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Stochastic Analysis and Non-Classical Random Processes

Organised by
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Abstract. The workshop focused on recent developments in the theory of stochastic processes and flows, with special emphasis on emerging new classes of processes, as well as new objects whose limits are expected to coincide with such processes. A prominent role was played by the SLE family of processes, motion in random media, non-classical noises and flows, and random planar maps.

Mathematics Subject Classification (2000): 60G05, 60G99, 60H99, 60J80, 60J99, 60K35, 60K37, 60K40, 60C05, 60D05, 05C30, 05C80, 82B20, 82B24, 82B41.

Introduction by the Organisers

The meeting was organized by J. D. Deuschel (Berlin), W. Werner (Orsay), and O. Zeitouni (Minneapolis). It was broadly divided into several distinct but related themes. The talks varied in length. There were two 70 minute overview talks, 12 talks of 45 minutes, and the rest were shorter 35 minutes talks.

The morning of Monday was devoted to an exposition of the topic of automorphisms of noises and Arveson systems, and their link with the Arratia flow and sticky Brownian motion. B. Tsirelson gave a longer talk that contained an overview of the field and exposed several open problems. This was followed with a talk by J. Warren on perturbations of Arratia flows.

In the afternoon, D. Aldous presented several problems related to the study of geometrical paths defined from a Poisson field in R^d, for example the study of shortest cycles involving a fraction of the points in the Poisson cloud. Several open problems were presented. P. Bougerol reported on his work relating representation theory to path transformations. Y. Le Jan presented a new class of processes called relativistic diffusions, with special emphasis on the example of the Schwarzschild geometry.

Most of Tuesday was devoted to the study of random planar maps (that is, random planar graphs embedded in the plane). The day begun with a longer overview talk by O. Angel, who presented the model and some of the techniques introduced to study it (Matrix integrals, explicit bijections, recursions for generating functions). Angel explained some of the conjectured universal exponents and presented his work on triangulations of the unit disc and the relation of percolation on this object and the Airy stable process. The rest of the morning was devoted to three shorter talks by P. Chassaing, J.-F. Marckert and G. Miermont, who considered various aspects of the coding of planar maps by labelled trees, and the consequences of such coding for random planar quadrangulations and related models, and their scaling limits.

Tuesday afternoon began with a talk by J.-F. Le Gall who studied the conditioning of continuous trees in the framework of the so called "real trees". He then applied his results to the study of objects appearing as scaling limits of encodings of planar maps. The rest of the afternoon was devoted to problems involving partial differential equations. J. Mattingly described his recent proof of ergodicity for stochastic Navier-Stokes under forcing by a finite number of modes. H. Owhadi discussed numerical schemes for the up-scaling of nonlinear divergence-form PDE's, based on metric deformations.

The subject of Wednesday's talks was the SLE and processes leading to SLE limits. G. Lawler discussed the Laplacian random walk, a generalization of the loop-erased random walk that should lead to an SLE limit, and sheds some light on how to define the SLE in non simply-connected domains. G. Kozma presented his proof of the existence of scaling limits for loop erased random walks in dimensions 2 and 3. J. Dubédat discussed aspects of the definition of multiple SLE's and their consequences for multiple crossing probabilities for some statistical physics models. R. Bauer described the possible generalizations of SLE in multiply connected domains via an analogue of Loewner's equations in such domains.

The weather on Wednesday afternoon was fine and the hike to St. Roman was pleasant.

The talks on Thursday focused on various models of motion in random media. In the morning, P. Mathieu presented his recent proof of diffusivity for random walk on the supercrtical percolation cluster, valid for any dimension greater than 2, and the two scale convergence ideas entering into the proof. M. Barlow, motivated by the study of random walk on the incipient infinite cluster of critical percolation in Z^d, studied this problem on trees, and described scaling limits for the distance traversed by the walk and related quantities. P. Mörters talked about the parabolic Anderson model and decsribed the four regimes that occur as a function of the tail of the driving noise. F. Castell described the limiting regimes for large deviations for random walks and Brownian motion in random sceneries.

In the afternoon, A.-S. Sznitman described the model of diffusions in random environments and presented a proof of diffusive behavior, in the perturbative regime and dimension at least 3, for environments whose laws satisfy an isotropy condition. Y. Hu discussed limits of partition functions for oriented random polymer models. The rest of the afternoon was wrapped up by two talks on edge-reinforced random walks. In the first, F. Merkl introduced a description of linearly edge reinforced random walk on Z^2 in terms of a random walk in random environments, which might be a step in proving recurrence for the reinforced random walk. V. Limic discussed strongly reinforced walks on finite graphs, focusing on the case of a triangle and announcing the attractiveness of an edge under monotonicity conditions.

On Friday morning, T. Funaki discussed the dynamics of the Winterbottom droplet in a lattice model of coupled diffusions. This was followed by three shorter talks. The first, by A. Klenke, discussed the multi-fractal spectrum of intersection local time of two Brownian motions in dimensions 2 and 3. M. Winkel described natural growth processes with values in Lévy forests, that lead to natural limits. W. Koenig described a limit result for non-nearest neighbor random walks conditioned to remain ordered. He described Karlin-McGregor like formulae and convergence toward Dyson limits.

After lunch, T. Lyons concluded the meeting with a talk on rough paths and their signatures.

In addition to the talks during the day and the lively discussions throughout the day, there were two evening sessions in which current Ph.D. students presented briefly their results. These sessions included talks by J. Trujello-Ferreras, G. Pete, L. Georgen, T. Schmitz, C. Boutillier and A. Hammond.

Workshop: Stochastic Analysis and Non-Classical Random Processes

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Automorphisms of noises and Arveson systems

Boris Tsirelson

1. Notes about morphisms of continuous products (CPs)

In discrete time, independence means product of probability spaces. By a morphism from one product to another I mean a measure preserving map that decomposes into product (of measure preserving maps between the factors). In continuous time, independence means *continuous* product (CP, for short) of probability spaces [1, 3c1, 3c6]; still, morphisms are defined naturally [1, 4a1]. They appear to be rare. For example, no morphism is possible from the Brownian motion to the Poisson process.

One may consider continuous products of various objects, especially

- probability spaces,
- measure class spaces (see [1, 10a3, 10a7],
- Borel spaces,
- singular spaces (in the sense of Kechris, see [1, 2e]),
- Hilbert spaces (see [1, 5a1]),

and morphisms between them. Some examples:

- from CP of probability spaces (namely, Brownian motion) to CP of Borel spaces (namely, locally finite subsets of the time axis): the trivial morphism (the empty set corresponds to everything) is the only morphism,
- from CP of probability spaces (namely, Brownian motion) to CP of singular spaces (namely, dense countable subsets of the time axis): I know of three examples only (local minima; local maxima; both). Anything else? I do not know [1, 2e3], but J. Warren has a promising idea.

We have absolutely no theory of morphisms to CPs of singular spaces!

2. Automorphisms of the Brownian motion

By an automorphism of a CP I mean its morphism to itself. In addition, I restrict myself to stationary (that is, commuting with time shifts) automorphisms.

The Brownian motion leads to a CP of probability spaces, therefore (by downgrading) also to a CP of measure class spaces, and (by further downgrading) a CP of Hilbert spaces. Their groups of automorphisms differ, namely:

- probability spaces: only two automorphisms, $B_t \mapsto \pm B_t$,
- measure class spaces: $B_t \mapsto \pm B_t + vt$ (drift),
- Hilbert spaces: basically, the group of motions of $\mathbb{C}$ (W. Arveson), see below.

Multiplicative functionals of the Brownian motion are well-known to be of the form $u_t = \exp(zB_t + z_1 t)$ for $z, z_1 \in \mathbb{C}$. Their definition can be formulated in terms of the CP of Hilbert spaces; in this context they are called 'units'. Normalized units (modulo trivial units) are of the form

$$u_t = \exp\big(zB_t - (\operatorname{Re} z)^2 t\big) \quad \text{for } z \in \mathbb{C}:$$

automorphisms transform this one-parameter set into itself. Corresponding transformations of $\mathbb{C}$ are isometric, since

$$\langle u_t^{(1)}, u_t^{(2)} \rangle = \exp\big(-\tfrac{1}{2}|z_1 - z_2|^2 + \mathrm{i}\dots \big)\,.$$

By the way, this formula holds also for the (compensated) Poisson process, which leads to an isomorphism between Brownian and Poissonian CPs of Hilbert spaces. (Both are generated by units, which is called type I.)

Explicit description of the automorphisms, according to motions of $\mathbb{C}$ (see [2, Sect. 1]):

- $z \mapsto z + \mathrm{i}\lambda$ ($\lambda \in \mathbb{R}$): multiplication by $\exp(\mathrm{i}\lambda B_t)$,
- $z \mapsto z + \lambda$ ($\lambda \in \mathbb{R}$): drift,
- $z \mapsto \mathrm{e}^{\mathrm{i}\lambda} z$ ($\lambda \in \mathbb{R}$): $dB_t \mapsto \mathrm{e}^{\mathrm{i}\lambda} dB_t$.

Clearly, the group of automorphisms acts transitively on the set of normalized units.

3. The question and the answer

The question: For arbitrary Arveson systems (that is, stationary CPs of Hilbert spaces), is the group of automorphisms transitive on normalized units?

The answer: No. A counterexample of type II_1 is constructed [2], in fact, a stochastic flow driven by the Brownian motion (a weak solution, of course).

The counterexample combines two ideas:

- a random variable 'from thin air' (A. Vershik ∼1960; B. Tsirelson 1975),
- new randomness attached to local minima of the Brownian motion (J. Warren 1999).

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Perturbing the Arratia flow

Jon Warren

(joint work with Chris Howitt)

The Arratia flow is a doubly indexed family $\big(X_{s,t}; s \le t\big)$ of maps of the real line. It has the properties of a stationary stochastic flow. The two point motion is that of a pair of Brownian motions who move independently prior to meeting and then coalesce.

The Arratia flow provides an example of black noise in Tsirelson's theory of continuous products of probability spaces, [1]. This means that there is no Brownian motion B whose increments are adapted to the flow in the sense that

$$B_t - B_s \text{ is measurable with respect to } \sigma\big(X_{u,v}; s \leq u \leq v \leq t\big) \text{ for all } s \leq t.$$

Being a black noise is equivalent to certain lack of stability which manifests itself through joinings of the flow. A joining is a pair $\big(X, X'\big)$ of Arratia flows defined on the same probability space having the property

$$\big(X_{t_1,t_2}, X'_{t_1,t_2}\big), \big(X_{t_2,t_3}, X'_{t_2,t_3}\big) \dots \big(X_{t_{n-1},t_n}, X'_{t_{n-1},t_n}\big) \text{ are independent}$$

whenever $t_1 \leq t_2 \dots \leq t_n$.

Fix $p \in [0,1]$. Then for each $n \geq 1$ we may construct a joining $\big(X, X^{(n)}\big)$ by first choosing a sequence $\big(\epsilon_k; k \in \mathbf{Z}\big)$ of independent Bernoulli(p) random variables. Then we take

$$\big(X_{s,t}; \tfrac{k}{n} \leq s \leq t \leq \tfrac{k+1}{n}\big) = \big(X^{(n)}_{s,t}; \tfrac{k}{n} \leq s \leq t \leq \tfrac{k+1}{n}\big) \text{ if } \epsilon_k = 1$$
$$\big(X_{s,t}; \tfrac{k}{n} \leq s \leq t \leq \tfrac{k+1}{n}\big) \text{ independent of } \big(X^{(n)}_{s,t}; \tfrac{k}{n} \leq s \leq t \leq \tfrac{k+1}{n}\big) \text{ if } \epsilon_k = 0.$$

According to Tsirelson the Arratia flow being a black noise is equivalent to the fact that for each fixed $p < 1$, as n tends to infinity, the joinings $\big(X, X^{(n)}\big)$ converge in distribution to the joining $\big(X, X'\big)$ in which X and X' are independent.

We introduce the θ-joining $\big(X, X^\theta\big)$ of the Arratia flow which is characterized by the fact that

$$t \mapsto \frac{1}{\sqrt{2}}\big|X_{0,t}(x) - X^\theta_{0,t}(y)\big|$$

is a sticky Brownian motion with parameter $\theta \in (0,\infty)$. We recall that the latter process is a diffusion Z on $[0,\infty)$ which behaves as a standard Brownian motion away from 0 and whose semi-martingale local time at 0 satisfies $L^0_t = 2\theta \int_0^t \mathbf{1}(Z_s = 0)ds$.

Returning to the construction of the sequence of joinings $\big(X, X^{(n)}\big)$ we now let p depend on n according to $p = p(n) = 1 - \theta\sqrt{\frac{\pi}{2n}}$. In this case the sequence of joinings converges as n tends to infinity to the θ-joining. Using this convergence we obtain the following Markovian property of θ-joinings. If the triple of Arratia flows $\big(X, X^{\theta_1}, X^{\theta_1+\theta_2}\big)$ is such that X and $X^{\theta_1+\theta_2}$ are conditionally independent given X^{θ_1}, and so that $\big(X, X^{\theta_1}\big)$ is a θ_1-joining while $\big(X^{\theta_1}, X^{\theta_1+\theta_2}\big)$ is a θ_2-joining, then $\big(X, X^{\theta_1+\theta_2}\big)$ is a $(\theta_1+\theta_2)$-joining.

We define a stochastic flow of kernels $\big(K_{s,t}; s \leq t\big)$ of the type studied by Le Jan and Raimond, [2], via

$$K_{s,t}(x, A) = \mathbf{P}\big(X_{s,t}(x) \in A | X^\theta\big), \qquad x \in \mathbf{R}, A \in \mathcal{B}(\mathbf{R}).$$

We may derive the two point motion of this flow of kernels with the help of the Markovian property of θ-joinings. Consider the triple $\big(X, X^\theta, X^{2\theta}\big)$. The semigroup $P_t^{(2)}$ of the two point motion satisfies

$$P_t^{(2)}\big((x,x'),A\times A'\big)=\mathbf{E}\big[K_{0,t}(x,A)K_{0,t}(x',A')\big]=$$
$$\mathbf{P}\big(X_{0,t}(x)\in A, X_{0,t}^{2\theta}(x')\in A'\big).$$

Thus the two point motion is the diffusion $t\mapsto \big(X_{0,t}(x),X_{0,t}^{2\theta}(x')\big)$. This is the same two point motion as that of a flow of kernels of constructed by Le Jan and Raimond in [3]. However we conjecture that the n point motions for $n\geq 4$ do not agree, and that the two flows of kernels are different.

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Percolating paths through random points

David J. Aldous

Take n random points in a d-dimensional cube of volume n. Let $L_n(1)$ be the length of the shortest cycle through all n points, i.e. the length of the solution of the traveling salesman problem (TSP). Almost 50 years ago, Beardwood-Halton-Hammersley [1] proved there exists a constant $c(1)$ (depending on dimension d) such that $EL_n(1)\sim c(1)n$ as $n\to\infty$. Subsequent work on related problems is described in the monographs by Steele [5] and Yukich [6]. Here we consider a problem which, conceptually speaking, links the random TSP to percolation theory. In continuum percolation [4], which involves a Poisson point process (ξ_i) on $\mathbb{R}^d$, one can informally describe the critical value for percolation as the smallest c such that there exists some infinite sequence $\xi_{j_1},\xi_{j_2},\xi_{j_3},\ldots$ of distinct points such that $\max_{i\geq 1}|\xi_{j_{i+1}}-\xi_{j_i}|\leq 2c$. Here $|\cdot|$ denotes Euclidean distance. What we study can analogously be described informally as the smallest c such that there exists some infinite sequence $\xi_{j_1},\xi_{j_2},\xi_{j_3},\ldots$ of distinct points such that $\lim_{n\to\infty} n^{-1}\sum_{i=1}^n|\xi_{j_{i+1}}-\xi_{j_i}|\leq c$. This could also be viewed as a continuum analog of the time constant in first passage percolation [2, 3], but where we use "distance along a path" in place of "end-to-end Euclidean distance".

In this talk we describe equivalence of several approaches to a rigorous definition of what we will call $c(0+)$. Fix dimension $d\geq 2$ and write "volume" for d-dimensional Lebesgue measure. Let (ξ_i) be a Poisson point process of rate 1 per unit volume in $\mathbb{R}^d$.

Approach 1: continuum first passage percolation across a diagonal. For $s>0$ define a random variable W_s as the minimum, over all $m\geq 0$ and all choices

$\{\xi_{j_1}, \ldots, \xi_{j_m}\} \subset [0,s]^d$ of m distinct points of the Poisson process, of

$$m^{-1} \sum_{i=1}^{m+1} |\xi_{j_i} - \xi_{j_{i-1}}|$$

where $\xi_{j_0} = (0, \ldots, 0)$ and $\xi_{j_{m+1}} = (s, \ldots, s)$.

Approach 2: TSP on sparse subsets of the cube. Let $\mathbb{C}_n = [0, n^{1/d}]^d$ be the cube of volume n in $\mathbb{R}^d$. Put n random (independent, uniformly distributed) points (η_j) into $\mathbb{C}_n$. Fix $0 < \delta \le 1$. Let $L_n(\delta)$ be the minimum, over all choices of cycles $(\eta_{j_1}, \eta_{j_2}, \ldots, \eta_{j_m}, \eta_{j_{m+1}} = \eta_{j_1})$ through any chosen $m = \lceil \delta n \rceil$ of the random points, of the cycle length $\sum_{i=1}^{m} |\eta_{j_{i+1}} - \eta_{j_i}|$.

Approach 3: Translation invariant distributions on infinite paths through Poisson points. This is a more abstract approach. Consider a locally finite set (x_i) of points in $\mathbb{R}^d$, together with a set $\mathcal{E}$ of edges whose endpoints are in (x_i), where the edges form a collection of doubly-infinite paths, each point of (x_i) appearing either once or never in the paths. Write $\mathbf{S}$ for the space of such points-and-paths configurations. The Euclidean translation group acts naturally on $\mathbf{S}$, so one can define a probability distribution μ on $\mathbf{S}$ to be *invariant* if it is invariant under the action of the Euclidean translation group. Let $\mathcal{M}$ be the set of invariant distributions on $\mathbf{S}$ under which the distribution of the points (ξ_i) is the Poisson point process of rate 1. Informally, a $\mu \in \mathcal{M}$ is just a way of collecting some subsets of the Poisson points into paths using a rule which doesn't depend on the location of the origin. For $\mu \in \mathcal{M}$ there is a constant $\delta(\mu) \in [0,1]$ specified informally as "the proportion of points which are in some infinite path" and formally via the formula: for every cube $\mathbb{C} \subset \mathbb{R}^d$,

$$E_\mu(\text{number of points } \xi_i \in \mathbb{C} \text{ which are in some infinite path}) = \delta(\mu) \text{ volume}(\mathbb{C}).$$

Similarly there is a constant $\ell(\mu)$ interpreted as "mean edge-length over all edges in the paths of μ" and formally via the formula: for every cube $\mathbb{C} \subset \mathbb{R}^d$,

$$E_\mu(\text{length of } \mathcal{E} \cap \mathbb{C}) = \delta(\mu)\ell(\mu) \text{ volume}(\mathbb{C}).$$

Finally define

$$\bar{c}(\delta) := \inf\{\ell(\mu) : \ \mu \in \mathcal{M}, \ \delta(\mu) = \delta\}.$$

Theorem 1. *(a) For $0 < \delta \le 1$ there exists a constant $c(\delta)$ such that $\frac{L_n(\delta)}{\delta n} \to c(\delta)$ in L^1 as $n \to \infty$.*
(b) The function $c(\delta), \ 0 < \delta \le 1$ is non-decreasing and continuous; the function $\delta c(\delta)$ is convex. The limit $c(0+) := \lim_{\delta \downarrow 0} c(\delta)$ is strictly positive.
(c) $W_s \to c(0+)$ in probability as $s \to \infty$.
(d) $\bar{c}(\delta) = c(\delta), \ 0 < \delta \le 1$.

The proof uses fairly straightforward subadditivity and weak convergence methods.

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Brownian motion and (asymptotic) representation theory

Philippe Bougerol
(joint work with Philippe Biane, Neil O'Connell)

A recent important progress in the theory of representation is the path model developped by Littelmann [6] to give a unified combinatorial setup, generalizing the theory of Young tableaux to semi-simple Lie algebras of type other than A. In this model, the multiplicity of a weight of a given irreducible representation is the number of continuous paths in a Cartan subalgebra obtained by the application of certain path transformations. It can be interpreted in terms of random walks with values in the weight lattice. In the rank one case, these path transformations appear in Pitman's theorem [8] which states that if $(B_t)_{t\geq 0}$ is a one-dimensional Brownian motion, then the stochastic process $R_t := B_t - 2\inf_{0\leq s\leq t} B_s$ is a three dimension Bessel process.

This has led us to define Pitman transforms. These transforms operate on the set of continuous functions π with values in a real vector space V, starting at 0, and are given by the formula

$$\mathcal{P}_\alpha \pi(t) = \pi(t) - \inf_{t\geq s\geq 0} \alpha^\vee(\pi(s))\alpha \qquad T \geq t \geq 0.$$

Here $\alpha \in V$ and $\alpha^\vee \in V^\vee$ (where $V^\vee$ is the dual space of V) satisfy $\alpha^\vee(\alpha) = 2$. These transforms satisfy braid relations. Consider a Coxeter system (W, S). To each fundamental reflection we associate a Pitman transform $\mathcal{P}_{\alpha_i}$. Then the braid relations imply that if $w \in W$ has a reduced decomposition $w = s_{i_1} \dots s_{i_n}$, then the operator $\mathcal{P}_w = \mathcal{P}_{\alpha_{i_1}} \dots \mathcal{P}_{\alpha_{i_n}}$ is well defined, i.e. it depends only on w and not on the reduced decomposition. Let $w_0 \in W$ be the longest element. For each continuous path π, $P_{w_0}\pi$ is a path in the Weyl chamber. One of the main result of Biane, Bougerol, O'Connell [2] is the following multidimensional generalization of Pitman's theorem:

Theorem 1. *Let B be the Brownian motion in V. Then $\mathcal{P}_{w_0}B$ is the Brownian motion in the Weyl chamber, i.e. the Brownian motion conditionned not to exit the Weyl chamber.*

It contains several representation results of the eigenvalue of the G.U.E., see [1], [3], [5], [7]. When W is a Weyl group, let η be a dominant path ending in the weight lattice, then the Littelmann module generated by η, is

$$\{\pi; \eta = \mathcal{P}_{w_0}\pi\}.$$

As mentioned, Littelman's model can be interpreted in terms of random walks. Asymptotic representation theory deals with the behaviour of the representation with highest weight λ, when λ goes to infinity. It can be analyzed with the Brownian motion: The conditional law of $B(1)$ when $\mathcal{P}_{w_0}B(t), 0 \leq t \leq 1$, is a given path is the Duistermatt Heckmann measure associated with the coadjoint orbit through $P_{w_0}B(1)$ (see [4]). An interesting feature of our approach is that it works for any Coxeter system, even when there is no representation theory. We don't need to have a Weyl group.

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Relativistic Diffusions and Schwarzschild Geometry

Yves Le Jan

(joint work with Jacques Franchi)

The purpose of this article is to introduce and study a relativistic motion whose acceleration, in proper time, is given by a white noise. We deal with general relativity, and consider more closely the problem of the asymptotic behaviour of paths in the Schwarzschild geometry example.

The classical theory of Brownian motion is not compatible with relativity, as it appears clearly from the fact that the heat flow propagates instantaneously to infinity. A Lorentz invariant generalized Laplacian was defined by Dudley on the tangent bundle of the Minkowski space, and it was shown that there is no other adequate definition than this one , as long as Lorentz invariance is assumed. An intuitive description of the associated diffusion (i.e. continuous Markov process) is that boosts are continuously applied in random directions of space. We show that this process is induced by a left invariant Brownian motion on the Poincaré group. The asymptotic behaviour of the paths of this process was studied .

Considering the importance of heat kernels in Riemannian geometry and the extensive use that is made of their probabilistic repesentation via sample paths, it is somewhat surprising that Dudley's first studies were not pursued and extended to the general context, namely to Lorentz manifolds. It is indeed easy to check that the "relativistic diffusion" can be defined on any Lorentz manifold using a development, as done below. The infinitesimal generator is the generator of the geodesic flow perturbed by the vertical Laplacian. But such an extension would have little appeal, if some natural questions such as the asymptotic behaviour and the nature of harmonic functions could not be solved in some exemples of interest.

Here we provide a rather complete study of this question in the case of Schwarzschild and Kruskal-Szekeres manifolds, which are used in physics to represent "black holes". The specific interest of these manifolds comes from the vanishing of Ricci curvature, their symmetry, and the integrability of the geodesic flow.

The picture that comes out in the Kruskal-Szekeres case appears quite remarkable, with paths confined in a neighborhood of the singularity, while their velocity increases, and an infinity of SO_3-invariant harmonic functions.

One difficulty of the study (and it might explain why Dudley had few followers) is that no explicit solution was found. The reason is that, even after reduction using the symmetries, the operator cannot involve less than three coordinates (even in Minkowski space), instead of one for the Laplacian on Riemann spaces of constant curvature. Estimations and comparison techniques of stochastic analysis are the main tools we use to prove our results. They do not include yet a full determination of the Poisson boundary, but they suggest that for general Lorentz manifolds bounded harmonic functions could be characterised by classes of light rays, i.e. null geodesics.

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Scaling of Percolation on Planar Maps

Omer Angel

Random planar maps have a dual history, as they have attracted interest from both combinatorialists and physicists. In combinatorics the study of random planar maps is rooted in the works of Tutte in the 60's [7, 8], in which he developed a method for enumerating various classes of planar maps. In physics planar maps are seen as a discrete approximation to a continuous manifold (e.g. by thinking of the map as a 2 dimensional simplicial complex), and have attracted much attention (see [1]). A central role is held by the KPZ relation [5], which relates critical exponent of various models on a Euclidean 2-dimensional lattice to the corresponding exponents on random graphs.

One of the key problems concerning random planar maps, which has been partially solved in [4, 6] is to understand the scaling limit of uniform planar maps. It is believed that there is some measure on random metric spaces, which is a scaling limit of the uniform sample of planar maps. Furthermore, this measure is canonical in that it does not depend (up to some minimal restrictions) on the class of planar maps used at the discrete level (as the Brownian motion is the scaling limit of random walks with weak restrictions.)

We consider the scaling limit of crossing probabilities for critical percolation on random planar triangulations. Triangulations are chosen because the techniques used are easiest to apply to them. The events under consideration take place on one of several types of maps: finite or infinite, and with a finite boundary, no boundary or an infinite boundary (i.e. half plane). Of particular interest is the half plane random map. As the name suggests this a measure on maps in the half plane, i.e. having a boundary that is an infinite line. It has the remarkable property, reminiscent of SLE curves, that when a piece of it is removed, the remaining map has the same law as the whole map.

Our main results express the crossing probabilities in a number of scenarios, in terms of hitting probabilities of a random walk with i.i.d. steps. In the case of triangulations the steps have distribution

$$X_i = \begin{cases} 1 & \text{with prob. } 2/3, \\ -k & \text{with prob. } 2\frac{(2k-2)!}{4^k(k-1)!(k+1)!}. \end{cases}$$

The simplest manifestation of the relation between crossing probabilities and the random walk is the following:

Theorem 1. *In the half plane triangulation, the probability that a black cluster connects a boundary segment of length a to a semi infinite part of the boundary at distance b from the first segment is given by*

$$Q_{a,b} = \mathbb{P}_a(S \text{ hits } \mathbb{Z}^- \text{ in } (-\infty, -b]),$$

where a, b describe the boundary conditions, for the crossing problem, and S is a sample of the above random walk started at a.

Crossing probabilities in more complex scenarios can similarly be expressed in terms of more complicated hitting probabilities of the same random walk, or multiple independent copies of the random walk. The technique works for several variations of crossing probabilities in the half plane and in other natural measures on planar triangulations, as well as for other events determined by critical percolation (e.g. multiple crossings) see [2, 3].

We also consider the limit under scaling of the crossing probabilities. The above random walks have a scaling limit given by a stable process with increment having the Airy distribution, and so the crossing probabilities have a scaling limit given by hitting distributions of such processes. For example, the above probability has

$$\lim Q_{\lambda a, \lambda b} = \mathbb{P}\left(|Y_{\tau_-}| > b\right),$$

where Y_t is an Airy-stable process started at $Y_0 = a$ and τ_- is the hitting time of $\mathbb{R}_-$.

Finally, natural symmetries of the triangulations are expressed in terms of several identities on the Airy-stable processes. The simplest of these identities states that if $\mathbb{P}_a$ is the probability measure describing an Airy-stable process Y_t started at a, and if τ_- is the hitting time of $\mathbb{R}_-$ by the process, then

$$\mathbb{P}_a\left(|Y_{\tau_-}| > b\right) = \mathbb{P}_b\left(|Y_{\tau_-}| < a\right),$$

and this is also equal to the probability that a process started at b hits $\mathbb{R}_-$ before an independent process started at a.

When the half plane map is replaced by a full plane map, similar results hold, but generally the sum of the involved Airy-stable processes is conditioned to remain positive.

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Local limit of labelled trees and expected volume growth in a random quadrangulation

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(joint work with Bergfinnur Durhuus)

As a first step, we describe a bijection between planar quadrangulations and well labelled trees, due to Gilles Schaeffer. Then we define an ensemble of infinite surfaces as a limit of uniformly distributed ensembles of quadrangulations of fixed finite volume. The limit random surface can be described in terms of a birth and death process and a sequence of multitype Galton Watson trees. As a consequence, we find that the expected volume of the ball of radius r around a marked point in the limit random surface is $\Theta(r^4)$.

The combinatorics of planar maps begin more or less with the seminal article [10] by W.T. Tutte, and have been a subject of continuing development, specially after the realization of its importance in quantum field theory [5], in string theory and two-dimensional quantum gravity [1, 2, 8]. In the latter case planar maps play the role of two-dimensional discretized (Euclidean) space-time manifolds, whose topology equals that of a sphere with a number of holes, see e.g. [2] for an overview.

Denoting by $B_r(\mathcal{M})$ the ball of radius r around a marked point in the surface $\mathcal{M}$, and by $|B_r(\mathcal{M})|$ its volume, i.e. the number of vertices in $B_r(\mathcal{M})$, then the growth α is determined by

$$\mathbb{E}\left[|B_r|\right] = \Theta(r^\alpha) \ , \tag{1}$$

assuming such a relation exists. Here the expectation value is understood with respect to the limiting probability measure on random maps, mentionned previously. It is implicit in the definition that the support of the measure consists of surfaces of infinite extent.

We define such a measure μ as a limit of uniformly distributed surfaces of fixed finite volume, following an approach recently proposed in [3, 4]. We adapt the technique of [3] to the case of *well labelled trees* (trees with positive labels on the vertices, to make it short) and thereby construct by simple combinatorial arguments a uniform probability measure μ on infinite well labelled trees. We show how to identify well labelled trees in the support of this measure with infinite quadrangulated planar surfaces through a mapping that shares the basic properties of Schaeffer's bijection [6, 9] between finite trees and finite quadrangulations. In particular, there is a one-to-one correspondence between the vertices in a tree and the vertices in the corresponding surface, except for a certain marked vertex in the surface, and the label $r \in \mathbb{N}$ of a vertex in a tree equals the (graph) distance between the corresponding vertex in the surface and the marked vertex.

Viewing, via this identification, μ as a measure on quadrangulated planar surfaces, we prove the relation (1) with $\alpha = 4$. Thus the exponent 4 does not appear only for the volume of large balls in large but finite random quadrangulations (cf. [6]). The result of the present paper, for balls with radius r in an ensemble of infinite quadrangulations, is closer in spirit to the result of [4] for triangulations.

Together these results corroborate the still unproven claim of universality of the exponent $\alpha = 4$ for planar random surfaces.

Let us give now a description of μ: μ-almost surely the trees have exactly one infinite branch, allowing a definition of the *spine* of a sample well labelled tree as the unique infinite non-self-intersecting path starting at the root. For a random μ–distributed element ω we denote by e_n the vertex at height n on its spine. The label of e_n is denoted by $X_n(\omega)$. Furthermore, we let $L_n(\omega)$ (resp. $R_n(\omega)$) be the *finite* subtree of ω attached to e_n on the left (resp. on the right) of its spine. Finally, set

$$w_k = 2\frac{k(k+3)}{(k+1)(k+2)}\ .$$

$$d_k = \frac{3}{280}\,\frac{k(k+3)}{(k+1)(k+2)}\,\left(5k^4+30k^3+59k^2+42k+4\right).$$

We have

Theorem. *The measure μ has the following probabilistic description.*

i) *$X = (X_n)_{n\geq 0}$ is a birth & death process with parameters (p_k, r_k, q_k) for $k \geq 1$ defined by*

$$q_k \doteq \frac{(w_k)^2}{12\,d_k}\,d_{k-1}, \quad r_k \doteq \frac{(w_k)^2}{12}, \quad p_k \doteq \frac{(w_k)^2}{12\,d_k}\,d_{k+1}, \tag{2}$$

where by convention $d_0 = 0$.

ii) *Conditionally, given that $X = (s_n)_{n\geq 0}$, the L_n's and R_n's form two independent sequences of independent random labelled trees, distributed according to the measures $\hat{\rho}^{(s_n)}$, supported by the set of well labelled trees whose root has label s_n, and defined by:*

$$\hat{\rho}^{(s_n)}(\omega) = 12^{-|\omega|}/w_{s_n}.$$

iii) *More precisely, conditionally, given that $X_n = k$, R_n and L_n are independent multitype Galton-Watson trees, in which the ancestor has type k, and a (type ℓ)-individual can only have progeny of type $\ell+\varepsilon$, $\varepsilon \in \{0, \pm 1\}$. In such multitype Galton-Watson trees, the progeny of a (type ℓ)-individual is determined by a sequence of independent trials with 4 possible outcomes, $\ell+1$, $\ell-1$, ℓ and e (for "extinction"), with respective probabilities $w_{\ell+1}/12$, $w_{\ell-1}/12$, $w_\ell/12$ and $\frac{1}{w_\ell}$ (that add up to 1), sequence stopped just before the first occurence of e: a (type ℓ)-individual has as many children of type $\ell+1$, $\ell-1$ or ℓ as there are occurences of $\ell+1$, $\ell-1$, ℓ in the sequence, before the first occurence of e.*

iv) *As a consequence, the average progeny of a (type ℓ)-individual, $w_\ell - 1 = 1 - O(\ell^{-2})$, has supremum in ℓ equal to 1, and the average size of the tree rooted at such an individual is $\left(3\ell^2+9\ell-2\right)/10$.*

As further consequences, the birth & death process is shown to behave like a discretized version of a 9-dimensional Bessel process (and it is transient). More important, the average number $\mathbb{E}\left[N_r\right]$ of vertices with a fixed label r is $\Theta(r^3)$ for r large, leading to (1) with $\alpha = 4$.

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Limit of Normalized Quadrangulations: the Brownian map

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(joint work with Abdelkader Mokkadem)

1. Introduction

A planar map is a proper embedding without edge crossing of a connected graph in the sphere. Two planar maps are identical if one of them can be mapped to the other by a homeomorphism that preserves the orientation of the sphere. A planar map is a quadrangulation if all faces have degree four. A quadrangulation may contain some multiple edges and any quadrangulation with n faces has $2n$ edges and $n+2$ vertices.

A planar map is said to be pointed (resp. rooted) if one node, called the origin or the root-vertex (resp. one oriented edge, called the root or root-edge) is distinguished. Two pointed (resp. rooted) quadrangulations are identical if the homeomorphism preserves also the distinguished node (resp. oriented edge). We denote by $Q_n^\bullet$ (resp. $\overrightarrow{Q}_n$) the set of pointed (resp. rooted) quadrangulations with n faces.

Since the pioneer work of Tutte [5], the combinatorial study of planar maps has received a considerable attention. Many statistical properties have been obtained for a number of classes of finite planar maps. A question arises does : does it exist a continuous limit object for some rescaled classes of planar maps? This question is important in combinatorics and in probability but also in theoretical physics. As a matter of fact, it has been realized in these last years that random planar structures have a leading role in quantum field theory, string theory and quantum gravity. Following the algebraic topology point of view, the physicists consider triangulations, quadrangulations (or other classes of maps) as discretized versions of 2-dimensional manifolds; they are mainly interested in a continuous limit for suitably normalized discretization. A limit behavior without any scaling has been investigated by Angel & Schramm [1]. They show that the uniform law on the set of finite planar triangulations with n faces converges to a law on the set of infinite planar triangulations (endowed with a non-Archimedean metric). They obtain a limit behavior of the triangulations in the ball of fixed radius k around the origin. Chassaing & Durhuus [2] show a similar result for the convergence of unscaled random quadrangulations with a different approach. The topology used in [1, 2] does not allow to consider a rescaled limit for triangulations or quadrangulations.

In other respects, Chassaing & Schaeffer [3] show that the radius of a random rooted quadrangulation taken uniformly in $\overrightarrow{Q}_n$ and scaled by $n^{1/4}$ converges in distribution, up to a multiplicative constant, to the range of the Brownian snake. This suggests that a good normalization for the edges is $n^{1/4}$. Our purpose in the present paper is to show that random quadrangulations, uniformly chosen in $Q_n^\bullet$ or chosen in $\overrightarrow{Q}_n$, normalized by $n^{1/4}$, endowed with the distribution $\mathbb{P}_n^D$ defined below, converge to a limit object, "a continuum random map", that we name the Brownian map.

2. Models

We consider mainly two random models of quadrangulations:

- $(Q_n^\bullet, \mathbb{P}_U^n)$ where $\mathbb{P}_U^n$ is the uniform distribution on $Q_n^\bullet$.
- $(\overrightarrow{Q}_n, \mathbb{P}_n^D)$ where $\mathbb{P}_n^D$ is defined, for each $q \in \overrightarrow{Q}_n$ with root degree $\deg(q)$, by

$$\mathbb{P}_n^D(q) = \frac{c_n}{\deg(q)} \quad \text{where} \quad c_n = \Big(\sum_{q' \in \overrightarrow{Q}_n} \frac{1}{\deg(q')} \Big)^{-1}.$$

The probability $\mathbb{P}_n^D$ gives, to each rooted quadrangulation, a weight proportional to the inverse of its root degree (it is not the uniform distribution on $\overrightarrow{Q}_n$, which is the law studied in [3]).

The two models are related thanks to the canonical surjection K from $\overrightarrow{Q}_n$ onto $Q_n^\bullet$: Let q' be a rooted quadrangulation in $\overrightarrow{Q}_n$ with root-edge $\overrightarrow{vw}$. The pointed quadrangulation $K(q')$ is the planar pointed map whose origin is v and which is identical to q' as unrooted map. The distance in variation between the image of $\mathbb{P}_n^D$ by K and the uniform distribution on $Q_n^\bullet$ goes to 0.

The Schaeffer's bijection Q between $\overrightarrow{Q}_n$ and $\mathcal{W}_n^+$, the set of well labeled trees with n edges, is the starting point of the work (see [3]). We describe the application Q in a slightly different way. We exhibit two trees, the doddeling tree $\mathcal{D}_n$ and the gluer tree $\mathcal{G}_n$, naturally associated with rooted quadrangulations. This leads us to a new description of Q : a rooted quadrangulation is shown to be "$\mathcal{D}_n$ folded around $\mathcal{G}_n$", in other words, a rooted quadranguation is shown to be $\mathcal{D}_n$ together with an identification of its nodes, with the help of $\mathcal{G}_n$. This yields a notion of rooted abstract map. The leading idea is to construct a notion of maps sufficiently robust to be compatible with rooted quadrangulations described with normalized version of $(\mathcal{D}_n, \mathcal{G}_n)$ and their limits, which are shown to exist. The limit, that we name the Brownian map, is described with the help of the Brownian snake with lifetime process the normalized Brownian excursion. A model of rooted quadrangulation with random edge lengths is also shown to converge to the Brownian map.

Using the surjection K, a pointed quadrangulation may be seen as an equivalence class of rooted quadrangulations. This is the point of view used to build the notion of pointed abstract map. The convergence of normalized pointed quadrangulations under $\mathbb{P}^n_U$ in the space of pointed abstract maps is shown. The limit is still the Brownian map.

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Invariance principles for random labeled mobiles and planar maps

Grégory Miermont

(joint work with Jean-François Marckert)

The enumeration of maps i.e. homeomorphism classes of proper embeddings of graphs in a surface, and the properties of randomly sampled maps has met considerable attention in the physicists community, as such objects can be interpreted as discretized versions of (still ill-defined) random surfaces [1]. One of the most important underlying ideas is that of universality, namely that the local structure of the maps, such as face degrees, should not influence global properties such as diameter or the rate of growth of balls in large randomly chosen maps.

We give an instance of such universality properties in the case when the maps are planar, i.e. are drawn on the 2-dimesional sphere, and *bipartite*, i.e. all faces have even degree. For technical reasons, we also suppose the maps are pointed (a vertex u is distinguished) and rooted (an oriented edge vw is distinguished). Let $\mathcal{M}$ be the set of such maps, to which has been added a cemetary point † (a "faceless" map). For a non-negative weight sequence $\mathbf{q} = (q_i, i \geq 1)$, let $W_\mathbf{q}$ be the measure on $\mathcal{M}$ assigning weight q_i per degree-$2i$ face. Supposing $Z_\mathbf{q} = W_\mathbf{q}(\mathcal{M}) < \infty$, we define a Boltzmann probability distribution $P_\mathbf{q} = W_\mathbf{q}/Z_\mathbf{q}$.

We then make use of a recent bijection due to [2] between $\mathcal{M}$ and a set $\mathcal{T}$ of labeled trees called *mobiles*, which are pairs $(\mathbf{t}, \ell)$ with $\mathbf{t}$ a rooted plane tree with a bipartite coloration of its vertices (say $\circ$ or $\bullet$), and ℓ is a $\mathbb{Z}$-valued labeling function on $\circ$ vertices, such that $\ell(\text{root}) = 0$ and $\ell(\sigma') \geq \ell(\sigma) - 1$ whenever σ, σ' are consecutive $\circ$ vertices when turning clockwise around a $\bullet$ vertex. The bijection has the nice property that each $\bullet$ with degree k vertex corresponds to a face with degree $2k$ of the initial map, each $\circ$ corresponds to a vertex $\neq u$, and $\ell(\sigma) - \min \ell + 1$ is the graph distance $d(x, u)$ from the vertex x associated with σ to u.

We observe that if M is $P_{\mathbf{q}}$-distributed, then the associated random mobile (T, L) is described as follows. The tree T is a (subcritical) two-type Galton-Watson process, in which $\circ$ vertices give birth to a geometric($Z_{\mathbf{q}}^{-1}$) number of $\bullet$ vertices, and $\bullet$ vertices give birth to k vertices of type $\circ$ with probability

$$\frac{Z_{\mathbf{q}}^{k+1}\binom{2k+1}{k+1} q_{k+1}}{Z_{\mathbf{q}} - 1}.$$

Conditionally on T, the labeling L is then uniform among possible labelings.

This leads us to show an invariance principle for a general class of two-type discrete snakes, which goes as follows. Let $\mu^\circ, \mu^\bullet$ be two distributions on $\mathbb{Z}_+$, and let T be the associated two-type Galton-Watson tree with anti-diagonal mean matrix with a $\circ$ root. Let H_n be the height of the $n+1$-th vertex $\sigma(n)$ of T in depth-first order (see e.g. [4]). For $k \geq 1$ let $\nu_k^\circ, \nu_k^\bullet$ be centered probability distributions on $\mathbb{R}^k$. Suppose that the L^p-norm of a r.v. with distribution ν_k° or $\nu_k^\bullet$ grows at most polynomially as $k \to \infty$, for some $p > 4$. Given T, for each $\sigma \in T$ with children $\sigma_1, \ldots, \sigma_k$ let $(Y_{\sigma_i}, 1 \leq i \leq k)$ have law ν_k° or $\nu_k^\bullet$ according to the type of σ, independently over σ's. Let R_σ be the sum of $Y_{\sigma'}$ over non-root ancestors σ' of σ, let $R_n = R_{\sigma(n)}$. Let $(H_s, R_s, s \geq 0)$ be obtained from H and R by linear interpolation. Then

Theorem 1. *Given T has n vertices of type $\bullet$, we have*

$$\left(\frac{H_{(|T|-1)s}}{\sqrt{n}}, \frac{R_{(|T|-1)s}}{n^{1/4}}\right)_{0 \leq s \leq 1} \xrightarrow[n \to \infty]{d} (Ce_s, Dr_s)_{0 \leq s \leq 1},$$

where e is a standard Brownian excursion and given e, r is a Gaussian process with covariance

$$\operatorname{cov}(r_s, r_{s'}) = \inf_{s \wedge s' \leq t \leq s \vee s'} e_t.$$

Last, C and D are two constants depending on $\mu^\circ, \mu^\bullet, \nu_k^\circ, \nu_k^\bullet, k \geq 1$.

This result improves over past results [5] in two ways. First, becausse several types are allowed, and second, because the increments of the spatial motion are allowed to be locally dependent and to depend on the local structure of the tree (here on the degree). Lifting this result back to random maps, we obtain

Theorem 2. *Define*

$$f_{\mathbf{q}}(x) = \sum_{k \geq 0} x^k \binom{2k+1}{k+1} q_{k+1},$$

and suppose $Z_{\mathbf{q}}^2 f'(Z_{\mathbf{q}}) = 1$*, and* $f_{\mathbf{q}}$ *is analytic on a neighborhood of* $Z_{\mathbf{q}}$*. Then conditionally on* M *having* n *faces,*

$$n^{-1/4} \max_{x \in M} d(x, u) \underset{n \to \infty}{\overset{d}{\to}} \left(\frac{8 + 4 Z_{\mathbf{q}}^3 f''_{\mathbf{q}}(Z_{\mathbf{q}})}{9(Z_{\mathbf{q}} - 1)} \right)^{1/4} \Delta,$$

where $\Delta = \sup r - \inf r$*,* r *being as in Theorem 1.*

This result encompasses in principle results obtained by Chassaing-Schaeffer [3] (see also Le Gall [6]), in the case of quadrangulations, although the latter authors rather consider *unpointed* rooted quadrangulations, which imposes a positivity constraint on the labels. Other results are possible, like convergence to the Brownian map in a similar fashion as [7].

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Conditioned Brownian trees

Jean-François Le Gall

Consider a Brownian tree consisting of a collection of one-dimensional Brownian paths started from the origin, whose genealogical structure is given by the Continuum Random Tree (CRT). This Brownian tree may be generated from the Brownian snake driven by a normalized Brownian excursion, and thus yields a convenient representation of the so-called Integrated Super-Brownian Excursion (ISE), which can be viewed as the uniform probability measure on the tree of paths. Different approaches can be given in order to define the Brownian tree conditioned to stay on the positive half-line. For instance, one can first condition the Brownian tree not to hit the half-line $]-\infty,-\varepsilon[$ and then let $\varepsilon \to 0$ in order to obtain a limiting distribution. Alternatively, a Verwaat-like theorem shows that this conditioned Brownian tree can be obtained by re-rooting the unconditioned one at the vertex corresponding to the minimal spatial position. In terms of ISE, this theorem yields the following fact: Conditioning ISE to put no mass on $]-\infty,-\varepsilon[$ and letting ε go to 0 is equivalent to shifting the unconditioned ISE to the right so that the left-most point of its support becomes the origin. A number of explicit estimates and formulas can be derived for our conditioned Brownian tree. In particular, the probability that ISE puts no mass on $]-\infty,-\varepsilon[$ behaves like $2\varepsilon^4/21$ when ε goes to 0. Also, for the conditioned Brownian tree with a fixed height h, we obtain a decomposition involving a spine whose distribution is absolutely continuous with respect to that of a nine-dimensional Bessel process on the time interval $[0,h]$, and Poisson processes of subtrees originating from this spine. The preceding results are developed in the paper [2] in collaboration with Mathilde Weill.

We also discuss invariance principle relating conditioned (discrete) trees to our conditioned Brownian tree. Consider a Galton-Watson tree associated with a critical offspring distribution (with finite variance) and conditioned to have exactly n vertices. Such a tree can be embedded in the real line by affecting spatial positions to the vertices, in such a way that the increments of the spatial positions along edges of the tree are independent variables distributed according to a symmetric probability distribution on the real line, called the spatial distribution. Then condition on the event that all spatial positions are nonnegative. Under suitable assumptions on the offspring distribution and the spatial distribution, one can prove that these conditioned spatial trees converge as $n \to \infty$, modulo an appropriate rescaling, towards the conditioned Brownian tree that is presented above. This invariance principle was motivated in part by the work of Chassaing and Schaeffer [1] concerning asymptotics for random quadrangulations. A key ingredient is Schaeffer's bijection between planar rooted quadrangulations with n faces and well-labeled trees with n edges. Since well-labeled trees are a special case of the conditioned discrete trees discussed above (with a geometric offspring distribution and a spatial distribution uniform over $\{-1,0,1\}$) one can use the invariance principle to give new proofs of asymptotics originally derived in [1] via a more combinatorial approach. These results are presented in [3].

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SPDEs with Highly degenerate forcing: densities and ergodicity

Jonathan C. Mattingly

1. Introduction

Consider the two dimensional Navier–Stokes equation driven by an additive stochastic forcing:

$$(1)\qquad \begin{cases} \frac{\partial w}{\partial t}(t,x) + B(w,w)(t,x) = \nu\Delta w(t,x) + \frac{\partial W}{\partial t}(t,x) \\ w(0,x) = w_0(x), \end{cases}$$

where $x=(x_1,x_2)\in\mathbb{T}^2$, the two-dimensional torus $[0,2\pi]\times[0,2\pi]$, $\nu>0$ is the viscosity constant, $\frac{\partial W}{\partial t}$ is a white-in-time stochastic forcing to be specified below, and $B(w,\tilde w)(x)=\sum_{i=1}^2(\mathcal{K}w)_i(x)\frac{\partial\tilde w}{\partial x_i}(x)$, where $\mathcal{K}$ is the Biot-Savart integral operator which will be defined next. First, we define a convenient basis in which we will perform all explicit calculations. Setting $\mathbb{Z}^2_+=\{(j_1,j_2)\in\mathbb{Z}^2: j_2>0\}\cup\{(j_1,j_2)\in\mathbb{Z}^2: j_1>0, j_2=0\}$, $\mathbb{Z}^2_-=-\mathbb{Z}^2_+$ and $\mathbb{Z}^2_0=\mathbb{Z}^2_+\cup\mathbb{Z}^2_-$, we define a real Fourier basis for functions on $\mathbb{T}^2$ with zero spatial mean by $e_k(x)$ where $e_k(x)=\sin(k\cdot x)$ if $k\in\mathbb{Z}^2_+$ and $e_k(x)=\cos(k\cdot x)$ if $k\in\mathbb{Z}^2_-$. Write $w(t,x)=\sum_{k\in\mathbb{Z}^2_0}\alpha_k(t)e_k(x)$ for the expansion of the solution in this basis. With this notation, in the two-dimensional periodic setting, $\mathcal{K}(w)=\sum_{k\in\mathbb{Z}^2_0}\frac{k^\perp}{|k|^2}\alpha_k e_{-k}$, where $k^\perp=(-k_2,k_1)$. We use the vorticity formulation for simplicity, but all of our results can easily be translated into statements about the velocity formulation of the problem. We solve (1) on the space $\mathbb{L}^2=\{f=\sum_{k\in\mathbb{Z}^2_0}a_k e_k:\sum|a_k|^2<\infty\}$. For $f=\sum_{k\in\mathbb{Z}^2_0}a_k e_k$, we define the norms $\|f\|^2=\sum|a_k|^2$ and $\|f\|_1^2=\sum|k|^2|a_k|^2$.

The emphasis of this note will be on forcing which directly excites only a few degrees of freedom. Such forcing is both of primary modeling interest and is technically the most difficult. Specifically we consider forcing of the form

$$(2)\qquad W(t,x)=\sum_{k\in\mathcal{Z}_*}\sigma_k W_k(t)e_k(x)\ .$$

Here $\mathcal{Z}_*$ is a finite subset of $\mathbb{Z}^2_0$, $\sigma_k>0$, and $\{W_k: k\in\mathcal{Z}_*\}$ is a collection of mutually independent standard scalar Brownian Motions on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. We will present results on the existence of a density for finite dimensional projection of the densities and the ergodic theory of equation (1). For historical remarks and more details, we refer the reader to [MHP04, HM04a, MP04, HM04b].

The geometry of the forcing is encoded in the structure of $\mathcal{Z}_*$. As observed in [EM01], its structure gives information about how the randomness is spread throughout the phase space by the nonlinearity. Define $\mathcal{Z}_0$ to be the symmetric, and hence translationally stationary part of the forcing set $\mathcal{Z}_*$, given by $\mathcal{Z}_0=\mathcal{Z}_*\cap(-\mathcal{Z}_*)$. Then define the collection

$$\mathcal{Z}_n=\left\{\ \ell+j\in\mathbb{Z}^2_0\ :\ j\in\mathcal{Z}_0,\ \ell\in\mathcal{Z}_{n-1}\quad\text{with}\quad \ell^\perp\cdot j\neq 0,\ |j|\neq|\ell|\ \right\}$$

and lastly, $\mathcal{Z}_\infty = \bigcup_{n=1}^\infty \mathcal{Z}_n$. $\mathcal{Z}_\infty$ captures the directions to which the randomness has spread. We also define the subspace spanned by these directions $S_\infty = \mathrm{Span}\big(e_k : k \in \mathcal{Z}_\infty \cup \mathcal{Z}_*\big)$.

One of the main results of [MP04] is the following:

Theorem 1. *For any $t > 0$ and any finite dimensional subspace S of S_∞, the law of the orthogonal projection $\Pi w(t,\cdot)$ of $w(t,\cdot)$ onto S is absolutely continuous with respect to the Lebesgue measure on S and has a C^∞ density.*

Again we refer the reader to [MHP04, HM04a, MP04] for more history, the complete references and more details. The proof proceeds by proving that the Malliavin Covariance matrix, projected onto the finite dimensional subspace, is non-degenerate with its smallest eigenvalues bounded away from zero with high probability. With additional assumptions on the controllability of (1) conditions are also given ensuring the strict positivity of the density. This extends results of Ben Arous and Léandre and Aida, Kusuoka and Stroock to this setting.

Recall that an *invariant measure* for (1) is a probability measure $\mu_\star$ on $\mathbb{L}^2$ such that $P_t^*\mu_\star = \mu_\star$, where P_t^* is the semigroup on measures dual to the Markov transition semigroup P_t defined by $(P_t\phi)(w) = \mathbb{E}_w\phi(w_t)$ with $\phi \in C_b(\mathbb{L}^2)$. While the existence of an invariant measure for (1) can be proved by "soft" techniques using the regularizing and dissipativity properties of the flow, showing its uniqueness is a more challenging problem that requires a detailed analysis of the nonlinearity. The following theorem is the main result of [HM04b].

Theorem 2. *If $\mathcal{Z}_\infty = \mathbb{Z}_0^2$, then, (1) has a unique invariant measure in $\mathbb{L}^2$.*

Combining this result with the following proposition one obtains easy to check conditions guaranteeing a unique invariant measure.

Proposition 1. *One has $\mathcal{Z}_\infty = \mathbb{Z}_0^2$ if and only if both:*

(1) *Integer linear combinations of elements of $\mathcal{Z}_0$ generate $\mathbb{Z}_0^2$.*
(2) *There exist at least two elements in $\mathcal{Z}_0$ with unequal euclidean norm.*

This characterization is sharp in the sense that if $\mathcal{Z}_* = -\mathcal{Z}_*$ and one of the above two conditions fails, then there exists a non-trivial subspace of $\mathbb{L}^2$ which is left invariant under the dynamics of (1). Also notice that if $\mathcal{Z}_0 = \{(0,1),(0,-1),(1,1),(-1,-1)\}$ then Proposition 1 implies that $\mathcal{Z}_\infty = \mathbb{Z}_0^2$. Hence forcing four well chosen modes is sufficient to have the randomness move through the entire system. Of course one can also force a small number of modes center elsewhere than at the origin and obtain the same effect.

The proof of Theorem 2 proceeds by proving the following estimate: there exists a locally bounded $C(w)$, a non-decreasing sequence of times $t_n \to \infty$, and a strictly decreasing sequence ϵ_n with $\epsilon_n \to 0$ so that

$$|\nabla(P_{t_n}\phi)(w)| \le C(w)\|\phi\|_\infty + \epsilon_n\|\nabla\phi\|_\infty \tag{3}$$

for all Fréchet differentiable functions $\phi : \mathbb{L}^2 \to \mathbb{R}$ and all $n \ge 1$. This estimate is a generalization of the idea of a strong Feller diffusion to one which possess the smoothing properties of a strong Feller diffusion but only once the system has settled onto its attractor. This estimate is proven by an approximate integration by parts formula using Malliavin Calculus.

While the above results apply to the stochastic Navier–Stokes equation, the techniques used can be applied to a wide range of stochastic partial differential equations. For example consider the following evolution equation

$$du(t) = F(u)dt + f(t)dt + \sum_{k=1}^{d} g_k dW_k(t) \tag{4}$$

where the g_k are some fixed functions and $F(y) = F_0 + F_1(y) + F_2(y, y) + \ldots + F_m(y, \ldots, y)$ with each of the F_n are n-linear symmetric operators. We will not specify the precise functional setting here. However, under some technical assumptions, it is proven in [BM] that the finite dimensional projections of the transition densities of (4) possess a density with respect to Lebesgue measure if conditions completely analogous to the Lie bracket condition in Hörmander's "sum of squares" theorem are satisfied.

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Metric Based Up-scaling

Houman Owhadi

Heterogeneous multi-scale structures can be found everywhere in nature. Can these structures be accurately simulated at a coarse level? Homogenization theory ([2], [3]) allows us to do so under the assumptions of ergodicity and scale separation by transferring bulk (averaged) information from sub-grid scales to computational scales. Can we get rid of these assumptions? Can we compress a PDE with arbitrary coefficients? Surprisingly the answer is yes; it is rigorous and based on a new form of compensation.

Let Ω be a bounded and convex domain of class C^2. We consider the following benchmark PDE

$$\begin{cases} -\operatorname{div}\big(a(x)\nabla u(x)\big) = g \quad \text{in} \quad \Omega \\ u = 0 \quad \text{in} \quad \partial\Omega. \end{cases} \tag{1}$$

Where g is a function in $L^\infty(\Omega)$. And $x \to a(x)$ is a mapping from Ω the space of positive definite symmetric matrices. We assume a to be symmetric, uniformly elliptic with entries in $L^\infty(\Omega)$.

Recall that F. Murat and L. Tartar's theory of H-convergence [5] provides a mathematical framework for analysis of composites in complete generality, without any need for geometrical hypotheses such as periodicity or randomness. This theory is based on a powerful tool called compensated compactness or div-curl lemma introduced in the 70's by Murat and Tartar [4], [7]. Here we consider up-scaling from a slightly different point of view: we want to solve (1) on a coarse mesh and we want to understand which information should be transferred from fine scales to coarse scales when the entries of a are arbitrary. For that purpose we need a new form of compensation.

Let F be the weak solution of the following boundary value problem

$$\begin{cases} \operatorname{div} a\nabla F = 0 \quad \text{in} \quad \Omega \\ F(x) = x \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2}$$

The new compensation phenomenon takes the following form: if a has only bounded entries, then the entries ∇F and ∇u are only in L^2 but surprisingly $(\nabla F)^{-1}\nabla u$ is Hölder continuous.

This higher regularity is controlled by the following object:

Definition 1. *We call* renormalization tensor *the tensor σ defined by*

$$\sigma := {}^t\nabla F a \nabla F. \tag{3}$$

We write μ_σ the anisotropic distortion of σ defined by

$$\mu_\sigma := \operatorname{esssup}_{x\in\Omega}\Big(\frac{\lambda_{\max}\big(\sigma(x)\big)}{\lambda_{\min}\big(\sigma(x)\big)}\Big). \tag{4}$$

Where $\lambda_{\max}(M)$ ($\lambda_{\min}(M)$) denote the maximal (minimal) eigenvalue of M.

Definition 2. *In dimension $n = 2$, we say that the* renormalization tensor *is stable if and only if $\mu_\sigma < \infty$ and there exist a constant $\epsilon > 0$ such that* $\big(\operatorname{Trace}(\sigma)\big)^{-1-\epsilon} \in L^1(\Omega)$

We call β_σ the Cordes parameter associated to σ defined by

$$\beta_\sigma := \operatorname{esssup}_{x\in\Omega}\Big(n - \frac{\big(\sum_{i=1}^n \lambda_{i,\sigma(x)}\big)^2}{\sum_{i=1}^n \lambda_{i,\sigma(x)}^2}\Big). \tag{5}$$

Where $(\lambda_{i,M})$ denotes the eigenvalues of M. Observe that β_σ is a measure of the anisotropy of σ.

Definition 3. *In dimension $n \geq 3$, we say that the* renormalization tensor *is stable if and only if, $\beta_\sigma < 1$ and if $n \leq 4$ that there exist a constant $\epsilon > 0$ such that* $\big(\mathrm{Trace}(\sigma)\big)^{\frac{n}{2}-2-\epsilon} \in L^1(\Omega)$

Definition 4. *We call*

$$\nabla_F u := \big(\nabla F\big)^{-1} \nabla u. \tag{6}$$

the gradient of u in the metric induced by F.

We can now introduce the new compensation phenomenon:

Theorem 1. *Assume that the* renormalization tensor *is stable. Then there exist constants $\alpha > 0$ and $C > 0$ such that $\nabla_F u \in C^\alpha(\Omega)$ and*

$$\big\|\nabla_F u\big\|_{C^\alpha(\Omega)} \leq C\|g\|_{L^\infty(\Omega)}. \tag{7}$$

Remark 1. *The constant α depends on $\Omega, \lambda_{\max}(a)/\lambda_{\min}(a)$ and μ_σ (β_σ if $n \geq 3$). The constant C depends on the constants above, $\lambda_{\min}(a)$ and if $n \leq 4$ on* $\big\|\big(\mathrm{Trace}(\sigma)\big)^{\frac{n}{2}-2-\epsilon}\big\|_{L^1(\Omega)}$.

Assume now that one needs to solve (1) for p different g (p large). It follows from this new compensation phenomenon that one needs to solve (1) on a fine mesh only n-times (to obtain F, n being the dimension of the space) instead of p (those equations can be solved on a coarse mesh). Indeed theorem 1 provides a rigorous justification of the multi-scale finite element method in its form refined by Allaire and Brizzi [1].

Let $\mathcal{T}_h$ be a coarse conformal mesh on Ω composed of n-simplices. We write $V_h \subset H^1(\Omega)$ the set of piecewise linear functions on the coarse mesh vanishing at the boundary of the tessellation. We write $\mathcal{N}_h$ the set of interior nodes of the tessellation and φ_i ($i \in \mathcal{N}_h$) the usual nodal basis function of V_h satisfying

$$\varphi_i(y_j) = \delta_{ij}. \tag{8}$$

We consider the elements $(\psi_i)_{i\in\mathcal{N}_h}$ defined by

$$\psi_i := \varphi_i \circ F(x). \tag{9}$$

Let us write u_h the solution of the Galerkin scheme associated to (2) based on the elements $(\psi_i)_{i\in\mathcal{N}_h}$. Observe that the number of elements is on the order of h^{-n} and we have the following theorem

Theorem 2. *Assume that the renormalization tensor is stable. Then there exist constants $\alpha, C > 0$ such that*

$$\|u - u_h\|_{H^1} \leq C h^\alpha \|g\|_{L^\infty(\Omega)}. \tag{10}$$

Remark 2. *The constant α depends only on n, Ω and μ_σ (β_σ for $n \geq 3$). The constant C depends on the objects mentioned above plus $\lambda_{\min}(a), \lambda_{\max}(a)$ and* $\big\|\big(\mathrm{Trace}(\sigma)\big)^{\frac{n}{2}-2-\epsilon}\big\|_{L^1(\Omega)}$ *if $n \leq 4$.*

An other point of view on up-scaling is the compression issue. Images can be compressed. Can the same thing be done with operators? To answer that question we can look at the operator (1) as a bilinear form on $H_0^1(\Omega)$

$$a : \begin{cases} H_0^1(\Omega) \times H_0^1(\Omega) & \to \mathbb{R} \\ (v, w) & \to \int_\Omega {}^t\nabla v a \nabla w. \end{cases} \tag{11}$$

The up-scaled or compressed operator, written $\mathcal{U}_h a$ will naturally be a bilinear form on the space of piecewise linear functions on the coarse mesh with Dirichlet boundary condition.

$$\mathcal{U}_h a : \begin{cases} V_h \times V_h & \to \mathbb{R} \\ (v, w) & \to \mathcal{U}_h a[v, w]. \end{cases} \tag{12}$$

The new compensation phenomenon allows us to obtain the following formula: for $v, w \in V_h$

$$\mathcal{U}_h a[v, w] := \sum_{K \in \mathcal{T}_h} \int_K {}^t\nabla v \langle a \nabla F \rangle_K \big(\nabla F(K)\big)^{-1} \nabla w. \tag{13}$$

The only information kept from the small scales in the compressed operator (13) are the bulk quantities $\langle a\nabla F\rangle_K$ (average of $a\nabla F$ over the triangle K) and the coarse gradients of F evaluated at the nodes of a, b, c of the triangles of the coarse mesh (obtained from the non averaged quantities $F(b)-F(a)$). The latter quantity can be interpreted as a deformation of the coarse mesh induced by the small scales (or a new distance defining coarse gradients). Let us write $\mathcal{I}_h u$ the linear interpolation of u over $\mathcal{T}_h$:

$$\mathcal{I}_h u := \sum_{i \in \mathcal{N}_h} u(x_i)\varphi_i(x). \tag{14}$$

Write u^m the solution in V_h of the following linear problem: for all $i \in \mathcal{N}_h$,

$$\mathcal{U}_h a[\varphi_i, u^m] = (\varphi_i, g)_{L^2(\Omega)}. \tag{15}$$

We have the following estimate

Theorem 3. *There exist constants $\alpha, C_m > 0$ such that*

$$\|\mathcal{I}_h u - u^m\|_{H^1(\Omega)} \le C_m h^\alpha \|g\|_{L^\infty(\Omega)}. \tag{16}$$

Remark 3. *The constant α depends only on n, Ω and μ_σ. The constant C_m can be written*

$$C_m := C\frac{\eta^*_{\min}\eta_{\max}}{S^m}. \tag{17}$$

where C depends on the objects mentioned above plus $\lambda_{\min}(a), \lambda_{\max}(a)$ and $\big\|\big(\mathrm{Trace}(\sigma)\big)^{-1-\epsilon}\big\|_{L^1(\Omega)}$. $\eta_{\max}$ is a standard aspect ratio of the triangles of the mesh, $\eta^*_{\min}$ is a weak aspect ratio of those triangles in the metric induced by F, S^m is a stability parameter of the scheme.

One can consider up-scaling from the transport point of view. The elliptic operator appearing in (1) can be seen as the generator of a random walk on a fine graph (disordered environment), can that random walk be simulated by an other one evolving on a coarse subset of nodes of the fine graph? The answer is given by a finite volume method using the compensation appearing in theorem 1. We refer to [6] for that method.

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The Laplacian random walk and SLE

Gregory F. Lawler

The Laplacian random walk with exponent $b \in \mathbb{R}$, the Laplacian-b walk, is a measure on self-avoiding paths on a subset of the integer lattice obtained by weighting steps of a simple random walk by the bth power of the probability of avoiding the path. To be more precise, suppose that V is a connected subset of $\mathbb{Z}^d$ and z, w are distinct points in ∂V. Let $H_V(x,w)$ denote the discrete Poisson kernel, i.e., for $x \in V$, $H_V(x,w)$ is the probability that a simple random walk starting at x leaves V at w. By definition, $H_V(w,w) = 1$ and $H_V(x,w) = 0$ for $x \in \partial V \setminus \{w\}$. The probability that the first step of the Laplacian-b walk from z to w in V goes to x is

$$\frac{H_V(x,w)^b}{\sum_{|y-z|=1} H_V(y,w)^b},$$

provided that $|z-x| = 1$. Conditioned on the first step going to x, the probability for the next step is obtained by replacing z with x and V with $V \setminus \{x\}$. The case $b = 1$ corresponds to the loop-erased random walk.

We consider possible scaling limits for the Laplacian-b walk in two dimensions, which we call the Laplacian motion, LM_b. Under the assumption of conformal invariance, the only possible limit is the Schramm-Loewner evolution (SLE_κ) for some value of κ. (See [1, 5] for the definition of SLE_κ.) In order to determine κ we postulate that the limit process should satisfy the following condition which is satisfied by the discrete process. Suppose D is a subdomain of the upper half plane $\mathbb{H}$. Then we postulate that LM_b in the smaller domain D is the same as LM_b in $\mathbb{H}$ weighted locally by $Q(D_t;U_t)^b$. Here U_t is the driving function of SLE_κ with corresponding conformal maps g_t and random curve γ, $U_t = g_t(\gamma(t))$, $D_t = g_t(D)$, and $Q(D;x)$ denotes the probability that a half-plane excursion starting at $x \in \mathbb{R}$ stays in D. If D is simply connected, then SLE_κ in D can also be defined by conformal invariance. By a simple application of Girsanov's Theorem, we see that the two definitions are compatible if

$$b = \frac{3a-1}{2} \qquad \text{where} \qquad a = \frac{2}{\kappa}.$$

Using this as motivation, we define the Laplacian-b walk in simply connected domains for $b > -1/2$ as the corresponding SLE_κ.

While there are many possible extensions of SLE_κ to non-simply connected domains, there is a particular definition of LM_b to non-simply connected domains that is compatible with the rule listed above. Suppose D is a subdomain of $\mathbb{H}$ such that $\mathbb{H} \setminus D$ is bounded and $\text{dist}(0, \mathbb{H} \setminus D) > 0$. Then LM_b in D is SLE_κ weighted by $Q(D;x)^b$ with $\kappa = \kappa(b)$ as above. More precisely, if $C_t = Q(D_t;U_t)^b$, then C_t satisfies

$$dC_t = b\, C_t\, [J_t\, dt + K_t\, dB_t],$$

where

$$J_t = a\,\Gamma(D_t;U_t) + \frac{1}{2}\,(1-a)\,Sq(D_t;U_t)\,;$$
$$K_t = [\log Q(D_t;U_t)]'$$

S denotes Schwarzian derivative; q is an antiderivative of Q; and $\Gamma(D_t;U_t)$ denotes the "Brownian bubble" measure of bubbles at U_t that leave D_t. Let

$$M_t = C_t\, \exp\left\{ -b \int_0^t J_s\, ds \right\}.$$

Then M_t is a local martingale and by Girsanov's theorem paths weighted by M_t satisfy the equation

$$dU_t = b\,[\log Q(D_t;U_t)]'\, dt + dB_t.$$

We can use this as a definition of LM_b. This process was previously introduced by Zhan [7] and he called it the harmonic random Loewner chain.

Two important examples are:

- When D is simply connected, the local martingale M_t was studied in [3]. In this case $Q(D_t;U_t) = \Phi_t'(U_t)$ where Φ_t is a conformal tranformation of D_t onto $\mathbb{H}$ with $\Phi_t(\infty) = \infty, \Phi_t'(\infty) = 1$. Also, $\Gamma(D_t;U_t) = -(1/6)\, Q(D_t;U_t)$ and the martingale can be written as

$$M_t = \Phi_{D_t}(U_t)^b\, \exp\left\{ -a\lambda \int_0^t \Gamma(D_s;U_t)\, ds \right\},$$

  with $\lambda = (3a-1)(4a-3)/(2a)$. The paramter λ, which is equal to $(-1/2)$ times the central charge, has an interpretation as the density of a loops added to the path.

- When $b = 1$ ($\kappa = 2$, $a = 1$), the martingale becomes just

$$M_t = Q(D_t; U_t)\ \exp\left\{-ab \int_0^t \Gamma(D_s; U_s)\, ds\right\},$$

which shows that we get an interpretation of the Laplacian-b walk in terms of Brownian loops. This works in non-simply connected domains and is related to work of Zhan [6] on loop-erased walk in non-simply connected domains.

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The scaling limit of loop-erased random walk in two and three dimensions

Gady Kozma

Loop-erased random walk is a model for a random simple path created by taking a simple random walk on a graph and erasing loops "chronologically" meaning that each loop is erased as it is created. It can also be viewed as a specific case ($b = 1$) of the Laplacian random walk which is the topic of Lawler's talk. Further, it is a critical model in statistical mechanics: in the Fortuin-Kasteleyn representation [FK72] of the q-Potts model taking $q \to 0$ gives the uniform spanning tree, which is closely related to loop-erased random walk. This rich set of symmetries and representations allowed to solve most major problems associated with statistical mechanics models. See [L99, S00] for surveys (with somewhat different focus) and [F01, BLPS01, BKPS04, BK, PR] for some recent results.

The aim of this talk is to give a sketch of a number of ideas involved in the proof of the following:

Theorem 1. *Let $\mathcal{D} \subset \mathbb{R}^d$, $d = 2, 3$ be a polyhedron and let $a \in \mathcal{D}$. Let $\mathbb{P}_n$ be the distribution of the loop-erasure of a random walk on $2^n\mathcal{D} \cap \mathbb{Z}^3$ starting from $2^n a$ and stopped when hitting $\partial 2^n\mathcal{D}$, multiplied by 2^{-n}. Then $\mathbb{P}_n$ converge in the space $\mathcal{M}(\mathcal{H}(\overline{\mathcal{D}}))$.*

where $\mathcal{H}$ stands for the space of closed subsets of $\overline{\mathcal{D}}$ with the Hausdorff distance. The two dimensional case was known, having been proved in [LSW04a] and later in [K]. The talk discusses the techniques of [K] and what is required to generalize it to three dimensions [Kb]. It also explains what these techniques tell us about the limit itself: it is a measure on the space of simple paths which is invariant to scalings and rotations.

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Multiple SLEs

Julien Dubédat

For critical models of statistical physics in the plane, it is generally conjectured that the scaling limit should define a continuous random object with conformal invariance properties. In the plane, this conformal equivalence requirements is very constraining. In [8], Schramm introduces the Stochastic (or Schramm-) Loewner Evolutions. They constitute a one-parameter (κ) family of probability laws on non-self traversing curves in the plane. These laws are the only ones satisfying two axioms: a "domain Markov" property, and conformal invariance. For background on SLE, see e.g. [10].

In general, configurations in discrete models constitute a family of interfaces. Under the two previous axioms, the distribution of a single interface is classified by the positive parameter κ. This parameter is a linear function of the a.s. Hausdorff dimension of the trace (if the curve is not space-filling).

In [2], we discuss the problem of classifying possible scaling limits for a finite number of interfaces. Each of these is absolutely continuous (at least away from the boundary) with respect to some SLE_κ. To restrict the form of the "interaction" between interfaces, we introduce a "commutation" axiom, which is of geometric nature.

Different examples of this general situation arise from the study of discrete models, such as the Uniform Spanning Tree ([6]), Loop-erased random walks ([6, 4]), critical percolation ([9]), and also from properties of SLE itself, such as restriction ([5, 7]). For instance, for percolation, one can proceed as follows: divide the boundary of a simply connected domain in $(2n)$ arcs of alternate colors. The collection of percolation interfaces contains simple closed loops (that disappear in the scaling limit) and n simple paths connecting the $(2n)$ marked points on the boundary. This defines a random pairing of the $(2n)$ points, and a collection of n (macroscopic) interfaces.

Using the restriction property and loop soups ([5, 7]), one can represent the density of a "commuting" system of n SLEs w.r.t. n independent SLEs in terms of the Brownian loop measure, at least when $\kappa \leq 8/3$. When $\kappa = 2$, this coincides with the scaling limit of n non-intersecting Loop-Erased Random Walks embedded in the same Uniform Spanning Tree via Wilson's algorithm. This last situation is that of Fomin's formulae ([4]).

Through the Loewner's equations, plane curves are encoded by real-valued process. The commutation requirement translates into an algebraic condition on the infinitesimal generators of diffusions associated with the different interfaces. We show that these (non-linear differential coupled) conditions can be recast as a holonomic system. Euler integral representations of solutions of this system (one can think of these as partition functions) are given in [3]. As a corollary, one gets extensions of Cardy's crossing formula ([1, 9]).

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On SLE in multiply connected domains

Robert O. Bauer
(joint work with Roland Friedrich)

In this presentation we discuss the possible candidates for a mathematically rigorous notion of conformally invariant random non-self-crossing curves which begin and end on the boundary of a multiply connected planar domain, and which satisfy a Markovian-type property. The Markovian type property means that the random curves can be developed dynamically as a (locally) growing family of random compacts. We aim to proceed in the spirit of Schramm, who deduced that, under an additional reflection symmetry, there is only a one parameter family of such random curves in simply connected domains, which he termed Stochastic Loewner Evolutions, see [10]. As such conformally invariant random growing compacts are conjectured to arise as scaling limits of interfaces of 2-dimensional statistical mechanical systems at criticality, Schramm had with one stroke identified what those limits can be. This has many consequences and applications, see [7], [8], [9], [11], and references therein.

Statistical mechanical systems have been studied in discrete approximations of multiply connected domains and Riemann surfaces, see [6], and [1], and the connections with conformal field theory (CFT) indicate that the stochastic Loewner evolution should also extend to multiply connected domains and Riemann surfaces.

For multiply connected domains the situation is already more subtle when compared to the simply connected case, because moduli spaces enter the picture and, as we will show, one has to consider interactions with these moduli.

Families of random compacts from the boundary to the boundary now come in two flavors, as the random compact may grow to either connect a boundary component to itself (the chordal case) or it may grow to connect two different boundary components. We call the latter the *bilateral* case.

The radial case, treated in [2], where the random compact grows from the boundary to an interior point, can be considered as a limit of the bilateral case, when the boundary component the random compact grows towards shrinks to a point. This can be made precise, see [5].

Our procedure rests on an appropriate extension of Loewner's equation to the multiply connected case. In the simply connected case, Loewner's equation allows to encode a simple curve in a domain D which has one endpoint on the boundary ∂D by a continuous motion on the boundary. In the multiply connected case, we can show [3] that a simple curve induces a motion on the boundary of the domain. To recover the curve inside the domain requires also the knowledge of the moduli $\mathbf{M}$ (which describe the conformal equivalence class), as the curve grows. We can show [3] that these moduli can be recovered from the boundary motion and thus, once these moduli have been obtained, the curve in the interior itself.

A growing random non-self-crossing curve in a multiply connected domain can then also be encoded into a random motion $\xi(t)$ on the boundary. However, if the connectivity is greater than one, then ξ cannot be Markov. We show that in the chordal case the boundary motion ξ together with the motion of moduli $\mathbf{M}$ is a Markov process, and that it satisfies Brownian scaling.

These facts dramatically reduce the number of possible diffusions. Indeed, in addition to a real parameter κ one is only free to choose a function A which is homogeneous of degree minus one in the variables ξ and $\mathbf{M}$. The term A measures the interaction of the random growing compact with the boundary (for example if it is desired that the random set avoids the interior boundary components).

SLE(κ, ρ), see [8], and [4], also fits naturally into this framework. There, the random compact grows into the upper half-plane, the boundary is the real axis, and the interaction is with a finite number of points on the real axis and given in terms of the simplest homogeneous function of degree minus one, $1/x$. Even though the upper half-plane is simply connected, the marked points on the boundary can serve as moduli and then SLE(κ, ρ) is given by a particular moduli diffusion.

For multiply connected domains it is natural to look for an interaction A which is expressed in terms of domain functionals such as the Green function. Appropriate combinations of derivatives of the Green function are homogeneous of degree minus one in ξ and the moduli. The 'harmonic random Loewner chains' studied in [12] are a particular example of this.

In our opinion the only further reduction in possible diffusions $(\xi, \mathbf{M})$ are regularity requirements on the homogenous function A. In particular we may wish to allow only functions which are analytic. We propose to call the growing family of random compacts obtained by solving the chordal Loewner equation for a diffusion $(\xi, \mathbf{M})$ associated to an analytic function A homogenous of degree minus one, *chordal stochastic Loewner evolution*.

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Quenched invariance principles for random walks on percolation clusters

Pierre Mathieu
(joint work with Andrey Piatnitski)

Consider super critical Bernoulli bond percolation in $\mathbb{Z}^d$, $d \geq 2$: for $x, y \in \mathbb{Z}^d$, we write: $x \sim y$ if x and y are neighbors in the grid $\mathbb{Z}^d$, and let $\mathbb{E}_d$ be the set of non-oriented nearest pairs (x, y). We identify a sub-graph of $\mathbb{Z}^d$ with an application $\omega \in \{0,1\}^{\mathbb{E}_d}$, writing $\omega(x, y) = 1$ if the edge (x, y) is present in ω and $\omega(x, y) = 0$ otherwise. Edges pertaining to ω are then called *open*. Connected components of such a sub-graph will be called *clusters* and the cluster of ω containing a point $x \in \mathbb{Z}^d$ is denoted with $\mathcal{C}_x(\omega)$.

Define now Q to be the probability measure on $\{0,1\}^{\mathbb{E}_d}$ under which the random variables $(\omega(e),\, e \in \mathbb{E}_d)$ are Bernouilli(p) independent variables and let

$$p_c = \sup\{p\, ;\, Q[\#\mathcal{C}_0(\omega) = \infty] = 0\}$$

be the critical probability. It is known that $p_c \in]0,1[$, see [3]. We assume that

$$p > p_c. \tag{0.1}$$

Then, Q almost surely, the graph ω has a unique infinite cluster there after denoted with $\mathcal{C}(\omega)$.

We are interested in the behaviour of the simple symmetric random walk on $\mathcal{C}_0(\omega)$: let $D(\mathbb{R}_+, \mathbb{Z}^d)$ be the space of càd-làg $\mathbb{Z}^d$-valued functions on $\mathbb{R}_+$ and $X(t)$, $t \in \mathbb{R}_+$, be the coordinate maps from $D(\mathbb{R}_+, \mathbb{Z}^d)$ to $\mathbb{Z}^d$. $D(\mathbb{R}_+, \mathbb{Z}^d)$ is endowed with the Skorohod topology. For a given sub-graph $\omega \in \{0,1\}^{\mathbb{E}_d}$, and for $x \in \mathbb{Z}^d$, let P^ω_x be the probability measure on $D(\mathbb{R}_+, \mathbb{Z}^d)$ under which the coordinate process is the Markov chain starting at $X(0) = x$ and with generator

$$\mathcal{L}^\omega f(x) = \frac{1}{n^\omega(x)} \sum_{y \sim x} \omega(x, y)(f(y) - f(x))\,,$$

where $n^\omega(x)$ is the number of neighbors of x in the cluster $\mathcal{C}_x(\omega)$.

The behaviour of $X(t)$ under P^ω_x can be described as follows: starting from point x, the random walker waits for an exponential time of parameter 1 and then chooses, uniformly at random, one of its neighbors in $\mathcal{C}_x(\omega)$, say y and moves to y. This procedure is then iterated with independent hoping times. The walker clearly never leaves the cluster of ω it started from. Since edges are not oriented, the measures with weights $n^\omega(x)$ on the possibly different clusters of ω are reversible.

Let Q_0 be the conditional measure $Q_0(.) = Q(.|\#\mathcal{C}_0(\omega) = \infty)$ and let $Q_0.P_x^\omega$ be the so-called *annealed* semi-direct product measure law defined by

$$Q_0.P_x^\omega[\, F(\omega, X(.))\,] = \int P_x^\omega[\, F(\omega, X(.))\,]\, dQ_0(\omega)\,.$$

Note that $X(t)$ is not Markovian anymore under $Q_0.P_x^\omega$. From [2], it is known that, under $Q_0.P_0^\omega$, the process $(X^\varepsilon(t) = \varepsilon X(\frac{t}{\varepsilon^2}), t \in \mathbb{R}_+)$ satisfies an invariance principle as ε tends to 0 i.e. it converges in law to the law of a non-degenerate Brownian motion. The proof is based on the *point of view of the particule.* It relies on the fact that the law of the environment ω, viewed from the current position of the Markov chain is reversible, when considered under the annealed measure. It does not give any information on the behaviour of the walk for a typical choice of ω. On the other hand, only partial results in dimension higher than 4 have been obtained for almost sure, also called *quenched*, invariance principles in the joint work of V. Sidoravicius and A-S. Sznitman, [8]. Our result holds for any dimension:

Theorem 1. *Q almost surely on the event $\#\mathcal{C}_0(\omega) = \infty$, under P_0^ω, the process $(X^\varepsilon(t) = \varepsilon X(t/\varepsilon^2), t \in \mathbb{R}_+)$ converges in law as ε tends to 0 to a Brownian motion with covariance matrix $\sigma^2 Id$ where σ^2 is positive and does not depend on ω.*

Our strategy of proof follows the classical pattern introduced by S.M. Kozlov for averaging random walks with random conductances in [6]. The method of Kozlov was successfully used under ellipticity assumptions that are clearly not satisfied here. We refer in particular to first part of [8] where random walks in elliptic environments are considered. The main idea is to modify the process $X(t)$ by the addition of a *corrector* in such a way that the sum is a martingale under P_0^ω and use a martingale invariance principle. Then one has to prove that, in the rescaled limit, the corrector can be neglected or equivalently that the corrector has sub-linear growth. For this second step, in a classical elliptic set-up, one would invoke the Poincaré inequality and the compact embedding of H^1 into L^2. For percolation models, a weaker, but still suitable form of the Poincaré inequality was proved in the paper of P. Mathieu and E. Remy [7], see also [1]. However another difficulty arises: our reference measure is the counting measure on the cluster at the origin. When rescaled, it does converge to Lebesgue measure on $\mathbb{R}^d$ but, for a fixed ε it is of course singular. Thus rather than using classical functional analysis tools, one has to turn to L^2 techniques in varying spaces or 2 scale convergence arguments as they have been recently developped for the theory of homogenization of singular random structures in the work of A. Piatnitski and V. Jikov, see [5]. The positivity of σ is a consequence of the results of [4].

Note: we recently learnt of a related work by N. Berger and M. Biskup where a quenched invariance principle is obtained for planar percolation. Their preprint is available on the arXiv (http://front.math.ucdavis.edu/).

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Random walks on critical percolation clusters

Martin T. Barlow

We consider bond percolation on $\mathbb{Z}^d$. Write $\mathcal{C}(x)$ for the open cluster containing x, and let $\theta(p) = \mathbb{P}_p(|\mathcal{C}(x)| = \infty)$. Recall that in the supercritical regime ($p > p_c$) there exists a unique infinite cluster, which we denote by $\mathcal{C}_\infty = \mathcal{C}_\infty(\omega)$.

Let $Y = (Y_t, t \geq 0, P^x_\omega, x \in \mathcal{C}_\infty)$ be the continuous time simple random walk (CTSRW) on $\mathcal{C}_\infty$: this is the Markov chain Y which stays in x for an exponential time with mean 1, then moves to each neighbour of x with equal probability. Write $\mu(x)$ for the vertex degree of $x \in \mathcal{C}_\infty$, and define the transition density of Y_t under P^x_ω by

$$q^\omega_t(x,y) = \frac{1}{\mu(y)} P^x_\omega(Y_t = y).$$

De Gennes [3] called Y (or more exactly its discrete time analogue) 'the ant in the labyrinth', and suggested that its properties would be related to 'dynamic' or 'transport' properties of $\mathcal{C}_\infty$ – such as heat flow, electrical resistance and eigenvalues. Most of what is known about the process Y on $\mathcal{C}_\infty$ is summarised in the following two theorems.

Theorem 1. [1] *Let $p > p_c$. For each $x \in \mathbb{Z}^d$ there exist r.v. $N_x(\omega)$ with $\mathbb{P}_p(N_x \geq n) \leq c\exp(-n^{\varepsilon_d})$ and (non-random) constants $c_i = c_i(d,p)$ such that the transition density of X satisfies, for $x, y \in \mathcal{C}_\infty(\omega)$:*

$$\frac{c_1}{t^{d/2}} e^{-c_2|x-y|^2/t} \leq q^\omega_t(x,y) \leq \frac{c_3}{t^{d/2}} e^{-c_4|x-y|^2/t}, \qquad (GE)$$

provided

$$t \geq N_x(\omega), \ \text{and } t \geq |x-y|.$$

Theorem 2. *(See [7], and papers in preparation by N. Berger and M. Biskup, P. Mathieu and A. Piatnitski). There exists a set $F \subset \Omega$ with $\mathbb{P}_p(F) = 1$ such that if $\omega \in F$, $x \in \mathcal{C}_\infty(\omega)$, then the processes*

$$Y^{(n)}(t) = n^{-1/2}(Y_{nt} - x), \quad t \geq 0,$$

under the law P^x_ω converge weakly to $(D^{1/2}W_t, t \geq 0)$ where W is a standard BM in $\mathbb{R}^d$.

The constants in Theorem 1 and Theorem 2 are not 'effective' except (possibly) for p close to 1, and one wishes to know how in particular $D = D(d,p)$ behaves as $p \downarrow p_c$. It is believed (and proved in some cases) that key percolation quantities have a power law behaviour as $p \downarrow p_c$. So one would also guess that

$$D(p) \approx (p - p_c)^\sigma \quad \text{as } p \downarrow p_c,$$

and would like to know how σ is related to 'known' percolation exponents.

One approach is through critical percolation. It is conjectured that there is no infinite cluster if $p = p_c$, but Kesten [5] introduced the 'incipient infinite cluster' (IIC) by conditioning $\mathcal{C}(0)$ to be infinite. This has been proved to exist in a few cases – for $d = 2$ in [5] and more recently in various high dimensional cases using lace expansion methods – see [4].

Now let $p > p_c$. One may guess that $\mathcal{C}_\infty$ is made out of pieces which look like the IIC of size $R = \xi(p)$ (the *correlation length*), which more or less cover $\mathbb{Z}^d$. Let R_c be the graph (chemical) distance of a path in $\mathcal{C}_\infty$ which covers a Euclidean distance of R: one expects that $R_c = R^\kappa$ where $\kappa \geq 1$. Let $T = (R_c)^{d_w}$ be the time it takes the CTSRW on $\tilde{\mathcal{C}}$ to move a (graph) distance R_c or Euclidean distance R. Then

$$E^0|Y_T|^2 \approx R^2, \quad E^0|Y_{nT}|^2 \approx nR^2,$$

and so

$$D(p) = \lim_n \frac{|Y_{nT}|^2}{nT} = \frac{R^2}{T} = R^{2-d_w\kappa} = \xi(p)^{(2-d_w\kappa)}.$$

So if $\xi(p) \approx (p - p_c)^{-\nu}$ then one deduces

$$\sigma = \nu(\kappa d_w - 2). \tag{1}$$

Note that one expects that $d_w > 2$, $\kappa > 1$, so $\sigma > 0$ and $D(p) \to 0$ as $p \downarrow p_c$. The exponent ν is 'classical', while d_w and κ probably relate to new features of the percolation process.

This (very) non-rigorous argument suggests that results on the CTSRW on the IIC may give information on $D(p)$ for p close to p_c. In 1982 Alexander-Orbach conjectured that for the IIC $\widetilde{\mathcal{C}}$ in $\mathbb{Z}^d$ for any $d \geq 2$ the transition density $q_t(x,y)$ satisfies $q_t(x,x) \approx t^{-2/3}$. Thus if

$$-\frac{1}{2}d_s(\tilde{\mathcal{C}}) = \lim_{t\to\infty} \frac{\log q_t(x,x)}{\log t}$$

then the A-O conjecture is that $d_s(\tilde{\mathcal{C}}) = 4/3$. It is now thought that this is unlikely to hold for small d, but it is probably true for large d, and has been proved for the tree case '$d = \infty$'.

This case is much easier. Let $\mathbb{B}$ be the (rooted) binary tree, and write 0 for the root. Kesten [6] constructed the IIC $\tilde{\mathcal{C}}$, which can be regarded as a branching process with $\texttt{Bin}(2, \frac{1}{2})$ offspring distribution, conditioned on non-extinction. Write $\tilde{\mathbb{P}}$ for the law of $\tilde{\mathcal{C}}$.

One finds that there is a single infinite path (*the backbone* $\mathcal{B}$) from 0 to infinity, and that there are $O(n^2)$ points within a distance n of the root. It is believed that the large scale structure of the IIC on the binary tree and $\mathbb{Z}^d$, for large d, will be very similar.

Let Y be CTSRW on $\tilde{\mathcal{C}}$. If one cuts out the time spent off the backbone then one expects Y to take time R^2 to reach height R. So time spent at a point $x \in \mathcal{B}$ with height less than $R/2$ will be $O(R)$. One expects Y to spend a similar amount of time at nearby points off the backbone. So if

$$T_R = \inf\{t : d(0, Y_t) = R\}$$

then $E^0(T_R) \approx R^2.R = R^3$. Also $\int_0^{R^3} q_t(0,0)dt \approx R$, suggesting that $q_t(0,0) \approx t^{-2/3}$.

Let P^0_ω be the law of Y started at the root 0, and $\mathbb{P}^* = \mathbb{P} \times P^0_\omega$. Define the rescaled height process $Z^{(n)}$ by

$$Z^{(n)}_t = n^{-1/3} d(0, Y_{nt}), \quad t \geq 0.$$

Theorem 3. *(Kesten) (a) For all* $\varepsilon > 0$ *there exist* λ_1, λ_2 *such that*

$$\mathbb{P}^*(\lambda_1 \leq R^{-3} T_R \leq \lambda_2) \geq 1 - \varepsilon, \qquad \textit{for all } R \geq 1.$$

(b) Under $\mathbb{P}^*$ *the processes* $Z^{(n)} \Rightarrow Z$*, where* $Z \neq 0$*.*

For x in the binary tree write $\tilde{\mathbb{P}}_x(\cdot) = \tilde{\mathbb{P}}(\cdot | x \in \tilde{\mathcal{C}})$, and write $q^\omega_t(x,y)$ for the transition density of Y.

Theorem 4. [2] *(a) For* $t > S_x$ *(where* $\tilde{\mathbb{P}}_x(S_x < \infty) = 1$*),*

$$c t^{-2/3} (\log\log t)^{-17} \leq q^\omega_t(x,x) \leq C t^{-2/3} (\log\log t)^3.$$

$$c t^{1/3} (\log\log t)^{-12} \leq E^x_\omega[d(x, Y_t)] \leq C t^{1/3} \log t.$$

(b) For $t \geq 1$*,*

$$c_3 t^{-2/3} \leq \mathbb{E}_x[q^\omega_t(x,x)] \leq c_4 t^{-2/3},$$

$$c_5 t^{1/3} \leq \mathbb{E}_x E^x_\omega d(x, Y_t) \leq c_6 t^{1/3}.$$

The logarithmic fluctuations in the quenched estimates in (a) are really there. For example there exist $t_n(\omega) \to \infty$ such that

$$q^\omega_{t_n}(0,0) \leq 2 t_n^{-2/3} (\log\log t_n)^{-1/6}.$$

Theorem 5. *(a) (Kesten) The rescaled height processes* $Z^{(n)}$ *are tight with respect to the annealed measure* P^**.*
(b) [2] *For* $\tilde{\mathbb{P}}$ *almost all* ω *the processes* $Z^{(n)}$ *are not tight with respect to* P^0_ω*.*

The explanation for (b) is that if $U(n) = n^{-2}|B_\omega(0,n)|$ then $U(n)$ converges weakly, but not a.s., to a non-constant r.v., and that an unusually small (or large) cluster at the length scale R gives rises to unusually small (or large) values of the exit time $E^0_\omega T_R$.

The proofs of these theorems use the connection between random walks and electrical resistance. The graph $\tilde{\mathcal{C}}$ is 'strongly recurrent' and resistance methods work particularly well in this case.

I conclude this survey by summarising the conjectures one can make on σ, the exponent for the diffusivity $D = D(p,d)$. Recall that a hyperscaling argument gives that $\sigma = \nu(\kappa d_w - 2)$. For large d it is known that $\nu = \frac{1}{2}$, and one expects paths in $\tilde{\mathcal{C}}$ to be random walks, so that $\kappa = 2$. The A-O conjecture is that $d_w = 3$, giving $\sigma = 2$.

For site percolation on triangular lattice connection with SLE means many exponents (including ν) are now known. But it is doubtful if κ, let alone d_w, can be obtained using SLE methods.

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The universality classes in the parabolic Anderson model

Peter Mörters

(joint work with Remco van der Hofstad, Wolfgang König)

We consider the unique continuous nonnegative solution $v\colon [0,\infty) \times \mathbb{Z}^d \to [0,\infty)$ to the Cauchy problem with random coefficients and localised initial datum,

$$\begin{aligned} \tfrac{\partial}{\partial t}\, v(t,z) &= \Delta^{\mathrm{d}} v(t,z) + \xi(z)\, v(t,z), && \text{for } (t,z) \in (0,\infty) \times \mathbb{Z}^d, \\ v(0,z) &= \mathbb{1}_0(z), && \text{for } z \in \mathbb{Z}^d. \end{aligned}$$

Here $\xi = (\xi(z) : z \in \mathbb{Z}^d)$ is an i.i.d. random potential with values in $[-\infty, \infty)$, and Δ^{d} is the discrete Laplacian. This parabolic problem is called the *parabolic Anderson model*, it is well-studied in the mathematics and mathematical physics literature because it exhibits an *intermittency effect.* This means, loosely speaking, that most of the total mass

$$U(t) = \sum_{z\in\mathbb{Z}^d} v(t,z), \qquad \text{for } t>0,$$

of the solution is concentrated on a small number of spatially well separated localised islands.

This talk is mainly concerned with the *annealed* behaviour, i.e. with the behaviour of the expectations $\langle v(t,z)\rangle$ with respect to the random potential. If the potential is truly random this behaviour is markedly different from the behaviour of the heat equation in the mean potential. This is due to the fact that exceptionally favourable potentials ξ dominate the average solutions. Important questions in the annealed case are:

- How fast (if at all) does the expected mass spread into space?
- What is the long-term behaviour of the expected total mass $\langle U(t)\rangle$?
- What is the shape of the favourable potentials ξ dominating the average?

These questions have been studied for *special classes* of potentials, for example in [GM98] for the double-exponential distribution, [A95, BK01] for potentials bounded from above, and [S98, GK00] for variants of the model in continuous space. See also [CM94, GK05] for surveys.

In [HKM05] we initiate the study of the precise dependence of the answers to these questions on the distribution of the potential ξ. We assume that the logarithmic moment generating function,

$$H(t) = \log \left\langle e^{t\xi(0)} \right\rangle$$

is finite for all $t>0$, so that all moments $\langle U(t)^p\rangle$ exist at all times. To simplify the presentation, we make the assumption that if ξ is bounded, then $\operatorname{esssup}\xi(0)=0$, so that $\lim_{t\to\infty} H(t)/t \in \{0,\infty\}$. This is no loss of generality, as additive constants in the potential appear as additive constants in the logarithmic asymptotics of $\langle U(t)^p\rangle$. To avoid 'mixed behaviour' we also make the mild technical assumptions that $H(t)/t$ is in the de Haan class, which implies that H is regularly varying with some index $\gamma \geq 0$, and that the auxiliary function has a limit $\kappa^* \in [0,\infty]$.

Under these assumptions we can identify precisely *four* qualitatively different classes of behaviour that can arise in the parabolic Anderson model. In all cases there is a scale function $\alpha(t)$ such that, in a suitable sense, the mass of the solution at time t is confined to the ball of radius of order $\alpha(t)$ about the origin. Moreover the first two terms in an expansion of $\langle U(t)^p\rangle$ are

$$\frac{1}{pt}\log\langle U(t)^p\rangle = \frac{H\big(pt\,\alpha(pt)^{-d}\big)}{pt\,\alpha(pt)^{-d}} - \frac{1}{\alpha(pt)^2}\big(\chi + o(1)\big), \qquad \text{as } t\uparrow\infty,$$

where χ is a real number given in terms of a variational problem encoding (at least on a heuristical level) the optimal shape of the potential fields and associated solutions that make the dominant contribution to the average. The four classes are given in terms of the parameters γ and κ^* as follows:

(1) $\boxed{\gamma > 1, \text{ or } \gamma = 1 \text{ and } \kappa^* = \infty.}$

This case is included in the discussion of [GM90, GM98]. Here $\chi = 2d$, $\alpha(t) = 1$ and the first term of the expansion dominates the sum, which diverges to infinity. The expected mass remains localised in the origin in the sense that

$$\lim_{t\uparrow\infty} \frac{1}{t} \log \frac{\langle v(t,0)\rangle}{\langle \sum_{z\in\mathbb{Z}^d} v(t,z)\rangle} = 0\,.$$

We therefore call this case the *single-peak case*, it comprises, for example, the case of Gaussian potentials.

(2) $\boxed{\gamma = 1 \text{ and } 0 < \kappa^* < \infty.}$

This case, covering distributions in the vicinity of the double-exponential distribution, is the main objective of [GM98]. Here $\alpha(t) \to 1/\sqrt{\kappa^*} \in (0,\infty)$, so that the expected solution does not spread into space, but remains essentially confined to a finite ball. The first term in the expansion dominates the sum, which goes to infinity. Moreover,

$$\chi = \min_{\substack{g\colon \mathbb{Z}^d\to\mathbb{R}\\ \sum g^2(z)=1}} \Big\{ \tfrac{1}{2} \sum_{\substack{x,y\in\mathbb{Z}^d\\ x\sim y}} \big(g(x)-g(y)\big)^2 - \rho \sum_{x\in\mathbb{Z}^d} g^2(x)\log g^2(x) \Big\},$$

where $x \sim y$ means that the points are neighbours, and ρ is an additional parameter. This variational problem is difficult to analyse. It has a solution, which is unique for sufficiently large values of ρ, and heuristically this minimizer represents the shape of the solution.

(3) $\boxed{\gamma = 1 \text{ and } \kappa^* = 0.}$

Potentials in this class are called *almost bounded* in [GM98]. The class contains both bounded and unbounded potentials with very light upper tails, and is discussed in detail for the first time in our paper [HKM05]. We show that $\alpha(t)$ is going to infinity, but is slowly varying. In particular $\alpha(t) \uparrow \infty$ more slowly than any polynomial in t. The first term of the expansion dominates the sum, which may go to infinity or zero. Moreover,

$$\chi = \min_{\substack{g\in H^1(\mathbb{R}^d)\\ \|g\|_2=1}} \Big\{ \|\nabla g\|_2^2 - \rho \int g^2(x)\log g^2(x)\,dx \Big\}\,.$$

This problem is the continuous variant of the problem in (2) and is much easier to solve. By the logarithmic Sobolev inequality $\chi = d\rho\big(1 - \frac{1}{2}\log\frac{\rho}{\pi}\big)$ and there is a unique minimiser,

$$g_*(x) = \big(\frac{\rho}{\pi}\big)^{d/4} \exp\big(-\frac{\rho}{2}|x|^2\big),$$

which heuristically represents the shape of the solution in the scale $\alpha(t)$.

(4) $\boxed{\gamma < 1.}$

This is the case of *completely bounded potentials*, which is treated in [BK01]. Here $\alpha(t) \uparrow \infty$ and $t \mapsto \alpha(t)$ is regularly varying of index $\frac{1-\gamma}{d+2-d\gamma} < \frac{1}{2}$. Moreover,

$$\chi = \inf_{\substack{g \in H^1(\mathbb{R}^d) \\ \|g\|_2 = 1}} \Big\{ \|\nabla g\|_2^2 - \rho \int_{\mathbb{R}^d} \frac{g^{2\gamma}(x) - g^2(x)}{\gamma - 1}\, dx \Big\}.$$

The two terms of the expansion are of the *same order*, and $\langle U(t)^p \rangle$ converges to zero.

Besides this classification, our main contribution is the detailed investigation of the almost bounded case (3). We use the Feynman-Kac formula to express $v(t,z)$ in terms of exponential functionals of the local times of a continuous-time random walk. Three major technical ingredients enter in our proofs:

- a compactification argument of Biskup and König [BK01] based on estimating Dirichlet eigenvalues in large boxes against maximal Dirichlet eigenvalues in small subboxes;
- a powerful inequality of Brydges, van der Hofstad and König [BHK05] which refines the classical Donsker-Varadhan large deviation upper bound and allows to study highly discontinuous functionals of the local time field of continuous time random walks;
- a local large deviation upper bound for q-norms of the normalised local time field of continuous time random walks based on the combinatorial approach of [KM02].

Full details of the proofs and additional results on the *quenched* problem, i.e. the behaviour of the *random* solutions $v(t,z)$, can be found in our preprint [HKM05].

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Large deviations for Brownian motion in a random scenery

Fabienne Castell
(joint work with Amine Asselah)

The Brownian motion in a random scenery is the process defined for all $t \geq 0$, by $Y_t = \int_0^t v(B_s)\,ds$, where $(B_t; t \geq 0)$ is a d-dimensional Brownian motion, and $(v(x), x \in \mathbb{R}^d)$ is a random stationary random field, called the scenery, which is independent from B. In a discrete time setting, this process is called a random walk in a random scenery, and is defined as $Y_n = \sum_{k=1}^n v(S_k)$, where S_k is now a random walk on $\mathbb{Z}^d$, and the scenery is a random field indexed by $\mathbb{Z}^d$. It has been introduced by Borodin [4], Kesten & Spitzer [7], and Matheron & de Marsily [8]. When we average both over the law of the Brownian motion and that of the scenery -the so-called annealed measure- the fluctuations have been studied in [4, 7] in dimension $d = 1$ or $d \geq 3$. The fluctuations in $d = 2$ have been treated by Bolthausen [3].

We are interested in large deviations asymptotics for Y_t (or Y_n in large time). We focus on a discrete white noise scenery. We assume that v is constant over unit cubes $(Q_j, j \in \mathbb{Z}^d)$ and that on Q_j, it is equal to ξ_j, where the variables $(\xi_j, j \in \mathbb{Z}^d)$ are i.i.d. centered, and satisfy $P[|\xi_j| \geq t] \asymp C' \exp(-Ct^\alpha)$ for large t. According to the values of α and d, two main regimes for the annealed large deviations are distinguished. When $\alpha > \max(1, d/2)$, a full large deviations principles for Y_t holds, with speed $t^{\frac{d}{d+2}}$, and a rate functional which is not convex in general. This large deviations principle has first been derived in the continuous time setting for $\alpha = +\infty$ (bounded scenery) by Asselah & Castell in [1]. The discrete time setting and the extension to $\alpha > \max(1, d/2)$ has then be obtained by König, Gantert & Shi in [5]. In this regime, the best strategy is for the Brownian motion to be confined on a ball of radius $r = t^{\frac{1}{d+2}}$, and for the scenery to perform a large deviation on the $t^{\frac{d}{d+2}}$ sites of this ball.

For $\alpha < 1$, a full large deviations principle in discrete time has been proved by van der Hofstad, Gantert & König in [6]. Denoting by $l_n(x)$ the total amount of time spent by the random walk on site x up to time n, and rewriting Y_n as $\sum_{x\in\mathbb{Z}^d} v(x) l_n(x)$, Y_n can be seen in this case as a weighted linear combination of variables with streched exponential tails. The large deviations behavior is then dominated by one term of the sum, and the optimal strategy is for the random walk to stay an infinitesimal amount of time (of order $n^{\frac{\alpha}{\alpha+1}}$ for $d \geq 3$) on **one** site, where the scenery is large (of order $n^{\frac{1}{\alpha+1}}$ in $d \geq 3$).

Turning to the case where $1 \leq \alpha < d/2$, we prove lower and upper bounds of the correct order $n^{\frac{\alpha}{\alpha+1}}$ in [2]. The optimal strategy is now for the random walk to spend a time of order $n^{\frac{\alpha}{\alpha+1}}$ on a **finite** number of sites where the scenery is of order $n^{\frac{1}{\alpha+1}}$.

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Invariance Principle and Isotropic Diffusions in Random Environment

Alain-Sol Sznitman
(joint work with Ofer Zeitouni)

In this talk we discuss some recent results obtained jointly with Ofer Zeitouni. We consider uniformly elliptic diffusions in a random environment on $\mathbb{R}^d$. The local characteristics are assumed to be bounded stationary functions, which are Lipschitz continuous and fulfill a finite range dependence condition. Under a restricted isotropy assumption, we show that when the diffusions are small perturbation of Brownian motion and the space dimension is three or more, for almost every environment:

- the diffusion satisfies an invariance principle with limit law corresponding to Brownian motion with a deterministic non-degenerate variance $\sigma^2 > 0$;
- the diffusion is transient;
- we provide applications to problems of homogenization in random media.

One difficulty of the model we analyze, stems from its genuine non self-adjoint character. It can be seen as a continuous counterpart of the discrete model of random walks in random environment studied by Bricmont-Kupiainen in [1]. Our strategy of proof is however different.

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Directed Polymers In Random Environments

Yueyun Hu
(joint work with Philippe Carmona)

Directed polymers can be presented as a random walk in a time-dependent random environment or equivalently a directed random walk in random environment: Let $((S_n), \mathbb{P})$ be a simple symmetric random walk in $\mathbb{Z}^d$ and $(g(i,x), i \geq 1, x \in \mathbb{Z}^d, \mathbf{P})$ be a family of i.i.d. variables. We assume for instance that $g(i,x)$ is a centered standard gaussian variable or more generally a centered r.v. having exponential moments.

The process (S_n) being independent of $(g(i,x))$, we define the polymers measure $\langle\cdot\rangle^{(n)}$ as follows: Fix $\beta > 0$. For any test function F,

$$\langle F(S)\rangle^{(n)} := \frac{1}{Z_n(\beta)}\, \mathbb{E}\Big(F(S)e^{\beta \sum_{i=1}^n g(i,S_i)}\Big), \quad \forall F(\cdot) \geq 0,$$

where $\mathbb{E}$ means the expectation with respect to S and $Z_n(\beta)$ is the normalization factor (partition function). See Comets-Shiga-Yoshida [7] for physical background and links to Kadard-Parisi-Zhang's equation.

The problem is to understand the behaviors of $(S_k)_{0\leq k\leq n}$ under $\langle\cdot\rangle^{(n)}$ as $n \to \infty$. Of particular interests are the following exponents:

- Volume exponent ξ: $\langle |S_n|\rangle^{(n)} \approx n^{\xi}$ as $n \to \infty$.
- Fluctuation exponent χ: $\text{var}(\log Z_n(\beta)) \approx n^{2\chi}$.

It has been conjectured that $\xi = 2/3$ when $d = 1$ and $\beta > 0$ is small and there is some Scaling identity:

$$\xi = \frac{1+\chi}{2}.$$

According to martingale convergence theorem, $\tilde{Z}_n(\beta) = \frac{Z_n(\beta)}{\mathbf{E}Z_n(\beta)} \to \tilde{Z}_\infty(\beta) \geq 0$, *a.s.* We shall say "Strong Disorder" if $Z_\infty(\beta) = 0$ a.s. and "Weak Disorder" if $Z_\infty(\beta) > 0$ a.s.

In this talk, we present some results and several questions related to the phenomenon of Strong or Weak Disorder and the localization property of polymers. Roughly saying, more recurrent is the underlying process (S_n), more stronger is the disorder:

- When $d \geq 3$ and β is small, we have "Weak Disorder" (see [11, 2, 16, 1]). Moreover, Comets and Yoshida [10] showed that "Weak Disorder" entails the diffusive behaviors of polymers.
- When $d = 1$ or $d = 2$, we have "Strong Disorder" for any $\beta > 0$ (see [3, 6]), moreover, in this case, the polymers shares a phenomenon of localization ([3]).
- When (S_n) is an exponentially recurrent Markov chain, $\tilde{Z}_n(\beta)$ decays exponentially to 0 for any $\beta > 0$ ([5]).

We also present a large deviation result concerning S_n under $\langle\cdot\rangle^{(n)}$ and a relationship between the rate function and the volume exponent ([4]).

For some related models, we mention for example Johansson [12] (model corresponding to $\beta = \infty$, rigourously proved that $\xi = 2/3$), Petermann [14], Méjane [13] (proved respectively that $\xi \geq 3/5$ and $\xi \leq 3/4$ where (S_n) is a Brownian motion and $g(\cdot,\cdot)$ is a gaussian fields) and Comets and Yoshida [8, 9] (S is a Brownian motion, $g(\cdot,\cdot)$ is a Poisson random measure and Piza [15], Wüthrich [17].

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A random environment for edge-reinforced random walk on $\mathbb{Z}^2$

Franz Merkl
(joint work with Silke Rolles)

Linearly edge-reinforced random walk on a graph $G = (V, E)$ is the following model: A non-Markovian random walker moves randomly on the vertices of the graph in discrete time. It starts in a fixed vertex v_0, and at every time step, it jumps to a neighboring vertex. The "memory" of the non-Markovian process is encoded as follows: The undirected edges $e \in E$ are assigned positive weights $w_e(t)$, which change in time. Initially, all weights equal a constant $a > 0$. Every time an edge is traversed, its weight increases by 1, and the weights of all other edges remain unchanged. Given the path $X_0, \ldots, X_t$ of the random walker up to time t, the transition probability to a neighboring vertex is proportional to the weight $w_e(t)$ of the traversed edge at that time.

This model was introduced by Diaconis in the 1980s [CD86]; see also [Pem88]. Diaconis asked whether reinforced random walk on $\mathbb{Z}^d$ is recurrent. For all $d > 1$, this question is still open.

Pemantle [Pem88] analyzed the model on infinite binary trees, finding a phase transition in the recurrence-transience behavior. His method does not apply to graphs with loops.

Linearly edge-reinforced random walks belong to the larger class of partially exchangable random walks. *Recurrent* processes in this class can be written as a mixture of Markov chains, as was proven by Diaconis and Freedman in [DF80]. This applies to linearly edge-reinforced random walks on *finite* graphs: These processes have the same law as certain reversible random walks in a complicated *time-independent* random environment. In an unpublished manuscript, Coppersmith and Diaconis [CD86] explicitly described the law of this random environment. An extended version of this result was proven by Keane and Rolles in [KR00].

The main theorem in the present talk claims that for all initial weights $a > 0$, reinforced random walk on $\mathbb{Z}^2$ starting in $0 \in \mathbb{Z}^2$ is also equivalent to a random walk in a *time-independent* random environment. The random environment is given by positive random weights x_e, $e \in E$, on the edges, such that the transition probability to jump to a neighboring edge is proportional to x_e. Furthermore, writing $x_v = \sum_{e \ni v, e \in E} x_e$ for $v \in V$, the random variables $\log(x_v/x_0)$ have finite expectation, and one has

$$c(a) := \limsup_{|v| \to \infty} \frac{1}{\log |v|} E\left[\log \frac{x_v}{x_0}\right] < 0,$$

and $c(a) \to -\infty$ as $a \to 0$.

The proof uses an approximation of the infinite lattice by finite boxes with periodic boundary conditions. An essential ingredient is the above-mentioned explicit representation of the random environment for reinforced random walk on these finite boxes. The proof makes use of entropy bounds and deformation arguments motivated by statistical mechanics. Such estimates have already been extremely useful to analyze linearly edge-reinforced random walk and the corresponding random environment for ladders of arbitrary width and, more generally, on graphs of the type $\mathbb{Z} \times T$, where T is a finite tree; see [MR05a], [Rol04], and [MR04].

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Triangle problem revisited

Vlada Limic

Reinforcement is observed frequently in nature and society, where beneficial interactions tend to be repeated. Edge reinforced random walker on a graph remembers the number of times each edge was traversed in the past, and decides to make the next random step with probabilities favouring places visited before.

More precisely, given the underlying graph $\mathcal{G}$ with vertex set V and edge set E, and given a *reinforcement weight* function $W : \mathbb{N}_0 \to (0, \infty)$, the edge W-reinforced random walk on $\mathcal{G}$ is the process $(I_n, (X_n^e, e \in E))_{n\geq 0}$, where $I_n \in V$ is the position of the walk at time n and

$$X_n^e = X_0^e + \#\{\text{undirected traversals of } e \text{ up to time } n\},$$

such that if u is a neighbor of v then on $\{I_n = v\}$

$$P(I_{n+1} = u, X_{n+1}^e = X_n + \delta_{\{v,u\}}(e)|\mathcal{F}_n) = \frac{W(X_n^{\{v,u\}}}{\sum_{w\sim v} W(X_n^{\{v,w\}})}.$$

Here $\mathcal{F}_n, n \geq 0$ is the natural filtration, $\delta_{\{v,u\}}(\cdot)$ is the Kronecker symbol, and $w \sim v$ means that w is a neighbor of v and $\{v, w\} \in E$ is then the edge connecting u and v. Assume that the initial configuration X_0^e is bounded, for concreteness let $X_0^{\cdot} \equiv 0$.

It is straightforward to check that the condition

$$\sum_k \frac{1}{W(k)} < \infty \tag{1}$$

is equivalent to

$$P(\text{the walker traverses only one edge forever}) > 0.$$

Call an edge e *attracting* for the walk if $e = \{I_n, I_{n+1}\}$ for all large n. It is not difficult to see that on any graph of bounded degree (1) is equivalent to $P(\text{walker stays in finite region}) = 1$ and to $P(\text{attracting edge exists}) > 0$.

Since $\{\text{attracting edge exists}\} = \{\{I_n, I_{n+1}\} = \{I_n, I_{n-1}\}$ for all large $n\}$ is a tail event for the natural filtration, it is interesting to ask whether it is a trivial event, or equivalently, whether (1) is equivalent to

$$P(\text{attracting edge exists}) = 1. \tag{2}$$

This question in the case when the underlying graph is a *triangle* is still open. However, a number of things are already known.

In this paragraph assume that the underlying graph $\mathcal{G}$ has bounded degree. A remarkably short argument of Sellke [S] shows that (1) is equivalent to (2) on $\mathbb{Z}^d$, and in fact this argument easily generalizes to a proof of the same statement on any bipartite graph $\mathcal{G}$. Using martingale techniques and comparison with the generalized Urn scheme, the main theorem in [L] says that if $W(k) = k^\rho$, $\rho > 1$, then (2) holds. Recently in [LT] it is shown that under (1) and additional technical assumptions (2) holds. A nice corollary of the main theorem in [LT] states that if W is non-decreasing then (1) implies (2).

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A random motion of Winterbottom-like shape

Tadahisa Funaki

We are interested in deriving a motion of crystals from certain microscopic models. The derivation of a crystal shape, called Wulff shape, from the Ising model goes back to Dobrushin, Kotecký and Shlosman [1] under the equilibrium situation. The natural choice of the corresponding dynamics would be the Kawasaki dynamics (a system of random walks on a lattice with exclusion rule). However, at present, mathematically rigorous results for the derivation of a crystalline motion from such dynamics are not known (cf. [2]) except the zero temperature limit for a system of interacting Brownian particles, which is a continuous analogue of Kawasaki dynamics, see [3]. Our goal is to discuss this kind of problem in the context of the so-called $\nabla\varphi$-interface model. The following topics were discussed in my lecture after introducing the $\nabla\varphi$-interface model with self potentials and the scaling from microscopic to macroscopic levels in this model.

(1) (Static theory) For the corresponding Gibbs measures, large deviation principles are obtained under several situations and the law of large numbers is deduced from them. The limits are characterized by variational principles, which determine Wulff shape, Winterbottom shape, free boundary problem of Alt-Caffarelli's type and others depending on the situations we are concerned.

(2) (Dynamic theory) The corresponding dynamics are defined through the Langevin equation of non-conservative or conservative types. The hydrodynamic limit, the equilibrium fluctuation and the dynamic entropic repulsion under the situation with hard walls are discussed.

(3) (Motion of Winterbottom-like shape) If the macroscopic equilibrium shapes (i.e. large deviation minimizers) are not unique, one can expect to observe a random motion of the equilibrium shapes under the properly chosen time scale beyond the hydrodynamic one. This is still unsolved in a desirable way, for instance, for the $\nabla\varphi$-interface dynamics of conservative type on a lattice torus. Instead, we consider related stochastic partial differential equations with smooth noise under a proper time scale, from which a diffusive motion of the equilibrium shapes is derived.

Recent results concerning the $\nabla\varphi$-interface model are reviewed in the lecture notes [4].

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The multifractal spectrum of Brownian intersection local times

Achim Klenke
(joint work with Peter Mörters)

Intersection local time. Let $W^1 = (W_t^1)_{t\in[0,1]}$ and $W^2 = (W_t^2)_{t\in[0,1]}$ be independent Brownian motions in $\mathbb{R}^d$, $d \geq 2$, both started in the origin. We are interested in the geometric properties of the intersection set of the paths

$$S := W^1_{[0,1]} \cap W^2_{[0,1]} = \{x \in \mathbb{R}^d : x = W_s^1 = W_t^2 \quad \text{for some } s,t \in [0,1]\}.$$

It is well known that $S = \{0\}$ a.s. if $d \geq 4$ and $\dim S = 4-d$ if $d=2$ or $d=3$. Hence the interesting dimensions are $d = 2,3$.

In order to get a better understanding of the set S we introduce a sort of uniform measure on S, the Hausdorff measure ℓ, which is called the intersection local time of W^1 and W^2. Le Gall ([LG87, LG89]) gave a construction of ℓ as the Hausdorff measure on S as well as via an approximation scheme using Wiener sausages. The typical point $x \in S$ has local dimension $4-d$, that is almost surely

$$\lim_{r\downarrow 0} \frac{\log \ell(B(x,r))}{\log r} = 4-d \quad \text{for a.a. } x \in S.$$

We call a point a thick point if the local dimension is smaller (sic!) than $4-d$ in the sense that

$$\liminf_{r\downarrow 0} \frac{\log \ell(B(x,r))}{\log r} < 4-d.$$

On the other hand a point is a thin point if

$$\limsup_{r\downarrow 0} \frac{\log \ell(B(x,r))}{\log r} > 4-d.$$

Dembo et al [DPRZ02] showed for $d=2$ and König and Mörters [KM02] for $d=3$ that there are no thick points in S. We establish here that there are, however, thin points and we measure the number of thin points by the multifractal spectrum.

Let

$$\mathcal{T}^s(a) = \Big\{x \in S \,:\, \limsup_{r\downarrow 0} \frac{\log \ell(B(x,r))}{\log r} = a\Big\}$$

be the sets of *a-thin points* for $a \geq 4-d$.

Intersection exponents. In order to formulate our main result we need to introduce the so-called intersection exponents. Consider $M+N$ Brownian motions $W^1,\ldots,W^{M+N}$ in $\mathbb{R}^d$. Start these Brownian motions (independently) uniformly distributed on the unit sphere $\partial B(0,1)$ and stop them upon leaving $B(0,R)$ for some large R that we let go to infinity. We define $\tau_R^i = \inf\{t \geq 0 : W_t^i \notin B(0,R)\}$ and

$$\mathfrak{B}^1(R) = \bigcup_{i=1}^{M} W^i([0,\tau_R^i]) \quad \text{and} \quad \mathfrak{B}^2(R) = \bigcup_{i=M+1}^{M+N} W^i([0,\tau_R^i]).$$

It is easy to show that there exists a constant $\xi_d(M,N)$ such that

$$\mathbf{P}\big[\mathfrak{B}^1(R)\cap\mathfrak{B}^2(R)=\emptyset\big]=R^{-\xi_d(M,N)+o(1)},\ \text{as } R\uparrow\infty.$$

The numbers $\xi_d(M,N)$ are called the *intersection exponents*.

Lawler, Schramm and Werner ([LSW01a, LSW01b, LSW02] proved the physicists' conjecture that in the plane some of these values are rational. More precisely they showed (using conformal invariance which does not apply in $d=3$)

$$\xi_2(M,N)=\frac{\big(\sqrt{24M+1}+\sqrt{24N+1}-2\big)^2-4}{48}.$$

This gives $\xi_2(2,2)=35/12$.

Results. **Theorem 1** *For $d=2,3$ almost surely*

$$\sup\{a:\mathcal{T}^s(a)\neq\emptyset\}=\frac{(4-d)\,\xi_d(2,2)}{\xi_d(2,2)-(4-d)}.$$

Furthermore, for $4-d\leq a\leq\frac{(4-d)\,\xi_d(2,2)}{\xi_d(2,2)-(4-d)}$ the Hausdorff dimension of $\mathcal{T}^s(a)$ is $(4-d)\frac{\xi_d(2,2)}{a}+(4-d)-\xi_d(2,2)$.

Note that the multifractal spectrum, i.e. the map $a\mapsto\dim\mathcal{T}^s(a)$ is convex for the values of a considered in the theorem, opposed to the physicists' formalism that predicts concave multifractal spectra.

One key to Theorem 1 is the following estimate on the the lower tails of the projected intersection local time $\ell_{M,N}$ of two packages of M and N Brownian motions. We define for these packages of Brownian motions

$$\ell_{M,N}:=\sum_{i=1}^{M}\sum_{j=M+1}^{M+N}\ell^{i,j},$$

where $\ell^{i,j}$ is the intersection local time of $W^i_{[0,1]}$ with $W^j_{[0,1]}$.

Theorem 2 $$\lim_{\delta\downarrow 0}\frac{\log\mathbf{P}\big[\ell_{M,N}(B(0,1))<\delta\big]}{-\log\delta}=-\frac{\xi_d(M,N)}{4-d}.$$

Note that the algebraic lower tail of $\ell_{M,N}(B(0,1))$ contrasts significantly with the behaviour of the upper tail $\mathrm{P}[\ell_{M,N}(B(0,1))>\delta]\approx\exp(-\theta\delta^{-1/2})$ as $\delta\to\infty$ for some $\theta>0$, as shown by König and Mörters [KM02]. Here lies the reason that there do exist *thin* points but there do not exist *thick* points.

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Growth of Lévy forests

Matthias Winkel
(joint work with Thomas Duquesne)

It is well-known that the only space-time scaling limits of Galton-Watson processes are continuous-state branching processes. Their genealogical structure is most explicitly expressed by discrete trees and $\mathbb{R}$-trees, respectively. Weak limit theorems have been recently established for some of these random trees. We study here a Markovian forest growth procedure that allows to construct the genealogical forest of any continuous-state branching process with immigration as an a.s. limit of Galton-Watson forests with edge lengths. Our method yields results in the general supercritical case that was excluded in most of the previous literature. Applications include superprocesses, ancestral decompositions of Poissonian samples, and snake-like constructions of continuum forests with edge lengths.

1. Introduction

Classical Galton-Watson trees are genealogical trees of ancestors and their (distinguishable) progeny where each individual produces offspring independently with common distribution ξ on $\{0, 1, 2, \ldots\}$. Combinatorially, one represents individuals by vertices and the parent-child relation by edges. Rooted plane trees allow to capture the order of siblings. The offspring mechanism induces a probability distribution on the set of rooted plane trees. Graphically, we can represent these e.g. in $[0, \infty)^2$, attaching ancestors to the horizontal axis in a given order.

Such objects have been studied for a long time, cf. Harris [10] for early references and Neveu [13] for a more formal treatment. As natural extensions, one can assign independent *edge lengths* with distribution ν, and study Galton-Watson *forests* as sequences of Galton-Watson trees. We can either consider these to grow from a single root or introduce a spatial structure by arranging them suitably on the half-line, say. See also Pitman [15].

We are interested in *growing* such forests in a Markovian way that preserves the class of forests described, and that allows to pass to continuum limits. Several possibilities for such schemes are more or less explicit in the literature. They are often described by their co-transition rules.

Firstly, Neveu [14] and Salminen [17] erase branches in general (non-explosive) Galton-Watson trees with exponential edge lengths continuously from their tips. Aldous [2], Le Jan [12], Abraham [1] and Pitman [15] reverse the procedure to grow stable/Brownian trees and forests from appropriate Galton-Watson trees/forests. Secondly, Aldous and Pitman [3] perform p-percolation (retain each *edge* with probability $1-p$) on the Galton-Watson trees (without edge lengths) and retain the connected component of the root, as a tree-valued Markov process as p varies. They call the procedure pruning of a Galton-Watson tree. The viewpoint is to gradually reduce the tree by consistently increasing p.

Geiger and Kaufmann [9] discount the offspring distribution to reduce a given Galton-Watson tree in a size-biased way. This is a special case of multiplicity-dependent pruning at vertices. In [6] we show that this is also related to a *third scheme* of percolation on the *leaves*. Thin (red) trees in Figure 1 have the discounted offspring distribution. [9] uses size-biasing and conditioning arguments to identify the combinatorial structure of normalised stable trees.

Pitman and Winkel [16] perform percolation on the leaves of *binary* Galton-Watson forests, in the sense that the subforest spanned by leaves retained after percolation is considered, cf. Figure 1. Again, as p varies, this gives a simple forest-valued Markov process. The emphasis is here on the *increasing* process $(\mathcal{F}_p)_{0\le p<1}$, which has independent "increments", expressed by a composition rule. It can be consistently extended to increase to the Brownian forest $\mathcal{F}_1$. This passage to the limit is understood by convergence of coding height processes via a Donsker type theorem.

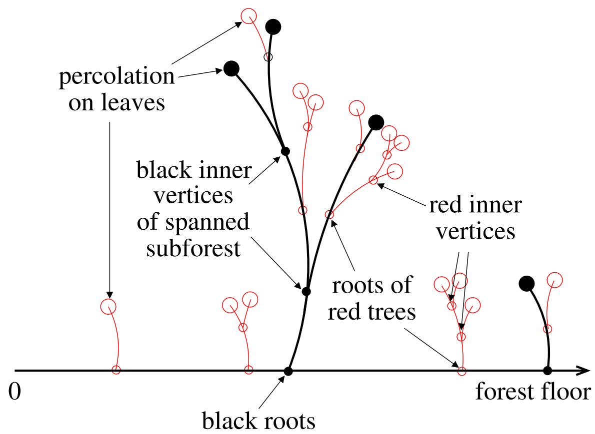

Figure 1. Percolation on the leaves

In [6] we develop this third scheme systematically for general Galton-Watson forests with exponential edge lengths, although, no doubt, similar studies can be carried out for any of the other two schemes. The limit objects are forests of Lévy trees. Unlike most previous studies on Lévy trees (see Duquesne and Le

Gall [4, 5, 11]), we do not work with height process coding, partly because these height processes are not nice objects to work with (in general non-Markovian, not semimartingales etc.), but also because they are not immediately applicable to some important classes such as supercritical trees that we do not wish to exclude.

An appropriate technical set-up for forest-valued processes with edge-lengths are random marked graphs. To pass to continuum limits, we use metric-space representations and Gromov-Hausdorff convergence as developed for a probabilistic context by Evans et al. [7] for the compact case. We extend this in [6] to (isometry classes of) locally compact real trees.

2. Some of the main results of [6]

We say that $\mathcal{F}_0$ is p-extensible if there is a forest $\mathcal{F}_p$ in which $\mathcal{F}_0$ is contained as p-percolation subforest. W.l.o.g. we then set $\mathcal{F}_{p'}$ as $1-(1-p')/(1-p)$-percolation of $\mathcal{F}_p$ for all $p' < p$.

Theorem 1. *ξ is p-extensible for all $p \in (0,1)$ if and only if $\sum_{n\geq 0} \xi_n r^n = r + \psi(1-r)$ for the branching mechanism ψ of a continuous-state branching process.*

Proposition 1. *$Z_t^p = \#\{v \in \mathcal{F}_p : d(0,v) = t\}$ is a continuous-time Galton-Watson process. As $p \uparrow 1$, $Z_t^p/c_p \to Z_t$, a continuous-state branching process with mechanism ψ, for some $c_p \to \infty$,*

Theorem 2. *If Z is non-explosive (in finite time) and has local extinction, then $\overline{\mathcal{F}}_p \to \overline{\mathcal{F}}_1$ as $p \uparrow 1$, a.s. in the Gromov-Hausdorff sense. $\overline{\mathcal{F}}_1$ is a forest of continuum random trees.*

If Z^p explodes in finite time, the trees are not locally compact. If Z^p does not get locally extinct, then the limit trees are not separable. We only need local extinction, not a.s. global extinction and explicitly include supercritical processes.

Theorem 3. *For ℓ_1-representatives $\mathcal{F}_p$, empirical measure on leaves $\mathrm{m}_p/\psi(c_p) \to \mathrm{m}$ "Lebesgue measure on $\mathcal{F}_1$", vaguely. Consistent Poisson($\psi(c_p)\mathrm{m}$) sampled $(\overline{\mathcal{G}}_p)_{0\leq p\leq 1} \sim (\overline{\mathcal{F}}_p)_{0\leq p\leq 1}$.*

ℓ_1-representations were first considered by Aldous [2]. Evans and Winter [8] consider Markov processes on a new Gromov-Hausdorff space of trees equipped with a probability measure.

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Ordered Random Walks

Wolfgang König
(joint work with P. Eichelsbacher)

We consider k i.i.d. random walks on the real line. Our goal is to construct the conditional version given that all walkers stay in a strict order at all times. This is the discrete analogue of Dyson's Brownian motions [Dy62d] (also called non-colliding Brownian motions), which is a conditional k-dimensional Brownian motion $B = (B(t))_{t\in[0,\infty)}$ given that it never leaves the *Weyl chamber*,

$$W = \{x \in \mathbb{R}^k : x_1 < x_2 < x_3 < \cdots < x_k\}. \tag{1}$$

In other words, B is conditioned on the event $\{T = \infty\}$, where

$$T = \inf\{t \in [0,\infty) : B(t) \notin W\} \tag{2}$$

is the first time of a collision of the particles. The construction of the conditional process is via a Doob h transform with h equal to the *Vandermonde determinant* $\Delta: \mathbb{R}^k \to \mathbb{R}$ given by

$$\Delta(x) = \prod_{1\leq i<j\leq k} (x_j - x_i) = \det\big[(x_j^{i-1})_{i,j=1,\dots,k}\big], \qquad x = (x_1,\dots,x_k) \in \mathbb{R}^k. \tag{3}$$

More precisely, $h = \Delta : W \to (0,\infty)$ is a positive harmonic function for the generator $\frac{1}{2}\sum_{i=1}^n \partial_i^2$ of $B = (B_1,\dots,B_k)$ on W. Hence we may consider the Doob h-transform of B on W with $h = \Delta$. Non-colliding diffusions have been considered for some few explicit cases, see, e.g., [Br91], [KO01], [HW96], [Gr99], and [KT04], to mention a few.

Non-colliding versions of nearest-neighbour random walks on the integers have been constructed for a few explicit examples, see [KOR02] and others. Like in the continuous case, the construction is in terms of a Doob h-transform using a positive regular function h, which again turns out to be the Vandermonde determinant in these particular cases. For simple random walks, the non-colliding version is sometimes called *vicious walkers* or *non-intersecting paths*. They play an important role in the study of the corner-growth model [Jo00a], the Arctic circle [Jo02] and series of queues in tandem [OC03].

It is our goal to construct, for general random walks, the non-colliding version via a Doob h-transform. Indeed, we identify a suitable positive regular function h, which is strongly related to the Vandermonde determinant. It reduces to that function in the above mentioned special cases and approaches it asymptotically in an appropriate sense in general.

For $k \in \mathbb{N}$, let $X_1, \dots, X_k$ be k independent copies of a random walk, $X_i = (X_i(n))_{n\in\mathbb{N}_0}$, on $\mathbb{R}$. Then $X = (X_1, \dots, X_k)$ is a random walk on $\mathbb{R}^k$ with i.i.d. components. We denote by $\mathbb{P}_x$ and $\mathbb{E}_x$ the probability and expectation when the process X starts in $x \in \mathbb{R}^k$. Now we introduce the main object, a function $V\colon W \to \mathbb{R}$ defined by

$$V(x) = \Delta(x) - \mathbb{E}_x\big[\Delta(X(\tau))\big], \qquad x \in W, \tag{4}$$

where

$$\tau = \inf\{n \in \mathbb{N}_0 \colon X(n) \notin W\} \tag{5}$$

is the first time that some component reaches or overtakes another one. We in particular have to assume that the $(k-1)$-st moment of the steps be finite, which ensures that $\Delta(X(n))$ is integrable under $\mathbb{P}_x$ for any $x \in \mathbb{R}^k$ and any $n \in \mathbb{N}$. However, a much more difficult question is whether or not $\Delta(X(\tau))$ is integrable, i.e., the question if V is well-defined. This question is trivially answered in the affirmative in the special case of nearest-neighbor walks on $\mathbb{Z}$ that start in an even distance to each other. In this case, $X(\tau)$ lies on the boundary of W and therefore $\Delta(X(\tau)) = 0$ with probability one, which shows that V is identical to Δ. However, in the general case, the integrability of $\Delta(X(\tau))$ seems subtle, and we succeeded in proving this only under some moment condition on the steps.

Theorem 1. *Assume that sufficiently large (depending only on k) moments of the steps are finite. Then the following holds.*

(i) *For any $x \in W$, the random variable $\Delta(X(\tau))$ is integrable under $\mathbb{P}_x$.*

(ii) *The function V defined in (4) is a positive regular function for the restriction of the transition kernel to W, i.e., $V > 0$ on W and*

$$\mathbb{E}_x[V(X(1))\mathbb{1}_{\{\tau>1\}}] = V(x), \qquad x \in W. \tag{6}$$

(iii) *The Doob h-transformation of X on W with $h = V$ is equal to the distributional limit of the conditional process given that $\tau > n$ as $n \to \infty$.*

(iv) *For any $x \in W$,*

$$\lim_{n\to\infty} n^{\frac{k}{4}(k-1)}\mathbb{P}_x(\tau > n) = KV(x), \qquad \text{where } K = \prod_{i=0}^{k-1}\frac{1}{l!}\int_W \frac{\mathrm{e}^{-\frac{1}{2}|y|^2}}{(2\pi)^{k/2}}\Delta(y)\,dy. \tag{7}$$

(v) *Uniformly in $x \in W$ on compacts, $\lim_{n\to\infty} n^{\frac{k}{4}(k-1)}V(\sqrt{n}\,x) = \Delta(x)$.*

(vi) *As $n \to \infty$, the distribution of the process $(n^{-\frac{1}{2}}X(\lfloor nt \rfloor)_{t\in[0,\infty)}$ under the transformed probabilities, i.e., under $\widehat{\mathbb{P}}^{(V)}_{\sqrt{n}x}$ for any $x \in W$, converges towards Dyson's Brownian motion started at x. In particular, the distribution of $n^{-1/2}X(n)$ converges towards the Hermite ensemble.*

The transition probability function of the Doob h-transform with $h = V$ is given by

$$\widehat{\mathbb{P}}^{(V)}_x(X(n) \in dy) = \mathbb{P}_x(\tau > n; X(n) \in dy)\frac{V(y)}{V(x)}, \qquad x, y \in W.$$

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Pure rough paths and properties of the signature

Terry Lyons

In many systems that evolve one can separate out as distinct components *the response*, and some *control* – the former reflecting the behaviour of the sytem responding to the fluctuating inputb or "control" for the evolution. Examples abound in pure and applied mathematics.

They include Cartan development and more generally development of a path from a manifold to a principal bundle using a connection, and in probability, stochastic differential equations. They can all be articulated as differential equations of the following form:

$$dy_t = \sum_i f^i\,(y_t)\,dx^i_t, \;\; \pi_{s,t}\,(y_s) := y_t$$

The map from $x.$-to $y.$ (or π) is uniformly continuous on bounded sets in a range of rough path metrics and so is defined on the completion of the bounded variation continuous $x..$ (The geometric rough paths). This allows one to substantially widen the class of $x.$ which can be used as controls in equations like the one above. The approach depends heavily on describing paths locally in terms of nilpotent elements instead of chords (elements of the abelian group $\mathbb{R}^n$)

The basic theorems extend results of L. C. Young (Acta 1936) and K. T. Chen (Annals 1957).

In this talk one examine the signature of a path. That is, if $x.$ is a path in $\mathbb{R}^d$ then

$$S(x) := 1 + \int_{0<u<T} dx_u + \int\int_{0<u<v<T} dx_u \otimes dx_v + \cdots$$

is an element in the algebra of formal tensor series. Chen showed that for regular paths, the signature $S(x)$ determines the path up to time re-parameterisation. Hambly and Lyons have proved that any bounded variation parth is characterised up to tree like modifications by its signature.

Many open questions remain. The obvious one is reconstruction. Others:

(1) Is the uniqueness theorem true for rough paths with $p > 1$
(2) Can one identify the range of the signature? For example, is it the case that

$$\lim \frac{1}{n} \log \left((n)! \left\| \int\int_{0<u<v<T} dx_u \otimes dx_v \right\| \right) = O(1)$$

implies that the path has finite length.
(3) Sidorova and Lyons have proved that the radius of convergence of the log signature is almost always finite, and conjecture that this will always be true if path is not a straight line segment.

The signature is a faithful representation of the group (modulo tree like equivalence) of bounded variation paths and injects it into the tensor algebra.

One can identify the range as a subset of the group-like elements in the algebra - in particular log is defined and is always an element of the free Lie subalgebra of the tensor algebra generated by $\mathbb{R}^n$. However, the range is only a $\mathbb{Z}$ module. Every path has a Lie log signature; however, the range is not closed under scalr mulitplication and there are elements

$$\begin{aligned} l &= \log(S(x)) \\ \frac{l}{2} &\in \{\log(S(x))\} \end{aligned}$$

implying thast our notion of Lie algebra as tangent space needs refining a bit as one goes to native infinite dimensional groups.- the standard approach where multiplication is continuous already seems unreasonable when the group elements are tree reduced paths of bounded variation.

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