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Nonlinear Evolution Problems

Organised by
Klaus Ecker (Berlin)
Jalal Shatah (New York)
Michael Struwe (Zürich)

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Abstract. In this workshop three types of nonlinear evolution problems— geometric evolution equations (essentially of parabolic type), nonlinear hyperbolic equations, and dispersive equations— were the subject of 22 talks.

Mathematics Subject Classification (2000): 47J35, 74J30, 35L70, 53C44, 35Q55, 35-06.

Introduction by the Organisers

The workshop 'Nonlinear Evolution Problems' focussed on three types of nonlinear evolution equations:

(1) Geometric evolution equations (essentially of parabolic type)
(2) Nonlinear hyperbolic equations
(3) Dispersive equations

The programme consisted of 22 talks presented by international specialists from Australia, France, Germany, Italy, Sweden, Switzerland and the United States.Generally, three lectures were delivered in the morning sessions and two in the late afternoon which left ample time for individual discussions.

In the first group of equations, in particular the Ricci flow and conformal flows such as the Yamabe flow and the Q-curvature flow were considered. A further focus was on curvature flows for hypersurfaces such as the Gauss and harmonic mean curvature flow and on geometric flows of higher order. Julie Clutterbuck presented a direct approach to certain fundamental estimates for a general class of parabolic equations. Mete Soner pointed out important relations between fully nonlinear parabolic equations and backward stochastic differential equations.

The class of hyperbolic equations was represented by wave maps, Einstein's equations of gravitation and nonlinear wave equations in waveguides and cones.

Among the dispersive equations the nonlinear Schroedinger equation and the KdV equation were considered.

The individual discussions covered connections between approaches to the different equations in each group but also on common techniques used across the three classes of equations. Among such techniques are for instance the careful examination of the algebraic and geometric structure of the nonlinear terms. Another common theme appears to be the phenomenon of blow-up (singularity formation) for solutions and methods to describe the solution near these. Selfsimilar solutions and blow-up rates play an important role here.

The organizers decided to particularly encourage the young researchers among theparticipants to present their work, including a PhD student and several recent post-doctoral fellows.

Workshop: Nonlinear Evolution Problems

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Abstracts

Second Order Backward Stochastic Differential Equations and Fully Nonlinear Parabolic PDEs

H. M. Soner
(joint work with Patrick Cheridito, Nizar Touzi, Nicolas Victoir)

In the probability literature, *backward stochastic differential equations* (BSDE) received considerable attention after their introduction by E. Pardoux and S. Peng [7,8] in 1990. During the past decade, interesting connections to partial differential equations (PDE) were obtained and the theory found wide applications in mathematical finance. The key property of the BSDEs is the *random terminal* data that the solutions are required to satisfy. Due to the usual adaptedness conditions the stochastic processes are required to satisfy, this condition satisfied in the future introduces additional difficulties in the stochastic setting. However, these difficulties were overcome and an impressive theory is now available. See for instance the survey of El Karoui, Peng and Quenez [5] and the references therein for this theory and its applications.

A backward stochastic differential equation (BSDE in short) is this. We are first given a diffusion process

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t), \tag{1}$$

with an invertible $d \times d$ matrix σ and a vector μ satisfying standard regularity assumptions. In addition, deterministic functions g and f are also given. Then, the solution is adapted processes $(Y(t), Z(t)) \in \mathcal{R} \times \mathcal{R}^d$ satisfying

$$dY(t) = f(t, X(t), Y(t), Z(t))dt + Z(t)dX(t),$$

with terminal data $Y(T) = g(X(T))$. Connection with the partial differential equations (PDE in short) is easily obtained if one assumes that the solution $Y(t)$ has a Markovian dependence on the driving diffusion process $X(t)$, i.e., if we assume $Y(t) = v(t, X(t))$ for some smooth function v. Then, it is a direct calculation with the use of Ito calculus to show that $Z(t) = \nabla v(t, X(t))$ and that v satisfies

$$-\mathcal{L}v(t,x) + f(t,x,v(t,x),Dv(t,x)) = 0\,, \quad \text{on } [0,T) \times \mathcal{R}^d,$$

with terminal condition $v(T,x) = g(x)$. Here $\mathcal{L}$ is the Dynkin operator of the diffusion X without the drift term and it is given by

$$\mathcal{L}v(t,x) = \varphi_t(t,x) + \frac{1}{2}\mathrm{Tr}\left[D^2\varphi(t,x)\sigma(x)\sigma(x)'\right],$$

and $D\varphi$, $D^2\varphi$ are the gradient and the matrix of second derivatives of φ with respect to the x variables. We should note that in the theory of BSDEs strong existence and uniqueness results were obtained without the a priori Markovian assumption.

Although, the above formula is an interesting stochastic representation for a nonlinear PDE, it is valid only for semi-linear PDEs. Namely, the nonlinearity in this equation is in the first order derivatives and the second order derivatives appear only linearly in $\mathcal{L}$. One possible extension is to allow the equation for the driving diffusion process $X(t)$ to carefully depend on the other processes Y and Z. Then, one has two stochastic differential equations, one for X and the other for Y. While the X process satisfies an initial condition, the Y process satisfies a terminal condition. For this reason these equations are known as *forward-backward* stochastic differential equations. However, even with this extension, the representation holds only for quasi-linear partial differential equations and not for all fully nonlinear equations.

In our paper [3], we to extend this representation result to fully nonlinear partial equations. In view of recent developments in Monte Carlo and other probabilistic methods, we believe that such a representation may prove to be useful in developing stochastic numerical methods for solving fully nonlinear PDEs.

The starting point of our analysis is the recent results of the first three authors [1, 2]. Motivated by these results, we restrict the Z process in BSDE to be a diffusion process also. Namely, the *second order* backward stochastic differential equation again starts with the diffusion process $X \in \mathcal{R}^d$ solving (1). Then, given deterministic functions g and f, we look for a quintuple (Y, Z, Γ, A) of adapted process with certain properties. Firstly, their ranges are

$$Y(\cdot) \in \mathcal{R}, \quad Z(\cdot) \in \mathcal{R}^d, \quad \Gamma(\cdot) \in \mathcal{S}^d, \quad A(\cdot) \in \mathcal{R}^d,$$

where $\mathcal{S}^d$ is the set of all $d \times d$ symmetric matrices. We require them to solve the stochastic differential equations, which we refer concisely as 2BSDE,

$$dY(t) = f(t, X(t), Y(t), Z(t), \Gamma(t))dt + Z(t)' \circ dX(t), \tag{2}$$

$$dZ(t) = A(t)dt + \Gamma(t)dX(t), \tag{3}$$

$$Y(T) = g(X(T)), \tag{4}$$

where $\circ$ is the Fisk–Stratonovich integral. Further technical growth and regularity conditions are needed and we refer to [3] for the precise definition of a solution to a 2BSDE.

In addition to the new equation (3), another important difference between 2BSDE and the standard BSDE is that in (2) the nonlinearity may also depend on the Γ process. This dependence of f on Γ together with (3), enables us to cover all fully nonlinear parabolic PDEs. Indeed, as in the BSDEs, let us formally assume that there is a solution to 2BSDE and that $Y(t) = v(t, X(t))$ for some deterministic function v. Then, we calculate directly that the Z process is again the gradient, and $\Gamma(t)$ is equal to $D^2v(t, X(t))$. Hence, the dependence of f on Γ translates into a nonlinear dependence on the Hessian of the related nonlinear PDE which is given by

$$-v_t(t,x) + f(t, x, v(t,x), Dv(t,x), D^2v(t,x)) = 0\,, \quad \text{on } [0,T) \times \mathcal{R}^d, \tag{5}$$

$$v(T, x) = g(x). \tag{6}$$

Notice that the above equation does not contain the $\mathcal{L}v$ term as in the standard BSDE. This is caused by our use of the Fisk–Stratonovich integral in (2) rather than the Ito integral. However, this choice is made only to simplify some of the arguments below and certainly is not an important difference. Indeed, if you used the Ito integral an additional Hessian term that would have appeared in $\mathcal{L}$, but we could absorb this by redefining f.

Above calculations imply the existence of a solution to 2BSDE when there exists smooth solutions to (5). Indeed, let us assume that the PDE (5) together with the terminal condition (6) has a solution $v : [0,T] \times \mathcal{R}^d \to \mathcal{R}$ which is continuous and

$$v_t, \nabla v, D^2 v, \mathcal{L}Dv \text{ exist and are continuous on } [0,T) \times \mathcal{R}^d,$$

where $\mathcal{L}$ is as before. Then, it is a direct calculation with the use of Itô formula that the quintuple

$$(Y(t), Z(t), \Gamma(t), A(t)) = (v(t,X(t)), \nabla v(t,X(t)), D^2 v(t,X(t)), \mathcal{L}D^2 v(t,X(t)))$$

satisfies the 2BSDE.

Now we turn to the question of uniqueness of solutions to 2BSDEs. We prove a uniqueness result under a condition on the PDE and on f. First we assume that the nonlinearity f is Lipschitz in Y and is non-increasing in the Hessian variable. The monotonicity assumption is very natural as it implies that (5) is a parabolic equation. Secondly, we assume that (5) together with the terminal condition (6) has comparison as in the theory of viscosity solutions. The notion of comparison is defined precisely in [3] and it simply states that any viscosity subsolution of (5), (6) is less than ay viscosity supersolution of the same equations. In view of the strong comparison results proved in the theory of viscosity solutions [4,6] this condition holds for a very large class of nonlinearities f.

Under the above assumptions, we prove a uniqueness result which we state only formally in this Introduction.

Main Theorem.

Suppose that the nonlinear PDE (5) *and the terminal condition* (6) *has comparison as din the theory of viscosity solutions. Assume further that f is elliptic and uniformly Lipschitz and g has polynomial growth. Then, 2BSDE has at most one solution. Moreover, if the 2BSDE has a solution, then* (5) *and* (6) *has a unique viscosity solution v and the solution (Y, Z, Γ, A) satisfies $Y(t) = v(t, X(t))$*

This theorem is precisely stated and proved in [3].

References

[1] Cheridito, P., Soner H.M. and Touzi, N. (2005). Small time path behavior of double stochastic integrals and applications to stochastic control. *Annals of Applied Probability*, to appear.

[2] Cheridito, P., Soner H.M. and Touzi, N. (2005). The multi-dimensional super-replication problem under Gamma constraints, *Annales de lŠInstitut Henri Poincaré: Analyse Non-Linéaire*, to appear.

[3] Cheridito, P., Soner H.M., Touzi, N. and Victoir, N. (2005). Second Order Backward Stochastic Differential Equations and Fully Nonlinear Parabolic PDEs. preprint.

[4] Crandall, M.G., Ishii, H. and Lions, P.L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.* 27(1), 1–67.

[5] El Karoui, N., Peng, S. and Quenez, M.C. (1997). Backward stochastic differential equations in finance. *mathematical Finance.* 7(1), 1–71.

[6] Fleming, W.H. and Soner, H.M. (1993). *Controlled Markov Processes and Viscosity Solutions*. Applications of Mathematics 25. Springer-Verlag, New York.
[7] Pardoux and Peng, s. (1990). Adapted solution of a backward stochastic differential equation. *System Control Letters*, 14, 55–61.
[8] Peng, S. (1990). A general maximum principle for stochastic optimal control problems. *SIAM J. Cont. Opt.*, 28, 966–976.

Finitely many surgeries for mean curvature flow and Ricci flow

Gerhard Huisken

The mean curvature flow of hypersurfaces and Hamilton's Ricciflow of Riemannian metrics are both weakly parabolic systems that lead to quasilinear reaction-diffusion equations for the extrinsic and intrinsic curvatures of the evolving geometries respectivey. In both cases the flow tends to have singularities for general initial data and the shape of the singularities is determined by a delicate balance between the diffusion and reaction forces of the relevant flow.

In high dimensions and without curvature conditions on the initial data both flows allow a huge variety of possible singularities including in particular pinching behaviour along higher dimensional, homothetically shrinking cylinders of type $S^{n-k} \times R^k$. With today's analytical technology there is no hope to control such singularities and to extend the flow beyond them by surgery if there is more than one flat direction, ie if $k > 1$. One therefore has to consider dimensions and curvature conditions that make sure that cylindrical neckpinching can only occur with $k = 1$, ie with necks of type $S^{n-1} \times R$. There are four such cases, two for each of the flows, where the same behaviour is expected: There is an algorithm depending only on the initial data that combines the smooth flow with a surgery procedure which occurs only at discrete times, which reduces the curvature on each occasion by a chosen large factor and replaces, either, a spherical cap with large curvature by a spherical cap with small curvature, or, replaces a cylinder $S^{n-1} \times [a, b]$ of large curvature with two spherical caps. In each case the flow then allows topological conclusions on the initial object.

For mean curvature flow the first example is the case $n \geq 3$ for 2-convex surfaces, that is surfaces with the sum of the two smallest principal curvatures being positive. In this case the positivity assumption, which is preserved by the flow of hypersurfaces in Euclidean space, prevents more than one flat direction near a singularity. Joint work of Huisken and Carlo Sinestrari (Universita di Roma, Tor Vergata), to appear, shows that a priori estimates for the roundness of the neck and for the gradient of the curvature can be directly obtained from the initial data and the evolution equation. It is crucial for the direct gradient estimate that $n \geq 3$ in order to exploit the Codazzi equations in the cross section of a neck.

If mean curvature flow of embedded 2-surfaces with positive mean curvature is considered, a similar behaviour is expected. However in this case the gradient estimate will have to exploit the embeddedness to prevent a singularity as known from the grim reaper solution. This can possibly be accomplished with techniques based on work of Brian White for the level set flow.

In the case of Hamilton's Ricci flow the singularity analysis of Hamilton and Perelman shows that only neckpinches with one flat direction can occur for $n = 3$, without making additional curvature assumptions on the initial metric. The proposed proof of Perelman for the surgery property as stated above depends on a priori estimates for the curvature combined with integral monotonicity formulae. Like in the 2-dimensional mean curvature flow the gradient estimate for the curvature can not be derived directly but follows from contracdiction arguments.

The fourth example is Hamilton's Ricci flow for 4-manifolds of positive isotropic curvature which is a preserved condition that also rules out singularities with more than one flat direction. It seems that the missing gradient estimate in Hamilton's original paper can either be provided by following Perelman's arguments or by obtaining it via a priori estimates similar to the case of 2-convex mean curvature flow for $n \geq 3$.

The blow-up behavior of the biharmonic map heat flow in four dimensions

Roger Moser

Let $\Omega \subset \mathbb{R}^4$ be an open set and N a compact Riemannian manifold which is embedded isometrically in a Euclidean space $\mathbb{R}^n$. We consider maps $u : \Omega \to N$. We consider two functionals for such maps: First we have the Dirichlet energy

$$E_1(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx,$$

which we call the first energy. Its L^2-gradient is minus the tension field

$$\tau_1(u) = \Delta u + A(u)(\nabla u, \nabla u),$$

where A denotes the second fundamental form of $N \subset \mathbb{R}^n$. The second energy is defined by

$$E_2(u) = \frac{1}{2}\int_\Omega |\tau_1(u)|^2\,dx.$$

If S is the tensor on N defined by the condition $\langle S(X,\nu), Y\rangle = \langle A(X,Y), \nu\rangle$, the L^2-gradient of E_2 is given by

$$\begin{aligned}\tau_2(u) \quad = \quad & \Delta\tau_1(u) + 2\operatorname{div} A(u)(\tau_1(u), \nabla u) - D^N_{\tau_1(u)} A(u)(\nabla u, \nabla u)\\ & - A(u)(\tau_1(u), \tau_1(u)) + S(u)(\tau_1(u), A(u)(\nabla u, \nabla u))\\ & - 2S(u)(\nabla u, A(u)(\tau_1(u), \nabla u)).\end{aligned}$$

Here D^N denotes covariant derivatives on N. The biharmonic map heat flow is the L^2-gradient flow belonging to E_2, that is, the flow given by the fourth order parabolic equation

$$\frac{\partial u}{\partial t} + \tau_2(u) = 0.$$

It can be regarded as a higher order counterpart to the harmonic map heat flow, which is given by the equation

$$\frac{\partial u}{\partial t} = \tau_1(u).$$

We study a solution of the flow which is initially smooth. It must be expected that this solution may become singular in finite time, even though no such example is known. We consider the behavior of the flow as the first singular time is approached.

It turns out that the set of points in Ω where singularities occur, is characterized by concentrations of either the first or the second energy. Moreover, at each singularity, there exists a sequence of rescalings of the flow about the respective point, such that the obtained rescaled maps converge either to a non-constant harmonic map (i. e., a map with $\tau_1(u) = 0$), or to a non-constant biharmonic map (a map satisfying $\tau_2(u) = 2$). Moreover, the harmonic maps arising here can be identified with either harmonic 2-spheres or stationary weakly harmonic 3-spheres in N. Certain conclusions about the structure and the size of the set where the singularities occur are also possible: For instance, this set corresponds to a generalized surface in Ω, and there exists a connection between the curvature of this generalized surface and the limiting behavior of E_2 at the singular time.

Asymptotic silence of inhomogenous cosmological singularities

Lars Andersson

(joint work with Henk van Elst, Claes Uggla, Woei-Chet Lim)

References: Gowdy phenomenology in scale-invariant variables, CQG 21 (2004) S29-S57; gr-qc/0310127

Asymptotic silence of generic cosmological singularities, Phys. Rev. Lett. 94 (2005) 051101; gr-qc/0402051

BKL and cosmic censorship. Consider spacetimes $(V, g_{\alpha\beta})$, signature $- + + \cdots +$ satisfying Einstein equations $R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = T_{\alpha\beta}$. Assume $(V, g_{\alpha\beta})$ is maximal, globally hyperbolic and that suitable energy conditions hold. Singularity theorems tell us that generic spacetimes are causally geodesically incomplete (singular), but give no information about the nature of the singularities. The Strong Cosmic Censorship Conjecture states that generic maximal globally hyperbolic spacetimes are inextendible. The Belinskiĭ, Khalatnikov and Lifshitz (BKL) proposal gives a heuristic scenario for generic cosmological singularities. Roughly speaking, the singularity is spacelike: observers near the singularity can't have communicated in the past; silence holds — particle horizons shrink to zero; the singularity is local: spatial derivatives are dynamically insignificant near the singularity

According to this scenario, non-stiff matter is dynamically insignificant near the singularity; the singularity is oscillatory in case matter is non-stiff and spacetime dimension $D < 11$ and non-oscillatory otherwise.

A non-oscillatory singularity is asymptotically Kasner along generic timelines; an generic timeline in an oscillatory singularity has Kasner epochs interspersed with bounces which change the Kasner parameters according to the BKL map.

Mixmaster vs. AVTD. According to BKL, in 3+1 dimensions, the spacetime geometry along the timeline of a spatial point moving in the direction of a singularity will during a large fraction of the time be close to a Kasner geometry $-dt^2 + t^{2p}dx^2 + t^{2q}dy^2 + t^{2r}dz^2$ (coasting), interspersed by bounces which change the Kasner exponents p, q, r. The sequence of bounces lead to a chaotic dynamics for the Kasner exponents — the BKL map. Bianchi IX, Mixmaster, is known to have an oscillatory singularity [Ringström, 2000]. In the presence of stiff matter or a scalar field, singularity is quiescent or "asymptotically velocity term dominated" (AVTD), i.e. the geometry along a generic timeline approaches a Kasner geometry, asymptotically near the singularity, [Andersson and Rendall, 2001].

Hierarchy of cosmological models. In order to understand the structure of singularities, it is useful to consider models with a spatial group of isometries. In dimension 3+1 one has in order of decreasing orbit dimension for the symmetry action, Bianchi or Kantowski-Sachs models with symmetry orbits of dimension 3, Surface symmetric or G_2 models with orbits of dimension 2, $U(1)$ models with orbits of dimension 1, and the full 3+1 Einstein equations without symmetry. The reduced system is an ODE in case the orbit dimension is 3, an 1+1 dimensional PDE in case orbit dimension is 2, and a 2+1 and 3+1 dimensional system in the last two cases.

Cosmic censorship is well understood in the Bianchi and surface symmetric case. In the case of G_2, the Gowdy subcase, with T^3 Cauchy surface and symmetry group $U(1) \times U(1)$ with hypersurface orthogonal Killing fields, is well understood. In that case the reduced system is semi-linear, while in the full G_2 case, the reduced system is quasi-linear. Generic Gowdy spacetimes have AVTD singularity and cosmic censorship holds [Ringström, 2004]. Further, a (finite number of) spikes form in generic Gowdy. At spike timelines, the spatial derivative is dynamically significant, while for generic timelines, the spatial derivative becomes dynamically insignificant.

Dynamical systems approach. Using scale invariant (Hubble normalized) frame variables one can write the Einstein equations in first order form. This allows one to classify fixed point sets, attractors etc. In particular, this approach gives a natural formulation of BKL proposal and identify an asymptotic dynamical system (the silent boundary system).

The G_2 case. For vacuum G_2 with Cauchy surface T^3, it is known that the area of the symmetry orbits is a time function, and in this case maximal vacuum G_2 spacetimes are globally foliated by level sets of the area function. Introducing a group-invariant orthonormal frame with one element the timelike normal, and one element aligned with one of the Killing fields, one may write the G_2 symmetric vacuum Einstein equations in terms of commutator and frame variables. Normalizing these variables with suitable powers of the Hubble expansion rate H, one arrives at an autonomous constrained first-order system of PDE's with state vector

$$\boldsymbol{X} = ({E_1}^1, \Sigma_+, \Sigma_-, \Sigma_\times, \Sigma_2, N_\times, N_-)^T = ({E_1}^1) \otimes \boldsymbol{Y} \ .$$

Here ${E_1}^1$ is a rescaled frame variable parametrizing the "spatial metric", the variables $\Sigma_+, \Sigma_-, \Sigma_\times, \Sigma_2$ parametrize the shear, i.e. the tracefree part of the second fundamental form, while the $N_\times, N_-$ parametrize the spatial connection.

The G_2 vacuum Einstein equations now becomes a constrained system of PDE's for these variables. The Hamiltonian constraint restricts $\boldsymbol{X}$ to lie on the unit sphere in $\mathbb{R}^6$. The variables ${E_1}^1, \Sigma_+, \Sigma_2$ satisfy transport equations with no spatial derivative, while the variables $\Sigma_-, N_\times, \Sigma_\times, N_\times$ satisfy pairs of 1:st order hyperbolic equations. The rescaled spatial frame derivative is of the form ${E_1}^1\partial_x$ which is what allows one to extract an asymptotic dynamical system.

The silent boundary. Our numerical studies indicate two important features of the G_2 system. First, asymptotic silence holds, i.e. ${E_1}^1 \to 0$ exponentially as $t \to \infty$. This has the consequence that for the reduced system, light cones collapse near the singularity and in particular the particle horizon is of a size proportional to ${E_1}^1$.

The spatial derivative in the Hubble normalized system is of the form ${E_1}^1\partial_x$. Setting ${E_1}^1 = 0$ one may extract an autonomous constrained system of ODE's. The set ${E_1}^1 = 0$ may be thought of as an unphysical boundary of phase space, the Silent Boundary. Going to the silent boundary corresponds to collapse of the light cones.

The dynamics on the silent boundary gives an asymptotic dynamical system, the SB system. The SB system is equivalent to a spatially self-similar model. Numerical studies indicate that the G_2 SB system has oscillatory behavior, analogous to the Bianchi IX system. However, while oscillatory behavior has been proved in the case of Bianchi IX, this has not been done in the case of the G_2 SB system.

Our numerical studies indicate that generic timelines have the property that spatial derivatives become dynamically insignificant and for those the dynamics is governed by the SB system. There are exceptional spike timelines where spatial derivatives are dynamically significant. These are analogous to the spike timelines for Gowdy, however for G_2 the spike timelines are "dynamic" in the sense that spikes recur along these timelines. Our work indicates that there is an effective dynamical system governing the asymptotic dynamics also along spike time lines.

The above, heuristic picture, now indicates that the full G_2 system should be governed asymptotically by the SB system, along generic timelines. Since the SB system has oscillatory behavior, this should also be the case for the full G_2 system. This is strongly supported by numerical experiments, which show that timelines for the full G_2 system and orbits of the SB system indeed have the same asymtpotic behavior.

A formal analysis of the rescaled Weyl tensor along non-spike and spike timelines indicates that for generic timelines in generic G_2 spacetimes, the Weyl scalars will take arbitrarily large values along a sequence of times approaching the singularity. This behavior is consistent with strong cosmic censorship.

References

[Andersson and Rendall, 2001] Andersson, L. and Rendall, A. D. (2001). Quiescent cosmological singularities. *Comm. Math. Phys.*, 218(3):479–511.

[Ringström, 2000] Ringström, H. (2000). Curvature blow up in Bianchi VIII and IX vacuum spacetimes. *Classical Quantum Gravity*, 17(4):713–731.

[Ringström, 2004] Ringström, H. (2004). Strong cosmic censorship in T^3-Gowdy spacetimes.

On a wave map equation arising in General Relativity

Hans Ringström

Due to the work of Choquet-Bruhat and Geroch [1, 3], it is possible to view the Einstein vacuum equations as an initial value problem. Given vacuum initial data, there is a maximal globally hyperbolic development (MGHD), which is unique up to isometry. Since there are examples for which the MGHD is extendible in inequivalent ways, one is naturally led to the strong cosmic censorship conjecture, stating that for generic initial data, the MGHD is inextendible. Since trying to prove this conjecture in all generality is too ambitious at this time, it is natural to consider a class of initial data satisfying a given set of symmetry conditions. One can then ask the question: is it possible to show that the MGHD is inextendible for initial data that are generic in this class? One way of proving that a spacetime is inextendible is to prove that, given a causal geodesic, there are two possible outcomes in a given time direction; either the geodesic is complete, or it is incomplete but the curvature is unbounded along it. Here we shall be interested in the T^3-Gowdy spacetimes, for which it is known that all causal geodesics are complete in one time direction (the expanding direction), cf. [6], and incomplete in the opposite direction (the singularity). What remains to be proved is thus that, for generic initial data, the curvature is unbounded along all causal geodesics that end on the singularity.

The essential part of the equations in the case of T^3-Gowdy, cf. [7], are

$$P_{\tau\tau} - e^{-2\tau}P_{\theta\theta} - e^{2P}(Q_\tau^2 - e^{-2\tau}Q_\theta^2) = 0 \tag{1}$$

$$Q_{\tau\tau} - e^{-2\tau}Q_{\theta\theta} + 2(P_\tau Q_\tau - e^{-2\tau}P_\theta Q_\theta) = 0. \tag{2}$$

In the above parametrization, the singularity corresponds to $\tau \to \infty$, and our main concern here is the asymptotic behaviour of solutions to (1)-(2) in this time direction.

The equations (1)-(2) constitute a wave map equation with hyperbolic space as a target, cf. [7]. The representation of hyperbolic space associated with the equations is $g_R = dP^2 + e^{2P}dQ^2$ on $\mathbb{R}^2$. By the wave map structure, isometries of hyperbolic space map solutions to solutions. One particular isometry which we shall need in order to state the results is the inversion, denoted Inv. In the upper half plane it corresponds to an inversion in the unit circle with center at the origin. Note that the map taking (Q,P) to (Q,e^{-P}) defines an isometry from $(\mathbb{R}^2, g_R)$ to the upper half plane model.

In the analysis of Gowdy spacetimes, the existence of expansions for the solutions close to the singularity in certain situations is the key starting point:

$$P(\tau,\theta) = v(\theta)\tau + \phi(\theta) + u(\tau,\theta) \tag{3}$$

$$Q(\tau,\theta) = q(\theta) + e^{-2v(\theta)\tau}[\psi(\theta) + w(\tau,\theta)]. \tag{4}$$

Here $w, u \to 0$ as $\tau \to \infty$ and $0 < v < 1$. In [4] and [5], the authors proved that given smooth v, ϕ, q, ψ with $0 < v < 1$, there are unique solutions to (1)-(2) with asymptotics of the form (3)-(4). The functions v, ϕ, q and ψ can thus be considered to be initial data on the singularity.

In our experience, the most important function appearing in the expansions is v. This object may seem devoid of geometric significance, but if one naively differentiates the expantions with respect to τ, assumes that u_τ and w_τ converge to zero and computes the pointwise limit of the *kinetic energy density* $\mathcal{K} = P_\tau^2 + e^{2P}Q_\tau^2$ as $\tau \to \infty$, one obtains v^2 as a result. Since $\mathcal{K}$ is a geometrically defined object, v^2 is thus geometrically defined. In [7], we proved that the pointwise limit of $\mathcal{K}(\tau,\theta)$ always exists. This leads to the following definition.

Definition 1. Let (Q,P) be a solution to (1)-(2) and let $\theta_0 \in S^1$. Then we define *the asymptotic velocity at* θ_0 to be $v_\infty(\theta_0) = [\lim_{\tau\to\infty} \mathcal{K}(\tau,\theta_0)]^{1/2}$.

The importance of the asymptotic velocity comes from the fact that one can prove that a causal curve ending at a point θ on the singularity experiences curvature blow up if $v_\infty(\theta) \neq 1$. In other words, the asympotic velocity can be used as an indicator for curvature blow up. It can also be used as an indicator for the existence of expansions.

Proposition 1. *Let* (Q,P) *be a solution to (1)-(2) and assume* $0 < v_\infty(\theta_0) < 1$. *If* $P_\tau(\tau,\theta_0)$ *converges to* $v_\infty(\theta_0)$, *then there is an open interval* I *containing* θ_0, $v, \phi, q, r \in C^\infty(I,\mathbb{R})$, $0 < v < 1$, *polynomials* Ξ_k *and a* T *such that for all* $\tau \geq T$

$$\|P_\tau(\tau,\cdot) - v\|_{C^k(I,\mathbb{R})} \leq \Xi_k e^{-\alpha\tau}, \tag{5}$$

$$\|P(\tau,\cdot) - p(\tau,\cdot)\|_{C^k(I,\mathbb{R})} \leq \Xi_k e^{-\alpha\tau}, \tag{6}$$

$$\left\|e^{2p(\tau,\cdot)}Q_\tau(\tau,\cdot) - r\right\|_{C^k(I,\mathbb{R})} \leq \Xi_k e^{-\alpha\tau}, \tag{7}$$

$$\left\|e^{2p(\tau,\cdot)}[Q(\tau,\cdot) - q] + \frac{r}{2v}\right\|_{C^k(I,\mathbb{R})} \leq \Xi_k e^{-\alpha\tau} \tag{8}$$

where $p(\tau,\cdot) = v\cdot\tau + \phi$ *and* $\alpha > 0$. *If* $P_\tau(\tau,\theta_0)$ *converges to* $-v_\infty(\theta_0)$, *then* $\mathrm{Inv}(Q,P)$ *has expansions of the above form in a neighbourhood of* θ_0.

Let us introduce some terminology.

Definition 2. Assume $0 < v_\infty(\theta_0) < 1$ for some $\theta_0 \in S^1$ and

$$\lim_{\tau\to\infty} P_\tau(\tau,\theta_0) = -v_\infty(\theta_0).$$

Let $(Q_1, P_1) = \text{Inv}(Q, P)$. By Proposition 1, $Q_1(\tau, \cdot) \to q_1$. We call θ_0 a *non-degenerate false spike* if $\partial_\theta q_1(\theta_0) \neq 0$.

Definition 3. Assume $1 < v_\infty(\theta_0) < 2$ for some $\theta_0 \in S^1$ and

$$\lim_{\tau \to \infty} P_\tau(\tau, \theta_0) = v_\infty(\theta_0).$$

Then Q converges to a smooth function q in a neighbourhood of θ_0. If $\partial_\theta^2 q(\theta_0) \neq 0$, we say that θ_0 is a *non-degenerate true spike.*

For a non-degenerate false spike, $\lim_{\tau \to \infty} P_\tau(\tau, \theta) = v_\infty(\theta)$ in a punctured neighbourhood of θ_0, cf. [7]. For a non-degenerate true spike, $0 < v_\infty(\theta) < 1$ in a punctured neighbourhood of θ_0 and $\lim_{\tau \to \infty} P_\tau(\tau, \theta) = v_\infty(\theta)$ in a neighbourhood of θ_0, cf. [7].

Definition 4. Let $\mathcal{G}_{l,m}$ be the set of solutions (Q, P) with l non-degenerate true spikes $\theta_1, ..., \theta_l$ and m non-degenerate false spikes $\theta'_1, ..., \theta'_m$ such that

$$\lim_{\tau \to \infty} P_\tau(\tau, \theta) = v_\infty(\theta),$$

for all $\theta \notin \{\theta'_1, ..., \theta'_m\}$ and $0 < v_\infty(\theta) < 1$ for all $\theta \notin \{\theta_1, ..., \theta_l\}$. Let

$$\mathcal{G} = \bigcup_{l=0}^{\infty} \bigcup_{m=0}^{\infty} \mathcal{G}_{l,m}.$$

Theorem 1. *$\mathcal{G}_{l,m}$ is open in the $C^2 \times C^1$-topology on initial data and $\mathcal{G}$ is dense with respect to the C^∞-topology on initial data.*

The proof of these statements are to be found in [7] and [8]. The main point is that all solutions in $\mathcal{G}$ have the property that all causal curves ending on the singularity experience curvature blow up. Since it is known that all causal geodesics are complete in the opposite time direction, cf. [6], this gives a proof of strong cosmic censorship in the class of T^3-Gowdy. Note however that the above result gives a very detailed description of the asymptotic behaviour of generic singularities; the true and false spikes are in fact very well understood.

References

[1] Y. Choquet-Bruhat and R. Geroch, *Global aspects of the Cauchy problem in general relativity*, Comm. Math. Phys. **14** (1969), 329–335

[2] R. H. Gowdy, *Vacuum spacetimes with two-parameter spacelike isometry groups and compact invariant hypersurfaces: Topologies and boundary conditions*, Ann. Phys. NY **83** (1974), 203–241.

[3] Y. Fourès-Bruhat, *Théorème d'existence pour certains systèmes d'équations aux derivées partielles non linéaires*, Acta Mathematica **88** (1952), 141.

[4] S. Kichenassamy and A. Rendall, *Analytic description of singularities in Gowdy spacetimes*, Class. Quantum Grav. **15** (1998), 1339–1355.

[5] A. Rendall, *Fuchsian analysis of singularities in Gowdy spacetimes beyond analyticity*, Class. Quantum Grav. **17** (2000), 3305–3316.

[6] H. Ringström, *On a wave map equation arising in General Relativity*, Comm. Pure Appl. Math. **57** (2004), 657–703.

[7] H. Ringström, *Existence of an asymptotic velocity and implications for the asymptotic behaviour in the direction of the singularity in T^3-Gowdy*, accepted for publication in Comm. Pure Appl. Math., cf. http://www.math.kth.se/˜hansr, title: *On the concept of an asympotic velocity in T^3-Gowdy.*
[8] H. Ringström, *Strong cosmic censorship in T^3-Gowdy*, cf. http://www.math.kth.se/˜hansr.

Finite Dimensional Approximations to (and non-squeezing for) the KdV flow with periodic boundary conditions

Markus Keel
(joint work with J.Colliander, G. Staffilani, H. Takaoka, T. Tao)

This talk is concerned with the symplectic behavior of the Korteweg-de Vries (KdV) flow

$$u_t + u_{xxx} = 6uu_x; \quad u(0,x) = u_0(x) \tag{1}$$

on the circle $x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, where $u(t,x)$ is real-valued. In particular we investigate how the flow may (or may not) be accurately approximated by certain finite-dimensional models, and then use such an approximation to conclude a symplectic non-squeezing property.

Unlike the work of Kuksin [9] which initiated the investigation of non-squeezing results for infinite dimensional Hamiltonian systems, the non-squeezing argument here does not construct a capacity directly, but relies rather on the aforementioned approximations and the finite-dimensional non-squeezing theorem of Gromov [6]. In this way our results are similar to those obtained for the NLS flow by Bourgain [2]. A major difficulty here though is the lack of any sort of smoothing estimate which would allow us to easily approximate the infinite dimensional KdV flow by a finite-dimensional Hamiltonian flow. To resolve this problem we invert the Miura transform and work at the level of the modified KdV (mKdV) equation, for which smoothing estimates can be established.

The material sketched here is described in detail in the paper [3].

On the circle we have the spatial Fourier transform

$$\widehat{u}(k) := \frac{1}{2\pi}\int_0^{2\pi} u(x)\exp(-ikx)\,dx$$

for all $k \in \mathbb{Z}$, and the spatial Sobolev spaces

$$\|u\|_{H_x^s} := (2\pi)^{1/2}\|\langle k\rangle^s \widehat{u}\|_{l_k^2}$$

for $s \in \mathbb{R}$, where $\langle k\rangle := (1+|k|^2)^{1/2}$. Define the mean operator P_0 by $\widehat{P_0 u}(k) = \chi_{k=0}\widehat{u}(k)$, and the mean-zero periodic Sobolev spaces (which are preserved by the KdV flow) H_0^s by $H_0^s := \{u \in H_x^s : P_0 u = 0\}$ endowed with the same norm as H_x^s.

If the initial datum u_0 for (1) is smooth, then there is a global smooth solution $u(t)$ (see e.g. [11]). We can thus define the non-linear flow map $S_{KdV}(t)$ on $C^\infty(\mathbb{T})$ by $S_{KdV}(t)u_0 := u(t)$. In particular this map is densely defined on every Sobolev space H_0^s. If $s \geq -1/2$, then the equation (1) is globally well-posed in H_0^s. (see

[4, 5, 7].) In fact, if one asks only that the flow be continuous in time, then global well-posedness for (1) has been established for all $s \geq -1$ in [8] using inverse scattering methods.

We describe now the low frequency approximation results which lie at the heart of our results. We let $b(k)$ be the restriction to the integers of a real even bump function adapted to $[-N, N]$ which equals 1 on $[-N/2, N/2]$, and consider the evolution

$$u_t + u_{xxx} = B(6uu_x); \quad u(0) = u_0 \tag{2}$$

where

$$\widehat{Bu}(k) = b(k)\widehat{u}(k).$$

Let S_{BKdV} denote the flow map associated to (2). Observe that this is a finite-dimensional flow on the space $P_{\leq N} H_0^s$, where $P_{\leq N}$ is the Fourier projection to frequencies $\leq N$: $\widehat{P_{\leq N} u}(k) = \chi_{|k| \leq N} \widehat{u}(k)$. While S_{BKdV} is not a symplectomorphism, we can conjugate a flow of the form (2) with a simple multiplier operator to arrive at the desired finite dimensional symplectomorphism (see [3]).

Theorem 1. *Fix $s \geq -1/2$, $T > 0$, and $N \gg 1$. Let $u_0 \in H_0^s$ have Fourier transform supported in the range $|k| \leq N$. Then*

$$\sup_{|t| \leq T} \| P_{\leq N^{1/2}} (S_{BKdV} u_0(t) - S_{KdV}(t) u_0) \|_{H_0^s} \leq N^{-\sigma} C(s, T, \|u_0\|_{H_0^s})$$

for some $\sigma = \sigma(s) > 0$.

Theorem 1 can be viewed as a statement that one can (smoothly) truncate the KdV evolution at the high frequencies without causing serious disruption to the low frequencies. Our second main result is in a similar vein:

Theorem 2. *Fix $s \geq -1/2, T > 0, N \geq 1$. Let $u_0, \tilde{u}_0 \in H_0^s$ be such that $P_{\leq 2N} u_0 = P_{\leq 2N} \tilde{u}_0$ (i.e. u_0 and $\tilde{u}_0$ agree at low frequencies). Then we have,*

$$\sup_{|t| \leq T} \| P_{\leq N} (S_{KdV}(t) \tilde{u}_0 - S_{KdV}(t) u_0) \|_{H_0^s} \leq N^{-\sigma} C(s, T, \|u_0\|_{H_0^s}, \|\tilde{u}_0\|_{H_0^s})$$

for some $\sigma = \sigma(s) > 0$.

We prove Theorem 1 and Theorem 2 by using the *Miura transform* $u = \mathbf{M}v$, defined by

$$u = \mathbf{M}v := v_x + v^2 - P_0(v^2). \tag{3}$$

As discovered in [10], this transform allows us to conjugate the KdV flow to the *modified Korteweg-de Vries* (mKdV) flow

$$v_t + v_{xxx} = F(v); \quad v(x, 0) = v_0(x) \tag{4}$$

where the non-linearity $F(v)$ is given by

$$F(v) := 6(v^2 - P_0(v^2))v_x. \tag{5}$$

The modified KdV equation has slightly better smoothing properties than the ordinary KdV equation, and in addition the process of inverting the Miura transform adds one degree of regularity (from $H_0^{-1/2}$ to $H_0^{1/2}$). By proving a slightly more refined trilinear estimate than those found in e.g. [5] we are able to prove the above two theorems by passing to the mKdV setting using the Miura transform.

To describe the non-squeezing result, we need some additional notation. For any $u_* \in H_0^{-1/2}(\mathbb{T})$, $r > 0$, $k_0 \in \mathbb{Z}^*$, and $z \in \mathbf{C}$, we consider the infinite-dimensional ball

$$\mathbf{B}^\infty(u_*; r) := \{u \in H_0^{-1/2}(\mathbb{T}) : \|u - u_*\|_{H_0^{-1/2}} \leq r\}$$

and the infinite-dimensional cylinder

$$\mathbf{C}_{k_0}^\infty(z; r) := \{u \in H_0^{-1/2}(\mathbb{T}) : |k_0|^{-1/2}|\widehat{u}(k_0) - z| \leq r\}.$$

Theorem 3. *Let $0 < r < R$, $u_* \in H_0^{-1/2}(\mathbb{T})$, $k_0 \in \mathbb{Z}^*$, $z \in \mathbf{C}$, and $T > 0$. Then*

$$S_{KdV}(T)(\mathbf{B}^\infty(u_*; R)) \not\subseteq \mathbf{C}_{k_0}^\infty(z; r).$$

In other words, there exists a global $H_0^{-1/2}(\mathbb{T})$ solution u to (1) *such that*

$$\|u(0) - u_*\|_{H_0^{-1/2}} \leq R$$

and

$$|k_0|^{-1/2}|\widehat{u(T)}(k_0) - z| > r.$$

Note that no smallness conditions are imposed on u_*, R, z, or T.

Roughly speaking, this theorem asserts that the KdV flow cannot squash a large ball into a thin cylinder. Notice that the balls and cylinders can be arbitrarily far away from the origin, and the time T can also be arbitrary. Note though that this result is interesting even for $u_* = 0, z = 0$ and smooth initial data u_0, as it tells us that the flow cannot at any time uniformly squeeze the ball $B^\infty(0, R)$ even at a fixed frequency k_0. A second immediate application of Theorem 3 to smooth solutions was highlighted in a different context already in [9], namely that such smooth solutions of (1) cannot uniformly approach some asymptotic state: for any neighborhood $B^\infty(u_0; R)$ of the initial data in $H^{-\frac{1}{2}}(\mathbb{T})$ and for any time t, the diameter of the set $S_{KdV}(t)(B^\infty(u_0; R))$ cannot be less than R.

References

[1] J. Bourgain, *Aspects of longtime behaviour of solutions of nonlinear Hamiltonian evolution equations*, GAFA **5** (1995), 105–140.

[2] J. Bourgain, *Approximation of solutions of the cubic nonlinear Schrödinger equations by finite-dimensional equations and nonsqueezing properties*, Int. Math. Res. Notices, **1994**, no. 2, (1994), 79–90.

[3] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Symplectic non-squeezing of the KdV flow*, submitted, 2005,

[4] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Sharp global well-posedness for periodic and non-periodic KdV and mKdV on* $\mathbb{R}$ *and* $\mathbb{T}$, J. Amer. Math. Soc. **16** (2003), 705–749.

[5] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Multilinear estimates for periodic KdV equations, and applications*, Journ. Funct. Analy. **211** (2004), no. 1, 173–218.

[6] M. Gromov, *Pseudo-holomorphic curves in symplectic manifolds*, Invent. math., **82** (1985), 307–347.

[7] C. Kenig, G. Ponce, L. Vega, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc. **9** (1996), 573–603.

[8] T. Kappeler, P. Topalov, *Global well-posedness of KdV in $H^{-1}(\mathbb{T},\mathbb{R})$*, preprint 2003.

[9] S. Kuksin, *Infinite-dimensional symplectic capacities and a squeezing theorem for Hamiltonian PDEs*, CMP **167** (1995), 521–552.

[10] R. Miura, *Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation*, J. Mathematical. Phys. **9** (1968), 1202–1204.

[11] A. Sjöberg, *On the Korteweg-de Vries equation: existence and uniqueness*, J. Math. Anal. Appl. **29** (1970), 569–579.

On the blow-up for the critical nonlinear Schrödinger equation in a domain

Fabrice Planchon

(joint work with Pierre Raphaël)

We consider the L^2-critical focusing nonlinear Schrödinger equation in a domain Ω with Dirichlet boundary condition:

$$(1)\qquad (NLS)\ \begin{cases} iu_t = -\Delta u - |u|^{\frac{4}{N}}u, & (t,x) \in [0,T)\times\Omega, \\ u_{|\partial\Omega} = 0, \\ u(0,x) = u_0(x), & u_0 : \Omega \to \mathbb{C} \end{cases}$$

with $u_0 \in H^1_0 = H^1_0(\Omega)$ in dimension $N \geq 1$.

We thereafter assume the domain Ω to be such that local well-posedness in the energy space $H^1_0 = H^1_0(\Omega)$ holds in the following sense: for all $u_0 \in H^1_0$, there exists $T > 0$ and a unique maximal solution $u(t) \in \mathcal{C}([0,T), H^1_0)$ to (1). Moreover, we assume that there is continuous dependence of the solution with respect to the initial data locally in time— note that uniform continuity is not required— and that the time of existence is lower bounded by a function of the H^1 norm of the initial data only. This is known to be the case when $\Omega = \mathbb{R}^N$ or if $N = 1, 2$, for any domain Ω with conveniently smooth boundaries [11], and when $N \geq 3$, recent progress has been made when Ω is an exterior domain [2].

The power in (1) is known to be the smallest one for which blow up may occur, and indeed finite time blow up does happen [4] from the virial identity in any star-shaped domain (generalizing the argument of [3]).

A lot of work has been devoted to the $\mathbb{R}^N$ case in recent years. We briefly summarize (some of) these below. At least two different blow up behaviors are known to occur:

- In dimension $N = 1, 2$, there exist a family of solutions with blow-up rate $|\nabla u(t)|_{L^2} \sim \frac{1}{T-t}$ near blow up time, by a result of Bourgain-Wang [1].
- On the other hand, numerical simulations, [5], and formal arguments, [14], suggest the existence of solutions blowing up like

$$|\nabla u(t)|_{L^2} \sim \left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}}$$

in dimension $N = 2$. In dimension $N = 1$, Perelman, [12], proves the existence of such a solution and its stability in some space $E \subset H^1$.

Then the situation was clarified by Merle and Raphaël in the series of papers [6–10,13], where they describe extensively the dynamics of (1) by classifying blow-up solutions and proving existence of *log − log* solutions in the energy space under natural assumptions.

Our aim is to address similar issues for a general domain Ω. One of the main outcome of the analysis by Merle and Raphaël in the $\mathbb{R}^N$ case is that *the singularity formation is in some sense a local in space phenomenon.*

Two main difficulties occur for the study of the blow-up dynamics of (1) on a domain when compared to the analysis in $\mathbb{R}^N$:
(i) First, we no longer retain the large symmetry group of (1) in $\mathbb{R}^N$, and in particular we have to do without the conservation of the momentum $\int Im(\nabla u \overline{u})(t)\, dx$.
(ii) Second, a new difficulty is the possibility for blow up to occur on the boundary of the domain.

Our first result is a classification of blow-up dynamics for a suitable class of solutions.

Theorem 1 (Rigidity of the blow up dynamics). *Let $N = 1$ or $N \geq 2$ assuming the Spectral Property from [6]. There exist universal constants $\alpha^* > 0$ and $C^* > 0$ such that the following holds true. Let an initial data $u_0 \in H_0^1$ and $u(t)$ the corresponding solution to (1) with $[0, T)$ its maximum time interval existence on the right in H_0^1. Assume the following:*
(H1) Small super critical mass: $u_0 \in \mathcal{B}_{\alpha^*} = \{u_0 \in H^1(\mathbb{R}^N) \text{ with } \int Q^2 \leq \int |u_0|^2 < \int Q^2 + \alpha^*\}$.
(H2) Blow-up in finite time: $0 < T < +\infty$; note that under these two hypothesis, $u(t)$ admits, for t close enough to T, a geometrical decomposition
$\left| \lambda_0(t)^{\frac{N}{2}} u(t, \lambda_0(t)x + x_0(t)) e^{i\gamma_0(t)} - Q(x) \right|_{H^1} \leq \delta(\alpha^*)$*, by the variational characterization of the ground state Q.*
(H3) Localization of the center of mass: For t close enough to T, we have

$$x_0(t) \in \Omega \quad \text{and} \quad \liminf_{t \to T} d(x_0(t), \partial\Omega) > 0. \tag{2}$$

(H4) Finite momentum assumption: There exists a fixed and smooth vector-valued cut-off function ϕ taking values between zero and one, with $\mathrm{Supp}(\phi) \subset_{\mathrm{comp}} \Omega$, $\phi(x)$ being the identity when $d(x, \partial\Omega) \geq \frac{1}{2} \liminf_{t \to T} d(x_0(t), \partial\Omega)$, and $|\nabla \phi|_{L^\infty} \leq \frac{10}{\liminf_{t \to T} d(x_0(t), \partial\Omega)}$, such that $u(t, x)$ verifies

$$\limsup_{t \to T} \left| Im\left(\int \phi \cdot \nabla u(t) \overline{u(t)}\right) \right| < +\infty. \tag{3}$$

Then:
(i) Universality of the singular structure in space: there exist parameters $(\lambda(t), x(t), \gamma(t)) \in \mathbb{R}_+^* \times \Omega \times \mathbb{R}$ and an asymptotic profile $u^* \in L^2$ such that

$$u(t) - \frac{1}{\lambda(t)^{\frac{N}{2}}} Q\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)} \to u^* \quad \text{in } L^2 \quad \text{as } t \to T. \tag{4}$$

Moreover, the blow-up point is finite and inside the domain:

$$x(t) \to x(T) \in_{\text{comp}} \Omega \quad as \quad t \to T.$$

(ii) Selection of the dynamics for $E_0 \leq 0$: if moreover

$$E(u_0) \leq 0,$$

then $u(t)$ blows up with the log-log law

$$\lim_{t \to T} \frac{|\nabla u(t)|_{L^2}}{|\nabla Q|_{L^2}} \left(\frac{T-t}{\log|\log(T-t)|} \right)^{\frac{1}{2}} = \frac{1}{\sqrt{2\pi}}. \tag{5}$$

(iii) Rigidity of the blow up speed: If $E_0 > 0$, then $u(t)$ either satisfies log-log law (5) or the following lower bound for t close enough to T:

$$|\nabla u(t)|_{L^2} \geq \frac{C^*}{(T-t)\sqrt{E_0}}. \tag{6}$$

(iv) Asymptotic of u^* on the singularity: The same conclusions on u^* as in the $\mathbb{R}^N$ case hold true.

In other words, the same classification results known in $\mathbb{R}^N$ hold in a domain, provided blow up is a priori assumed to occur in the interior, assumption (H3), and the modified momentum is uniformly bounded, assumption (H4).

Whenever the domain Ω is a ball, and under spherical symmetry of the datum, assumption (H4) is automatically fulfilled. A direct corollary of Theorem 1 is thus:

Corollary 1 (Log-log blow up for $E_0 < 0$ on a ball)**.** Let $N = 1$ or $N \geq 2$ assuming the Spectral Property. There exists a universal constant $\alpha^* > 0$ such that the following holds true. Let $R > 0$ and $\Omega = B(0, R)$. Let u_0 be a radial— or even in dimension $N = 1$— initial data with

$$u_0 \in \mathcal{B}_{\alpha^*} \quad \text{and} \quad E(u_0) < 0,$$

then the corresponding solution $u(t)$ to (1) blows up in finite time $0 < T < +\infty$ in the log-log regime (5).

Note that this result proves the existence of log-log dynamics for the special case $\Omega = B(0, R)$. Existence and stability of log-log dynamics may be generalized to an arbitrary domain, by carefully constructing data which are very well localized inside the domain. Such data yield solutions which are in the log-log regime and for which one may bootstrap all the a priori control, especially the momentum.

Theorem 2 (Existence and stability of the log-log dynamics)**.** *Let $N = 1$ or $N \geq 2$ assuming Spectral Property. There exists a universal constant $\alpha^* > 0$ such that the following holds true.*
(i) Existence of log-log dynamics: For all $\tilde{x} \in \Omega$, there exists a time $T(\tilde{x}) > 0$ such that for all $T \in (0, T(\tilde{x}))$, there exists a solution $u(t) \in H_0^1(\Omega)$ to (1) which satisfies assumptions (H1), (H2), (H3), (H4) of Theorem 1 and blows up at time $0 < T < +\infty$ and at blow-up point $x(T) = \tilde{x}$ in the log-log regime (5).

(ii) Stability of the log-log dynamics: The set of initial data $u_0 \in \mathcal{B}_{\alpha^*}$ such that the corresponding solution $u(t)$ to (1) satisfies assumptions (H1), (H2), (H3), (H4) of Theorem 1 and blows up in the log-log regime (5) is open in $H^1_0(\Omega)$.

A simple but remarkable corollary of Theorem 2 is the existence of a two points log-log blow up solution in $\mathbb{R}^N$.

Corollary 2 (Existence of a two point log-log blow up solution in $\mathbb{R}^N$)**.** Let $N = 1$ or $N \geq 2$ assuming the Spectral Property. There exists an initial data $u_0 \in H^1(\mathbb{R}^N)$ such that the corresponding solution $u(t)$ to (1) with $\Omega = \mathbb{R}^N$ blows up in finite time $0 < T < +\infty$ in the log-log regime (5) at exactly two points in space.

References

[1] J. Bourgain and W. Wang. Construction of blowup solutions for the nonlinear Schrödinger equation with critical nonlinearity. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 25(1-2):197–215 (1998), 1997. Dedicated to Ennio De Giorgi.

[2] N. Burq, P. Gérard, and N. Tzvetkov. On nonlinear Schrödinger equations in exterior domains. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 21(3):295–318, 2004.

[3] R. T. Glassey. On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations. *J. Math. Phys.*, 18(9):1794–1797, 1977.

[4] O. Kavian. A remark on the blowing-up of solutions to the Cauchy problem for nonlinear Schrödinger equations. *Trans. Amer. Math. Soc.*, 299(1):193–203, 1987.

[5] M. J. Landman, G. C. Papanicolaou, C. Sulem, and P.-L. Sulem. Rate of blowup for solutions of the nonlinear Schrödinger equation at critical dimension. *Phys. Rev. A (3)*, 38(8):3837–3843, 1988.

[6] F. Merle and P. Raphael. Blow up dynamic and upper bound on the blow up rate for critical nonlinear schrödinger equation. to appear in Annals of Math.

[7] F. Merle and P. Raphael. Profiles and quantization of the blow up mass fro critical non linear schrödinger equation. to appear in Comm. Math. Phys.

[8] F. Merle and P. Raphael. Sharp lower bound on the blow up rate for critical nonlinear schrödinger equation. preprint.

[9] F. Merle and P. Raphael. Sharp upper bound on the blow-up rate for the critical nonlinear Schrödinger equation. *Geom. Funct. Anal.*, 13(3):591–642, 2003.

[10] F. Merle and P. Raphael. On universality of blow-up profile for L^2 critical nonlinear Schrödinger equation. *Invent. Math.*, 156(3):565–672, 2004.

[11] T. Ogawa and T. Ozawa. Trudinger type inequalities and uniqueness of weak solutions for the nonlinear Schrödinger mixed problem. *J. Math. Anal. Appl.*, 155(2):531–540, 1991.

[12] G. Perelman. On the formation of singularities in solutions of the critical nonlinear Schrödinger equation. *Ann. Henri Poincaré*, 2(4):605–673, 2001.

[13] P. Raphael. Stability of the log-log bound for blow up solutions to the critical nonlinear schrödinger equation. to appear in Math. Annalen.

[14] C. Sulem and P-L. Sulem. *The nonlinear Schrödinger equation*, volume 139 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1999. Self-focusing and wave collapse.

Stable manifolds for all monic supercritical focusing NLS in 1-d

Joachim Krieger
(joint work with Wilhelm Schlag)

Consider the nonlinear focusing NLS

$$i\partial_t\psi + \partial_x^2\psi = -|\psi|^{2\sigma}\psi,\ (t,x) \in \mathbf{R}^{1+1}, \tag{1}$$

where $\sigma > 2$, i. e. the L^2-supercritical case. It is well known that this equation admits standing waves of the form

$$\psi(t,x) = e^{i\alpha^2 t}\phi(x,\alpha),\ \alpha > 0$$

where the function $\phi(x,\alpha)$ solves the associated elliptic problem

$$-\phi''(.,\alpha) + \alpha^2\phi(.,\alpha) = \phi^{2\sigma+1}(.,\alpha)$$

Indeed, we explicitly have

$$\phi(x,1) = \frac{(\sigma+1)^{\frac{1}{2\sigma}}}{\cosh^{\frac{1}{\sigma}}(\sigma x)}$$

>From the work of Berestycki and Cazenave, it is well-known that these standing waves are unstable, in the sense that certain smooth initial data arbitrarily close to any $\phi(x,\alpha)$ result in finite-time blow-up. An analysis of the spectrum of the linearization of (1) around any standing wave reveals the existence of a point spectrum consisting of a 4 dimensional root space and two imaginary eigenvalues, one of which leads to exponential growth for the linear evolution at $t \to +\infty$. This intuitively suggests that choosing initial data in at least a co-dimension 5 manifold of initial data should result in smooth solutions as $t \to +\infty$. Indeed, applying the inherent symmetries of (1) consisting of Galilei transforms, re-scalings and phase-shifts should allow one to recoup four of the missing dimensions, resulting in a co-dimension 1 manifold of good initial data.
To construct these solutions, one tries an ansatz

$$\psi(t,x) = W(t,x) + R(t,x),$$

where $W(t,x)$ is a Galilei-transformed, phase-shifted and re-scaled version of the standing wave $e^{it}\phi(x,1)$:

$$W(t,x) := e^{i(v(t)x - \int_0^t (v^2(s)-\alpha^2(s))ds + \gamma(t))}\phi(x - 2\int_0^t v(s)ds - D(t), \alpha(t)) \tag{2}$$

depending on the parameters $\pi(t) := \{\gamma(t), v(t), D(t), \alpha(t)\}$, and $R(t,x)$ represents the dispersive radiation part. The purpose of these parameters is to ensure that the radiation part $R(t,x)$ is controlled as far as its root part is concerned. We have

Theorem 1.1. Fix some $\sigma > 2$ in (1) and any $\alpha_0 > 0$. Then there exist a real-linear subspace $\mathcal{S} \subset L^2(\mathbf{R})$ of co-dimension five and a small $\delta > 0$ with the following properties: Let

$$\mathcal{B} := \Big\{ R_0 \in L^2(\mathbb{R}) \mid \|\!|R_0|\!\| := \|R_0\|_{H^1} + \|\langle x\rangle R_0\|_{L^1\cap L^2} + \|\langle x\rangle \partial_x R_0\|_{L^1} < \delta \Big\} \tag{3}$$

and let $\Sigma := \{f \in L^2(\mathbb{R}) \mid \|\!|f|\!\| < \infty\}$. Then there exists a map[1] $\Phi : \mathcal{B} \cap \mathcal{S} \to \Sigma$ with the properties

$$\|\!|\Phi(R_0)|\!\| \lesssim \|\!|R_0|\!\|^2 \qquad \forall R_0 \in \mathcal{B} \cap \mathcal{S} \tag{4}$$

$$\|\!|\Phi(R_0) - \Phi(\tilde{R}_0)|\!\| \lesssim \delta \|\!|R_0 - \tilde{R}_0|\!\| \qquad \forall R_0, \tilde{R}_0 \in \mathcal{B} \cap \mathcal{S} \tag{5}$$

and so that for any $R_0 \in \mathcal{B} \cap \mathcal{S}$ the NLS (1) has a global H^1 solution $\psi(t)$ for $t \geq 0$ with initial condition $\psi(0) = \phi(\cdot, \alpha_0) + R_0 + \Phi(R_0)$. Moreover,

$$\psi(t) = W(t, \cdot) + R(t)$$

where W as in (2) is governed by a path $\pi(t)$ of parameters which converges to some terminal vector $\pi(\infty)$ such that $\sup_{t\geq 0} |\pi(t) - \pi(\infty)| \lesssim \delta$ and so that

$$\|R(t)\|_{H^1} \lesssim \delta, \quad \|R(t)\|_\infty \lesssim \delta \langle t\rangle^{-\frac{1}{2}}, \quad \|\langle x - y(t)\rangle^{-\frac{1}{2}-\varepsilon} R(t)\|_\infty \lesssim \delta \langle t\rangle^{-1-\varepsilon} \tag{6}$$

for all $t > 0$ and some $\varepsilon > 0$. The solution $\psi(t)$ is unique amongst all solutions with these initial data and satisfying the above decay assumptions as well as certain orthogonality relations and certain decay assumptions on the path. Finally, there is scattering:

$$R(t) = e^{it\partial_x^2} f_0 + o_{L^2}(1) \quad \text{as } t \to \infty$$

for some $f_0 \in L^2(\mathbb{R})$.

As mentioned earlier, one regains 4 dimensions here by letting the inherent symmetries act on S:

Theorem 1.2. Fix any $\alpha_0 > 0$. Then there exist a small $\delta > 0$ and a Lipschitz manifold $\mathcal{N}$ inside the space Σ of size[2] δ and codimension one so that $\phi(\cdot, \alpha_0) \in \mathcal{N}$ with the following property: for any choice of initial data $\psi(0) \in \mathcal{N}$ the NLS (1) has a global H^1 solution $\psi(t)$ for $t \geq 0$. Moreover,

$$\psi(t) = W(t, \cdot) + R(t)$$

where W as in (2) is governed by a path $\pi(t)$ of parameters so that $|\pi(0) - (0, 0, 0, \alpha_0)| \lesssim \delta$ and which converges to some terminal vector $\pi(\infty)$ such that $\sup_{t\geq 0} |\pi(t) - \pi(\infty)| \lesssim \delta$. The solution is unique under the same conditions as in the preceding theorem. Finally, (6) holds and there is scattering:

$$R(t) = e^{it\partial_x^2} f_0 + o_{L^2}(1) \quad \text{as } t \to \infty$$

for some $f_0 \in L^2(\mathbb{R})$.

[1] $\mathcal{B} \cap \mathcal{S}$ is L^2-dense in $\mathcal{S}$

[2] This means that $\mathcal{N}$ is the graph of a Lipschitz map Ψ with domain $\mathcal{B} \cap \tilde{\mathcal{S}}$ where $\tilde{\mathcal{S}}$ is a subspace of codimension one and with $\mathcal{B}$ as in (3). As before, $\mathcal{B} \cap \tilde{\mathcal{S}}$ is L^2-dense in $\tilde{\mathcal{S}}$.

The proof of these theorems relies heavily on an analysis of the spectral properties of the operator

$$\mathcal{H} = \mathcal{H}(\alpha) = \begin{pmatrix} -\partial_x^2 + \alpha^2 - (\sigma+1)\phi^{2\sigma}(\cdot,\alpha) & -\sigma\phi^2(\cdot,\alpha) \\ \sigma\phi^{2\sigma}(\cdot,\alpha) & \partial_x^2 - \alpha^2 + (\sigma+1)\phi^{2\sigma}(\cdot,\alpha) \end{pmatrix}$$

The spectrum is located on $\mathbf{R} \cup i\mathbf{R}$ with continuous spectrum consisting of the intervals $(-\infty, -\alpha^2] \cup [\alpha^2, \infty)$, while the discrete spectrum consists of $\{0, \pm i\gamma(\alpha)\}$ for suitable $\gamma(\alpha) \in \mathbf{R}_+$. Important further properties of this spectrum are the absence of resonances at the endpoints of the continuous spectrum. Control over the radiation part then relies on the following linear estimates:

$$||e^{it\mathcal{H}} P_{dis} \begin{pmatrix} R \\ \bar{R} \end{pmatrix} ||_{L^2_x} \lesssim ||R||_{L^2_x}, \; ||e^{it\mathcal{H}} P_{dis} \begin{pmatrix} R \\ \bar{R} \end{pmatrix} ||_{L^\infty_x} \lesssim \langle t \rangle^{-\frac{1}{2}} ||R||_{L^1_x}$$

$$||\langle x \rangle^{-1} e^{it\mathcal{H}} P_{dis} \begin{pmatrix} R \\ \bar{R} \end{pmatrix} ||_{L^\infty_x} \lesssim \langle t \rangle^{-\frac{3}{2}} ||\langle x \rangle R||_{L^1_x}$$

Of these inequalities the last is the most delicate, and represents a significant departure from the behavior of the free evolution $e^{it\mathcal{H}_0}$ where

$$\mathcal{H}_0 := \begin{pmatrix} -\partial_x^2 + \alpha^2 & 0 \\ 0 & \partial_x^2 - \alpha^2 \end{pmatrix}$$

Indeed, this is where the absence of resonances at the edges of the continuous spectrum comes in.

The intriguing question remains as to what happens for initial data close to but away from the initial data manifold $\mathcal{N}$.

References

[1] Agmon, S. *Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators.* Mathematical Notes, 29. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1982.

[2] Berestycki, H., Cazenave, T. *Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires.* C. R. Acad. Sci. Paris Sér. I Math. 293 (1981), no. 9, 489–492.

[3] Buslaev, V. S., Perelman, G. S. *Scattering for the nonlinear Schrödinger equation: states that are close to a soliton.* (Russian) Algebra i Analiz 4 (1992), no. 6, 63–102; translation in St. Petersburg Math. J. 4 (1993), no. 6, 1111–1142.

[4] Cazenave, T., Lions, P.-L. *Orbital stability of standing waves for some nonlinear Schrödinger equations.* Comm. Math. Phys. 85 (1982), 549–561

[5] Christ, M., Kiselev, A. *Maximal functions associated with filtrations,* J. Funct. Anal. 179 (2001), 409-425.

[6] Cuccagna, S. *Stabilization of solutions to nonlinear Schrödinger equations.* Comm. Pure Appl. Math. 54 (2001), no. 9, 1110–1145.

[7] Flügge, S. *Practical quantum mechanics.* Reprinting in one volume of Vols. I, II. Springer-Verlag, New York-Heidelberg, 1974.

[8] Goldberg, M., Schlag, W. *Dispersive estimates for Schrödinger operators in dimensions one and three.* Comm. Math. Phys. 251 (2004), no. 1, 157–178.

[9] Grillakis, M. *Analysis of the linearization around a critical point of an infinite dimensional Hamiltonian system.* Comm. Pure Appl. Math. 41 (1988), no. 6, 747–774.

[10] Kato, T. *Wave operators and similarity for some non-selfadjoint operators.* Math. Ann. 162 (1965/1966), 258–279.
[11] Perelman, G. *On the formation of singularities in solutions of the critical nonlinear Schrödinger equation.* Ann. Henri Poincaré 2 (2001), no. 4, 605–673.
[12] Reed, M., Simon, B. *Methods of modern mathematical physics. IV.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1979.
[13] Rodnianski, I., Schlag, W. *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials.* Invent. Math. 155 (2004), no. 3, 451–513.
[14] Schlag, W. *Stable manifolds for an orbitally unstable NLS.* preprint, 2004.
[15] Schlag, W. *Recent developments on dispersive estimates for Schrödinger equations.* preprint 2004, to appear in Conference proceedings IAS.
[16] Smith, H., Sogge, C. *Global Strichartz estimates for nontrapping perturbations of the Laplacian.* Comm. Partial Differential Equations 25 (2000), no. 11-12, 2171–2183.
[17] Soffer, A., Weinstein, M. *Multichannel nonlinear scattering for nonintegrable equations.* Comm. Math. Phys. 133 (1990), 119–146
[18] Soffer, A., Weinstein, M. *Multichannel nonlinear scattering, II. The case of anysotropic potentials and data.* J. Diff. Eq. 98 (1992), 376–390
[19] Weinstein, Michael I. *Modulational stability of ground states of nonlinear Schrödinger equations.* SIAM J. Math. Anal. 16 (1985), no. 3, 472–491.
[20] Weinstein, Michael I. *Lyapunov stability of ground states of nonlinear dispersive evolution equations.* Comm. Pure Appl. Math. 39 (1986), no. 1, 51–67.

A geometric convergence proof for the Yamabe flow

Michel Grüneberg

We study a geometric evolution equation which is derived from the total scalar curvature functional on the space of Riemannian metrics on a compact manifold, and prove a general convergence result in the case of manifolds that have positive conformal Yamabe invariant. In particular, our method does not require that the manifold be conformally inequivalent to the standard round n-sphere (S^n, g_{round}).

Specifically, we consider the *Yamabe flow*

$$\begin{cases} \dfrac{\partial}{\partial t} g(t) = \big(r_{g(t)} - R_{g(t)}\big)\, g(t), \\ \quad g(0) = g^0 \text{ given,} \end{cases}$$

on a compact Riemannian manifold (M^n, g_{back}) of dimension $n \geq 3$, starting from an arbitrary initial metric $g^0 \in [g_{\text{back}}]$, the conformal class of the background metric. Here, $r_{g(t)} := \operatorname{vol}\big(g(t)\big)^{-1} \int_M R_{g(t)}\, d\mu_{g(t)}$ is the average of the scalar curvature of $g(t)$ over M.

This geometric evolution equation was originally introduced by R. Hamilton shortly after the Ricci flow as an alternative approach to solving the Yamabe problem (see [4] for an account) on manifolds of positive conformal Yamabe invariant; however it did not appear in the literature before 1988 (see [3]). It arises as the negative L^2-gradient flow for the (normalized) total scalar curvature functional when restricted to a conformal class, which is given by

$$\mathscr{R}(g) = \frac{1}{\mathrm{vol}(g)^{\frac{n-2}{n}}} \int_M R_g \, d\mu_g \quad \text{for } g \in [g_{\mathrm{back}}].$$

As such, it can be viewed as a natural geometric deformation of a Riemannian metric to a conformal metric of constant scalar curvature. Therefore the convergence question for this flow constitutes the "parabolic version" of Yamabe's problem.

The goal of this talk is to outline a proof of the general convergence result for this flow on compact three-manifolds, given arbitrary initial metrics. We first give a new, *local* proof (compare with [8]) of the general convergence result for the Yamabe flow on compact locally conformally flat manifolds of positive conformal Yamabe invariant: We show that on such manifolds, the evolving metric produced by the flow converges smoothly from any initial metric to a unique limit metric of constant scalar curvature. We then outline how this proof can be modified to prove convergence from arbitrary initial data on any compact three-manifold of positive conformal Yamabe invariant.

The main idea of the proof is to use the local Weil-Petersen geometry near the $(n+2)$-dimensional submanifold $\mathscr{M}_{\mathrm{cc}}$ of constant curvature metrics in the conformal class $[g_{\mathrm{round}}]$ of the standard metric on S^n to describe the evolution of large solutions $g(t)$ of the Yamabe flow by means of a projected path of constant curvature metrics $g_0(t)$. We carefully estimate the projection error that occurs in this procedure, and based on this estimate we show how the Riemannian Pohozaev Identity (see [5]) and the Riemannian Positive Mass Theorem (see [6]) can be used to deduce that the scale parameter of $g_0(t)$ (measuring the concentration of the constant curvature metric) improves once the solution is greater than some threshold value.

By making this construction canonical, we succeed in applying this fact to carefully chosen collections of bubbles of the solution in order to conclude that $g(t)$ remains uniformly bounded, which implies convergence of the flow.

This work uses in its first step (where we show that the scalar curvature function produced by the flow converges uniformly to a constant) an integral decay estimate that was obtained by H. Schwetlick and M. Struwe (see [7, Lemma 3.3]), and is based on quite different methods than S. Brendle's work [1].

References

[1] Simon Brendle, *Convergence of the Yamabe flow for arbitrary initial energy*, Preprint, 2004.

[2] Michel Grueneberg, *The Yamabe flow on three-manifolds*, Ph.D. thesis, Stanford University, 2004.

[3] Richard S. Hamilton, *The Ricci flow on surfaces*, Mathematics and general relativity (Santa Cruz, CA, 1986), Contemp. Math., vol. 71, Amer. Math. Soc., Providence, RI, 1988, pp. 237–262.

[4] John M. Lee and Thomas H. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. (N.S.) **17** (1987), no. 1, 37–91.

[5] Richard M. Schoen, *The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation*, Comm. Pure Appl. Math. **41** (1988), no. 3, 317–392.

[6] Richard M. Schoen and Shing Tung Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), no. 1, 45–76.
[7] Hartmut Schwetlick and Michael Struwe, *Convergence of the Yamabe flow for "large" energies*, J. Reine Angew. Math. **562** (2003), 59–100.
[8] Rugang Ye, *Global existence and convergence of Yamabe flow*, J. Differential Geom. **39** (1994), no. 1, 35–50.

Global Convergence of the Yamabe Flow

Simon Brendle

Let M be a compact manifold of dimension $n \geq 3$. Along the Yamabe flow, the Riemannian metric is deformed such that

$$\frac{\partial g}{\partial t} = -(R_g - r_g)\, g,$$

where R_g is the scalar curvature associated with the metric g and r_g denotes the mean value of R_g. Since the velocity is a scalar multiple of the metric, the Yamabe flow preserves the conformal structure.

The Yamabe flow can be reduced to a nonlinear partial differential equation of parabolic type. Indeed, if we write the metric in the form $g = u^{\frac{4}{n-2}}\, g_0$ for a fixed background metric g_0, then the function u satisfies the equation

$$\frac{\partial}{\partial t} u^{\frac{n+2}{n-2}} = \frac{n+2}{4}\Big(\frac{4(n-1)}{n-2}\,\Delta_{g_0} u - R_{g_0}\, u + r_g\, u^{\frac{n+2}{n-2}}\Big).$$

It is known that the Yamabe flow exists for all time. Moreover, if $3 \leq n \leq 5$ or M is locally conformally flat, then the flow approaches a metric of constant scalar curvature as $t \to \infty$. I discuss how this result can be generalized to higher dimensions ($n \geq 6$) under a technical condition on the Weyl tensor. The proof requires the construction of an appropriate family of test functions.

Stability of Minkowski space in harmonic gauge

Igor Rodnianski
(joint work with H. Lindblad)

The talk discussed a new proof of global stability of Minkowski space for the Einstein-vacuum and Einstein-scalar field equations for general asymptotically flat initial data. The new approach relies on the use of the harmonic gauge previously believed to be unsuitable for this problem. The problem is viewed as a "small data global existence result" for a system of quasilinear wave equations for the components of a metric and uses a notion of the weak null condition.

Linear and non-linear waves in general relativity

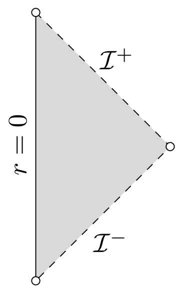
MIHALIS DAFERMOS
(joint work with Igor Rodnianski)

Let us begin our discussion with the wave equation

$$\Box\phi = 0 \tag{1}$$

on Minkowski space $\mathbb{R}^{3+1}$.

To understand the issue at hand, we need a good picture of the causal structure of $\mathbb{R}^{3+1}$. This can be provided by its so-called *Penrose diagram*:



What is pictured above is in fact the 2-dimensional Lorentzian quotient $\mathcal{Q} = \mathbb{R}^{3+1}/SO(3)$, conformally embedded as a bounded subset of $\mathbb{R}^{1+1}$. The point of these diagrams is two-fold: On the one hand, we can read off by inspection the causal structure, i.e. we can identify spacelike, (future and past-directed) timelike, and null curves, and, from these concepts, for each point $p \in \mathcal{Q}$, its *causal future*,[1] denoted $J^+(p)$, and its *causal past*, denoted $J^-(p)$. On the other hand, as the embedding is bounded, it induces a boundary $\partial\mathcal{Q} \subset \mathbb{R}^{1+1}$ to which causal relations can also be applied. In particular, we can identify a subset $\mathcal{I}^+ \subset \partial\mathcal{Q}$ called *future null infinity*. This set comprises of limit points of future-directed null rays in $\mathcal{Q}$ for which $r \to \infty$, wher r denotes the *area radius* function[2].

Let ϕ now be an $SO(3)$-invariant solution of the wave equation (1) such that ϕ and $\nabla\phi$ are of compact support on a complete Cauchy hypersurface with projection in $\mathcal{Q}$ to be denoted Σ. Writing the equation (1) as

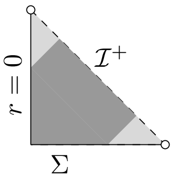
$$\partial_u\partial_v(r\phi) = 0$$

where u and v are arbitrary null coordinates with respect to the induced metric of $\mathcal{Q}$, it is clear that, to the future of Σ, ϕ is supported in the darker-shaded region

[1]The set of all points accessible from p by a future directed causal (i.e. timelike or null) curve.

[2]i.e. the function $r : \mathcal{Q} \to \mathbb{R}$ defined by $r(p) = \sqrt{Area(\pi_1^{-1}(p))/4\pi}$, where π_1 denotes the natural projection $\pi_1 : \mathbb{R}^{3+1} \to \mathcal{Q}$

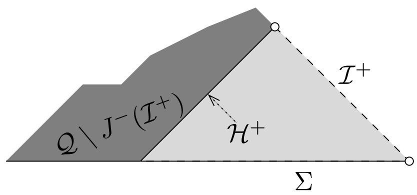
below:



This is the celebrated *strong Huygens principle*. Moreover, $r\phi$ extends regularly to $\mathcal{I}^+$, along which it is clearly compactly supported.

The subject of this talk is: What is the analogue of the above properties of ϕ when Minkowski space is replaced by the exterior of a **black hole**?

First we must understand when a spacetime $(\mathcal{M}, g)$ contains a black hole: Let us restrict consideration to spherically symmetric spacetimes, so–as with $\mathbb{R}^{3+1}$–we may talk of the 2-dimensional Lorentzian quotient $\mathcal{Q} = \mathcal{M}/SO(3)$, conformally embedded into a bounded domain of $\mathbb{R}^{1+1}$, and the area function r. We can define as before $\mathcal{I}^+$, and in the asymptotically flat case, this set will be non-empty. If $\mathcal{Q} = J^-(\mathcal{I}^+)$, we say that $\mathcal{Q}$ does *not* contain a black hole. This is clearly the case for $\mathbb{R}^{3+1}$, by inspection of its Penrose diagram. Otherwise, we call $\mathcal{Q} \setminus J^-(\mathcal{I}^+)$ the *black hole* of $\mathcal{Q}$, and $J^-(\mathcal{I}^+)$ the *exterior*. The future boundary $\mathcal{H}^+$ of $J^-(\mathcal{I}^+)$ in $\mathcal{Q}$ is known as the *event horizon*. These concepts are illustrated below:



Why is one interested in the behaviour of waves on black hole exteriors? The Einstein equations[3] themselves

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 2T_{\mu\nu} \tag{2}$$

can be viewed as a system of non-linear wave equations for $g_{\mu\nu}$, with respect to the Lorentzian metric defined again by $g_{\mu\nu}$. The behaviour of these waves on $\mathcal{I}^+$ is precisely what is observable to detection by the astrophysicist. But moreover, the very tenability of the notion of black holes rests on the conjecture that the explicit Kerr family[4] of black-hole solutions is dynamically stable. As in the case with the proof of stability of Minkowski space, one expects that this stability depends crucially on the decay properties of waves. Finally, decay rates along $\mathcal{H}^+$ are intimately related to the internal structure of black holes, in particular,

[3]These are the fundamental equations which define the dynamics of the gravitational field in general relativity.

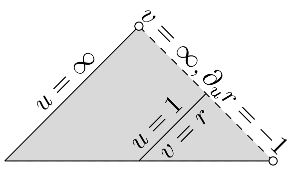
[4]These comprise a 2-parameter family which contain all axisymmetric stationary one-black hole solutions to the Einstein vacuum equations and are presently assumed by astrophysicists to describe the gravitational field surrounding millions of objects in the visible universe.

the existence and nature of "apparent" horizons, and the validity of the so-called strong cosmic censorship conjecture, i.e. the unique predictability of the future for all surviving observers.

In this talk, I report on fairly complete understanding of the issue of decay in what is essentially the simplest situation where it can be posed, namely the collapse of a spherically symmetric self-gravitating scalar field, i.e. the Cauchy problem for the coupled nonlinear system comprising of (2) and $\Box_g \phi = 0$, where

$$T_{\mu\nu} = \phi_\mu \phi_\nu - \frac{1}{2} g_{\mu\nu} \phi^\alpha \phi_\alpha,$$

for asymptotically flat, spherically symmetric initial data. Definitions of all these notions can be found in [6]. The study of the above system was initiated by Christodoulou [2]. In [3], he showed that for generic initial data, the causal structure of the solution spacetime is either as depicted above, or as in the case of Minkowski space. One can define a null coordinate system covering $J^-(\mathcal{I}^+)$ by declaring some arbitrary null ray reaching $\mathcal{I}^+$ to be $u = 1$, normalizing $v = r$ along that ray, and normalizing u asymptotically along $\mathcal{I}^+$ by the condition $\partial_u r = -1$.[5] With this normalization, it turns out that u will take all real values. This latter fact is known as *weak cosmic censorship*, i.e. astrophysical observers observe a regular past for infinite proper time:



Our first theorem [6] is

Theorem 1. *Let $(\mathcal{M}, g, \phi)$ be a spherically symmetric solution to the Einstein-scalar field system*[6]*, such that ϕ, $\nabla\phi$ are of compact support on a complete Cauchy surface, and assume we are in the case where the Penrose diagram of $\mathcal{Q}$ is as above. With respect to the above-defined coordinates we have*

$$|r\phi(u,\infty)| \leq Cu^{-2}, |\partial_u(r\phi)(u,\infty)| \leq C_\epsilon u^{-3+\epsilon}, \tag{3}$$

and, for $r(u,v) \leq R < \infty$, we have

$$|\phi(u,v)| \leq C_{R,\epsilon} v^{-3+\epsilon}, |\partial_v \phi(u,v)| \leq C_{R,\epsilon} v^{-3+\epsilon}. \tag{4}$$

These decay rates were discovered by R. Price [8,10], and go by the name *Price's law.* They are thought to be sharp (modulo ϵ). Actually, Price's heuristic analysis

[5] Coordinates u and v are known as retarded (Bondi) and advanced time, respectively.

[6] Our results also apply in the presence of an additional gravitationally-coupled electromagnetic field. In this case, the theorem, together with the results of [4], imply that strong cosmic censorship is false for this system.

was applied to the decoupled problem where a Schwarzschild metric[7] is fixed and ϕ is the zero'th spherical harmonic of the wave equation. Our results also apply to this problem:

Theorem 2. *The decay rates* (3)–(4) *apply for the* 0*'th spherical harmonic of a solution* ϕ *of the wave equation on a fixed Schwarzschild or Reissner-Nordström*[8] *background, where* ϕ, $\nabla\phi$ *are of compact support on a complete Cauchy surface.*

Here, we mention that if $r_0 > 2m$, then in the region $r_0 \leq r(u,v) \leq R$, the coordinate v is equivalent to the static time t. Similar decay rates in t for such fixed r have also been announced by Machedon and Stalker [9].

Theorem 1 does not include a smallness condition. In the absense of symmetry, one only expects to prove results perturbatively around known solutions— Kerr in the black hole context— and it is almost certain that precise understanding of decay must be used even to show that the solution does not break down. A natural question is whether the decay rates of (3)–(4) are in principle sufficient to handle a non-linearity perturbatively.

A model problem to pose in this context is

$$\Box_g \phi = |\phi|^p, \tag{5}$$

for some power p, on a Schwarzschild or Reissner-Nordström exterior background. The associated energy for this field is not necessarily positive. In view of the staticity of the background metric, it is natural to write the solution to (5) as a superposition of solutions to the linear homogeneous equation, with the nonlinearity appearing in the data. This representation is just the well-known Duhammel's principle. It should be clear that *a priori*, proving stability appears potentially problemetic, as one must integrate in t, and yet, there is *no uniform decay* in t for the linear homogeneous problem. Indeed, all points on $\mathcal{H}^+$ are accessible as a limit $t \to \infty$, while ϕ will not vanish there.

It turns out that for large enough p, this problem can be overcome by a careful analysis of the black hole geometry, together with use of the red-shift effect (see [6]), directly for the non-linear problem. We obtain [7]

Theorem 3. *Let* ϕ *be a spherically symmetric solution of* (5) *with compactly supported initial data on a Cauchy hypersurface* Σ *in Schwarzschild or Reissner-Nordström. For* $p > 4$*, if the* C^1 *norm of the data is sufficiently small, and spherically symmetric, the domain of existence of* ϕ *includes all of* $J^-(\mathcal{I}^+) \cap J^+(\Sigma)$*, and, defining a* v *coordinate as before, we have* $|\phi| \leq Cv^{-1}$ *in this region.*

We note that for $p < 1 + \sqrt{2}$, blow-up has been shown [1] for arbitrarily small-data solutions of (5). This agrees with the situation in Minkowski space, first

[7]The Schwarzschild family is a one-parameter subfamily of the Kerr family, with parameter m, and describes all spherically symmetric solutions of (2) with $T_{\mu\nu} = 0$.

[8]The Reissner-Nordström family is a two-parameter family describing all spherically symmetric solutions of (2) with $T_{\mu\nu}$ the energy momentum tensor of an electromagnetic field; by convention, we mean here a globally hyperbolic spacetime.

studied by F. John. Understanding the behaviour in the range $[1+\sqrt{2}, 4]$ remains an open problem.

References

[1] D. Catania and V. Georgiev *Blowup for the semilinear equation in Schwarzschild metric*, math-ph/0407001, preprint 2004

[2] D. Christodoulou *The problem of a self-gravitating scalar field* Comm. Math. Phys. **105** (1986), no. 3, 337–361

[3] D. Christodoulou *The instability of naked singularities in the gravitational collapse of a scalar field* Ann. of Math. **149** (1999), no 1, 183–217

[4] M. Dafermos *The interior of charged black holes and the problem of uniqueness in general relativity* Comm. Pure Appl. Math. **58** (2005), 445–504

[5] M. Dafermos *Stability and instability of the Cauchy horizon for the spherically-symmetric Einstein-Maxwell-scalar field equations*, Ann. Math. **158** (2003), 875–928

[6] M. Dafermos and I. Rodnianski *A proof of Price's law for the collapse of a self-gravitating scalar field*, to appear in Invent. Math.

[7] M. Dafermos and I. Rodnianski *Small amplitude non-linear waves on a black hole background*, to appear in Jour. Math. Pures Appl.

[8] C. Gundlach, R. H. Price, and J. Pullin *Late-time behavior of stellar collapse and explosions. I. Linearized perturbations* Phys. Rev. D **49** (1994), 883–889

[9] M. Machedon and J. Stalker *Decay of solutions to the wave equation on a spherically symmetric background*, preprint

[10] R. Price *Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations* Phys. Rev. D (3) **5** (1972), 2419-2438

Q-curvature flow on S^4

Andrea Malchiodi

(joint work with Michael Struwe, ETH, Zürich)

Let M be a closed four-manifold (compact without boundary) with metric g. If Ric_g denotes the Ricci Tensor of (M, g) and R_g the scalar curvature, the Q-curvature Q_g of M is defined by the expression

$$Q_g = -\frac{1}{12}\left(\Delta_g R_g - R_g^2 + 3|Ric_g|^2\right). \tag{1}$$

In complete analogy with the Gauss curvature on surfaces, in four space dimensions the Q-curvature of a metric $g = e^{2w} g_0$ is related to the Q-curvature Q_0 of the background metric g_0 via the equation

$$P_{g_0} w + 2Q_0 = 2Q_g e^{4w}, \tag{2}$$

where P_{g_0} is the *Paneitz operator* in the metric g_0. For any given g the operator P_g acts on a smooth function φ on M via

$$P_g(\varphi) = \Delta_g^2 \varphi - div\left((\frac{2}{3} R_g g - 2Ric_g) d\varphi \right). \tag{3}$$

Similar to the Laplace-Beltrami in two dimensions, the Paneitz operator is conformally invariant in the sense that

$$P_g = e^{-4w} P_{g_0} \tag{4}$$

for any conformal metric $g = e^{2w} g_0$. For the proofs of the above formulas we address the reader to the references in [1].

As for the Nirenberg's problem on S^2, our goal is to prescribe the Q-curvature of the standard sphere (S^4, g_{S^4}) as a given smooth and positive function f. Since $Q_{g_{S^4}} \equiv 3$, by (2) this is equivalent to finding a solution u of the equation

$$P_{g_{S^4}} u + 6 = 2 f e^{4u}, \tag{5}$$

where $P_{g_{S^4}} = \Delta^2_{g_{S^4}} - 2\Delta_{g_{S^4}}$. The problem is variational; solutions can be characterized as critical points of the functional

$$E_f(u) = \fint_{S^4} \left(u P_{g_{S^4}} u + 4 Q_{g_{S^4}} u \right) d\mu_{g_{S^4}} - 3 \log \left(\fint_{S^4} f e^{4u} d\mu_{g_{S^4}} \right) \tag{6}$$

on $H^2(S^4)$, where $\fint_{S^4} \varphi d\mu_{g_{S^4}}$ denotes the average of any function φ on S^4.

We prove the following result, which is based on the Morse inequalities and which extends a previous result in [2].

Theorem 1. *Suppose $f : S^4 \to \mathbb{R}$ is positive with only non-degenerate critical points with Morse indices $ind(f,p)$ and such that $\Delta_{g_{S^4}} f(p) \neq 0$ at any such point p. Let*

$$m_i = \#\{p \in S^4; \nabla f(p) = 0, \Delta_{g_{S^4}} f(p) < 0, ind(f,p) = 4 - i\}. \tag{7}$$

Then, if there is no solution with coefficients $k_i \geq 0$ to the system of equations

$$m_0 = 1 + k_0,\ m_i = k_{i-1} + k_i,\ 1 \leq i \leq 4,\ k_4 = 0, \tag{8}$$

there exists a solution of (5).

The theorem is proved using a flow in the conformal class of g_{S^4}. Given $g_0 = e^{2u_0} g_{S^4}$ with total volume equal to $\frac{8}{3}\pi^2$, we define $u(t)$ by

$$u_t = \frac{du}{dt} = \alpha f - Q, \tag{9}$$

with initial data $u(0) = u_0$. Here $Q = Q_g$ denotes the Q-curvature of $g = g(t)$, given by

$$Q = \frac{1}{2} e^{-4u} (P_{g_{S^4}} u + 6) \quad \text{on } S^4, \tag{10}$$

and α is chosen in such a way that

$$\alpha \int_{S^4} f \, d\mu = \int_{S^4} Q \, d\mu = 8\pi^2 \tag{11}$$

for all $t \geq 0$, where $d\mu = d\mu_g = e^{4u} d\mu_{g_{S^4}}$.

One can show that this flow is globally defined, preserves the total volume and decreases the functional E_f. Moreover, an accurate analysis show that either the flow is compact and converges to a solution of (5), or the metric $g(t) = e^{2u(t)} g_{S^4}$

becomes *round and concentrated* near some point $p(t) \in S^4$. In this case the point $p(t)$ approaches as $t \to +\infty$ a critical point of f for which $\Delta_{S^4} f \leq 0$.

Given a large number β, we consider then the set L_β consisting of the conformal metrics with total volume $\frac{8}{3}\pi^2$, and for which E_f is less than β. One can prove that if β is chosen sufficiently large L_β is contractible. Moreover, assuming non-existence of solutions to (5) and using the asymptotic analysis just described, one finds that L_β evolves into a contractible set N with some *handles* attached (in the sense of Morse). For each critical point p_i of f with $\Delta f(p_i) < 0$ there is a corresponding handle of dimension $4 - ind(f, p_i)$. Finally, using the assumptions of Theorem 1 and the Morse inequalities, one arrives to a contradiction to the fact that L_β and N are homotopic. Therefore problem (5) admits a solution.

References

[1] A. Malchiodi, M. Struwe, *Q-curvature flow on S^4*, preprint.

[2] J. Wei, X. Xu, *On conformal deformations of metrics on S^n*, J. Funct. Anal. **157-1** (1998), 292–325.

A Geometric Flow for Kirchhoff Elastic Rods

Hartmut R. Schwetlick

(joint work with Chun-Chi Lin)

Recently, rod theory has been applied to the mathematical modeling of bacterial fibers and biopolymers (e.g. DNA) to study their mechanical properties and shapes (e.g. supercoiling). In static rod theory, an elastic rod in equilibrium is the critical point of an elastic energy. This induces a natural question of how to find elasticae. In our project, we ask the question: starting from a given rod configuration Γ in $\mathbb{R}^3$, can we find the critical points of a Kirchhoff elastic energy, or the so called elasticae, by means of geometric gradient flows? In order to keep the model problem in this paper simple, we only consider a special isotropic Kirchhoff elastic energy. For more general rod theory, readers are referred to [1].

Suppose $f : I = \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ is the centerline of a closed rod. Let $\gamma = |\partial_x f|$, $ds = \gamma\, dx$ the arclength element, and $\partial_s = \gamma^{-1}\partial_x$ the arclength differentiation. Denote by $T = \partial_s f$ the unit tangent vector, and $\kappa = \partial_s^2 f$ the curvature vector of f. A rod configuration Γ is a framed curve described by $\{f(s); T(s), M_1(s), M_2(s)\}$, where the material frame $\{T, M_1, M_2\}$ forms an orthonormal frame field along f. Thus, we can write the skew-symmetric system

$$\begin{pmatrix} T' \\ M_1' \\ M_2' \end{pmatrix} = \begin{pmatrix} 0 & m_1 & m_2 \\ -m_1 & 0 & m \\ -m_2 & -m & 0 \end{pmatrix} \begin{pmatrix} T \\ M_1 \\ M_2 \end{pmatrix},$$

with arbitrary functions $m_1(s)$, $m_2(s)$, and $m(s)$. Consider the Kirchhoff elastic energy $\mathcal{E}$ of an isotropic rod Γ, defined by

$$\mathcal{E}[\Gamma] := \int_I \left(\alpha\cdot(m_1^2+m_2^2)+\beta\cdot m^2\right)\, ds,$$

with material constants $\alpha > 0$ and $\beta \geq 0$. The term involving α gives the bending energy, while the term involving β gives the twisting energy.

Whenever a smooth curve f has no inflection points, the Frenet frame field $\{T, N, B\}$ along f is well-defined. By using the Frenet frame field, it can be easily verified that

$$\mathcal{E}[\Gamma] = \int_I \left(\alpha|\kappa|^2+\beta m^2\right)\, ds, \tag{1}$$

(e.g., see [7]). A natural frame is an orthonormal frame field along a given curve f, which is uniquely determined by its initial data at a point and the skew-symmetric system,

$$\begin{pmatrix} T' \\ U' \\ V' \end{pmatrix} = \begin{pmatrix} 0 & u & v \\ -u & 0 & 0 \\ -v & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ U \\ V \end{pmatrix},$$

(see [3] or [7] p. 607). A natural frame can be thought as a frame without twisting. As we denote by θ the angle from U to M_1 with $\theta(0) = 0$, one can verify that m is equal to the twisting rate, i.e., $m(s) = \theta'(s)$. Whenever f contains no inflection points, the Frenet frame is well defined along f. Denote by ϕ the angle from U to N, then it is easy to verify that the torsion of the curve satisfies $\tau = \phi'$. Denote by $\Psi := \theta - \phi$ the angle from N to M_1 and let $\triangle\Psi := \Psi(L) - \Psi(0)$, where L is the total length of f. By these notations, we have

$$Tw[\Gamma] = \int_I m\, ds = \triangle\Psi + \int_I \tau\, ds. \tag{2}$$

We thus set up the boundary value problem by prescribing a real number, $\triangle\Psi$, which is called the end point condition of rod configurations in the rest of this paper. From above, we would like to emphasize that the bending energy and twisting energy interact as rod configurations achieving the critical points of the elastic energy. More precisely, the twisting depends on the centerlines of rods as well. Otherwise, the twisting energy and bending energy can be considered separately and the resulting centerlines of rod elasticae would simply be curve elasticae.

In [7], Langer and Singer proposed to study the generalized elastic curves by introducing the geometric functional $\widetilde{\mathcal{F}}$ of curves $f : I \to \mathbb{R}^3$,

$$\widetilde{\mathcal{F}}[f] := \lambda_3\mathcal{K}[f] + \lambda_2\mathcal{T}[f] + \lambda_1\mathcal{L}[f], \tag{3}$$

where

$$\mathcal{K}[f] := \int_I \frac{1}{2}|\kappa|^2 \ ds,\ \mathcal{T}[f] := \int_I \tau \ ds,\ \mathcal{L}[f] := \int_I ds,$$

and the λ_i are Lagrange multipliers for $i = 1, 2$. According to their formulation, a generalized elastic curve f in equilibrium is a critical point of the elastic energy $\widetilde{\mathcal{F}}$ among the class of curves with fixed total torsion $\mathcal{T}[f] = T_0$ and length $\mathcal{L}[f] = L$. As long as λ_i together with the fixed total torsion T_0 fit certain relations, they showed that f is the centerline of an isotropic elastic rod in equilibrium. The problems considered here and in Eq. (3) are closely related to curve straightening flow. To the authors' knowledge, curve straightening flows have been studied by Wen [10], Polden [9], Koiso [6] and Dziuk, Kuwert, Schätzle [4]. At the beginning, we tried to apply such methods to the geometric flow related to $\widetilde{\mathcal{F}}$ from [7]. However, we face an essential difficulty coming from the constraint of fixing the total torsion. Namely, after multiplying the term of the first variation of the total torsion $\mathcal{T}[f]$ by its Lagrange multiplier, the method of L^2 curvature estimates combined with Gagliardo-Nirenberg-type interpolation inequalities fails, because this term has higher power of derivatives in total than those coming from $\mathcal{K}[f]$.

In order to resolve this difficulty we propose another approach. We learn from [5] and [7] that a symmetric elastic rod must have a constant twisting rate. Observe that among all isotropic rod configurations Γ with constant twisting rate $m = \frac{\mathcal{T}[f]+\triangle\Psi}{L}$, fixed length L, but without inflection points, we have the identity,

$$\mathcal{E}[\Gamma] = \mathcal{G}_{\triangle\Psi,L}[f] := 2\alpha\mathcal{K}[f] + \frac{\beta}{L}\left(\mathcal{T}[f] + \triangle\Psi\right)^2.$$

Our first result, [8, Theorem 1], states that the equilibrium elastic rods must stay in the subclass of rod configurations with constant twisting rate and fixed length L. It turns out that in our geometric approach working with the functional

$$\mathcal{F}[f] := \mathcal{G}_{\triangle\Psi,L}[f] + \lambda_1 \cdot (\mathcal{L}[f] - L), \tag{4}$$

of curves with fixed length L is more suitable than working directly with the rod energy $\mathcal{E}$.

For smooth initial data we consider the length preserving L^2 gradient flow of $\mathcal{F}$ reading

$$\partial_t f = \lambda_3 \cdot (-\nabla_s^2\kappa - \frac{|\kappa|^2}{2}\kappa) + \lambda_2(t) \cdot \nabla_s(T \times \kappa) + \lambda_1(t) \cdot \kappa, \tag{5}$$

where $\lambda_2(t) := \frac{2\beta}{L}(\mathcal{T}[f] + \triangle\Psi)$, $\lambda_3 := 2\alpha$, and $\lambda_1(t)$ is chosen to fix the length.

Our main result, [8, Theorem 2], shows that for any real number $\triangle\Psi$ and any smooth initial closed curve f_0, there exists a smooth solution of (5), until the appearance of inflection points. With the assumption of no inflection points appearing during the flow, the curves sub-converge to f_∞, an equilibrium of the energy functional $\mathcal{F}$, after reparametrization by arclength and translation. Furthermore, if f_∞ contains no inflection points, then f_∞ is the centerline of an equilibrium Kirchhoff elastic rod with constant total twisting rate given by $\frac{\mathcal{T}[f_\infty]+\triangle\Psi}{L}$.

References

[1] S. S. Antman, *Nonlinear problems of elasticity*, Applied Mathematical Sciences, 107. Springer-Verlag, New York, 1995.
[2] T. Aubin, *Nonlinear analysis on manifolds. Monge-Ampère equations*, Grundlehren der Mathematischen Wissenschaften, 252. Springer-Verlag, New York, 1982.
[3] R. Bishop, *There is more than one way to frame a curve*, Amer. Math. Monthly, 82 (1975), 118-133.
[4] G. Dziuk, E. Kuwert, R. Schätzle, *Evolution of elastic curves in $\mathbb{R}^n$, existence and computation,* SIAM J. Math. Anal. 33 (2002), no. 5, 1228–1245.
[5] T. Ivey, D. Singer, *Knot types, homotopies and stability of closed elastic rods*, Proc. London Math. Soc. (3) 79 (1999), no. 2, 429–450.
[6] N. Koiso, *On the motion of a curve towards elastica*, Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), 403–436, Sémin. Congr., 1, Soc. Math. France, Paris, 1996.
[7] J. Langer, D. Singer, *Lagrangian aspects of the Kirchhoff elastic rod*, SIAM Rev. 38 (1996), no. 4, 605–618.
[8] C.C. Lin, H. Schwetlick, *On the geometric flow of Kirchhoff elastic rods*, SIAM J. Appl. Math. 65 (2005), no. 2, 720–736.
[9] A. Polden, *Curves and surfaces of least total curvature and fourth-order flows*, Ph.D. dissertation, Universität Tübingen, Tübingen, Germany, 1996.
[10] Y. Wen, *Curve straightening flow deforms closed plane curves with nonzero rotation number to circles*, J. Differential Equations 120 (1995), no. 1, 89–107.

Time-interior gradient estimates for quasilinear evolution equations

Julie Clutterbuck
(joint work with Ben Andrews)

This talk concerned an oscillation bound for a class of parabolic equations of the form

$$u_t = \text{trace}\{A(Du,t)D^2u\} + b(Du,t), \tag{1}$$

characterised by the positivity of

$$\alpha(r) := r^2 \inf\left\{ \frac{v^T A(q,t) v}{(v\cdot q)^2} \ : q\in\mathbb{R}^n, |q| = r, v\in\mathbb{R}^n\backslash(q)^\perp \right\}. \tag{2}$$

Our method is descended from one used by Kruzhkov [4] to find estimates for such equations in only one spatial dimension. The essential part of our argument is as follows: Let u be a smooth solution of (1) that is spatially periodic over a lattice $\Gamma \in \mathbb{R}^n$ (similar techniques lead to estimates for the Dirichlet and Neumann problems on a domain $\Omega \subset \mathbb{R}^n$). Define a new quantity on $\{(x,y,t) \in \mathbb{R}\times\mathbb{R}\times[0,T] : y \neq x\}$ by $Z(x,y,t) := u(y,t) - u(x,t) - 2\varphi(|y-x|,t)$, for φ to be chosen later so that $Z(\cdot,\cdot,0) < 0$. We then calculate the evolution equation for Z at the first positive maximum:

$$\begin{aligned}Z_t &= \text{trace}\left\{A(\varphi' e_1)D^2u(y,t)\right\} - \text{trace}\left\{A(\varphi' e_1)D^2u(x,t)\right\} - 2\varphi_t \\ &\qquad + b(\varphi' e_1, t) - b(\varphi' e_1, t) \\ &= \text{trace}\left\{\begin{bmatrix} A(\varphi' e_1) & C \\ C^T & A(\varphi' e_1)\end{bmatrix}\begin{bmatrix} D_{xx}Z & D_{xy}Z \\ D_{xy}Z & D_{yy}Z\end{bmatrix}\right\} - 2\varphi_t \\ &\qquad + \varphi''(A_{11} - C_{11}) + \frac{2\varphi'}{|y-x|}\left(\sum_{i=2}^{n}(A_{ii} - C_{ii})\right),\end{aligned}$$

where we have introduced new coordinates with $e_1 = (y-x)/|y-x|$, and C is an $n \times n$ matrix. If A satisfies the condition (2), then we may choose C so that the matrix $\begin{bmatrix} A(\varphi' e_1) & C \\ C^T & A(\varphi' e_1)\end{bmatrix}$ is positive definite, the coefficient of φ' is zero, and the coefficient of φ'' is maximised. Consequently, we find that $Z_t \le -2\varphi_t + 2\alpha(|\varphi'|)\varphi''$.

Now, φ is chosen to satisfy the one-dimensional parabolic equation $\varphi_t \ge \alpha(|\varphi'|)\varphi''$, with singular initial data given by a Heaviside function; this φ captures the worst behaviour of the oscillation of u. We then have $Z_t \le 0$ at a first positive maximum, so the maximum principle gives the estimate

$$|u(y,t) - u(x,t)| \le 2\varphi\left(|y-x|/2, t\right).$$

This leads to gradient estimates $|Du(\cdot,t)| \le C(t)$ for equations such as the mean curvature flow (here the estimate derived is similar to that by Evans and Spruck [3]), the anisotropic mean curvature flow, and the p-heat flow. The estimate for the anisotropic mean curvature flow has also been established using other methods in [2].

As an example, the following theorem applies to cases where A scales in a similar way to the coefficients for mean curvature flow for large values of Du:

Theorem. If there are positive constants a and P so that

$$\alpha(|p|)|p|^2 \ge a \text{ for } |p| \ge P$$

then there is a $T' > 0$ such that for $t \in (0, T']$,

$$|Du| \le C_1(1 + t^{3/2}\exp(C_2/t)),$$

where T', C_1 and C_2 depend on n, a, P and $\text{osc}u$.

None of gradient estimates depend on an initial gradient bound, but rather on an oscillation bound. This work may be found in the forthcoming article [1].

References

[1] Ben Andrews and Julie Clutterbuck, *Time-interior gradient estimates for quasilinear parabolic equations*, Preprint (2005).

[2] Julie Clutterbuck, *Interior gradient estimates for anisotropic mean curvature flow*, Preprint (2005).

[3] L.C. Evans and J. Spruck, *Motion of level sets by mean curvature. III.*, J. Geom. Anal. **2** (1992), no. 2, 121–150.

[4] S. N. Kružkov, *Quasilinear parabolic equations and systems with two independent variables*, Trudy Sem. Petrovsk. (1979), no. 5, 217–272. MR **81a:**35046

The Degenerate Gauss and Harmonic Mean Curvature Flows

Panagiota Daskalopoulos

We consider the evolution of weakly convex surfaces Σ in $\mathbb{R}^3$ with flat sides by the fully nonlinear geometric flows: the Gauss Curvature flow and the Harmonic Mean Curvature flow. The equations become degenerate at each flat side, whose free boundary moves with finite speed.

We study the following questions: (i) The short time existence and regularity of the interface and the solutions up to the interface (smoothing effect of the interface); (ii) The all time regularity of the interface; (iii) The evolution of the interface; (iv) The final shape of the interface.

The Gauss Curvature Flow: We consider the evolution of a weakly convex surface $\Sigma(t)$ with flat sides in $\mathbb{R}^3$ by the *Gauss Curvature Flow* (GCF)

$$\frac{\partial P}{\partial t} = K\, N$$

where each point P moves in the inward direction N with velocity equal to the Gaussian Curvature K of the surface. Expressing the surface $\Sigma(t)$ near the flat side as a graph $z = f(x, y, t)$, the function f evolves by

$$f_t = \frac{\det D^2 f}{(1 + |Df|^2)^{3/2}} \tag{1}$$

a *fully-nonlinear* equation which becomes *degenerate* at the interface $\Gamma(t)$ between the flat and strictly convex sides.

Assuming that $g = \sqrt{f}$ satisfies at $t = 0$ the non-degeneracy condition

$$0 < c \le |Dg| \le C < \infty \quad \text{and} \quad g_{\tau\tau} \ge c > 0$$

near the interface, we have shown : (a) (with R. Hamilton) the short time existence of a C^∞-smooth up to the interface solution of the GCF; (b) (with K. Lee) the C^∞-regularity of the interface $\Gamma(t)$, and the surface up to the interface, on $0 < t < T_c$, with T_c denoting the time when the area of the flat side shrinks to zero ; (c) (with K. Lee) that the flat side will shrink to a point. The final shape of the flat side (most probably an ellipse) is still an open problem.

The Harmonic Mean Curvature Flow: We consider the evolution of a surface $\Sigma(t)$ in $\mathbb{R}^3$ by the *Harmonic Mean Curvature Flow* (HMCF)

$$\frac{\partial P}{\partial t} = \frac{K}{H}\, N$$

where each point P moves in the inward direction N 2 with velocity equal to the Harmonic Mean Curvature of the surface, namely the quotient K/H of the Gaussian Curvature $K$of the surface over the its Mean Curvature H.

Expressing the surface Σ near the flat side as a graph $z = f(x, y, t)$, the function f evolves by

$$f_t = \frac{\det D^2 f}{(1+f_x^2)f_{yy} - 2\, f_x f_y f_{xy} + (1+f_y^2) f_{xx}}$$

a *fully-nonlinear* equation which becomes *degenerate* at the interface $\Gamma(t)$ between the flat and strictly convex sides. Also, since $H \equiv 0$ on the flat side the flow is not defined there.

Assuming that there exists $0 < p < 1$ and $c > 0$, $C < \infty$ such that at time $t = 0$:

$$|\nabla f^p| \le C \quad \text{and} \quad c \le f^{p-1} D^2 f \le C \tag{2}$$

we have shown (with M.C. Caputo) that (a) a unique solution of the HMCF satisfying condition (2) exists ; (b) the free-boundary $\Gamma(t)$ evolves by the Curve Shortening Flow;

Our approach, in both flows, is based on sharp a priori estimates on appropriately defined Hölder Spaces which are scaled according to the singular metric, which governs the evolution of the corresponding degenerate problem.

Wave Equations in Waveguides and Cones

Reinhard Racke
(joint work with P. Lesky)

1. Introduction

We report on our joint work with P. Lesky [1] (Part I), and announce a recent result obained with M. Dreher and P. Lesky [2] (Part II). We consider Klein-Gordon or wave equations:

$$u_{tt} - \Delta u + mu = f(u, u_t, \nabla u, \nabla u_t, \nabla^2 u) \tag{1.1}$$

$$u(t=0) = u_0, \quad u_t(t=0) = u_1 \tag{1.2}$$

$$u(t, \cdot) = 0 \quad \text{on } \partial\Omega \tag{1.3}$$

$$u = u(t,x), t \in \mathbb{R},\ x \in \Omega \subset \mathbb{R}^n, m \ge 0$$

The domain Ω has an infinite smooth boundary, hence is neither bounded nor an exterior domain (domain with bounded complement), but represents in Part I infinite cylinders (waveguides) or domains like strips in $\mathbb{R}^2$ or domains between two parallel planes in $\mathbb{R}^3$. More generally, Ω is assumed to satisfy

$$\Omega = \mathbb{R}^l \times B, \quad B \subset \mathbb{R}^{n-l} \text{ bounded,} \tag{1.4}$$

where $1 \le l \le n-1$. The aim is to get sharp L^p–L^q-decay rates for the associated linearized problem ($f \equiv 0$ or f at most depending on t and x), and a discussion of the nonlinear system — global existence and asymptotics for (1.1)–(1.3). For the

Cauchy problem, i.e. $\Omega = \mathbb{R}^n$, see e.g. Klainerman (1980, 1982, 1985, 1986), Klainerman and Ponce (1983), Christodoulou (1986), Shatah (1982, 1985) and others (see the references in [1]. For exterior domains, see e.g. Shibata and Tsutsumi (1986), Hayashi (1995), Keel, Smith and Sogge (2002), Sogge (2002) and others. Wave guides are different, e.g. the decay of solutions to the linear wave equation ($m = 0$, $f = f(t,x)$) in whole of $\mathbb{R}^3$ (Cauchy problem) is the same as for the region between two planes in $\mathbb{R}^3$. To get L^p-L^q-decay rates for the solutions we use a partial eigenfunction expansion in the bounded direction and information on the growth of eigenvalues of elliptic operators in general domains. For the fully nonlinear system (Part I only), we follow the strategy for exterior domains as given in by Shibata and Tsusumi (1986); this was not yet optimal with respect to the degree of vanishing of the nonlinearity near zero in space dimensions less than 6, recent work of Metcalfe, Sogge and Stewart overcomes this. For the situation of conical sets $\Omega = \{r\omega \in \mathbb{R}^n \mid 0 < r, \infty,\ \omega \in \Omega_0\}$, where $\Omega_0 \subset S^{n-1} \neq \emptyset$ smooth, $n \geq 2$, in Part II, we only study linear wave equations.

2. Part I: L^p–L^q-estimates in wave guides — linear case

Theorem 2.1. Let $f = 0$. Let u be the unique solution to (1.1)–(1.3), and let $2 \leq q \leq \infty$, $\quad 1/p + 1/q = 1$, then for $t \geq 0$,

$$\|(u(t), u_t(t), \nabla u(t))\|_{L^q(\Omega)} \leq \frac{c}{(1+t)^{(1-\frac{2}{q})\frac{l}{2}}} \|(u_0, u_1, \nabla u_0)\|_{W^{\tilde{N}_p,p}(\Omega)},$$

for some $\tilde{N}_p$, and c depends at most on q and m.

For the general linearized case, $f = f(t,x)$, one uses the transformation $v := (\Delta + m)^{-1} u_{tt}$.

3. Part I: Global solutions and asymptotics for nonlinear systems

We follow the following strategy of Shibata and Tsutsumi (1986): Let f be smooth and satisfy

$$f(W) = O(|W|^{\alpha+1}) \quad \text{as } |W| \to 0, \tag{3.1}$$

where

$$\alpha = \alpha(l) := \begin{cases} 3 & \text{if } l = 1, \\ 2 & \text{if } 2 \leq l \leq 4, \\ 1 & \text{if } l \geq 5. \end{cases} \tag{3.2}$$

Let $q(l) := 2\alpha(l)+2$ with associated Hölder exponent $p(l)$, and let $d(l) := \frac{\alpha(l)}{\alpha(l)+1}\,\frac{l}{2}$.

Theorem 3.1. Assume (3.1). Then there are $K \in \mathbb{N}$ and $\varepsilon > 0$ such that if $u_0 \in W^{2K,2}(\Omega) \cap W^{2K-1,p(l)}(\Omega)$, $u_1 \in W_2^{2K-1,2}(\Omega) \cap W^{2K-2,p(l)}(\Omega)$ and (u_0, u_1, f) satisfies the compatibility condition of order 2K, and

$$\|u_0\|_{W^{2K,2}(\Omega)} + \|u_0\|_{W^{2K-1,p(l)}(\Omega)} + \|u_1\|_{W^{2K-1,2}(\Omega)} + \|u_1\|_{W^{2K-2,p(l)}(\Omega)} < \varepsilon$$

then there exists a unique solution

$$u \in \bigcap_{j=0}^{2K} C^j\big([0,\infty), W^{2K-j,2}(\Omega)\big) \subset C^2([0,\infty) \times \overline{\Omega})$$

satisfying

$$\sup_{t\geq 0}\Big(\|(u(t), u_t(t), \nabla u(t)\|_{L^2(\Omega)} + (1+t)^{d(l)}\|u(t), u_t(t), \nabla u(t)\|_{L^{q(l)}(\Omega)}\Big) \leq c_1,$$

where the constant c_1 depends at most on l, Ω.

4. Part II: L^p-L^q-estimates for linear wave equations in cones

We use polar co-ordinates, Fourier expansion in the bounded region, and explicit representations of solutions to the radial wave equation, deriving appropriate sophisticated estimates in weighted Sobolev spaces.

Theorem 4.1. Let u be the solution to the initial boundary value problem for the linear wave equation, i.e. to (1.1)–(1.3) with $m = 0$ and $f = 0$ (and $u_0 = 0$), in the conical set Ω. Let $d := [\frac{n-1}{2}]$, and let A_S denote the Laplace-Beltrami operator on Ω_0. Then u satisfies

$$|u(t,x)| \leq Ct^{-(n-1)/2}\sum_{k=0}^{d}\|(s^{-2}A_S)^{(n-1-k)/2}\partial_s^k(s^{(n-1)/2}u_1(s,\phi))\|_{L^1(\Omega)}.$$

where the constant C does not depend on t, x or u_1.

Interpolation and nonlinear systems have not yet been discussed.

References

[1] Lesky, P.H., Racke, R.: Nonlinear wave equations in infinite wave guides. *Commun. PDE* **28** (2003), 1265–1301.

[2] Dreher, M., Lesky, P.H., Racke, R.: Asymptotic behavior of solutions to wave equations in conical sets. *In preparation.*

The equivalence of linear and dynamical stability of Ricci flat metrics

Natasa Sesum

We can talk about two kinds of stability of the Ricci flow at Ricci flat metrics. One of them is a linear stability, defined with respect to Perelman's functional $\mathcal{F}$. The other one is a dynamical stability and it refers to a convergence of a Ricci flow starting at any metric in a neighbourhood of a considered Ricci flat metric. The precise definitions of these two notions of stability are as follows.

Definition 1. Let g_0 be a geometry whose Ricci flow $g(t)$ converges in C^k norm $(k \geq 3)$. We will say that g_0 is *dynamicaly stable* if there exists a C^k neighbourhood $\mathcal{U}$ of a metric g_0 such that the Ricci flow $\tilde{g}(t)$ of every metric $\tilde{g} \in \mathcal{U}$ exists for all times $t \in [0, \infty)$ and converges to g_0. We will say that g_0 is *weakly dynamicaly stable* if there exists a neighbourhood $\mathcal{U}$ of a metric g_0 such that the Ricci flow $\tilde{g}(t)$ of every metric $\tilde{g} \in \mathcal{U}$ exists for all times $t \in [0, \infty)$ and converges.

The other kind of stability is related to Perelman's functional $\mathcal{F}$ introduced in [2]. It is given by

$$\mathcal{F}(g, f) = \int_M e^{-f}(|\nabla f|^2 + R)dV_g.$$

We can consider functional $\lambda(g) = \inf\{\mathcal{F}(g, f) \mid \int_M e^{-f} dV_g = 1\}$. It turns out that Ricci flat metrics are the critical points of functional λ. We can define a linear stability of a Ricci flat metric g_0 with respect to a second variation of $\mathcal{F}$. More precisely,

Definition 2. Let M be compact, with $\mathrm{Ric}(g_0) = 0$. We will say that g_0 is *linearly stable* iff $\mathcal{D}^2_{g_0}\lambda(h, h) \leq 0$.

It turns out that studying the linear stability of Ricci flat metrics reduces to studying the spectrum of the Lichnerowicz laplacian $\Delta_L h_{ij} = \Delta h_{ij} + 2R_{ipqj}h^{pq}$. We want to relate a linear and a dynamical stability of a Ricci flat metric g_0. More precisely, the main theorem we want to prove is the following.

Theorem 1. *Let g_0 be a Ricci flat metric on a closed manifold M. Then if g_0 is dynamicaly stable, it is linearly stable as well. If g_0 is linearly stable and integrable then it is weakly dynamicaly stable.*

The fact that dynamical stability implies linear stability is an easy corollary of the monotonicity formula for Perelman's functional λ along the Ricci flow. To prove the other way around we have to study the spectrum of Δ_L. Since the Ricci flow equation is not a strictly parabolic equation, we use DeTurck's trick fisrt to make it strictly parabolic. We show that by choosing sufficiently small neighbourhood around g_0 we can construct a gauge for every arbitrarily large, but fixed time interval so that our solution stays as close to g_0 as we want. The linearization of the right hand side of the Ricci DeTurck flow

$$\begin{aligned} \frac{d}{dt}\tilde{g}_{ij} &= -2\tilde{R}_{ij} + \nabla_V \tilde{g}, \\ \tilde{g}(0) &= g_0, \end{aligned}$$

around g_0 is just the Lichnerowicz laplacian. The linear stability condition on g_0 implies there are no positive eigenvalues of Δ_L. The negative ones give us exponentially decaying modes which imply the long time existence and the convergence of a flow starting nearby g_0 at the same time. To deal with the neutral modes (that come from the zero eigenfunctions of Δ_L) we use the integrability condition imposed on g_0. This condition tells us the set of Ricci flat metrics around g_0 has a smooth manifold structure. We use this condition to find a sequence g_i of new reference Ricci flat metrics on time intervals of fixed length so that there are no zero directions in a projection of $\tilde{g}(t) - g_i$ onto the set of solutions of a linear equation $\frac{d}{dt}F = \Delta_L F$. This will imply exponential decay and the exponential convergence of $\tilde{g}(t)$ to a Ricci flat metric. This yields the exponential convergence of the original Ricci flow equation as well.

As a corollary of Theorem 1 we get the following result about Calabi-Yau manifolds.

Theorem 2. *Let g_0 be a Kähler Ricci flat metric on a $K3$ surface M. There exists a neighbourhood $\mathcal{N}_{g_0}$ of g_0 so that the Ricci flow of any initial metric in $\mathcal{N}_{g_0}$ converges to a unique Ricci flat metric on M, that is g_0 is weakly dynamicaly stable. Moreover, any Calabi-Yau metric on a compact manifold M of an arbitrary dimension is dynamicaly stable.*

References

[1] C. Guenther, J. Isenberg, D. Knopf: Stability of the Ricci flow at Ricci-flat metrics; Communications in Analysis and Geometry 10 (2002), no 4 741–777.
[2] G. Perelman: The entropy formula for the Ricci flow and its geometric applications; arXiv:math.DG/0211159.

Evolution of an extended Ricci Flow system

Bernhard List

In my talk I discussed the following initial value problem

$$(*) \qquad \begin{cases} \partial_t g = -2Rc(g) + 4\, du \otimes du \\ \partial_t u = \Delta^g u \end{cases}$$

for a Riemannian metric g and a function $u \in C^\infty(M)$ on some closed 3-dimensional manifold M for given smooth initial data (g_0, u_0). I restricted the discussion to dimension three, although the presented results are also true for $n \geq 3$ with a slight modification of the numerical constants. Some reasons to look at this system are given by the following properties of $(*)$:

- Stationary points are solutions of the static Einstein vacuum equations. This provides an interesting link to general relativity.
- Setting $u \equiv const$, $(*)$ reduces to Hamilton's Ricci Flow (c.f. [5,6]).
- The system is weakly parabolic, so short time solutions exist following the arguments in [1].
- There is a variational structure for $(*)$ using recent ideas of Perelman in [3], namely the system
$$(**) \quad \begin{cases} \partial_t g = -2Rc(g) + 4\, du \otimes du - \mathcal{L}_{\nabla f} g \\ \partial_t u = \Delta^g u - \mathcal{L}_{\nabla f} u \\ \partial_t f = -\Delta^g f - T \end{cases}$$
where $T := R - 2|\nabla u|^2$, is the gradient flow of the entropy
$$E(g,u,f) := \int_M \big(|\nabla f|^2 + T\big)e^{-f} dV_g \ .$$
Given a solution of $(*)$, one can solve backwards for $f(t)$. Then $(**)$ is equivalent to $(*)$ via the diffeomorphisms generated by $\nabla f(t)$.

Since $(**)$ is a gradient flow one can compute

Lemma 1. $\partial_t E(t) \geq 0$ and equality precisely holds on gradient solitons.

A gradient soliton is a special solution changing in time only by diffeomorphisms generated by a gradient vector field. In addition there is the definition:

Definition 1. (g,u) is called breather, iff $\exists t_1 < t_2$ and $\alpha > 0$ such that $g(t_2) = \alpha \cdot (\phi^* g)(t_1)$ and $u(t_2) = (\phi^* u)(t_1)$ for some diffeomorphism ϕ. $\alpha = 1, > 1, < 1$ is called steady, expanding and shrinking breather.

So in contrast to solitons there are only two points in time such that the geometry of the solutions is more or less the same. As a first consequence of the monotonicity I proved in a similar way to [3, §2]:

Theorem 1. Steady/expanding breathers are gradient solitons. Moreover u is constant and g is Ricci flat/Einstein in this case.

For the proof one uses the monotonicity of the infimum of E over all normalized smooth f, the existence of minimizers and the fact that volume is nondecreasing.

Because of the last point this approach doesn't work for shrinking breathers. Therefore I looked at the modified entropy
$$W(g,u,f,\tau) := \int_M \big[\tau\big(|\nabla f|^2 + T\big) + f - n\big](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV$$
inspired by [3, §4]. A computation gives

Lemma 2. If $(g,u)(t)$ satisfy $(*)$ and $\partial_t f = -\Delta^g f - T + \frac{n}{2\tau}$ and $\partial_t \tau = -1$ holds, then $\partial_t W(t) \geq 0$ and equality precisely holds on homothetic gradient solitons.

Homothetic solitons can differ in time not only by diffeomorphisms but also by scaling. One can check now the monotonicity properties of the infimum of W over f as before and also over positive τ, which are inherited from W, to prove

Theorem 2. Shrinking breathers are gradient solitons.

A second important application of W is

Theorem 3. Solutions of $(*)$ cannot collapse at finite T.

Here collapse is defined as follows:

Definition 2. $(g,u)(t)$ collapses at T iff $\exists\,(t_k) \nearrow T$ and $B_k := B^{t_k}_{r_k}(x_k)$, s.t. $r_k^2/t_k \le C$ uniformly in k, $\sup_{B_k} |Rm|(t_k) \le r_k^{-2}$ and $r_k^{-n} Vol(B_k) \to 0$ for $k \to \infty$.

The proof of Theorem 3 follows the same lines as the proof in [3, §4.1]. Assuming to the contrary the existence of sequences of times and collapsing balls one can construct a sequence of test functions w_k supported on B_k and concentrating there s.t. for $\tau(t) := (t_k + r_k^2) - t$ one proves $W(g(t_k), u(t_k), w_k, r_k^2) \to -\infty$ for $k \to \infty$. Using Bishop-Gromov volume comparison and the monotonicity for $\mu(g,u,\tau) := \inf_f \{W(g,u,f,\tau) \big| \int (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV = 1\}$ one gets the contradiction

$$-\infty < C \le \mu\big(g,u,\tau\big)(0) \le \mu(g,u,\tau)(t_k) \le W(g(t_k), u(t_k), w_k, r_k^2) \to -\infty$$

where the lower bound C (which can be negative and large) follows from the fact that $g(0)$ and $u(0)$ are fixed and smooth and that $\tau(0) = t_k + r_k^2$ is bounded.

The second important tool I presented was the following regularity result. I used the scaling function $\varphi(t) := \frac{R^2 t}{R^2+t}$ and collected all relevant component functions in a vector $\Phi = (R_{ijkl}, \nabla_p \nabla_q u)_{i,j,k,l,p,q=1\ldots3}$ with norm $|\Phi|^2 := |Rm|^2 + |\nabla^2 u|^2$ and similar for higher derivatives. I denoted the domain of interest for fixed R and T by $\Gamma := \big\{(t,x) \big| 0 \le t < T, d_t(x_0,x) \le R\big\}$ and proved

Theorem 4. Assume $\sup_\Gamma \varphi^{k+2} |\nabla^k \Phi|^2 \le \tilde{C}_k \quad \forall\, k = 0 \ldots m$, where $m \ge 1$ is fixed, $R \le \sqrt{T}$, $x_0 \in M$. Then

$$\sup_{B^t_{\theta R}(x_0)} |\nabla^{m+1}\Phi|^2 \le \tilde{C}(1-\theta)^{-2} \left(\frac{1}{R^2} + \frac{1}{t}\right)^{m+3}$$

holds for all $t \in (0,T]$ and $\theta \in [0,1)$, where $\tilde{C} = \tilde{C}(n, m, \tilde{C}_0, \ldots, \tilde{C}_m)$.

The proof of Theorem 4 uses ideas from [4] and [2] applied to the test function $f = \varphi^{m+3} |\nabla^{m+1}\Phi|^2 \big(\lambda + \varphi^{m+2} |\nabla^m \Phi|^2\big)$.

The theorem allows to deduce smoothness of the solution from bounds on $|\Phi|$ by iteration. This can be improved:

Proposition 1. Let Γ be as above and suppose (g,u) is a solution of $(*)$. Then

$$\sup_{B^t_R(x_0)} |\nabla u|^2 \le \frac{1}{4} \left(\frac{1}{R^2} + \frac{1}{t}\right)$$

and if in addition $\sup_\Gamma \big(\varphi^2 |Rm|^2\big) \le \tilde{C}_0$ then

$$\sup_{B^t_{\theta R}(x_0)} |\nabla^2 u|^2 \le C(n, \tilde{C}_0)(1-\theta)^{-2} \left(\frac{1}{R^2} + \frac{1}{t}\right)^2$$

for all $t \in (0,T]$ and $\theta \in [0,1)$.

This means that the gradient of u is always bounded and that the curvature of $g(t)$ controls the Hessian of $u(t)$, i.e. boundedness of Rm implies smoothness of the solution. I deduced the following consequences from Theorems 3 and 4:

Corollary 1. Curvature blows up at singularities, i.e.

$$\limsup_{t \nearrow T_{max}} \left[\sup_{x \in M} |Rm|^2(t,x) \right] = \infty \ ,$$

where T_{max} is the maximal existence time of the solution.

Corollary 2. Let $(g,u)(t)$ be a solution on $[0,T) \times M$ for closed M and finite T. Assume there are sequences $t_k \to T$ and $x_k \in M$ s.t. $S_k := |Rm|(t_k, x_k) \to \infty$ for $t \to T$ and $\exists C > 0$ such that $\sup_{[0,t_k)\times M} |Rm|(t,x) \leq CS_k$. Define the rescalings $\tilde{g}_k(t) := S_k \cdot g(t_k + \frac{t}{S_k})$ and $\tilde{u}_k(t) := u(t_k + \frac{t}{S_k})$. Then a subsequence of $(\tilde{g}_k, \tilde{u}_k)$ converges smoothly on compact subsets to a complete ancient noncollapsed solution.

Corollary 3. Let M be complete. Then for all smooth initial data (g_0, u_0) s.t. $|Rm_0|_0^2 + |u_0|_0^2 + |\nabla u_0|_0^2 \leq k$, there is a smooth solution $(g,u)(t)$ to $(*)$ on $[0,T]$ for some $T = T(n,k) > 0$, satisfying $g(0) = g_0$ and $u(0) = u_0$.

References

[1] D. DeTurck, *Deforming metrics in the direction of their Ricci tensors*, Journal of Differential Geometry **18** (1983), 157–162.
[2] K. Ecker and G. Huisken, *Interior estimates for hypersurfaces moving by mean curvature*, Inventiones mathematicae **105** (1991), 547–569.
[3] G. Perelman, *The entropy formula for the Ricci Flow and its geometric applications*, `arXiv:math.DG/0211159v1 11Nov2002`.
[4] W. Shi, *Deforming the metric on complete Riemannian manifolds*, Journal of Differential Geometry **30** (1989), 223–301.
[5] R. Hamilton, *Three-manifolds with positive Ricci curvature*, Journal of Differential Geometry **17** (1982), 255–306.
[6] R. Hamilton, *The formation of singularities in the Ricci Flow*, Surveys in Differential Geometry **2** (1995)

Optimal Transportation Metrics for a Class of Nonlinear Wave Equations

Alberto Bressan

The Camassa-Holm equation can be written as a first order integro-differential PDE.:

$$u_t + \left(\frac{u^2}{2}\right)_x + P_x = 0\,, \tag{1}$$

where

$$P \doteq \frac{1}{2} e^{-|x|} * \left(u^2 + \frac{u_x^2}{2}\right) .$$

It is well known that this equation provides an example of completely integrable PDE. It admits solitary waves, called **peakons**. Special solutions consisting of finitely many peakons can be constructed in the form

$$u(t,x) = \sum_{i=1}^{n} p_i(t)\, e^{-|x-q_i(t)|}.$$

The strengths p_i and the locations of the peakons satisfy the Hamiltonian system of ODEs:

$$\left\{ \begin{aligned} \dot p_i &= \sum_{j\neq i} p_i p_j \,\mathrm{sign}(q_i - q_j)\, e^{-|q_i-q_j|} \\ \dot q_i &= \sum_{j} p_j\, e^{-|q_i-q_j|}. \end{aligned} \right\}$$

As long as the solutions remain smooth, the total energy

$$E(t) \doteq \frac{1}{2}\int_{-\infty}^{\infty} \big[u^2(t,x) + u_x^2(t,x)\big]\, dx$$

remains constant in time. However, solutions might lose regularity in finite time. In this case, they remain uniformly Hölder continuous but their gradient u_x may become infinite. One then has various ways for prolonging solutions, after singularity formation: either conserving the total energy, or dissipating part of it.

For a given initial condition $u_0 \in H^1(\mathbb{R})$, various methods are known for constructing global solutions. In [3] the authors constructed a continuous semigroup of dissipative solutions as limits of vanishing viscosity approximations. Alternatively, in [1] the conservative solution was obtained as the fixed point of a contractive transformation, using a new set of independent and dependent variables.

Concerning uniqueness and continuous dependence on the initial data, we remark that couples of solutions do not satisfy any a priori estimate of the form

$$\frac{d}{dt}\big\|u(t) - v(t)\big\|_{H^1} \le C\cdot \big\|u(t) - v(t)\big\|_{H^1},$$

$$\big\|u(t) - v(t)\big\|_{H^1} \le L\cdot \big\|u(0) - v(0)\big\|_{H^1}.$$

An effective tool to study continuous dependence on the initial data is provided by a certain distance, defined as the optimal cost of a transportation problem. We now review this construction, in connection with spatially periodic solutions of the equations (1).

Consider the space $X = \mathbb{R}\times\mathbb{R}\times\mathbf{T}$, where $\mathbf{T}$ is the unit circle. Given $u \in H^1_{per}(\mathbb{R})$, with $u(x) = u(x+1)$ for all $x \in \mathbb{R}$, define the graph

$$\mathrm{Graph}(u) = \Big\{\big(x,\ u(x),\ 2\arctan u_x(x)\big)\,;\ x\in\mathbb{R}\Big\} \subset X.$$

Define μ^u as the measure supported on Graph(u), having density $1+u_x^2$ w.r.t. Lebesgue measure, i.e.

$$\mu^u(A) = \int_{(x,u(x),2\arctan u_x(x))\in A} \big(1 + u_x^2(x)\big)\, dx.$$

Given $u, v \in H^1_{per}$, consider the corresponding measures μ^u, μ^v. Let $\psi : I\!R \mapsto I\!R$ be an absolutely continuous, increasing map, with inverse ψ^{-1} also absolutely continuous, and such that $\psi(x+n) = n + \psi(x)$ by spatial periodicity. Define a **transportation plan**, from μ^u to μ^v, moving mass according to

$$\big(x,\ u(x),\ 2\arctan u_x(x)\big) \ \mapsto\ \Big(\psi(x),\ v\big(\psi(x)\big),\ 2\arctan v_x\big(\psi(x)\big)\Big).$$

Notice that in this case the amount of mass $\big(1+u_x^2(x)\big)\,dx$ on the graph of u is put in correspondence to the mass $\Big(1+v_x^2\big(\psi(x)\big)\Big)\,\psi'(x)\,dx$ on the graph of v. The cost of this transportation is defined as

$$\begin{aligned} &J^\psi(u,v)\\ &\doteq \int_0^1 d_X\Big(\big(x,\, u(x),\, \arctan u_x(x)\big)\,,\ \big(\psi(x),\, v(\psi(x)),\, \arctan v_x(\psi(x))\big)\Big)\cdot \phi(x)\,dx\\ &\qquad + \int_0^1 \Big|\big(1+u_x^2(x)\big) - \big(1+v_x^2(\psi(x))\big)\psi'(x)\Big|\,dx\\ &= \int [\text{distance}]\cdot[\text{transported mass}] \ +\ \int [\text{excess mass}] \end{aligned}$$

Here $\phi(x) = \min\big\{\big(1+u_x^2(x)\big)\,,\ \big(1+v_x^2(\psi(x))\psi'(x)\big\}$. We then define the distance functional J by taking the infimum of the cost over all transportation plans:

$$J(u,v) = \inf_\psi J^\psi(u,v) \qquad\qquad u,v \in H^1_{per}.$$

As proved in [2], this new functional has the key properties

$$\frac{1}{C}\|u-v\|_{L^\infty} \le J(u,v) \le C\cdot\|u-v\|_{H^1}\,,$$

$$\left|\frac{d}{dt}J\big(u(t),v(t)\big)\right| \le \kappa\cdot J\big(u(t),v(t)\big)\,,$$

where the constants C, κ remain uniformly valid as u, v range over bounded subsets of H^1. The inequality $J\big(u(t),\,v(t)\big) \le e^{|\kappa|t}J\big(u(0),\,v(0)\big)$ now provides a sharp estimate on the dependence of solutions on the initial data.

References

[1] A. Bressan and A. Constantin, *Global conservative solutions of the Camassa-Holm equations*, preprint 2005.

[2] A. Bressan and M. Fonte, *An optimal transportation metric for solutions of the Camassa-Holm equation*, Methods and Applications of Analysis, to appear.

[3] G. Coclite, H. Holden and K. Karlsen, *Global weak solutions to a generalized hyperelastic-rod wave equation*, preprint 2004.

Semi-Relativistic NLS of Critical Type: The Boson Star Problem

Enno Lenzmann

We study nonlinear Schrödinger equations with kinetic energy part originating from special relativity. That is, we consider

$$i\partial_t u = \sqrt{-\Delta + m^2}\, u + F(u), \qquad (t,x) \in \mathbb{R}^{1+3}, \tag{1}$$

where $u(t,x)$ is complex-valued, $m \geq 0$ denotes a given mass parameter, and $F(u)$ is some nonlinearity. Here the operator $\sqrt{-\Delta + m^2}$ is defined via its symbol $\sqrt{\xi^2 + m^2}$ in Fourier space.

Equation (1) arises in the theory of *boson stars*, see, e. g., [1,2], where $F(u)$ is a focusing Hartree type nonlinearity given by

$$F(u) = -\big(\frac{1}{|x|} * |u|^2\big)u, \tag{2}$$

with $*$ as convolution. Motivated by this physical example, which leads to an L^2-critical equation, we address the Cauchy problem for equation (1) with nonlinearity (2). In fact, we prove local well-posedness for initial data $u(0,x) = u_0(x)$ in $H^s(\mathbb{R}^3)$, $s \geq 1/2$. Moreover, these solutions extend to all times, i. e., we have global well-posedness, if the initial datum u_0 satisfies

$$\int_{\mathbb{R}^3} |u_0(x)|^2\, dx < \int_{\mathbb{R}^3} |Q(x)|^2\, dx, \tag{3}$$

where $Q \in H^{1/2}(\mathbb{R}^3)$ is a ground state solution for

$$\sqrt{-\Delta}\, Q - \big(\frac{1}{|x|} * |Q|^2\big)Q = -Q. \tag{4}$$

We remark that criterion (3) implying global-in-time solutions is valid irrespectively of $m \geq 0$ in (1). In physical terms, the quantity

$$N_c = \int_{\mathbb{R}^3} |Q(x)|^2\, dx \tag{5}$$

corresponds to the "Chandrasekhar limit mass" for boson stars, and we find that $N_c > 4/\pi$ holds by using suitable estimates.

In addition to well-posedness, we also address solitary wave solutions for (1), just called *solitons*, with focusing nonlinearity given by (2). It turns out that if $m > 0$ holds, then orbitally stable ground state solitons $u(t,x) = e^{i\omega t}\varphi_\omega(x)$ exist for every $0 < \|\varphi_\omega\|_2^2 < N_c$, where

$$\sqrt{-\Delta + m^2}\, \varphi_\omega - \big(\frac{1}{|x|} * |\varphi_\omega|^2\big)\varphi_\omega = -\omega\varphi_\omega, \tag{6}$$

for some $\omega > 0$. On the other hand, when $m = 0$ the corresponding soliton ground states, φ_ω, necessarily have L^2-norm $\|\varphi_\omega\|_2^2 = N_c$ and are expected to be unstable.

We refer to [3] for extensions and proofs of parts of the material discussed above.

References

[1] Elgart, A. and Schlein, B., *Mean field dynamics of boson stars*, arXiv:math-ph/0504051 (2005).
[2] Lieb, E. H. and Yau, H.-T., *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics*, Comm. Math. Phys. **112** (1987), 147–174.
[3] Lenzmann, E., *Well-Posedness for semi-relativistic Hartree equations of critical type*, math.AP/0505456 (2005).

On stability of standing waves for NLS with perturbed periodic potential

Scipio Cuccagna

In this talk we consider standing waves of the Nonlinear Schrödinger Equation

$$i\partial_t u(t,x) + (\partial_x^2 - P(x) - \epsilon q(x))u(t,x) - (|u|^{p-1}u)(t,x) = 0 \quad \text{for} \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}$$

with $P(x)$ a periodic smooth potential, $q(x) \in C_0^\infty(\mathbb{R})$ a nonzero function, either nonpositive or nonnegative, $\epsilon > 0$ small. It is well known that the spectrum $\sigma(H_0)$ of $H_0 = -\frac{d^2}{dx^2} + P(x)$ is formed by bands, separated by gaps. Here we consider potentials with two bands $\sigma(H_0) = [E_0, E_1] \cup [E_2, +\infty)$, with $P(x)$ derived by the Jacobian elliptic function $\mathrm{sn}(x, \kappa)$. Associated to each value of the energy E there is a quasimomentum k with $\mathrm{Im} k \geq 0$. For each $E \in \sigma(H_0)$ there is an appropriately normalized pair of functions $\phi_\pm(x, E)$, called Bloch functions, of the form $\phi_\pm(x,E) = e^{\pm ik} m_\pm(x, E)$ with $m_\pm(x+1, E) = m_\pm(x, E)$, such that $\phi_\pm(x, E)$ can be used to define an analogue of the Fourier transform. The spectrum of $H = H_0 + \epsilon q(x)$ is formed by the same bands plus a certain number of eigenvalues. If $q(x) \geq 0$ there is an eigenvalue $\lambda_1 > E_1$ with approximately $\sqrt{\lambda_1 - E_1} \approx \epsilon \int_{\mathbb{R}} q(x)|\phi_+(x, E_1)|^2 dx$. By bifurcation the Nonlinear Schrödinger Equation admits small standing waves $u(t, x) = e^{-it\omega}\phi_\omega(x)$, with $\phi_\omega(x)$ smooth in (x, ω) and where as $\omega \to \lambda$ we have $\phi_\omega \to 0$ in any Sobolev space. There are two main notions of stability. We can write nearby solutions using the Ansatz

$$u(t, x) = e^{-i\int_0^t \omega(s)ds + i\gamma(t)} \left(\phi_{\omega(t)}(x) + R(t, x)\right)$$

and ask if the Ansatz is true for all times, with R always small in $H^1(\mathbb{R})$ and $\omega(t)$ close to a fixed value for all time. This is the notion of orbital stability, very much explored in the literature. The standard references are Cazenave and Lions [2], Weinstein [6], and [3, 4]. Unfortunately, none of these applies to this case. The other notion is that of Asymptotic Stability, where one asks whether R scatters and $\omega(t)$ converges to a fixed value. We proved this last fact, for solutions $u(t, x)$ such that $u(0, x)$ decays rapidly to 0 as $x \to \infty$. The proof is based, like [5], which deals with the case $P(x) \equiv 0$, on linearization around the standing wave, and on a perturbative argument which exploits the following two Linear Dispersive Estimates, for $P_c(H)$ the projection on the continuous spectrum of H,

$$\|P_c(H)e^{itH}\colon L^1(\mathbb{R}) \to L^\infty(\mathbb{R})\| \le C \max\left\{t^{-\frac{1}{2}}, \langle t\rangle^{-\frac{1}{3}}\right\}$$

and

$$\|P_c(H)e^{itH}\colon L^2(\langle x\rangle^8 dx) \to L^2(\langle x\rangle^{-8} dx)\| \le c\langle t\rangle^{-\frac{3}{2}}.$$

Specifically, inserting the Ansatz into the Nonlinear Schrödinger Equation we obtain the Equation for the Remainder

$$iR_t(t,x) - HR(t,x) + (\omega(t) - \dot\gamma(t))R(t,x) = Error.$$

By the right choice of $\gamma(t)$ and $\omega(t)$ in the Ansatz we have $R(t) = P_c(H)R(t)$. It turns then out that the Linear Dispersive Estimates allow to reduce to the case $Error = 0$ in the Equation for the Remainder, at least in the case when $p \gg 1$ in the Nonlinear Schrödinger Equation.

There is some beautiful mathematics behind the two Linear Dispersive Estimates and estimates of this type have yet to be proved for generic smooth and periodic potentials $P(x)$.

We have discussed $q(x) \ge 0$. Now let us turn to If $q(x) \le 0$ with $q(x) \not\equiv 0$. Now there are eigenvalues $\lambda_j > E_j$ for $j = 0, 2$ with approximately $\sqrt{E_j - \lambda_j} \approx \epsilon \int_{\mathbb{R}} q(x)|\phi_+(x, E_j)|^2 dx$. As before, by bifurcation the Nonlinear Schrödinger Equation admits small standing waves $u(t,x) = e^{-it\omega}\phi_\omega(x)$, where ω belong to intervals of the form $[\lambda_j - \eta, \lambda_j]$ for small $\eta > 0$. These are called Ground States for $j = 0$ and Excited States for $j = 2$. Ground States are orbitally stable by [3, 6], while Excited States are orbitally unstable. We discuss a proof of Asymptotic Stability of the Ground States. Once again the proof is based on inserting the Ansatz into the Nonlinear Schrödinger Equation to obtain the Equation for the Remainder. This time though H has two eigenvalues, and by modulation we can only assume that R has 0 ground state component. However there is no way to exclude that R has nonzero excited state component. This means that the Equation for the Remainder is in reality a system involving a continuous component and a discrete component. It turns out that the discrete component leaks slowly into the continuous component where it disperses. This implies that now the Equation for the Remainder involves long range perturbative terms and that decay is hard to prove. Fortunately the situation here is very similar to one considered by Buslaev and Perelman [1] which requires a normal forms argument for the discrete mode and the use of a Fermi Golden Rule. To implement these ideas one is forced to write the Equation for the Remainder in a vectorial form and to prove Linear Dispersive Estimates for a certain class of vectorial, and not selfadjoint, Schrödinger operators.

References

[1] Buslaev, Perelman , *On the stability of solitary waves for nonlinear Schrödinger equations*, Transl. AMS Ser. 2, 164, ed. N.N. Uraltseva (1995), 75–98.

[2] Cazenave, Lions, *Orbital stability of standing waves for nonlinear Schrödinger equations*, Comm. Math. Phys. **85** (1982), 549–561.

[3] Grillakis, Shatah, Strauss, *Stability of solitary waves in the presence of symmetries, I*, Jour. Funct. An. **74** (1987), 160–197.

[4] Grillakis, Shatah, Strauss, *Stability of solitary waves in the presence of symmetries, II*, Jour. Funct. An. **94** (1990), 308–348.

[5] Soffer, Weinstein, *Multichannel nonlinear scattering II. The case of anisotropic potentials and data*, J. Diff. Eq. **98** (1992), 376–390.

[6] Weinstein, *Lyapunov stability of ground states of nonlinear dispersive equations* , Com. Pure Appl. Math. **39** (1986), 51–68.

Reporter: Julie Clutterbuck

Participants

Prof. Dr. Luigi Ambrosio
luigi.ambrosio@sns.it
Scuola Normale Superiore
Piazza dei Cavalieri, 7
I-56100 Pisa

Prof. Dr. Lars Andersson
larsa@math.miami.edu
Lars.Andersson@aei.mpg.de
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Simon Brendle
brendle@math.princeton.edu
Department of Mathematics
Princeton University
1104 Fine Hall
Washington Road
Princeton NJ 08544
USA

Prof. Dr. Alberto Bressan
bressan@math.psu.edu
Department of Mathematics
Pennsylvania State University
University Park, PA 16802
USA

Dr. John Buckland
john.buckland@maths.anu.edu.au
Centre for Mathematics and its
Applications
Australian National University
Canberra ACT 0200
AUSTRALIA

Prof. Dr. Luca Capogna
lcapogna@uark.edu
Department of Mathematics
University of Arkansas
SCEN
Fayetteville AR 72701
USA

Dr. Julie Clutterbuck
juliec@math.fu-berlin.de
Fachbereich Mathematik
und Informatik
Freie Universität Berlin
Arnimallee 2-6
14195 Berlin

Prof. Dr. Scipio Cuccagna
cuccagna@math.princeton.edu
cuccagna.scipio@unimore.it
DISMI (Dipartimento di Scienze e
Metodi dell'Ingegneria)
University Modena & Reggio Emilia
Via Fogliani 1
I-42100 Reggio Emilia

Prof. Dr. Piero D'Ancona
dancona@mat.uniroma1.it
Dipartimento di Matematica
Universita di Roma "La Sapienza"
Istituto "Guido Castelnuovo"
Piazzale Aldo Moro, 2
I-00185 Roma

Dr. Mihalis Dafermos
M.Dafermos@dpmms.cam.ac.uk
Dept. of Pure Mathematics and
Mathematical Statistics
University of Cambridge
Wilberforce Road
GB-Cambridge CB3 OWB

Prof. Dr. Panagiota Daskalopoulos
pdaskalo@math.columbia.edu
Department of Mathematics
Columbia University
2990 Broadway, Math. Building 509
MC 4406
New York NY 10027
USA

Prof. Dr. Klaus Ecker
ecker@math.fu-berlin.de
Fachbereich Mathematik
und Informatik
Freie Universität Berlin
Arnimallee 2-6
14195 Berlin

Prof. Dr. Joachim Escher
escher@ifam.uni-hannover.de
Institut für Angewandte Mathematik
Universität Hannover
Welfengarten 1
30167 Hannover

Prof. Dr. Mikhail Feldman
feldman@math.wisc.edu
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Drive
Madison, WI 53706-1388
USA

Prof. Dr. Vladimir S. Georgiev
georgiev@dm.unipi.it
Dipartimento di Matematica
Universita di Pisa
Largo Bruno Pontecorvo,5
I-56127 Pisa

Dr. Michel Grüneberg
Michel.Grueneberg@aei.mpg.de
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Prof. Dr. Gerhard Huisken
gerhard.huisken@uni-tuebingen.de
Gerhard.Huisken@aei.mpg.de
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Prof. Dr. James Isenberg
jim@newton.uoregon.edu
Department of Mathematics
University of Oregon
Eugene, OR 97403-5203
USA

Prof. Dr. Markus Keel
keel@math.umn.edu
School of Mathematics
University of Minnesota
127 Vincent Hall
206 Church Street S. E.
Minneapolis MN 55455-0436
USA

Dr. Joachim Krieger
jkrieger@math.harvard.edu
Dept. of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
USA

Prof. Dr. Ernst Kuwert
ernst.kuwert@math.uni-freiburg.de
kuwert@mathematik.uni-freiburg.de
Mathematisches Institut
Universität Freiburg
Eckerstr.1
79104 Freiburg

Ralf-Enno Lenzmann
enno.lenzmann@math.ethz.ch
Mathematik Department
ETH Zentrum
HG G33.1
Rämistr. 101
CH-8092 Zürich

Bernhard List
bernhard.list@aei.mpg.de
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Prof. Dr. Andrea Malchiodi
malchiod@sissa.it
S.I.S.S.A.
Via Beirut 2 - 4
I-34014 Trieste

Dr. James McCoy
James.McCoy@maths.anu.edu.au
james.mccoy@anu.edu.au
School of Mathematics and Applied Statistics
University of Wollongong
Wollongong, N.S.W. 2522
AUSTRALIA

Prof. Dr. Frank Merle
frank.merle@math.u-cergy.fr
Departement de Mathematiques
Universite de Cergy-Pontoise
Site Saint-Martin
2, avenue Adolphe Chauvin
F-95302 Cergy-Pontoise Cedex

Dr. Jan Metzger
jan@everest.mathematik.uni-tuebingen.de
Jan.Metzger@aei.mpg.de
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Dr. Roger Moser
rmoser@cims.nyu.edu
Courant Institute
Warren Weaver Hall
251 Mercer Street
New York N.Y. 10012
USA

Reto Müller
reto.mueller@math.ethz.ch
Departement Mathematik
ETH-Zentrum
HG G 36.1
Rämistr. 101
CH-8092 Zürich

Prof. Dr. Andrea R. Nahmod
nahmod@math.umass.edu
Dept. of Mathematics & Statistics
University of Massachusetts
710 North Pleasant Street
Amherst, MA 01003-9305
USA

Prof. Dr. Fabrice Planchon
fab@math.univ-paris13.fr
Departement de Mathematiques
Institut Galilee
Universite Paris XIII
99 Av. J.-B. Clement
F-93430 Villetaneuse

Prof. Dr. Reinhard Racke
reinhard.racke@uni-konstanz.de
FB Mathematik und Statistik
Universität Konstanz
78457 Konstanz

Prof. Dr. Alan Rendall
rendall@aei.mpg.de
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Dr. Hans Ringström
hansr@math.kth.se
Dept. of Mathematics
Royal Institute of Technology
Lindstedtsvägen 25
S-100 44 Stockholm

Prof. Dr. Igor Rodnianski
irod@math.princeton.edu
irod@Princeton.EDU
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
USA

Prof. Dr. Reiner Schätzle
rsch@everest.mathematik.uni-tuebingen.de
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen

Dr. Oliver C. Schnürer
schnuere@math.fu-berlin.de
Fachbereich Mathematik
und Informatik
Freie Universität Berlin
Arnimallee 2-6
14195 Berlin

Dr. Felix Schulze
schulzef@math.fu-berlin.de
Fachbereich Mathematik
und Informatik
Freie Universität Berlin
Arnimallee 2-6
14195 Berlin

Dr. Hartmut Schwetlick
schwetlick@maths.bath.ac.uk
Department of Mathematical Sciences
University of Bath
Claverton Down
GB-Bath BA2 7AY

Prof. Dr. Natasa Sesum
natasas@cims.nyu.edu
natasas@math.mit.edu
Department of Mathematics
Massachusetts Institute of
Technology
Cambridge, MA 02139-4307
USA

Prof. Dr. Jalal Shatah
shatah@cims.nyu.edu
Courant Institute of
Mathematical Sciences
New York University
251, Mercer Street
New York, NY 10012-1110
USA

Dr. Miles Simon
msimon@mathematik.uni-freiburg.de
Mathematisches Institut
Universität Freiburg
Eckerstr.1
79104 Freiburg

Prof. Dr. H. Mete Soner
msoner@ku.edu.tr
Department of Mathematics
Koc University
Rumelifener Yolu
Sariyer
80910 Istanbul
TURKEY

Prof. Dr. Michael Struwe
michael.struwe@math.ethz.ch
struwe@math.ethz.ch
Departement Mathematik
ETH-Zentrum
Rämistr. 101
CH-8092 Zürich

Prof. Dr. Sijue Wu
sijue@umich.edu
Department of Mathematics
University of Michigan
1863 East Hall
Ann Arbor MI 48109-1109
USA