yoshida [6] proved that the multiplier ideal $\mathcal{J}(t \cdot \mathfrak{a}_\bullet)$ in a normal $\mathbb{Q}$-Gorenstein ring of characteristic zero coincides, after reduction to characteristic $p \gg 0$, with the ideal $\widetilde{\tau}(t \cdot \mathfrak{a}_\bullet)$. Also in fixed characteristic $p > 0$, the ideals $\widetilde{\tau}(t \cdot \mathfrak{a}_\bullet)$ satisfy several nice properties analogous to those of asymptotic multiplier ideals.

Lemma 3 ([11, Lemma 4.5])**.** *Let $\mathfrak{a}_\bullet = \{a_m\}$ be a graded family of ideals on a Noetherian reduced ring R of characteristic $p > 0$ and let $t \geq 0$ be a real number. Then for all integers $k, l \geq 0$,*
$$\mathfrak{a}_k \widetilde{\tau}(l \cdot \mathfrak{a}_\bullet) \subseteq \widetilde{\tau}((k+l) \cdot \mathfrak{a}_\bullet).$$

Proposition 4 ([5, Theorem 4.2])**.** *Let R be an F-finite reduced ring of characteristic $p > 0$. Let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$ and assume that $\mathfrak{a}$ has a reduction generated by l elements. Then*
$$\widetilde{\tau}(\mathfrak{a}^l) = \widetilde{\tau}(\mathfrak{a}^{l-1})\mathfrak{a} \subseteq \mathfrak{a}.$$

We have a formula for the ideals $\widetilde{\tau}(t \cdot \mathfrak{a}_\bullet)$ which gives a generalization of Demailly-Ein-Lazarsfeld's subadditivity formula to the case of singular varieties.

Theorem 5 ([11, Theorem 2.7])**.** *Let R be an equidimensional reduced affine algebra over a perfect field K of characteristic $p > 0$ and let $J = \mathfrak{J}(R/K)$ be the Jacobian ideal of R over K. Let $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}$ be a graded family of ideals on R. Fix positive integers k and l, and a real number $t > 0$. Then*
$$J\widetilde{\tau}(t(k+l) \cdot \mathfrak{a}_\bullet) \subseteq \widetilde{\tau}(tk \cdot \mathfrak{a}_\bullet)\widetilde{\tau}(tl \cdot \mathfrak{a}_\bullet).$$
In particular,
$$J^{l-1}\widetilde{\tau}(tkl \cdot \mathfrak{a}_\bullet) \subseteq \widetilde{\tau}(tk \cdot \mathfrak{a}_\bullet)^l.$$

Employing Ein-Lazarsfeld-Smith's strategy, as a consequence of Theorem 5, we can generalize Theorem 1 to the case of singular affine algebras.

Theorem 6 ([11, Theorem 4.6])**.** *Let R be an equidimensional reduced affine algebra over a perfect field K of characteristic $p > 0$ and $J = \mathfrak{J}(R/K)$ be the Jacobian ideal of R over K. Let $P \subseteq R$ be any prime ideal of height $h > 0$. Then, for every integer $n \geq 1$,*
$$\widetilde{\tau}(R)J^{n-1}P^{(hn)} \subseteq P^n.$$
In particular, one has for all $n \geq 1$
$$J^n\mathfrak{a}^{(hn)} \subseteq P^n.$$

Proof. We consider the graded family $P^{(\bullet)} = \{P^{(m)}\}$ of symbolic powers of P. By Lemma 3 and Theorem 5,

$$\widetilde{\tau}(R)J^{n-1}P^{(hn)} \subseteq J^{n-1}\widetilde{\tau}(hn \cdot P^{(\bullet)}) \subseteq \widetilde{\tau}(h \cdot P^{(\bullet)})^n.$$

Therefore it suffices to show that $\widetilde{\tau}(h \cdot P^{(\bullet)}) \subseteq P$. We may assume without loss of generality that the residue field of the ring R_P is infinite, and it follows that PR_P has a reduction ideal generated by at most h elements. Since $\widetilde{\tau}(h \cdot P^{(\bullet)})$ commutes with localization, by Proposition 4,

$$\widetilde{\tau}(h \cdot P^{(\bullet)})R_P = \widetilde{\tau}(h \cdot P^{(\bullet)}R_P) = \widetilde{\tau}((PR_P)^h) \subseteq PR_P,$$

because after localization at P the symbolic and ordinary powers of P are the same. Thus one has $\widetilde{\tau}(h \cdot P^{(\bullet)}) \subseteq P$, as required. □

Remark 7. Theorem 6 is an improvement of Hochster-Huneke's result [8, Theorem 3.7] concerning the growth of symbolic powers of ideals in singular affine algebras. Also, the exponent $n-1$ used on the Jacobian ideal in Theorem 6 cannot be replaced by $n-2$ in general: the exponent $n-1$ in Theorem 6 is best possible (see [8, Example 3.8]).

Next we think about Eisenbud-Mazur's conjecture concerning the behavior of symbolic square of ideals in regular local rings.

Conjecture 8 ([3])**.** *Let $(R, \mathfrak{m})$ be a regular local ring of equal characteristic zero and $P \subset R$ be a prime ideal. Then $P^{(2)} \subseteq \mathfrak{m}P$?*

When R is of positive characteristic or of mixed characteristic, counterexamples to their conjecture are known (see [3] and [9]). Nevertheless, using techniques similar to the proof of Theorem 6, we can prove some result in positive characteristic related to their conjecture.

Theorem 9. *Let $(R, \mathfrak{m})$ be a regular local ring of positive characteristic. Let $P \subseteq R$ be any prime ideal of height $h > 0$.*

(1) One has $P^{(h+1)} \subseteq \mathfrak{m}P$.
(2) If R/P is F-pure, then $P^{(2h-1)} \subseteq P^2$.

References

[1] Demailly, J.-P., Ein, L. and Lazarsfeld, R., *A subadditivity property of multiplier ideals*, Michigan Math. J. **48** (2000), 137–156.

[2] Ein, L., Lazarsfeld, R. and Smith, K., *Uniform bounds and symbolic powers on smooth varieties*, Invent. Math. **144** (2001), no.2, 241–252.

[3] Eisenbud, D. and Mazur, B., *Evolutions, symbolic squares, and Fitting ideals*, J. Reine Angew. Math. **488** (1997), 189–201.

[4] Hara, N., *A characteristic p analog of multiplier ideals and applications*, preprint, to appear in Comm. Algebra.

[5] Hara, N. and Takagi, S., *On a generalization of test ideals*, Nagoya Math. J. **175** (2004), 59–74.

[6] Hara, N. and Yoshida, K., *A generalization of tight closure and multiplier ideals*, Trans. Amer. Math. Soc. **355** (2003), no.8, 3143–3174.

[7] Hochster, M. and Huneke, C., *Tight closure, invariant theory and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), no.1, 31–116.

[8] ———, *Comparison of symbolic and ordinary powers of ideals*, Invent. Math. **147** (2002), no. 2, 349–369.

[9] Kurano, K. and Roberts, P., *The positivity of intersections multiplicities and symbolic powers of prime ideals*, Compositio. Math. **122** (2000), no.2, 165–182.

[10] Lazarsfeld, R., Positivity in Algebraic Geometry II, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, A Series of Modern Surveys in Mathematics, Vol. 49, Springer-Verlag, Berlin, 2004.

[11] Takagi, S., *Formulas for multiplier ideals on singular varieties*, preprint.

Reporter: Tim Römer

Participants

Prof. Dr. Shreeram S. Abhyankar
ram@cs.purdue.edu
Dept. of Mathematics
Purdue University
West Lafayette, IN 47907-1395
USA

Prof. Dr. Teruo Asanuma
asanuma@edu.toyama-u.ac.jp
Faculty of Education
Toyama University
3190 Gofuku
Toyama-shi 930-8555
JAPAN

Prof. Dr. Luchezar Avramov
avramov@math.unl.edu
Department of Mathematics
University of Nebraska, Lincoln
Lincoln, NE 68588
USA

Prof. Dr. Manuel Blickle
manuel.blickle@uni-essen.de
FB 6 - Mathematik
Universität Duisburg-Essen
Standort Essen
45117 Essen

Dr. Holger Brenner
H.Brenner@sheffield.ac.uk
Dept. of Pure Mathematics
Hicks Building
University of Sheffield
GB-Sheffield S3 7RH

Prof. Dr. Winfried Bruns
winfried@math.uos.de
Fachbereich Mathematik/Informatik
Universität Osnabrück
49069 Osnabrück

Prof. Dr. Ragnar-Olaf Buchweitz
ragnar@math.utoronto.ca
Dept. of Mathematics
University of Toronto
100 Saint George Street
Toronto, Ontario, M5S 3G3
CANADA

Prof. Dr. Giulio Caviglia
caviglia@msri.org
Mathematical Sciences Research
Institute
1000 Centennial Drive
Berkeley CA 94720-5070
USA

Prof. Dr. Marc Chardin
chardin@math.jussieu.fr
Institut de Mathematiques
UMR 7586 du CNRS and
Universite Pierre et Marie Curie
4 place Jussieu
F-75252 Paris Cedex 05

Prof. Aldo Conca
conca@dima.unige.it
Dipartimento di Matematica
Universita di Genova
Via Dodecaneso 35
I-16146 Genova

Dr. Anthony J. Crachiola
ajcrachi@loyno.edu
crachiola@member.ams.org
Department of Mathematics
Loyola University New Orleans
6363 St. Charles Ave., Box 35
New Orleans, LA 70118
USA

Prof. Dr. Stephen Dale Cutkosky
dale@cutkosky.math.missouri.edu
Dept. of Mathematics
University of Missouri-Columbia
201 Mathematical Science Bldg.
Columbia, MO 65211
USA

Prof. Dr. Lawrence Ein
ein@uic.edu
ein@math.uic.edu
Dept. of Mathematics, Statistics
and Computer Science, M/C 249
University of Illinois at Chicago
851 S. Morgan Street
Chicago, IL 60607-7045
USA

Prof. Dr. David Eisenbud
de@msri.org
Mathematical Sciences Research
Institute
1000 Centennial Drive
Berkeley CA 94720-5070
USA

Florian Enescu
fenescu@mathstat.gsu.edu
Dept. of Mathematics
Georgia State University
Atlanta, GA 30303
USA

Prof. Dr. Hubert Flenner
Hubert.Flenner@ruhr-uni-bochum.de
Fakultät für Mathematik
Ruhr-Universität Bochum
44780 Bochum

Prof. Dr. Gene Freudenburg
freudenb@usi.edu
Department of Mathematics
University of Southern Indiana
8600 University Boulevard
Evansville, IN 47712-3534
USA

Prof. Dr. Shiro Goto
goto@math.meiji.ac.jp
Meiji University
Department of Mathematics
School of Science and Technology
Higashi-mita 1-1-1, Tama-ku
Kawasaki-shi 214
JAPAN

Dr. Joseph Gubeladze
soso@math.sfsu.edu
Dept. of Mathematics
San Francisco State University
1600 Holloway Avenue
San Francisco, CA 94132
USA

Prof. Dr. Rajendra V. Gurjar
gurjar@math.tifr.res.in
School of Mathematics
Tata Institute of Fundamental
Research
Homi Bhabha Road
Mumbai 400 005
INDIA

Prof. Dr. Robin Hartshorne
robin@math.berkeley.edu
Department of Mathematics
University of California
Berkeley, CA 94720-3840
USA

Prof. Dr. Mitsuyasu Hashimoto
hasimoto@math.nagoya-u.ac.jp
Graduate School of Mathematics
Nagoya University
Chikusa-ku
Nagoya 464-8602
Japan

Prof. Dr. Raymond C. Heitmann
heitmann@math.utexas.edu
Department of Mathematics
University of Texas at Austin
1 University Station C1200
Austin, TX 78712-1082
USA

Prof. Dr. Jürgen Herzog
juergen.herzog@uni-essen.de
FB 6 - Mathematik
Universität Duisburg-Essen
Standort Essen
45117 Essen

Prof. Dr. Craig Huneke
huneke@math.ku.edu
Department of Mathematics
University of Kansas
405 Snow Hall
Lawrence, KS 66045-7567
USA

Prof. Dr. Eero Hyry
Eero.hyry@helsinki.fi
Department of Mathematics
PL 68
FIN-00014 Helsinki Yliopisto

Prof. Dr. Srikanth B. Iyengar
iyengar@math.unl.edu
Department of Mathematics
University of Nebraska, Lincoln
Lincoln, NE 68588
USA

Prof. Dr. Shulim Kaliman
kaliman@math.miami.edu
s.kaliman@math.miami.edu
Dept. of Mathematics
University of Miami
P.O. Box 249085
Coral Gables, FL 33124-4250
USA

Prof. Dr. Mordechai Katzman
M.Katzman@sheffield.ac.uk
Dept. of Pure Mathematics
Hicks Building
University of Sheffield
GB-Sheffield S3 7RH

Prof. Dr. Takesi Kawasaki
kawasaki@comp.metro-u.ac.jp
Dept. of Mathematics
Tokyo Metropolitan University
Minami-Ohsawa 1-1
Hachioji-shi
Tokyo 192-0397
JAPAN

Prof. Dr. Ernst Kunz
ernst.kunz@mathematik.uni-regensburg.de
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg

Prof. Dr. Kazuhiko Kurano
kurano@math.meiji.ac.jp
Department of Mathematics
Meiji University
1-1-1 Higashi-Mita, Tama-ku
Kawasaki 214-8571
Japan

Prof. Dr. Graham Leuschke
gjleusch@syr.edu
Mathematics Department
Syracuse University
Syracuse NY 13244-1200
USA

Prof. Dr. Leonid Makar-Limanov
lml@macs.biu.ac.il
lml@math.wayne.edu
Department of Mathematics
Wayne State University
656 West Kirby Avenue
Detroit, MI 48202
USA

Prof. Dr. Claudia Miller
cmille06@syr.edu
Dept. of Mathematics
Syracuse University
Syracuse, NY 13244-1150
USA

Prof. Dr. Masayoshi Miyanishi
miyanisi@ksc.kwansei.ac.jp
School of Science & Technology
Kwansei Gakuin University
Gakuen 2-1
Sanda 669-1337
Japan

Dr. Uwe Nagel
Dept. of Mathematics
University of Kentucky
715 Patterson Office Tower
Lexington, KY 40506-0027
USA

Prof. Dr. Dorin Mihail Popescu
dorin.popescu@imar.ro
Institute of Mathematics
University of Bucharest
P.O. Box 1-764
Bucharest 014700
ROMANIA

Dr. Tim Römer
troemer@mathematik.uni-osnabrueck.de
Fachbereich Mathematik/Informatik
Universität Osnabrück
Albrechtstraße 28
49069 Osnabrück

Dr. Sean Sather-Wagstaff
swagstaff@math.unl.edu
1209 S. Anderson St.
Urbana, IL 61801
USA

Dr. Peter Schenzel
schenzel@mathematik.uni-halle.de
schenzel@informatik.uni-halle.de
Fachbereich Mathematik u.Informatik
Martin-Luther-Universität
Halle-Wittenberg
Theodor-Lieser-Str. 5
06120 Halle

Prof. Dr. Stefan Schröer
schroeer@math.uni-duesseldorf.de
Mathematisches Institut
Heinrich-Heine-Universität
Universitätsstr. 1
40225 Düsseldorf

Dr. Liana M. Sega
lsega@math.msu.edu
Department of Mathematics
Michigan State University
Wells Hall
East Lansing, MI 48824-1027
USA

Prof. Dr. Anurag Singh
singh@math.utah.edu
Department of Mathematics
University of Utah
1155 South 1400 East, Room 233
Salt Lake City, UT 84112-0090
USA

Prof. Dr. Uwe Storch
Uwe.Storch@ruhr-uni-bochum.de
Fakultät für Mathematik
Ruhr-Universität Bochum
44780 Bochum

Prof. Dr. Jürgen Stückrad
stueckrad@mathematik.uni-leipzig.de
Fakultät für Mathematik/Informatik
Universität Leipzig
Augustusplatz 10/11
04109 Leipzig

Prof. Dr. Bernd Sturmfels
bernd@math.berkeley.edu
Department of Mathematics
University of California
Berkeley, CA 94720-3840
USA

Prof. Dr. Irena Swanson
iswanson@nmsu.edu
Department of Mathematical Sciences
New Mexico State University
P.O.Box 30001; 3MB
Las Cruces, NM 88001
USA

Prof. Dr. Shunsuke Takagi
stakagi@math.kyushu-u.ac.jp
Faculty of Mathematics
Graduate School
Kyushu University
Hakozaki 6-10-1, Higashi-ku
Fukuoka 812-8581
JAPAN

Prof. Dr. Ngo-Viet Trung
nvtrung@thevinh.ncst.ac.vn
Dept. of Mathematics
University of Hanoi
Dai hoc Tong hop
Box 631, BoHo
Hanoi 10000
VIETNAM

Prof. Dr. Bernd Ulrich
ulrich@math.purdue.edu
Dept. of Mathematics
Purdue University
West Lafayette, IN 47907-1395
USA

Prof. Dr. Mikhail Zaidenberg
Mikhail.Zaidenberg@ujf-grenoble.fr
Laboratoire de Mathematiques
Universite de Grenoble I
Institut Fourier
B.P. 74
F-38402 Saint-Martin-d'Heres Cedex