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Topological and Variational Methods for Differential Equations

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Abstract. These notes contain the extended abstracts of the talks presented at the workshop. The range of topics includes nonlinear Schrödinger equations, singularly perturbed equations, symmetry and nodal properties of solutions, long-time dynamics for parabolic equations, Morse theory.

Mathematics Subject Classification (2000): 35xx, 47xx, 58xx.

Introduction by the Organisers

Topological and variational methods have been at the core of nonlinear analysis for a long time and are still experiencing major new developments. They have had enormous new applications in the study of boundary value problems for nonlinear differential equations, in analyzing complicated (possibly infinite-dimensional) dynamics, phase transition and pattern formation, to name a few.

The workshop was mainly dedicated to variational methods for nonlinear elliptic and parabolic differential equations and systems with a special emphasis on

- Morse theory, Lusternik-Schnirelmann theory
- nonlinear Schrödinger equations
- singularly perturbed equations and their stable solutions
- multi-peak type solutions, both positive and sign-changing
- symmetry and nodal properties of solutions to elliptic boundary value problems
- long-time dynamics for semilinear parabolic equations

The workshop was attended by 47 mathematicians from 17 countries (from the Americas, Asia, Australia, Europe). During the five days 27 talks were delivered, both from leading researchers as well as young mathematicians. There was plenty of time for discussions and cooperative work in small groups outside the scheduled lecture time. This led to a fruitful and intensive scientific exchange between the participants. Many open problems were discussed and many collaborative projects were started or continued during this week. Many of the abstracts below list some open problems, thus guiding future research. We received a great deal of positive feedback from the participants and are sure that this workshop will lead to further collaboration between the participants.

It is our pleasure to thank the administration and staff of the MFO for their efficient work and their hospitality which was essential for the stimulating atmosphere during the workshop.

Workshop: Topological and Variational Methods for Differential Equations

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Abstracts

Nonlinear Schrödinger Equations with Potentials Vanishing at Infinity

Antonio Ambrosetti

We discuss some recent results dealing with the existence of solutions of the following elliptic problem on $\mathbb{R}^n$

$$(NLS_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \Delta u + V(x)u = K(x)u^p, \\ u \in W^{1,2}(\mathbb{R}^n),\ u > 0, \end{cases}$$

In the sequel we will always assume that $n \geq 3$.

We will focus on potentials V, K which decay to zero at infinity, referring for results dealing with other cases to the forthcoming book [2] and the references therein. Precisely we assume

$$(\mathrm{V}) \qquad \exists\ \alpha, a_1, a_2 > 0 \quad : \quad \frac{a_1}{1+|x|^\alpha} \leq V(x) \leq a_2,$$

$$(\mathrm{K}) \qquad \exists\, \beta, a_3 > 0 \quad : \quad 0 < K(x) \leq \frac{a_3}{1+|x|^\beta}\ .$$

Our first result deals with the existence of solutions of (NLS_ε) for every $\varepsilon > 0$. Set

$$\sigma = \sigma_{n,\alpha,\beta} = \begin{cases} \frac{n+2}{n-2} - \frac{4\beta}{\alpha(n-2)}, & \text{if } 0 < \beta < \alpha \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 1. [1, Thm. 1] *Let (V) hold with $0 < \alpha < 2$, (K) hold with $\beta > 0$ and suppose that $\sigma < p < \frac{n+2}{n-2}$. Then for every $\varepsilon > 0$, (NLS_ε) has a solution which is a ground-state, namely has minimal energy.*

For the application to Quantum Mechanics, it is also interesting to study the existence of solutions to (NLS_ε) for $\varepsilon \ll 1$, and their behavior as $\varepsilon \to 0$.

Theorem 2. [1, Thm. 3] *Under the same hypotheses of Theorem 1, the ground states concentrate at the global minimum of the auxiliary potential*

$$Q(x) := [V(x)]^\theta\, [K(x)]^{-2/(p-1)}, \qquad \theta = \frac{p+1}{p-1} - \frac{n}{2}.$$

Remark. If $\sigma < p < \frac{n+2}{n-2}$ then Q has indeed a global minimum.

Theorem 2 can be improved handling a broader class of potentials.

Theorem 3. [4] *Let $1 < p < \frac{n+2}{n-2}$ and suppose that V and K are smooth and satisfy*

$$(V_1) \qquad \exists\, a, A > 0, \text{ such that } \quad \frac{a}{1+|x|^2} \leq V(x) \leq A,$$

$$(V_2) \qquad \exists\ A_1 > 0\ :\ |V'(x)| \leq A_1, \quad \forall\, x \in \mathbb{R}^n.$$

$$(K_1) \qquad \exists\, \kappa > 0 \;:\; 0 < K(x) \le \kappa, \quad |K'(x)| \le \kappa, \quad \forall\, x \in \mathbb{R}^n.$$

Moreover, let x_0 be an isolated stable stationary point of Q. Then for $\varepsilon \ll 1$, (NLS_ε) has a solution concentrating at x_0.

We conclude this presentation with a result dealing with the case of radial potentials. Consider the problem

$$(NLS_{\varepsilon,r}) \qquad \begin{cases} -\varepsilon^2 \Delta u + V(|x|)u = u^p, \\ u \in W^{1,2}_{\text{rad}}(\mathbb{R}^n),\; u > 0, \end{cases}$$

and set

$$M(r) = r^{n-1} V^\ell(r), \qquad \ell = \frac{p+1}{p-1} - \frac{1}{2}.$$

Theorem 4. [5] *Let $p > 1$ and let $V \ge 0$ be radial, smooth, with bounded V'. Suppose there exist $r_0 \ge 0$ and $a_1, a_2 > 0$ such that*

$$\frac{a_1}{r^2} \le V(r) \le a_2, \qquad \forall\, r > r_0.$$

Moreover, let $r^* > r_0$ be a strict maximum or minimum of M. Then for $\varepsilon \ll 1$, $(NLS_{\varepsilon,r})$ has a radial solution concentrating at the sphere of radius r^*.

This latter result improves a previous one [3] because V is allowed to be zero for $r < r_0$ and to decay to zero at infinity.

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On Some Weighted Hardy-Sobolev Inequality

Zhi-Qiang Wang

Consider a family of weighted Hardy-Sobolev type inequalities due to Caffarelli, Kohn and Nirenberg (1984): There is $S(a, b) > 0$ such that for all $u \in C_0^\infty(\mathbb{R}^N)$, it holds

$$(1) \qquad \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx \ge S(a,b) \left(\int_{\mathbb{R}^N} |x|^{-bq} |u|^q \, dx \right)^{2/q}$$

for $N \geq 3$:

$$-\infty < a < \frac{N-2}{2}, 0 \leq b - a \leq 1, \quad q = \frac{2N}{N-2+2(b-a)}. \tag{2}$$

These inequalities extend to $\mathcal{D}_a^{1,2}(\mathbb{R}^N) := \overline{C_0^\infty(\mathbb{R}^N)}^{||\cdot||}$ with respect to the norm

$$||u||_a^2 = \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 dx,$$

and have the associated Lagrange equation

$$-div(|x|^{-2a}\nabla u) = |x|^{-bq} u^{q-1}, x \in \mathbb{R}^N \tag{3}$$

which is a prototype of more general anisotropic type nonlinear elliptic PDEs with multiple singularities and degeneracies. The L^p $(p > 1)$ version of the inequalities are given by replacing each 2 with p in (1) and (2). The questions we are concerned with include symmetry property of extremal functions (i.e., ground state solutions of the PDEs) and Hardy-Sobolev inequalities with remainder terms.

• **Symmetry and symmetry breaking of extremal functions.** Due to the work of Aubin (1976), Talenti (1976), Gidas-Ni-Nirenberg (1981), Lieb (1983), Caffarelli-Gidas-Spruck (1989) and Chou-Chu (1993), for $a \geq 0, a \leq b < a+1$, all extremal functions of the inequalities are radially symmetric. Some recent work have partially clarified the symmetry property of extremal functions for the remaining parameter region. Horiuchi (1997) gives the existence of extremal functions for $a < b < a+1$ and $a = b \geq 0$, and nonexistence of extremal functions for $a = b < 0$.

Theorem 1. *(Catrina-Wang, 2000, 2001) There is a function $h(a)$ defined for $a \leq 0$, satisfying $h(0) = 0$, $a < h(a) < a+1$ for $a < 0$, and $a + 1 - h(a) \to 0$ as $-a \to \infty$, such that for (a, b) satisfying $a < 0$ and $a < b < h(a)$, the extremal functions for $S(a, b)$ are non-radial.*

The symmetry breaking of extremal functions was observed independently by Willem (2003) for the case $0 < b - a << 1$. The curve $h(a)$ was sharpened by Felli-Schneider (2003) to $h(a) = 1 + a - \frac{N}{2}(1 - \frac{N-2-2a}{\sqrt{(N-2-2a)^2+4(N-1)}})$.

Theorem 2. *(Catrina-Wang, 2001) All positive solutions in $D_a^{1,2}(\mathbb{R}^N)$ of (3) satisfy, up to dilations, a modified inversion symmetry: $|x|^{N-2-2a}u(x) = u(|x|^{-2}x)$.*

Theorem 3. *(Catrina-Wang, 2001) Let $c \in (0, 1)$ fixed. For sufficiently large $-a$, the extremal function to $S(a, a+c)$ is unique up to dilations and rotations, and is axially symmetric about a line through the origin.*

Theorem 4. *(Lin-Wang, 2004) For (a, b) satisfying $a < 0$ and $a < b < h(a)$, any extremal function u to $S(a, b)$ is axially symmetric about a line through the origin. Moreover, up to a rotation, $u(x)$ only depends on the radius r and the angle θ_N between the x_N-axis and $\vec{ox}$, and on each sphere $\{x \in \mathbb{R}^N \mid |x| = r\}$, u is strictly decreasing as the angle θ_N increases.*

The symmetry property of the nonradial extremal functions were also studied by Smets-Willem (2003) by using the polarization method.

The symmetry breaking property has been proved for the L^p-version ($N > p > 1$) of the Caffarelli-Kohn-Nirenberg inequalities by Byeon-Wang (2002) and a result similar to Theorem 1 still holds. This was also studied by Smets and Willem (2003) by using a different method.

• **Sharp versions of the improved Hardy inequalities.** When restricted to bounded domains, the right hand side of (1) can add additional terms, i.e., Hardy-Sobolev inequalities with remainder terms. The following is the improved weighted Hardy inequality which gives the sharp version of the improved Hardy inequality due to Brezis-Vazquez (1997) and Vazquez-Zuazua (2000), as well as generalizes theirs to the weighted versions. These inequalities are useful tools for elliptic and parabolic equations having singular potentials.

Theorem 5. *(Wang-Willem, 2003) Let $N \geq 1$, $a < \frac{N-2}{2}$, and $\Omega \subset\subset B_R(0)$ for some $R > 0$. Then there exists $C = C(a, \Omega) > 0$ such that for all $u \in C_0^\infty(\Omega)$*

$$\int_\Omega |x|^{-2a}|\nabla u|^2 dx - \left(\frac{N-2-2a}{2}\right)^2 \int_\Omega |x|^{-2(a+1)} u^2 dx \geq C \int_\Omega \left(\ln \frac{R}{|x|}\right)^{-2} |x|^{-2a}|\nabla u|^2 dx. \tag{4}$$

When $0 \in \Omega$ the inequality is sharp in the sense that $\left(\ln \frac{R}{|x|}\right)^{-2}$ can not be replaced by $g(x) \ln\left(\frac{R}{|x|}\right)^{-2}$ with g satisfying $|g(x)| \to \infty$ as $|x| \to 0$.

Theorem 6. *(Wang-Willem, 2003) Let $N \geq 1$, $a \leq \frac{N-2}{2}$, and $\Omega \subset B_R(0)$ or $\Omega \subset \mathbb{R}^N \setminus B_R(0)$ for some $R > 0$. Then for all $u \in C_0^\infty(\Omega)$*

$$\int_\Omega |x|^{-2a}|\nabla u|^2 dx - \left(\frac{N-2-2a}{2}\right)^2 \int_\Omega |x|^{-2(a+1)} u^2 dx \geq \frac{1}{4} \int_\Omega \left(\ln \frac{R}{|x|}\right)^{-2} |x|^{-2(a+1)} u^2 dx. \tag{5}$$

When $0 \in \Omega$ (Ω is an exterior domain, reps.) the inequality is sharp in the sense that $\left(\ln \frac{R}{|x|}\right)^{-2}$ can not be replaced by $g(x) \ln\left(\frac{R}{|x|}\right)^{-2}$ with g satisfying $|g(x)| \to \infty$ as $|x| \to 0$ (as $|x| \to \infty$, reps.). The best constant $\frac{1}{4}$ is sharp.

• **Further questions.** We close up by proposing a few concrete open questions.

1.) The symmetry of extremal functions for parameters $a \leq 0, h(a) \leq b < a+1$. The conjecture is that all extremal functions in this region are radially symmetric.

2.) For the L^p version, $N > p > 1$, when $a = b < 0$ it is not known whether extremal functions exist. The conjecture is that a non-existence result holds.

3.) For the L^p ($p > 1$) case, it seems a sharp version of the improved Hardy inequality like (4) is not known yet.

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A variational problem for the spatial segregation of reaction–diffusion systems and related problems

Susanna Terracini

What follows is the extended abstract of the talk based on results published in the following joint papers:

1. M. Conti, S. Terracini, G. Verzini, *Nehari's Problem and Competing Species Systems*, Ann. Inst. H. Poincaré, AN 19, 6 (2002) 871–888
2. M. Conti, S. Terracini, G. Verzini, *An optimal partition problem related to nonlinear eigenvalues*, Journal of Funct. Anal. 198, 1 (2003) 160-196
3. M. Conti, S. Terracini, G. Verzini, *A variational problem for the spatial segregation of reaction–diffusion systems*, Indiana Univ. Math. J., to appear
4. M. Conti, S. Terracini, G. Verzini, *On a class of optimal partition problems related to the Fučík spectrum and to the monotonicity formulae*, Calc. Var. Partial Differential Equations 22 (2005), no. 1, 45–72s
5. M. Conti, S. Terracini, G. Verzini, *Asymptotic estimates for the spatial segregation of competitive systems*, preprint

The minimization problem

Let Ω be a bounded open, regular, connected subset of $\mathbb{R}^N$ ($N \geq 2$) and let us call *segregated state* a k–tuple $U = (u_1, \ldots, u_k) \in (H^1(\Omega))^k$ where

$$u_i(x) \cdot u_j(x) = 0 \qquad i \neq j, \text{ a.e. } x \in \Omega.$$

We define the *internal energy* of U as

$$J(U) = \sum_{i=1,\cdots,k} \left\{ \int_\Omega \left(\frac{1}{2} \, d_i^2(x) \, |\nabla u_i(x)|^2 - F_i(x, u_i(x)) \right) dx \right\},$$

Our first goal is to minimize J among a class of segregated states subject to some boundary and positivity conditions. We consider the following assumptions:

- On the boundary data ϕ_i's: $\phi_i \in H^{1/2}(\partial\Omega)$, $\phi_i \geq 0$, and

$$\phi_i \cdot \phi_j = 0, \qquad \forall i \neq j, \text{a.e.} on\partial\Omega$$

- On the diffusions d_i's and the f_i's: $d_i \in W^{2,\infty}(\Omega)$, $d_i > 0$ on $\overline{\Omega}$

(A1) $f_i(x, s)$ is Lipschitz in s, uniformly in x,

$$f_i(x, 0) \equiv 0$$

(A2) there exists $b_i \in L^\infty(\Omega)$ such that both

$$|f_i(x, s)| \leq b_i(x)s \qquad \forall\, x \in \Omega\ s \geq \bar{s} >> 1,$$

$$\int_\Omega \left(d_i^2(x)|\nabla w(x)|^2 - b_i(x)w^2(x)\right) dx > 0 \qquad \forall\, w \in H_0^1(\Omega).$$

The last assumption ensures the *coercivity* of the functional.

Surprisingly enough, the minimizer of J turns out to be unique in the *globally convex* case:

Theorem 1. *Assume moreover that*

$$d_i \equiv d_j\ , \forall i, j$$

$$\frac{\partial^2 F_i}{\partial s^2}(x, s) < 0, \forall x \in \Omega$$

Then, for each fixed boundary data, there is an unique minimizer.

Now we seek the extremality conditions associated to the minimization problem. Denote

$$\widehat{u}_i = u_i - \sum_{h \neq i} u_h$$

and similarly

$$\widehat{f}(x, \widehat{u}_i) = \sum_j f_j(x, \widehat{u}_i)\chi_{\text{supp}(u_j)} = \begin{cases} f_i(x, u_i) & \text{if } x \in \text{supp}(u_i) \\ -f_j(x, u_j) & \text{if } x \in \text{supp}(u_j), \\ & j \neq i. \end{cases}$$

Theorem 2. *Let U be a minimizer. Then, for every i, we have, in distributional sense,*

$$-\Delta\widehat{u}_i \geq \widehat{f}(x, \widehat{u}_i)$$

The class $\mathcal{S}$

Next we introduce a special functional class, which will be shown to contain both the extremals of the minimization problem and the limits of reaction–diffusion systems as the interspecific competition rate tends to infinity.

$$\mathcal{S} = \left\{ \begin{array}{l} (u_1, \cdots, u_k) \in (H^1(\omega))^h : u_i \geq 0,\ u_i \cdot u_j = 0 \text{ if } i \neq j \\ -\mathit{\Delta}\widehat{u}_i \geq \widehat{f}(x, \widehat{u}_i), \forall i = 1, \ldots, k \end{array} \right\}$$

Proposition 3. *If* $m(x) = 2$

$$\lim_{\substack{y \to x \\ y \in \mathrm{supp}(u_i)}} \nabla u_i(y) = - \lim_{\substack{y \to x \\ y \in \mathrm{supp}(u_j)}} \nabla u_j(y) \,.$$

We first consider the problem of regularity. To this aim, we define the following subclass of $\mathcal{S}$:

$$\mathcal{S}^*_{M,h}(\omega) = \left\{ (u_1, \cdots, u_h) \in (H^1(\omega))^h : \begin{array}{c} u_i \geq 0, \\ u_i \cdot u_j = 0 \text{ if } i \neq j \\ -\Delta u_i \leq M, \\ -\Delta \widehat{u}_i \geq -M \end{array} \right\}$$

Theorem 4. *Let* $M > 0$ *and* k *be a fixed integer. Let* $U \in \mathcal{S}^*_{M,k}(\Omega)$*: then* U *is Lipschitz continuous in the interior of* Ω*.*

Suitable variants and extensions of the the monotonicity formula by Alt, Caffarelli and Friedman turn out to be the key point in this analysis.

A class of competition–diffusion systems

Consider a system of k competing densities:

$$\begin{cases} -\Delta u_i(x) &= -\kappa u_i(x) \sum_{j \neq i} u_j(x) + f_i(x, u_i) & x \in \Omega \\ u_i(x) &= \phi_i(x) & x \in \partial\Omega \\ u_i(x) &> 0 & x \in \Omega \end{cases}$$

We are interested in the asymptotics as $\kappa \to \infty$.

Theorem 5. *Let* $U_\kappa = (u_{1,\kappa}, ..., u_{k,\kappa})$ *be a solution of the system at fixed* κ*. Let* $\kappa \to \infty$*: then, there exists* U *such that, for all* $i = 1, \ldots, k$*:*

(1) *up to subsequences,* $u_{i,\kappa} \to u_i$ *strongly in* H^1
(2) *if* $i \neq j$ *then* $u_i \cdot u_j = 0$ *a.e. in* Ω
(3) $-\Delta \widehat{u}_i \geq \widehat{f}(x, \widehat{u}_i)$*.*

- The class $\mathcal{S}$ is the limiting class of the solutions of competition–diffusion systems, as the interspecific competition rates tends to infinity

A perturbed monotonicity lemma

Let$f$ be a smooth function $f(r) : [0, \infty) \to (0, \infty)$ such that $f(r) = 1/r^{N-2}$ when $r > 1$ and $\Delta f(|x|) := 2m(|x|)$ is bounded and vanishes outside the ball of radius 1. Now we are ready to prove

Lemma 6. *Let* $N \geq 2$ *and let* $(u_1, \ldots, u_k)$ *be a solution of the system such that* $u_i > 0$ *for all* i*. Let* $h \leq k$ *be any integer, let* $h' < \beta(h, N)$ *and define*

$$\Phi(r) = \prod_{i=1}^{h} \frac{1}{r^{h'}} \Phi_i(r)$$

where

$$\Phi_i(r) = \int_{B(0,r)} \Big[f(|x|) \Big(|\nabla u_i(x)|^2 + u_i^2(x) \sum_{\substack{1 \leq j \leq k \\ j \neq i}} a_{ij} u_j(x) \Big) - m(|x|) u_i^2(x) \Big] dx \, .$$

Then there exists $r' = r(h') > 1$ *such that* $\Phi_i > 0$ *and* Φ *is an increasing function in* $[r', \infty)$.

As a consequence of the perturbed monotonicity formula we have a Liouville–type result:

Theorem 7. *Let* $k \geq 2$ *and let* $U = (u_1, \dots, u_k)$ *be a solution of*

$$\begin{cases} -\Delta u_i(x) &= -u_i(x) \sum_{j \neq i} a_{ij} u_j(x) & x \in \mathbb{R}^N \\ u_i(x) &\geq 0 & x \in \mathbb{R}^N \end{cases}$$

for every i. *Let* $\alpha \in (0, \beta(k, N))$ *such that*

$$\max_{i=1,\dots,k} \sup_{x \in \mathbb{R}^N} \frac{|u_i(x)|}{1 + |x|^\alpha} < \infty.$$

Then, $k-1$ *components annihilate and the last is a nonnegative constant.*

As a further consequence we can prove equi–hölderianity with respect to κ and obtain asymptotic estimates for the both the two–density and the many–density problems.

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Solution Set Splitting at Low Energy Levels in Nonlinear Schrödinger Equations with Periodic and Symmetric Potentials

Nils Ackermann

Schrödinger Equations of the type

$$-\Delta u(x) + V(x)u(x) = |u(x)|^{p-2}u(x) \tag{1}$$

with p in the subcritical range $(2, 2^*)$ are known to admit solutions u lying in $H^1(\mathbb{R}^N)$ if $V \in L^\infty(\mathbb{R}^N)$ satisfies $\operatorname{ess\,inf} V > 0$ and V is 1-periodic in x^i for $i = 1, 2, \ldots, N$ [4]. The associated variational functional

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u(x)|^2 + V(x)u(x)^2)\, dx - \frac{1}{p}\int_{\mathbb{R}^N} |u(x)|^p\, dx$$

is bounded below on the set K of nonzero solutions of (1), which coincides with the set of nonzero critical points of J .

Denote $c_0 := \inf J(K)$ and

$$K^c := \{\, u \in K \mid J(u) \le c \,\}$$

for c in $\mathbb{R}$. Note that the periodicity of V implies invariance of J under the action $\star$ of $\mathbb{Z}^N$ on $H^1(\mathbb{R}^N)$, defined by translation: $(a \star u)(x) := u(x-a)$ for a in $\mathbb{Z}^N$ and x in $\mathbb{R}^N$. Assuming that

$$\text{for some } \varepsilon > 0 \text{ the set } K^{c_0+\varepsilon}/\mathbb{Z}^N \text{ is finite,} \tag{2}$$

In 1991 Coti Zelati and Rabinowitz proved existence of solutions that are near to sums of translates of a certain initial solution, where the translations must be sufficiently large [2]. This class of solutions is commonly known as *multibump solutions.*

The only example where (2) is proved to hold was given by Kabeya and Tanaka [3]: They build a parameter dependent potential V_μ such that (2) is true for μ large enough. No explicit estimate is given for this range.

If $c \in \mathbb{R}$ we say that *the solution set splits at the level* c if the following holds:

(S_c) There is a compact subset $\mathcal{K}$ of K^c such that $K^c = \mathbb{Z}^N \star \mathcal{K}$ and $\mathcal{K} \cap (\mathbb{Z}^N \backslash \{0\}) \star \mathcal{K} = \emptyset$.

Note that (S_{c_1}) implies (S_{c_2}) if $c_2 \le c_1$. Our result in [1] states that two-bump solutions with a sign change can be constructed if the solution set splits at the level c_0. In that paper we also gave two parameter dependent examples that satisfy (S_c) for some $c > c_0$. Again we could not estimate the range of admissible parameters.

In the work presented here we give an explicit condition on V that implies (S_c) for some $c > c_0$. Namely we prove:

Theorem 1. *Suppose that $V \in C^2$. Fix some $\varepsilon > 0$. There are positive constants C_1, C_2 and C_3 that depend only on ε, $\min V$, $\max V$ and p, and that can be estimated explicitly, with the following property: Define $g\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ by*

$$g(r) := C_3 \frac{\int_{B_r(0)} \mathrm{e}^{-C_1|x|}\, dx}{\int_{\mathbb{R}^N \backslash B_r(0)} \mathrm{e}^{-C_2|x|}\, dx}\ .$$

Also define for $i \in \{1, 2, 3, \dots, N\}$ *and* $r \geq 0$

$$\Gamma_i := \sup |\partial_i^2 V|, \qquad \gamma_i(r) := \min\{ -\partial_i^2 V(x) \mid |x^i| \leq r \}.$$

If for every $i \in \{1, 2, 3, \dots, N\}$

- V *is even in* x^i
- *there is* $R_i > 0$ *such that*

$$\Gamma_i \leq \gamma_i(R_i) g(R_i) \ ,$$

then the solution set splits at the level $2c_0 - \varepsilon$.

We also present a simple way to construct examples that satisfy these assumptions.

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Uniqueness theorems in the calculus of variations by the method of transformation groups

Wolfgang Reichel

A classical problem in the calculus of variations is to determine if a given functional $\mathcal{L} : A \to \mathbb{R}$ has at most one critical point in the class A of admissible elements. A prototype functional is

$$\mathcal{L}[u] = \int_\Omega L(x, u, \nabla u).$$

In [2], [3] uniqueness results are given. Consider a one-parameter group $\{g_\epsilon\}_{\epsilon\in\mathbb{R}}$ of transformations $g_\epsilon : A \to A$ with $g_0 = \mathrm{Id}$ and $g_{\epsilon+\delta} = g_\epsilon \circ g_\delta$. The group is called a *variational symmetry* if $\mathcal{L}[g_\epsilon u] = \mathcal{L}[u]$ for all $u \in A, \epsilon \in \mathbb{R}$ or equivalently $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} = 0$ for all $u \in A$. Emmy Noether's [1] famous result of 1918 says that a variational symmetry leads to a conservation law.

As an obvious generalization one defines a *variational subsymmetry* as a one-parameter group s.t. $\mathcal{L}[g_\epsilon u] \leq \mathcal{L}[u]$ for all $\epsilon \geq 0$ and all $u \in A$ or equivalently $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} \leq 0$ for all $u \in A$. Simple examples are the negative gradients flow or, if available, the parabolic flow with ϵ as time. For the purpose of uniqueness theorems one sharpens this concept to a *strict variational subsymmetry with respect to* $u_0 \in A$, i.e., $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} < 0$ for all $u \in A \setminus \{u_0\}$. This is the basis for showing that u_0 is the only (possible) critical point of the functional $\mathcal{L}$.

Specific examples of one parameter groups are transformation groups described as follows. Consider a flow in $\mathbb{R}^{n+1}$ given by

$$\dot{X} = \xi(X,U), \quad X(0) = x \in \mathbb{R}^n, \qquad \dot{U} = \phi(X,U), \quad U(0) = u \in \mathbb{R}$$

with a smooth vectorfield ξ and a smooth function ϕ. The solution at time $\epsilon \in \mathbb{R}$ is denoted by $(\chi_\epsilon(x,u), \psi_\epsilon(x,u))$. If $u : \overline{\Omega} \to \mathbb{R}$ is a given function then the flow maps $(x, u(x))$ onto $(\tilde{x}, \tilde{u}) = (\chi_\epsilon(x,u(x)), \psi_\epsilon(x,u(x)))$. In order to find the functional between $(\tilde{x}, \tilde{u})$ one needs to invert the expression $\tilde{x} = \chi_\epsilon(\mathrm{Id} \times u)(x)$ as shown

$$x \xrightarrow{\chi_\epsilon(\mathrm{Id}\times u)} \tilde{x}, \qquad x \xleftarrow{[\chi_\epsilon(\mathrm{Id}\times u)]^{-1}} \tilde{x}.$$

This leads to the following formula for the transformed function $g_\epsilon u : \overline{\Omega}_\epsilon \to \mathbb{R}$

$$g_\epsilon u(\tilde{x}) = \tilde{u}(\tilde{x}) = \psi_\epsilon(\mathrm{Id}\times u)[\chi_\epsilon(\mathrm{Id}\times u)]^{-1}(\tilde{x}), \quad \tilde{x} \in \Omega_\epsilon, \tag{1}$$

where Ω_ϵ is the flow-deformation of Ω at time ϵ. If $u \in \mathrm{Lip}(\overline{\Omega})$ then $g_\epsilon u \in \mathrm{Lip}(\overline{\Omega}_\epsilon)$. From now on we will work in the space $A = \mathrm{Lip}_0(\overline{\Omega})$ of Lipschitz-functions with zero-boundary data.

Definition 1. *Let ξ, ϕ be the infinitesimal generators of the transformation group $\{g_\epsilon\}_{\epsilon\in\mathbb{R}}$.*

- (i) $u_0 \in \mathrm{Lip}_0(\overline{\Omega})$ *is called a fixed point of the group $\{g_\epsilon\}_{\epsilon\in\mathbb{R}}$ if $g_\epsilon u_0 = u_0$ on their common domain of definition. This is equivalent to $\phi(x, u_0(x)) - \nabla u_0(x) \cdot \xi(x, u_0(x)) = 0$ in Ω.*
- (ii) *The group $\{g_\epsilon\}_{\epsilon\in\mathbb{R}}$ is called domain contracting if $\Omega_\epsilon \subset \Omega$, $\Omega \neq \Omega_\epsilon$ for $\epsilon > 0$. This is equivalent to $\xi(x,0)\cdot\nu(x) \le 0, \not\equiv 0$ for all $x \in \partial\Omega$.*

In the presence of a fixed point u_0 the group-action of a domain contracting (half)-group $\{g_\epsilon\}_{\epsilon\ge 0}$ can be extended to map $\mathrm{Lip}_0(\overline{\Omega})$ into itself by setting $g_\epsilon u(x) = u_0(x)$ if $x \in \overline{\Omega}\setminus\Omega_\epsilon$ and $\epsilon > 0$.

Theorem 2 (cf. [2])**.** *Let $\mathcal{L}[u] = \int_\Omega L(x,u,\nabla u)$ for $u \in A = \mathrm{Lip}_0(\overline{\Omega})$. Suppose $\{g_\epsilon\}_{\epsilon\ge0}$ is a transformation group with fixed point u_0 which is moreover domain contracting and a strict variational subsymmetry w.r.t. u_0. Let furthermore $L(x,u,p)$ as a function $(x,u,p) \in \overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ be convex in the gradient-variable p. Then u_0 is the only strong $C^1_0(\overline{\Omega}) \cap W^{2,1}_0(\Omega)$-critical point of $\mathcal{L}$, where strong means that the Euler-Lagrange equation holds pointwise a.e. in Ω.*

The proof is based on the rate-of-change formula for strong solutions of the Euler-Lagrange equation. This identity is sometimes called Noether's identity or Pohožaev's identity:

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]\Big|_{\epsilon=0} = \int_\Omega \xi\cdot\nabla_x L + \phi\partial_u L + (\nabla\phi - D\xi^*\nabla u)\cdot\nabla_p L + L\,\mathrm{Div}\,\xi\,dx$$

$$= \int_\Omega \mathrm{div}\left(\xi L + (\phi - \xi\cdot\nabla u)\nabla_p L\right)dx = \oint_{\partial\Omega} \underbrace{(\xi\cdot\nu)}_{\le 0}\underbrace{\left(L - \nabla_p L\cdot(\nabla u - \nabla u_0)\right)}_{\le 0}ds$$

Example A: Let Ω, $\partial\Omega \neq \emptyset$ be a bounded open subset of an n-dimensional Riemannian manifold (M, g) without boundary. Consider the problem

$$-\Delta u = \lambda u + |u|^{p-1} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{2}$$

Theorem 3 (cf. [2])**.** *Suppose there exists a conformal vectorfield ξ on Ω, i.e., a vector-field such that $D\xi + D\xi^* = mult.\operatorname{Id}|_{T_xM}$ and suppose $\operatorname{div}\xi \leq 0, \not\equiv 0$. Moreover, let Ω be conformally contractible, i.e., pos. invariant under the flow $\dot{X} = \xi(X)$. If $p \geq \frac{n+2}{n-2}$ and $\lambda \leq \frac{1}{p-1}\frac{\Delta \operatorname{div}\xi}{\operatorname{div}\xi}$ then $u \equiv 0$ is the only strong $C_0^1(\overline{\Omega}) \cap W^{2,1}(\Omega)$-solution of (2).*

Remarks. (a) In $\mathbb{R}^n$ the class of conformally contractible domains is larger than the class of star-shaped domains. Examples can be found in [2], [3].
(b) If the scalar curvature R of M is constant then the quantity $\frac{\Delta \operatorname{div}\xi}{\operatorname{div}\xi} = R/(1-n)$. The uniqueness results obtained in this way for the sample manifolds $\mathbb{R}^n, \S^n, \mathbb{H}^n$ are sharp for $p = \frac{n+2}{n-2}$ and $n \geq 4$.
(c) The condition of ξ being conformal can be relaxed to $D\xi + D\xi^* \geq -2M(x)\operatorname{Id}$, where $M(x)$ is some scalar function. Uniqueness holds for $p \geq \frac{-\operatorname{div}\xi + 2M(x)}{-\operatorname{div}\xi - 2M(x)}$ provided $-2M(x) \geq \operatorname{div}\xi$. For Euclidean $\mathbb{R}^n$ this idea is due to Schaaf, [4].

Example B: The following example is similar to Example A. Consider for $m \in \mathbb{N}$

$$-\Delta^m u = \lambda u + |u|^{p-1} u \text{ in } \Omega, \qquad u = \nabla u = \ldots = D^{m-1} u = 0 \text{ on } \partial\Omega \tag{3}$$

Theorem 4 (cf. [3])**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded and conformally contractible.*

1. *If $\lambda < 0$ and $p \geq \frac{n+2m}{n-2m}$ then $u \equiv 0$ is the only $C^{2m}(\overline{\Omega})$-solution of (3).*
2. *If $\lambda < 0$ and $1 < p < \frac{n+2m}{n-2m}$ and u is a $C^{2m}(\overline{\Omega})$-solution of (3) then either $u \equiv 0$ or*
$$\|u\|_\infty^{p-1} \geq -\lambda \frac{2m(p+1)}{2n - (p+1)(n-2m)}.$$

Example C: Finally let us consider an example with uniqueness of the nontrivial solution.

$$-\Delta^m u = \lambda(1 + |u|^{p-1} u) \text{ in } \Omega, \qquad u = \nabla u = \ldots = D^{m-1} u = 0 \text{ on } \partial\Omega \tag{4}$$

Theorem 5 (cf. [3])**.** *Let $\Omega \subset \mathbb{R}^n$, $n > 2m$, be a bounded conformally contractible domain with conformal vector-field ξ s.t. $\operatorname{div}\xi \leq 0$ in Ω. Let $p > \frac{n+2m}{n-2m}$ and $\lambda \geq 0$.*

1. *If $m = 1$ or $m \geq 2$ and $\Omega = B_1(0)$ then there exists $\bar{\lambda} > 0$ such that (4) has a unique* **positive** *solution for $\lambda \in [0, \bar{\lambda}]$.*
2. *Suppose $p \geq 2$. If $m = 1$ and $\operatorname{div}\xi < 0$ in $\overline{\Omega}$ or $m \geq 2$ and no further restriction on $\operatorname{div}\xi$ then there exists $\bar{\lambda} > 0$ such that (4) has a unique solution for $\lambda \in [0, \bar{\lambda}]$.*

Remarks. (a) Under the same restrictions on n, m, p and λ the result holds for $f(x, s) = 1 + \lambda|s|^{p-1}s$ and $f(x, s) = \lambda e^s$. In particular (2) holds for $f(x, s) = \lambda e^s$ if $n > 2m$.
(b) Part (1) of the theorem generalizes to all those bounded domains Ω where the positivity preserving property of $(-\Delta)^m$ holds.

Open Problem. Can one extend the result of Theorem 5 to the critical case $p = \frac{n+2m}{n-2m}$ or not? For $m = 1$ results were obtained by Schaaf [4].

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Spikes for the Gierer-Meinhardt System – Analysis and Numerical Simulation

Matthias Winter

We consider two typial types of Turing systems [9] which have both been suggested by Gierer and Meinhardt [2].

First, the **Activator-Inhibitor Model**

$$\text{(GM1)} \quad \begin{cases} a_t = \epsilon^2 \Delta a - a + \frac{a^2}{h} \\ \tau h_t = D\Delta h - h + a^2 \end{cases}$$

where $a = a(x,t)$, $h(x,t)$ are the concentrations of two morphogens, the **activator** and **inhibitor**; $\epsilon^2, D > 0$ are the **diffusivities** and $\tau \geq 0$ is a **time-relaxation constant**. We study (GM1) on a bounded, smooth domain $\Omega \subset \mathbb{R}^2$ with Neumann boundary conditions.

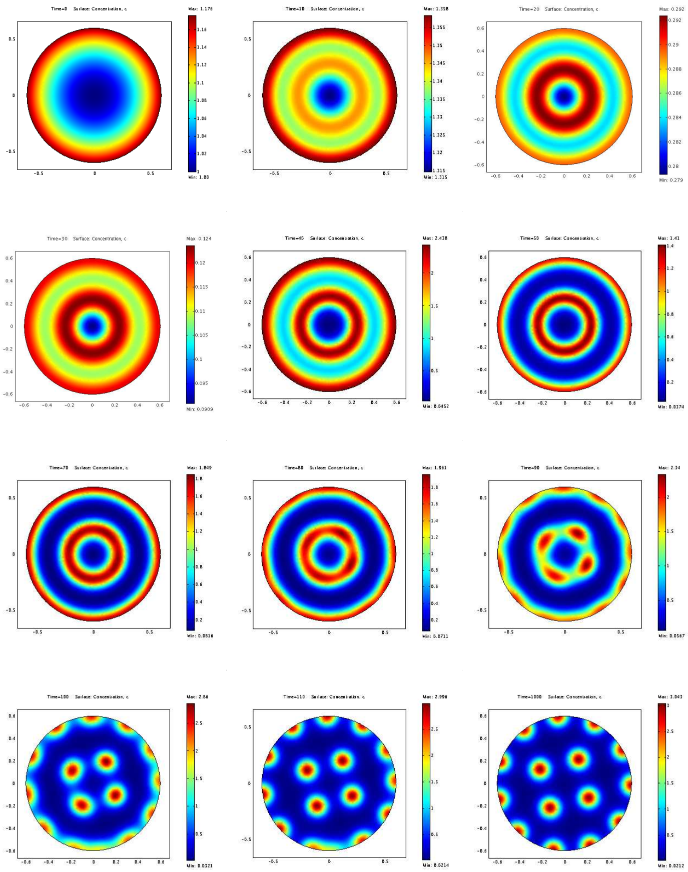
Second, the **Activator-Substrate Model**

$$\text{(GM2)} \quad \begin{cases} a_t = \epsilon^2 \Delta a - a + Aha^2 \quad \text{in } \Omega, \\ \tau h_t = D\Delta h + 1 - \mu h - ha^2 \quad \text{in } \Omega, \end{cases}$$

where $a = a(x,t)$ and $h(x,t)$ are the concentrations of **activator** and **substrate**. In addition to the constants above, here we have the **feedrate** A. Special cases are the Schnakenberg Model ($\mu = 0$) and the Gray-Scott Model ($\mu = 1$).

Numerical simulations frequently show **spikes**, i.e. steady states, where the **activator** is concentrated in **narrow regions** around **finitely many points** for $\frac{\epsilon^2}{D}$ small enough. Often these spiky patterns are numerically stable. Some references for such numerical simulations are H. Meinhardt [3], T. Kolokolnikov, M.J. Ward et. al. [6], [7]. J.E. Pearson [8]. We now present one time-dependent numerical simulation for the activator-inhibitor system using the Finite Element Simulation Software FEMLAB (Winter 2005).

We choose the system (GM1) with $\epsilon^2 = 0.001, D = 0.01, \tau = 1$.



In this simulation we observe unstable rings and the final state consists of 9.5 spikes.

What can be said about the number of spikes in general?

We analyze the weak-coupling case

$$\frac{\epsilon^2}{|\Omega|} << 1, \quad \frac{D}{|\Omega|} >> 1.$$

Our first results gives existence, asymptotic profile and positions of multiple spikes. Set $\eta_\epsilon = \frac{|\Omega|}{2\pi D} \ln \frac{\sqrt{|\Omega|}}{\epsilon}$.
Assume that $\lim_{\epsilon \to 0} \eta_\epsilon = \eta_0 \in [0, \infty] \setminus \{K\}$.

Theorem 1. *([12]) For ϵ small enough and D large enough, there exists a* **steady state** *(a_ϵ, h_ϵ) of (GM1) with*
(1) $a_\epsilon(x) = \xi_\epsilon \left(\sum_{j=1}^K w\left(\frac{x - P_j^\epsilon}{\epsilon}\right) + o(1) \right)$ uniformly for $x \in \bar{\Omega}$.

(2) $h_\epsilon(x) = \xi_\epsilon(1 + o(1))$ *uniformly for* $x \in \bar{\Omega}$.

- **Profile:** w *is the* **unique solution** *(ground state) of*

$$\begin{cases} \Delta w - w + w^2 = 0, \quad w > 0 \quad \text{in } \mathbb{R}^2, \\ w(0) = \max_{y \in \mathbb{R}^2} w(y), \quad w(y) \to 0 \text{ as } |y| \to \infty. \end{cases}$$

- **Amplitude:** $\xi_\epsilon \to \infty$ *as* $\epsilon \to 0$.
- **Positions:** $P_j^\epsilon \to P_j^0$ *as* $\epsilon \to 0$, $j = 1, \ldots, K$.

The **Proof of Theorem 1** uses Green's function, Liapunov-Schmidt reduction and asymptotic analysis.

The second theorem answers the question of how many spikes can be (linearly) stable. Assume that

$$\lim_{\epsilon \to 0} \eta_\epsilon = \lim_{\epsilon \to 0} \frac{|\Omega|}{2\pi D} \ln \frac{\sqrt{|\Omega|}}{\epsilon} = \eta_0 \in (0, \infty) \setminus \{K\}.$$

Theorem 2. *([12]) For* ϵ *small enough, we have for the* $K-$*spike solution* (a_ϵ, h_ϵ) *of Theorem 1:*

If $K < \eta_0$, *there exist* $0 < \tau_1 \leq \tau_2$, *such that* (a_ϵ, h_ϵ) *is* **stable** *for* $\tau < \tau_1$ *or* $\tau > \tau_2$.

If $K > \eta_0$ *and* $K > 1$, *the solution* (a_ϵ, h_ϵ) *is* **unstable** *for all* $\tau \geq 0$.

If $1 = K > \eta_0$, *the solution* (a_ϵ, h_ϵ) *is* **unstable** *for* τ *large enough.*

A generalization of **Theorem 2** to (GM2) is now given. The formula for the maximum number K of stable spikes (for $\tau = 0$) is as follows:

(GM1)	(GM2) Gray-Scott	(GM2) Schnakenberg
η_0	$\left(\frac{\eta_0}{\alpha_0}\right)^{1/2} - 2\eta_0$	$\left(\frac{\eta_0}{\alpha_0}\right)^{1/2}$

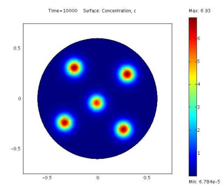
with

$$\eta_0 = \lim_{\epsilon \to 0} \frac{|\Omega|}{2\pi D} \ln \frac{\sqrt{|\Omega|}}{\epsilon}, \qquad \alpha_0 = \lim_{\epsilon \to 0} \frac{\epsilon^2 \int_{R^2} w^2}{A^2 |\Omega|}.$$

This has been proved in [14] for the **Gray-Scott** Model and in Wei/Winter (2005) for the **Schnakenberg** Model.

The **Proof of Theorem 2** consists of deriving and analyzing a nonlocal eigenvalue problem (NLEP) to understand the $O(1)$ eigenvalues. Here [10] and [1] play an important role. Further, some conditions for the positions of the spikes are explored which are related to $o(1)$ eigenvalues.

We have also prove existence and stability of **asymmetric spikes** (small and large amplitude) [15], [17]. Compare the simulation



where the central spike is much smaller than the rest.

The strong-coupling case $D = 1$ has been studied in [11], [13].

Finally, instabilities of spikes arising by **saturation** have been investigated in the shadow system case [16].

Recent surveys of the state of the art have been given in [4], [5].

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Existence of Conformal Metrics with Constant Q-Curvature

Andrea Malchiodi
(joint work with Zindine Djadli and Cergy Pontoise)

On four dimensional manifolds there exists a conformally covariant operator, the Paneitz operator, which enjoys analogous properties to the Laplace-Beltrami operator on surfaces, and to which is associated a natural concept of curvature. This operator, introduced by Paneitz, [5], [6], and the corresponding Q-curvature, introduced in [1], are defined in terms of Ricci tensor Ric_g and scalar curvature R_g of the manifold (M, g) as

$$P_g(\varphi) = \Delta_g^2 \varphi + div_g \left(\frac{2}{3} R_g g - 2Ric_g \right) d\varphi; \tag{1}$$

$$Q_g = -\frac{1}{12} \left(\Delta_g R_g - R_g^2 + 3|Ric_g|^2 \right), \tag{2}$$

where φ is any smooth function on M. The behavior (and the mutual relation) of P_g and Q_g under a conformal change of metric $\tilde{g} = e^{2w} g$ is given by

$$P_{\tilde{g}} = e^{-4w} P_g; \qquad P_g w + 2Q_g = 2Q_{\tilde{g}} e^{4w}. \tag{3}$$

Apart from the analogy between (3) and the transformation law of the Gauss curvature in 2D (after a conformal transformation), we have an extension of the Gauss-Bonnet formula which is the following

$$\int_M \left(Q_g + \frac{|W_g|^2}{8} \right) dV_g = 4\pi^2 \chi(M). \tag{4}$$

Here W_g denotes the Weyl tensor of (M, g). In particular, since $|W_g|^2 \, dV_g$ is a pointwise conformal invariant, it follows that the integral of Q_g over M is also a conformal invariant, and we denote it by k_P.

We consider the problem of finding conformal metrics on M with constant Q-curvature. By (3) (and after a normalization of the volume), we have to solve the following equation

$$P_g u + 2Q_g = k_P e^{4u} \qquad \text{on } M. \tag{5}$$

This equation is indeed variational and solutions can be found as critical points of the following functional

$$II(u) = \int_M u P_g u dV_g + 4 \int_M Q_g u dV_g - k_P \log \int_M e^{4u} dV_g, \tag{6}$$

defined on the Sobolev space $H^2(M)$.

Concerning (5), the only result in the literature was given in [3] (see also [2] for a different proof), where the authors prove existence under the assumptions that P_g is positive-definite (apart on the constants) and that $k_P < 8\pi^2$. The key ingredient for the proof is the following inequality, which extends a result due to D.Adams for the case of flat domains

$$\log \int_M e^{4(u-\overline{u})} dV_g \leq C + \frac{1}{8\pi^2} \int_M u P_g u dV_g. \tag{7}$$

Here $\overline{u}$ denotes the average of u on M and P_g is assumed to be positive definite.

We are able to extend the result in [3] proving the following theorem.

Theorem 1. *Suppose ker* $P_g = \{constants\}$*, and assume that* $k_P \neq 8k\pi^2$ *for* $k = 1, 2, \dots$. *Then* M *admits a conformal metric with constant* Q*-curvature.*

We remark that our assumptions are conformally invariant and generic, so the result applies to a large class of four manifolds. Also, by a result in [4], the set of solutions (which is non-empty) is bounded in $C^m(M)$ for any integer m. We also notice that Theorem 1 does *not* cover some cases of locally conformally flat manifolds with positive Euler characteristic, by (4).

Our assumptions include those made in [3] and one (or both) of the following two possibilities

$$k_P \in (8k\pi^2, 8(k+1)\pi^2), \qquad \text{for some } k \in \mathbb{N}; \tag{8}$$

(9) P_g possesses $\overline{k}$ (counted with multiplicity) negative eigenvalues.

In these cases the functional II is unbounded from below, and hence it is necessary to find extrema which are possibly saddle points. This is done using a new minimax scheme, which depends on k_P and the spectrum of P_g (in particular on the number of negative eigenvalues $\overline{k}$).

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A class of self-dual partial differential equations and its variational principles

Nassif Ghoussoub

Most equations arising on the interface between Riemmanian geometry and quantum field theory (e.g. Yang-Mills, Chern-Simon, Seiberg-Witten and Ginzburg-Landau) have dual or/and selfdual versions which enjoy very special features: They are obtained variationally as minima of the corresponding action functionals, yet they are not derived as Euler-Lagrange equations but as zeroes of certain non-negative Lagrangians. We show that a similar phenomenon of self-duality is quite prevalent among partial differential equations and systems, including some of the most basic ones.

We then use such self-dual features, to develop a systematic approach for a variational resolution of these equations. We establish a general variational principle which identifes the minimal value for this class of equations, and therefore allow for their derivation and their resolution even though they are outside the scope of Euler-Lagrange theory.

Our concept of self-duality extend the examples from QFT in many ways. For one, squares are replaced by general convex functions, and one can substitute the elementary method for completing squares by the Legendre-Fenchel duality formula: i.e., for a convex lower semi-continuous function ϕ on a Banach space X and its Legendre transform ϕ^* on X^*, we have $\phi(x) + \phi^*(p) - \langle x, p\rangle \geq 0$ with equality if and only if $p \in \partial\phi(x)$, where $\partial\phi$ denotes the subdifferential operator. Our *self- (or antiself-) dual equations* can be written in the following form:

$$\begin{cases} L(u, \Gamma u + \Lambda u) + \langle Bu, \Lambda u\rangle = 0 \\ \ell_i(\mathcal{B}_i u) - \frac{1}{2}\langle \mathcal{B}_i u, R_i \mathcal{B}_i u\rangle = 0 \quad \text{for } i = 1, ..., m \end{cases} \tag{1}$$

where L is a Lagrangian on $X \times X^*$ with X being a reflexive Banach space, $(\ell_i)_{i=1}^m$ are boundary Lagrangians on Hilbert spaces $H_1, \dots, H_m$ on which act self-adjoint operators $R_1, \dots, R_m$. The pair $B : X \to X$ and $\Gamma : D(\Gamma) \subset X \to X^*$ are linear operators while the operator $\Lambda : D(\Lambda) \subset X \to X^*$ is not necessarily linear. Our framework requires the following conditions:

(1) The pair (B, Γ) is *skew-adjoint* modulo a "boundary operator"

$$\mathcal{B} := (\mathcal{B}_i)_{i=1}^m : D(\mathcal{B}) \subset X \to \Pi_{i=1}^m H_i.$$

That is, if we have a Stokes-type formula of the form:

$$\langle Bx, \Gamma y\rangle + \langle By, \Gamma x\rangle = \sum_{i=1}^m \langle \mathcal{B}_i x, R_i \mathcal{B}_i y\rangle \qquad \text{for } x, y \in D(\Gamma) \cap D(\mathcal{B}). \tag{2}$$

(2) The operator $\Lambda : D(\Lambda) \subset X \to X^*$ is weak-to weak continuous on its domain in such a way that $x \to \langle Bx, \Lambda x\rangle$ is weakly lower semi-continuous on $D(\Lambda)$.

(3) The Lagrangian L is a convex lower semi-continuous functional on $X \times X^*$ which satisfies the following B-antiselfduality: $L(x, p) = L^*(-B^*p, -Bx)$

for $(x,p) \in X \times X^*$, where L^* denotes the Fenchel-Legendre transform in both variables.

(4) Each ℓ_i is convex lower semi-continuous on the Hilbert space H_i satisfying the R_i-selfduality property $\ell_i(s_i) = \ell_i^*(R_i s_i)$ for $s_i \in H_i$ $i = 1, ..., m$.

Equations (1) are true differential equations because selfduality and the limiting case in the Legendre-Fenchel duality, allow us to rewrite them as:

$$\text{(3)} \qquad \begin{cases} -(B^*\Lambda u + B^*\Gamma u, Bu) & \in & \partial L(u, \Gamma u + \Lambda u) \\ R_i \mathcal{B}_i u & \in & \partial \ell_i(\mathcal{B}_i u) \qquad i = 1, \ldots, m \end{cases}$$

where ∂L is the subdifferential of L. We obtain solutions for (1) by simply minimizing the functional $I(u) = L(u, \Gamma u + \Lambda u) + \langle Bu, \Lambda u \rangle + \sum_{i=1}^{m} \ell_i(\mathcal{B}_i u)$ on an appropriate subset of X. The key here is that under these conditions, the functional I can be written –after completing the squares– as a sum of non-negative terms: $I_0(u) = L(u, \Gamma u + \Lambda u) + \langle Bu, \Gamma u + \Lambda u \rangle$, and $I_i(u) = \ell_i(\mathcal{B}_i u) - \frac{1}{2}\langle \mathcal{B}_i x, R_i \mathcal{B}_i x \rangle$ for $i = 1, \ldots, m$ where the cross product $\langle Bu, \Gamma u \rangle$ has been added to complete the square in I_0, then subtracted in its boundary form. The boundary Lagrangians ℓ_i are then used to complete the squares (in the opposite direction) for the boundary cross-products $\langle \mathcal{B}_i x, R_i \mathcal{B}_i x \rangle$. Under appropriate coercivity conditions, the minimal value of I is zero and is attained and so the same holds for each one of the functionals I_0, I_i ($i = 1, .., m$). The identities thus obtained then lead to the equations (1) above which include naturally occuring boundary conditions. The required variational principles are normally easy to formulate, as the basic self (or anti-self) dual Lagrangian is often of the form $L(x,p) = \phi(x) + \phi^*(\pm p)$ where ϕ is a convex lower semi-continuous function. But the richness of the theory comes from iterating the above Lagrangians with appropriate operators and from the diversity of boundary conditions that one can capture. Here is a sample of self and anti-selfdual equations.

1) Transport equation: Consider

$$\text{(4)} \qquad \begin{cases} \vec{a} \cdot \vec{\nabla} v + a_0 v & = & v|v|^{p-2} + f \quad \text{on } \Omega \subset \mathbb{R}^n \\ v & = & 0 \qquad\qquad \text{on } \Sigma_+ \end{cases}$$

where $p > 1$, $\vec{a}$ is a smooth vector field such that $\frac{1}{2}\vec{\nabla} \cdot \vec{a} - a_0 \geq \delta > 0$ on Ω, and where $\Sigma_\pm = \{x \in \partial\Omega; \pm\vec{a}(x) \cdot \hat{n}(x) \geq 0\}$ are the entrance and exit sets. The minimum on $L^2(\Omega)$ of the functional

$$I(u) = \phi(u) + \phi^*(\vec{a} \cdot \vec{\nabla} u + a_0 u) + \frac{1}{2}\int_\Omega (\frac{1}{2}\vec{\nabla} \cdot \vec{a} - a_0)|u|^2 dx + \frac{1}{2}\int_{\Sigma_+} |u|^2 |\vec{a} \cdot \hat{n}| d\sigma + \frac{1}{2}\int_{\Sigma_-} |u|^2 |\vec{a} \cdot \hat{n}| d\sigma$$

– where $\phi(u) := \frac{1}{p}\int_\Omega |u|^p dx + \int_\Omega uf dx$ – is equal to zero, and is attained at a solution for (4).

2) Incompressible Stationary Navier-Stokes equation: Consider

$$\begin{cases} (u\cdot\nabla)u - f &= \nu\Delta u - \nabla p \quad \text{on } \Omega \\ \text{div} u &= 0 \quad \text{on } \Omega \\ u(x) &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{5}$$

where $\nu > 0$ and $f \in L^p(\Omega;\mathbb{R}^3)$. The minimum on $X = \{u \in H_0^1(\Omega); div(u) = 0\}$ of the functional $I(u) = \Psi(u) + \Psi^*(-(u\cdot\nabla)u + f) - \int_\Omega \langle f, u\rangle dx$ – where $\Phi(u) = \frac{\nu}{2}\int_\Omega \Sigma_{j,k=1}^3 (\frac{\partial u_j}{\partial x_k})^2\, dx$ – is equal to zero and is attained at a solution of (14).

3) Cauchy-Riemann equations: Consider

$$\begin{cases} (\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}) &= \partial\phi(\frac{\partial v}{\partial y}, -\frac{\partial u}{\partial y}) \quad \text{on } \Omega \subset \mathbb{R}^2 \\ -J\vec{n}\cdot\vec{\nabla}v &\in \partial\psi(u) \quad \text{on } \partial\Omega. \end{cases} \tag{6}$$

where Ω is simply connected, ϕ is a convex lsc function on $L^2(\Omega)\times L^2(\Omega)$ and ψ is any convex lsc function on $L^2(\partial\Omega)$. The minimum on $W^{1,2}(\Omega;\mathbb{R}^2)$ of the functional

$$I(u,v) = \int\int_\Omega \{\phi(\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}) + \phi^*(\frac{\partial u}{\partial x}, -\frac{\partial v}{\partial x})\}dxdy + \int_{\partial\Omega}(\psi(u) + \psi^*(-J\vec{n}\cdot\vec{\nabla}v))d\sigma.$$

is equal to zero and is attained at a solution of (6).

4) Nonlinear Laplace equation: Consider

$$\begin{cases} \Delta u \in \partial\phi(u) & \text{on} \quad \Omega \subset \mathbb{R}^n \\ -\frac{\partial u}{\partial n} \in \partial\psi(u) & \text{on} \quad \partial\Omega \end{cases} \tag{7}$$

where ϕ (resp., ψ) are convex functions on $L^2(\Omega)$ (resp., $L^2(\partial\Omega)$). The minimum of $I(u) = \int_\Omega(\phi(u) + \phi^*(\Delta u) + |\nabla u|^2)dx + \int_{\partial\Omega}(\psi(u) + \psi^*(-\frac{\partial u}{\partial n}))d\sigma$ is again zero and is attained at a solution of (7).

What is remarkable is that evolution equations associated to selfdual equations –such as the following– are themselves selfdual.

$$\frac{\partial u}{\partial t} + \Gamma u(t) + \Lambda u(t) + f(t) \in -\partial\phi(t, Bu(t)) \quad \text{on } [0,T]\times\Omega \tag{8}$$

$$R_i u(t) \in \partial\psi_i(u(t)) \quad \text{on } [0,T]\times\partial\Gamma_i \tag{9}$$

$$S_1 u(0) \in \partial\chi_1(u(0)) \quad \text{on } \Omega \tag{10}$$

$$S_2 u(T) \in \partial\chi_2(u(T)) \quad \text{on } \Omega \tag{11}$$

Examples of self-dual evolution equations include Gradient flows for convex functions such as the Heat equation and porous media [5], but also:

5) Anti-Hamiltonian systems: Consider

$$\begin{cases} -\dot{x}(t) &\in \partial_2\Phi(x(t), y(t)) \\ -\dot{y}(t) &\in \partial_1\Phi(x(t), y(t)) \\ -y(0) - A_1x(0) &\in \partial\psi_1(x(0)) \\ y(T) - A_2x(T) &\in \partial\psi_2(x(T)). \end{cases} \tag{12}$$

where Φ (resp., ψ_1, ψ_2) are convex lower semi-continuous on $\mathbb{R}^n \times \mathbb{R}^n$ (resp., $\mathbb{R}^n$) and A_1, A_2 are anti-symmetric matrices on $\mathbb{R}^n$. The minimum of the functional

$$\begin{aligned} I(x,y) = \int_0^T & \Phi((x(t),y(t)) + \Phi^*(-\dot{y}(t), -\dot{x}(t))dt \\ & + \psi_1(x(0)) + \psi_1^*(-y(0) - A_1x(0)) + \psi_2(x(T)) + \psi_2^*(y(T) - A_2x(T)). \end{aligned}$$

on $A^2([0,T];\mathbb{R}^{2n}) = \{u = (x,y) : [0,T] \to \mathbb{R}^{2n}; \dot{u} \in L^2_{\mathbb{R}^{2n}}\}$, is zero and is attained at a solution of (12).

6) Hamiltonian systems: Consider

$$\left\{ \begin{array}{rcl} \dot{x}(t) & \in & \partial_2\Phi(x(t),y(t)) \\ -\dot{y}(t) & \in & \partial_1\Phi(x(t),y(t)) \\ y(0) & \in & \partial\psi_1(x(0)) \\ -x(T) & \in & \partial\psi_2(y(T)). \end{array} \right. \tag{13}$$

where Φ (resp., ψ_1, ψ_2) are convex lower semi-continuous on $\mathbb{R}^n \times \mathbb{R}^n$ (resp., $\mathbb{R}^n$) and A_1, A_2 are anti-symmetric matrices on $\mathbb{R}^n$. The minimum of the following functional

$$\begin{aligned} I(x,y) = \int_0^T & \Phi((x(t),y(t)) + \Phi^*(-\dot{y}(t), \dot{x}(t)) + 2\langle x(t), \dot{y}(t)\rangle)dt \\ & + \psi_1(x(0)) + \psi_1^*(y(0)) + \psi_2(y(T)) + \psi_2^*(-x(T)). \end{aligned}$$

on $A^2([0,T];\mathbb{R}^{2n}) = \{u = (x,y) : [0,T] \to \mathbb{R}^{2n}; \dot{u} \in L^2_{\mathbb{R}^{2n}}\}$, is zero and is attained at a solution of (13).

7) The incompressible Navier-Stokes evolution: Consider

$$\left\{ \begin{array}{rcl} \frac{\partial u}{\partial t} + (u \cdot \nabla)u - f & = & \nu\Delta u - \nabla p \quad \text{on } [0,T] \times \Omega \\ \text{div} u & = & 0 \quad \text{on } [0,T] \times \Omega \\ u(t,x) & = & 0 \quad \text{on } [0,T] \times \partial\Omega \\ u(0,x) & = & u_0(x) \quad \text{on } \Omega. \end{array} \right.$$

where $\nu > 0$ and $f \in L^p(\Omega;\mathbb{R}^3)$. The minimum of

$$\begin{aligned} I(u) \quad = \quad & \int_0^T \left\{ \Psi(u(t)) + \Psi^*(-(u(t)\cdot\nabla)u(t) + f - \dot{u}(t)) - \int_\Omega \langle f(t), u(t)\rangle dx \right\} dt \\ & + \int_\Omega \left\{ \frac{1}{2}(|u(0,x)|^2 + |u(x,T)|^2) - 2\langle u(0,x), u_0(x)\rangle + |u_0(x)|^2 \right\} dx \end{aligned}$$

–where Ψ is as in (5)– is zero and is attained and is attained at a solution of (14).

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Sign Changing Solutions of Semilinear Elliptic Equations: Symmetry and Critical Exponent Problems

Filomena Pacella

We consider semilinear elliptic problems of the type

$$\begin{cases} -\Delta u = f(|x|, u) & \text{in } B \\ u = 0 & \text{in } \partial B \end{cases} \tag{1}$$

where B is either a ball or an annulus centered at the origin in $\mathbb{R}^N$, $N \geq 2$ and $f : \overline{B} \times \mathbb{R} \to \mathbb{R}$ is a $C^{1,\alpha}$-function.

We are interested in symmetry properties of classical solutions of (1).

A famous result in this direction was obtained in 1979 by Gidas-Ni-Nirenberg in [3] using the moving plane device [6]. However to be applied this method requires some convexity assumptions on the domain, the positivity of the solution and some monotonicity of the nonlinearity f in the x-variable. Indeed counterexamples to the radial symmetry of solutions can be given if some of these conditions fail. Nevertheless for some nonlinearities and for certain types of solutions some partial symmetry is anyway expected.

A first result in this direction was obtained in [4] in a simple way, exploiting the Morse index of the solution. More precisely in [4] it is proved that if f is strictly convex than any solution of (1) with Morse index is foliated Schwarz symmetric which means essentially that it is axially symmetric and monotone in the angular coordinate. Though the result of [4] applies in a variety of problems and does not require the positivity of the solution there are cases when one would like to have information on the symmetry of solutions of higher Morse index.

Moreover there are problems when the nonlinear term is not strictly convex. A sample case of this situation arises if $f(|x|, s) = p|s|^{p-1}$, $p > 1$ and we want to study sign changing solutions. Indeed these solutions have Morse index greater than one.

A result in collaboration with T. Weth ([5]) which covers this case, at least if $p \geq 2$, has been recently obtained and allows to consider solutions of any Morse index, if the dimension is sufficiently large.

More precisely the results are the following

Theorem 1. *If $f(|x|, s)$ has a convex derivative $f'(|x|, s) = \frac{\partial f}{\partial s}(|x|, s)$, for every $x \in B$, then every solution of* (1) *with Morse index $j \leq N$ has at least $N - (j - 1)$ orthogonal symmetry hyperplanes.*

Theorem 2. *If $f'(|x|, s)$ is strictly convex in the second variable then every solution of* (1) *with Morse index less than or equal to* $N - 2$ *has infinitely many symmetry hyperplanes.*

Theorem 3. *Under the same hypothesis of Theorem 1, any solution of* (1) *with Morse index less than or equal to* 2 *is foliated Schwarz symmetric.*

If the nonlinearity does not depend explicitly on the x-variable then some results on the nodal sets are also obtained ([5]).

Several other questions remain open. Among these we would like to single out the following two:

i) What symmetry results can be proved in symmetric domains other than the ball or the annulus, in particular if the domain is only invariant for a discrete group of symmetries?

ii) If u has Morse index two and changes sign, is it also antisymmetric?

As far as we know the second question has been answered, up to now, only in some particular cases, concerning asymptotic problems ([2], [7]).

In the work in progress [2] we consider an almost critical problem of the type

$$\begin{cases} -\Delta u = |u|^{p^*-2-\varepsilon}u & \text{in } B \\ u = 0 & \text{in } \partial B \end{cases} \tag{2}$$

where $p^* = \frac{2N}{N-2}$, $\varepsilon > 0$, and B is the ball in $\mathbb{R}^N$, $N \geq 3$.

Using a careful blow-up analysis made in [1] we are able to prove that, for ε sufficiently small, any nodal solution of (2) with low energy is foliated Schwarz symmetric and antisymmetric with respect to the hyperplane orthogonal to the symmetry axis.

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Boundary blow-up solutions with interior layers and spikes

Yihong Du

Consider the problem

$$-\epsilon^2 \Delta u = f(x, u) := u(u - a(x))(1 - u) \text{ in } \Omega, \; u|_{\partial\Omega} = \infty, \tag{1}$$

where Ω is a bounded smooth domain in R^N, $0 < a(x) < 1$ on $\overline{\Omega}$, and $\epsilon > 0$ is a (small) constant. We want to understand the solutions and their properties of problem (1) when ϵ is small.

Problem (1) arises from a number of applications, such as mathematical biology (a limiting case of the FitzHugh-Nagumo model) and material science (Allen-Cahn equation). Usually Neumann boundary conditions are considered. In such a situation, the trivial solutions 0 and 1 are stable (as steady-states of the corresponding parabolic problem), and hence it is generally called a "bistable problem".

Problem (1) with Dirichlet boundary conditions has also been extensively studied. We consider here the explosive boundary condition and our results demonstrate that for small ϵ, the solutions and their properties are not affected greatly by the boundary conditions. In the following, we describe some recent results of Y.Du, Zongming Guo and Feng Zhou [DGZ].

Definition. We say u_ϵ is a minimizer solution to (1) if it is a solution to (1) and there is a sequence $\{\beta_n\}$ with $\beta_n \to \infty$ as $n \to \infty$ such that $u_\epsilon^{\beta_n} \to u_\epsilon$ in $C^1_{loc}(\Omega)$, where $u_\epsilon^{\beta_n}$ is a minimizer of

$$\inf \Big\{ \frac{\epsilon^2}{2} \int_\Omega |Dw(x)|^2 - \int_\Omega F(x,w)dx, \quad w - \beta_n \in H_0^1(\Omega) \Big\},$$

where $F(x,t) = \int_0^t f(x,s)ds$. Clearly $u_\epsilon^{\beta_n}$ is a solution of the problem

$$-\epsilon^2 \Delta u = f(x,u) \ \text{ in } \Omega, \quad u|_{\partial\Omega} = \beta_n.$$

Remark: We can show that for any $\epsilon > 0$, (1) has a minimizer solution.

Theorem 1. *Let u_ϵ be a minimizer solution to (P_ϵ) and denote $A = \{x \in \Omega : a(x) < 1/2\}$, $B = \{x \in \Omega : a(x) > 1/2\}$. Then, as $\epsilon \to 0$,*

$$u_\epsilon \to \begin{cases} 1, & \text{uniformly on any compact subset of } A, \\ 0, & \text{uniformly on any compact subset of } B. \end{cases}$$

Remark: Theorem 1 implies that, if $\partial A \cap \partial B \neq \emptyset$, then any minimizer solution of (1) undergoes a sharp transition near $\partial A \cap \partial B$.

Theorem 2. *Let Ω_1 and Ω_2 be two open sets so that $\Omega_1 \cap \Omega_2 = \emptyset$, $\Omega_i \subset\subset \Omega$, $i = 1, 2$, $a(x) < 1/2$ if $x \in \partial\Omega_1$ and $a(x) > 1/2$ if $x \in \partial\Omega_2$. Here Ω_1 or Ω_2 can be empty. Then (1) has a solution v_ϵ satisfying, as $\epsilon \to 0$,*

$$v_\epsilon \to \begin{cases} 1, & \text{uniformly on any compact subset of } (A \backslash \overline{\Omega}_2) \cup \overline{\Omega}_1, \\ 0, & \text{uniformly on any compact subset of } (B \backslash \overline{\Omega}_1) \cup \overline{\Omega}_2. \end{cases}$$

Remark: Theorem 2 tells us that (1) has solutions which have no transition layers near some designated components of the set $\{x \in \Omega : a(x) = 1/2\}$.

Remark: Note that if $\Omega_1 = \Omega_2 = \emptyset$, Theorem 2 becomes Theorem 1. In fact, v_ϵ is a "local" minimizer solution of (1).

To describe our results on solutions with spikes, we need some preparations. Assume that $b \in (0, 1/2)$ is a constant. It is known that the following problem has a unique solution U_b:

$$\begin{cases} -\Delta u = f_b(u) := u(u-b)(1-u),\ u > 0 \text{ in } R^N, \\ u(0) = \max_{x\in R^N} u(x),\ \lim_{|x|\to\infty} u(x) = 0. \end{cases}$$

We denote $U_{\epsilon,x_0,b}(x) = U_b(\frac{x-x_0}{\epsilon})$.

Suppose that $a(x) > 1/2$ for $x \in \partial\Omega$. Then using Theorem 2 we can show that (1) has a positive solution w_ϵ^* such that $w_\epsilon^* \to 0$ as $\epsilon \to 0$. If $a(x) < 1/2$ for $x \in \partial\Omega$, then by Theorem 2, we can show that (1) has a solution w_ϵ^{**} satisfying $w_\epsilon^{**} \to 1$. Note that both w_ϵ^* and w_ϵ^{**} have boundary layers but not interior layers.

Our next results show that there are new solutions of (1) with interior peaks superimposed on w_ϵ^* or w_ϵ^{**}.

Theorem 3. *Suppose that $a(x) > 1/2$ for $x \in \partial\Omega$ and $A = \{x \in \Omega :\ a(x) < 1/2\} \neq \emptyset$. Let $x_1, \dots, x_k \in A$ be any sequence of strict local maximum points of $a(x)$, or a sequence of strict local minimum points of $a(x)$, and denote $a_i = a(x_i)$. Then there is an $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0]$, (1) has a solution of the form*

$$u_\epsilon^* = w_\epsilon^* + \Sigma_{i=1}^k U_{\epsilon,x_i^\epsilon,a_i} + \omega_\epsilon,$$

where as $\epsilon \to 0$,

$$x_i^\epsilon \to x_i,\quad i = 1, \dots, k,$$

$$\int_\Omega (\epsilon^2 |D\omega_\epsilon|^2 + |\omega_\epsilon|^2) = o(\epsilon^N).$$

Theorem 4. *Suppose that $a(x) < 1/2$ for $x \in \partial\Omega$ and $B = \{x \in \Omega :\ a(x) > 1/2\} \neq \emptyset$. Let $x_1, \dots, x_k \in B$ be any sequence of strict local maximum points of $a(x)$, or a sequence of strict local minimum points of $a(x)$, and denote $a_i = a(x_i)$. Then there is an $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0]$, (1) has a solution of the form*

$$u_\epsilon^{**} = w_\epsilon^{**} - \Sigma_{i=1}^k U_{\epsilon,x_i^\epsilon,1-a_i} + \omega_\epsilon,$$

where as $\epsilon \to 0$,

$$x_i^\epsilon \to x_i,\quad i = 1, \dots, k,$$

$$\int_\Omega (\epsilon^2 |D\omega_\epsilon|^2 + |\omega_\epsilon|^2) = o(\epsilon^N).$$

Remark: Theorems 3 and 4 tell us that we can construct a new solution for (1) by putting a peak upward near a strict extremum point of $a(x)$ in A to the solution w_ϵ^*, or by putting a peak downward near a strict extremum point of $a(x)$ in B to w_ϵ^{**}.

The proofs of Theorems 1 and 2 use ideas of Dancer and Yan [DY1]: For small ϵ, the global minimizer $u_\epsilon(x)$ of the corresponding energy functional for the problem

$$-\epsilon^2 \Delta u = f(x,u) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0,$$

must be close to 1 when $a(x) < 1/2$, and close to 0 when $a(x) > 1/2$.

But the corresponding functional is usually undefined for (1) as the solution has infinite boundary values. To overcome this difficulty, we use the notion of minimizer solution of (1) defined above. We can show that for each fixed n, $u_\epsilon^{\beta_n}$

develops certain interior layers as $\epsilon \to 0$, and the formation of these layers is uniform in n as $\epsilon \to 0$.

For the proof of Theorems 3, our strategy is to find a solution for (1) of the form $w_\epsilon^* + w$ with $w \in H_0^1(\Omega)$. This amounts to solving the problem

$$-\epsilon^2 \Delta w = h(x,w) \ \text{ in } \Omega, \quad w|_{\partial\Omega} = 0, \tag{8}$$

where

$$h(x,t) = h_\epsilon(x,t) = f(x, w_\epsilon^*(x) + t) - f(x, w_\epsilon^*(x)).$$

We firstly use a **variational approach**: For each small ϵ, let $H = H_\epsilon$ be the completion of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_\epsilon := \Big(\int_\Omega \big(\epsilon^2 |Du|^2 - f_t(x, w_\epsilon^*) u^2\big) dx \Big)^{1/2}.$$

We can show that $H \subset H_0^1(\Omega)$ and critical points of $I(u)$ in H are solutions of (8), where

$$I(u) = (1/2) \int_\Omega (\epsilon^2 |Du|^2 - f_t(x, w_\epsilon^*) u^2) dx - \int_\Omega G(x,u) dx,$$

$G(x,t) = \int_0^t g(x,s) ds$ and $g(x,u)$ is a suitable modification of $h(x,u) - f_t(x, w_\epsilon^*)$.

The proof is then completed by the so called **reduction method**, which reduces the problem to finding critical points of some C^1-map $Z \to \omega(Z) \in H$ defined for $Z \in D_\delta$, where

$$D_\delta = \{Z = (z^1, z^2, \ldots, z^k) : \ z^i \in B_\delta(x_i), \ i = 1, 2, \ldots, k\},$$

and x_i, $i = 1, \ldots, k$, are strict local minimum (or maximum) points of $a(x)$ in A. The proof of Theorem 4 is similar.

Open Question: Construct peak solutions off solutions with interior layers. This requires a much better understanding of the interior layers of solutions obtained in Theorems 1 and 2, which seems a rather challenging problem in 2 or higher dimensions, and is an important question in its own right.

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Stability of Poisson Equilibria

Claudia Wulff
(joint work with George W. Patrick and Mark Roberts)

This is a short introduction to some results published in [8].

Energy methods for determining the stability of equilibria of Hamiltonian systems $\dot{x} = \mathbb{J}\nabla h(x)$ are based on the general principle that, since the symplectic structure matrix $\mathbb{J}$ is invertible, an equilibrium x_e of a Hamiltonian system is a critical point of the Hamiltonian. Due to energy conservation it is therefore Lyapunov stable if the Hessian of the Hamiltonian is definite at the equilibrium. The *Energy-Casimir* method is an extension of this principle to equilibrium points of Poisson systems (defined below). It originated with the work of Arnold on the stability of equilibria of incompressible fluids [1]. Since then it has been used very extensively in applications to rigid bodies [4], elasticity theory [10] and fluids [3]. In this paper we first present a topological generalisation of the energy method and then use this to obtain a significant generalisation of the Energy-Casimir method. We apply our method to obtain new stability results for 'underwater vehicles', modelled as rigid bodies in ideal irrotational fluids.

A Poisson manifold is a manifold X with a Poisson bracket $\{\cdot,\cdot\}$ defined on the space of smooth functions on X, see e.g. [6]. A Poisson system with Hamiltonian $h : X \to \mathbb{R}$ is characterized by the fact that the time-evolution of any smooth function $f : X \to \mathbb{R}$ along trajectories of the Poisson system satisfies $\dot{f} = \{f, h\}$. Poisson systems arise by symmetry reduction of Hamiltonian systems with symmetry. Examples include the Euler equations of ideal fluid dynamics, for which the symmetry group is the particle relabelling group [3], and the Kirchhoff equations for the symmetry reduced dynamics of underwater vehicles in ideal irrotational fluids, for which the symmetry group is a Euclidean group (see e.g. [3, 4]). Relative equilibria of symmetric Hamiltonian systems are equilibria of the symmetry-reduced Poisson system and hence their stability modulo the symmetry group can be studied with the methods described below.

Locally Poisson systems take the form $\dot{x} = \mathbb{J}(x)\nabla h(x)$, like Hamiltonian systems, but the matrix $\mathbb{J}(x)$ is x-dependent and in general not invertible. Hence an equilibrium of a Poisson system is not necessarily a critical point of the Hamiltonian. Therefore the energy method for Hamiltonian systems described is in general not applicable. Other conserved quantities of Poisson systems have to be taken into account as well:

The flow of a Poisson system on a Poisson manifold X generated by a Hamiltonian h preserves both h and the symplectic leaves of X. The symplectic leaf of $x \in X$ is the set of all points in X which can be connected to x by concatenations of trajectories of Poisson systems. In the case of a Poisson system which is a symmetry reduced Hamiltonian system the invariant symplectic leaves originate from the conserved quantities associated to continuous symmetries of Hamiltonian systems by Noether's Theorem [6]. To test for stability the Energy-Casimir method can be applied if there is a Casimir C, ie a function which is constant on symplectic leaves, such that x_e is a critical point of $h+C$. Stability follows if this critical point is a local extremum.

When is it possible to find a Casimir C such that x_e is a critical point of $h+C$? One case is when x_e is a *regular* point of X, which means that locally the foliation into symplectic leaves is non-singular. Using this fact Arnold [1, 2] and Libermann and Marle [5] show that if x_e is regular and is a local extremum of the restriction of h to the symplectic leaf $L(x_e)$ through x_e, then x_e is stable on all of X. Examples show that this is not true in general, see [5]. In such cases it is natural to ask whether there exists a space between $L(x_e)$ and X such that x_e is stable if it is an extremal point of the restriction of h to this intermediate space. In this paper we show that there is such a space. More generally we answer a challenge posed by Weinstein [11] when, referring to the interaction between Poisson structures and stability, he wrote: "As yet there is no general theory for this kind of analysis".

Our results in this paper are based on a topological generalisation of the energy method extending a results by Montaldi [7]. This topological method can be applied to a continuous flow on a locally compact topological space X which has conserved quantities with values in another topological space. In the case of Poisson systems the conserved quantities are the Hamiltonian h and the quotient map to the space of symplectic leaves. An equilibrium x_e is stable if the leafspace is Hausdorff at $L(x_e)$ and x_e is an isolated point in the fibre of the restriction of h to $L(x_e)$. Thus the condition that x_e be regular in the result of Arnold, Libermann and Marle can be relaxed to the leafspace being Hausdorff at $L(x_e)$. However in applications, such as Poisson systems which arise by symmetry reduction of Hamiltonian systems with non-compact, non-abelian symmetry groups, this condition is often not satisfied. In particular it is violated in the examples from fluid dynamics we mentioned earlier. If the leafspace is not Hausdorff then h must isolate x_e in a larger subset $T_2(x_e)$ which depends only on the "non-Hausdorff-ness" of the leafspace.

We recover and generalise the Energy-Casimir method for Poisson equilibria: we will see that it suffices to make the assumptions of the Energy-Casimir method on a subset of the Poisson manifold X which contains $T_2(x_e)$.

Moreover we identify a necessary condition for the Energy-Casimir method to apply, namely that the Poisson equilibrium must be *tame*. If $T_2(x_e)$ is a manifold this means that $\nabla h|_{T_2(x_e)}(x_e) = 0$. However Poisson systems obtained by symmetry-reduction of Hamiltonian systems with Euclidean symmetry generally have some equilibria which are not tame (*wild*). The Energy-Casimir method can not be applied to these. Other methods then have to be used which only yield weaker stability (Nekhoroshev stability) [9].

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Asymptotic behaviour of the Kazdan-Warner solution in the annulus

Massimo Grossi

Let us consider the following problem

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geq 2$. It is a well known fact that if $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$, there exists a solution to (1) in any domain $\Omega \subset \mathbb{R}^N$. On the other hand, if $p \geq \frac{N+2}{N-2}$, using the Pohozaev identity (1) does not have any solution in $H_0^1(\Omega)$ provided Ω is starshaped with respect to some point. However, if Ω is not starshaped, we can have solution for any $p > 1$, as stated in the following classical result,

Theorem, (Kazdan and Warner, [KW]): *Let Ω be an annulus. Then (1) admits a radial solution for any $p > 1$.*

In ([NN]) Ni and Nussbaum proved the uniqueness of this solution in the class of the radial functions.

Here we study the asymptotic behaviour of this solution as $p \to \infty$.

One of the main result of the paper is that there is no concentration phenomenon as p goes to infinity. This is in sharp contrast with similar semilinear perturbed problems and also with problems involving the same nonlinearity in $\mathbb{R}^2$ (see ([AG]), ([EG]), ([RW1])).

Our first result concerns the convergence of the solution u_p of (1). In the rest of the paper Ω will denote the annulus $\Omega = \{x \in \mathbb{R}^N : 0 < a < |x| < b\}$.

Theorem 1. *Let u_p the unique radial solution of (1). Then, as $p \to \infty$,*

$$u_p(|x|) \to \omega(|x|) \quad \textit{in } C^0(\overline{\Omega}), \tag{2}$$

with

$$\omega(|x|) = \frac{2}{a^{2-N} - b^{2-N}} \begin{cases} a^{2-N} - |x|^{2-N} & \textit{for } a \le |x| \le r_0 \\ \\ |x|^{2-N} - b^{2-N} & \textit{for } r_0 \le |x| \le b \end{cases} \quad \textit{for } N \ge 3 \tag{3}$$

and

$$\omega(|x|) = \frac{2}{\log b - \log a} \begin{cases} \log|x| - \log a & \textit{for } a \le |x| \le r_0 \\ \\ \log b - \log|x| & \textit{for } r_0 \le |x| \le b \end{cases} \quad \textit{for } N = 2 \tag{4}$$

Finally r_0 *is given by*

$$r_0 = \begin{cases} \left(\frac{a^{2-N}+b^{2-N}}{2}\right)^{\frac{1}{2-N}} & \textit{if } N \ge 3 \\ \\ \sqrt{ab} & \textit{if } N = 2 \end{cases} \tag{5}$$

Note that ω is not differentiable at r_0 and $\omega(r_0) = \max_{r\in[a,b]} \omega(r)=1$.

¿From Theorem 1 we deduce the following sharp Sobolev inequality for radial functions in the annulus,

Theorem 2. *Let* Ω *be the annulus* $\Omega = \{x \in \mathbb{R}^N : 0 < a < |x| < b\}$. *Then, for any radial function* $u \in H_0^1(\Omega)$ *the following inequality holds,*

$$\int_\Omega |\nabla u|^2 \ge C_p \left(\int_\Omega u^p\right)^{\frac{2}{p}}, \quad \textit{for any } p > 1 \tag{6}$$

where

$$C_p \to \begin{cases} \omega_N \frac{4(N-2)}{a^{2-N}-b^{2-N}} & \textit{if } N \ge 3 \\ \frac{8\pi}{\log b - \log a} & \textit{if } N = 2 \end{cases} \tag{7}$$

as $p \to \infty$. *Here* ω_N *denotes the area of the unit sphere in* $\mathbb{R}^N$.

Observe that Theorem 1 implies that $||u_p||_\infty \to 1$. Next results gives a more precise estimate.

Theorem 3. *The following estimate holds*

$$||u_p||_\infty = 1 + \frac{\log p}{p} + \frac{\gamma}{p} + o\left(\frac{1}{p}\right) \tag{8}$$

where $\gamma = \lim_{r\to r_0} \frac{1}{2}\omega'(r)^2 = \begin{cases} \log\left[(N-2)^2 2^{\frac{2}{2-N}} \frac{\left(a^{2-N}+b^{2-N}\right)^{2\frac{N-1}{N-2}}}{(a^{2-N}-b^{2-N})^2}\right] & \textit{if } N \ge 3 \\ \\ \log\left[\frac{2}{ab(\log b - \log a)^2}\right] & \textit{if } N = 2 \end{cases}$

We point out that Theorems 2 and 3 are proved using the "global" convergence result in Theorem 1. Moreover the proofs of these results just use elementary arguments.

We remarked that the limit function $\omega(r)$ in Theorem 1 is not differentiable at $r = r_0$. Actually,

it is interesting to study more carefully the behaviour of the solution $u_p(r)$ near the maximum r_0. This leads to analyze the "local" convergence of the solution $u_p(r)$ near its maximum. In order to do this we use a blow-up procedure used in [AG]. Then, up to a suitable scaling, it is possible to associate to (1) the following limit problem defined in all $\mathbb{R}$,

$$-u'' = e^u \text{ in } \mathbb{R} \tag{9}$$

The existence of a "limit problem" could suggest to use perturbative methods in order to deduce existence results to (1) for p large when Ω is a non-spherical domain (for example the case where Ω has one hole). This leads to the following open problem,

Open problem:
Let $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ be a domain with one hole. Then, for p large enough, does it exist a solution u_p to (1) with the following properties:
i)u_p converges to a harmonic function in $\Omega \setminus M$, where M is a suitable $N-1$-dimensional manifold contained in Ω,
ii)$u_p \to 1$ on M?

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On The Number of Interior Peak Solutions for a Singularly Perturbed Neumann Problem

Juncheng Wei

The main theme of this talk is the **concentration phenomena** of the following singularly perturbed elliptic problem

$$\epsilon^2 \Delta u - u + u^p = 0, \quad u > 0 \text{ in } \Omega, \; \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \tag{1}$$

where $\epsilon > 0$ is a constant, Ω is a bounded smooth domain in R^N with its unit outward normal ν, and p satisfies $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$ and $1 < p < \infty$ for $N = 2$.

Although problem (1) takes a classical form of singular perturbations, the traditional techniques in that area did not seem helpful as the error terms appeared in the inner and outer expansions are exponentially small in $\epsilon > 0$. In the pioneering work [10, 11], Ni and Takagi studied the following "energy" functional in $H^1(\Omega)$

$$E_\epsilon[u] = \frac{1}{2}\int_\Omega (\epsilon^2|\nabla u|^2 + u^2) - \frac{1}{p+1}\int_\Omega u_+^{p+1}, \text{ where } u_+ = \max\{u, 0\} \tag{2}$$

and showed that the least energy solution u_ϵ exists and that *for each $\epsilon > 0$ sufficiently small, u_ϵ has exactly one (local) maximum point P_ϵ in $\bar{\Omega}$, and $H(P_\epsilon) \to \max_{P\in\partial\Omega} H(P), where H(P) is the mean curvature function$.*

Since the publication of [10] and [11], problem (1) has received a great deal of attention and significant progress has been made. More specifically, solutions with multiple boundary peaks as well as multiple interior peaks have been established. It turns out that *a general guideline is that while multiple boundary spikes tend to cluster around the local minimum points of the boundary mean curvature $H(P)$, the location of the interior spikes are governed by the distance between the peaks as well as the boundary of $\partial\Omega$.*In particular, it was established in Gui and Wei [3] that *for any two given integers $k \geq 0, l \geq 0$ and $k + l > 0$, problem (1) has a solution with exactly k interior spikes and l boundary spikes* for every ϵ sufficiently small. (For $k = 0$ or $l = 0$, this has been established independently in [1], [2] and [4].)

Ni ([8], [9]) made the following conjecture:

Ni's Conjecture: *Under some generic conditions on Ω, for each fixed integer $m = 1, ..., N-1$, there exists $p_m > 1$ such that for $1 < p < p_m$ and ϵ sufficiently small, problem (1) admits a solution u_ϵ concentrating on a $m-$dimensional subset of Ω.*

A first breakthrough towards Ni's conjecture came from [7], where they showed that along a subsequence of $\epsilon_n \to 0$, there exists solutions concentrating on the whole $\partial\Omega$. This shows that Ni's conjecture is true when $m = N - 1$. Later Malchiodi [6] constructed concentrating solutions on a geodesics of the boundary of a three-dimensional domain, and Wei and Yang [12] showed the existence of concentrating solutions in a line intersecting the boundary of a two-dimensional domain.

Two basic questions remain open:

Open Q1: Suppose we have the following energy bound: $E_\epsilon[u_\epsilon] \leq C\epsilon^{N-m}$, can we show that the concentration set is $m-$dimensional?

Open Q2: Given ϵ small, can we obtain a lower bound on the number of the solutions in terms of ϵ?

We answer both questions in the following theorem. In fact, we shall include a slightly more general equation than (1), namely,

$$\epsilon^2\Delta u - u + f(u) = 0, \quad u > 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{3}$$

where f satisfies the following conditions (f1)-(f2):
(f1): $f \in C^{1,\sigma}$ for some $0 < \sigma \leq 1$ and $f(u) \equiv 0$ for $u \leq 0$, $f(0) = f'(0) = 0$.
(f2): The following equation

$$(4) \quad \Delta w - w + f(w) = 0, \ w > 0 \quad \text{in } R^N, \ \ w(0) = \max_{y \in R^N} w(y), \ w \to 0 \quad \text{at } \infty$$

has a solution $w(y)$ and w is nondegenerate.

We now state our main result

Theorem 1. *(Lin-Ni-Wei, [5]) Let f satisfy assumptions (f1)-(f2). Then there exists an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ and any positive integer K satisfying*

$$(5) \qquad 1 \leq K \leq \frac{\alpha_{N,\Omega,f}}{\epsilon^N (|\ln \epsilon|)^N},$$

where $\alpha_{N,\Omega,f}$ is a constant depending on N, Ω and f only, problem (3) has a solution u_ε which possesses exactly K local maximum points $Q_1^\varepsilon, ..., Q_K^\epsilon$ such that $u_\epsilon(x) = \sum_{j=1}^K w(\frac{x - Q_j^\epsilon}{\epsilon}) + o(1)$, and we have the following energy estimate

$$(6) \qquad E_\epsilon[u_\epsilon] \sim \epsilon^N K.$$

The Morse index of u_ϵ is at least K.

Remark: The constant $\alpha_{N,\Omega,f}$ can be made more precise in terms of N, σ and the packing constant of Ω. See [5].

Our first corollary answers Q1 negatively:

Corollary 2. *For each real number $m \in (0, N)$, there exists a solution u_ϵ to (3) with the following energy bound: $E_\epsilon[u_\epsilon] \sim \epsilon^{N-m}$. When $m = N$, we have a solution u_ϵ with the following energy estimate: $E_\epsilon[u_\epsilon] \sim (|\ln \epsilon|)^{-N}$.*

Our second corollary answers Q2 positively:

Corollary 3. *For ϵ sufficiently small, problem (3) has at least $\left[\frac{\alpha_{N,\Omega,f}}{\epsilon^N |\ln \epsilon|^N}\right]$ number of positive solutions.*

Our results suggest the following revised questions:

Revised Open Question 1: Suppose we have the following energy bound

$$E_\epsilon[u_\epsilon] \leq C\epsilon^{N-m},$$

and that the concentration set is **connected**, can we show that the (limiting) concentration set is $m-$dimensional?

Revised Open Question 2: Given ϵ small, can we obtain an **optimal** bound on the number of the solutions in terms of ϵ?

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Braids and parabolic dynamics

Robert C.A.M. Vandervorst
(joint work with R.W. Ghrist, J.B. van den Berg)

1.1. Consider a scalar uniformly parabolic PDE, $u_t = u_{xx} + g(x, u, u_x)$, or more generally $u_t = f(x, u, u_x, u_{xx})$, where $0 < \lambda \leq \partial_{u_{xx}} f \leq \lambda^{-1}$ uniformly. Assume that g is of smoothness class C^1. For simplicity we use periodic boundary conditions; hence $x \in S^1$. We view the equation as an evolution equation on the curve $u(\cdot, t)$: as t increases, the graph of u evolves in the (x, u) plane.

It is a well-known fact (going back to Sturm, but revived and extended considerably by Matano [22], Brunovsky and Fiedler [9], Angenent [2], and others) that there is a **comparison principle**. Specifically, let $u^1(t)$ and $u^2(t)$ be solutions. Then the number of intersections of the graphs of u^1 and u^2, $z(t) := \#\left\{x : u^1(x, t) = u^2(x, t)\right\}$, is a weak Lyapunov function for the dynamics: z is non-increasing. Furthermore, at those particular times t for which the graphs of $u^1(t)$ and $u^2(t)$ are tangent, the function z decreases strictly, even in the case where the tangencies are of arbitrarily high order [2]. These facts are all at heart an application of classical maximum principle arguments which have a geometric interpretation; *Parabolic dynamics separates tangencies monotonically.*

Using this comparison principle (also known as *lap number* or *zero crossing* techniques), numerous authors have analyzed its dynamics in varying degrees of generality. We note in particular the papers of Angenent and Henry [5], [16], in which the comparison principle is used to show that the dynamics is often Morse-Smale, and the paper of Fiedler and Rocha [13], in which the global attractor for the dynamics is roughly classified. In this context we also mention the work by Fiedler and Mallet-Paret [12].

1.2. Our contribution is a globalization of the comparison principle using topological braid theory. For a motivating example, consider again a pair of evolving curves $u^1(t)$ and $u^2(t)$ in the (x, u) plane. If we lift these curves to the three-dimensional (x, u, u_x) space, we no longer have intersecting curves, unless t is such that the planar graphs of u^1 and u^2 intersect tangentially. The graphs of u^1 and u^2 in the (x, u, u_x) space are instead an example of a **closed braid** on two strands. What was the intersection number of their projections is now the **linking number** of the pair of strands.

We therefore see that the comparison principle takes on a linking number interpretation (a fact utilized in a discrete setting by LeCalvez [20] and Angenent [4] and for geodesics by Angenent in [3]). After lifting solutions u^1 and u^2 to the (x, u, u_x) space, the comparison principle says that the linking number is a non-increasing function of time which decreases strictly at those times at which the curves are tangent.

1.3. Our goal is to produce a forcing theory for the dynamics of parabolic equations, and, as we shall relate, more general discrete systems. For simplicity, we focus on forcing stationary solutions, though periodic and connecting orbits are likewise accessible. Say that one has found a **skeleton** of stationary curves $\{v^1, v^2, \ldots, v^m\}$ for a particular representative of the above equation. How many and which types of other stationary curves are forced to be present?

Since the skeleton of known fixed curves $\mathbf{v} = \{v^1\}_{i=1}^m$ lifts to a braid, the problem is naturally couched in braid-theoretic terms: given a braid $\mathbf{v}$ fixed by a particular uniform parabolic PDE, which other classes of braids $\mathbf{u}$ are forced to exist as stationary curves? In this context, the forcing theory is reminiscent of Boyland's theory of "braid types" for periodic orbits of two-dimensional homeomorphisms [8]: a braid-theoretic version of the Nielsen-Thurston theory for surface homeomorphisms. In [3] Angenent develops a similar theory using flat knot types to find closed geodesics on closed 2-manifolds.

The spirit of our forcing theory is as follows:

- Given a fixed braid $\mathbf{v}$, construct the configuration space of all n-strand braids $\mathbf{u}$ which have $\mathbf{v}$ as a sub-braid.
- Use the braid-theoretic comparison principle to decompose this space into isolating blocks for the parabolic dynamics corresponding to distinct braid classes.
- Define a Conley index for these relative braid classes which depends only on the topology of the braids, and not on the analytic details of the dynamics.
- Prove Morse-type inequalities for forcing stationary curves in the PDE from a nontrivial braid index.

This is the basic recipe, modulo the frequent discretization needed to ensure the compactness necessary for the Conley index. Section 1 gives a rough outline of the index definition. Topological and dynamical features are outlined in Section 2.

1. Definitions and discretizations

2.1. There are two types of braids to consider: **topological** and **discretized**. Roughly speaking, a **topological braid** on n strands is an embedding of n disjoint arcs into $D^2 \times [0,1]$ transverse to the discs $D^2 \times \{x\}$ for all x. Given a braid β, its **braid class** $\{\beta\}$ is the equivalence class of isotopic braids. Braid classes possess a group structure for which generators are strand crossings in a planar projection and concatenation of the braids forms the group operation [7].

The class of **discretized braids** are best visualized as piecewise-linear braid diagrams. A discretized braid, $\mathbf{u}$, on n strands of period p, is determined by np **anchor points**: $\mathbf{u} = \{u_i^\alpha\}$. Superscripts $\alpha = 1 \dots n$ refer to strand numbers, and subscripts $i = 1 \dots p$ refer to spatial discretizations. One connects the anchor point u_i^α to u_{i-1}^α and u_{i+1}^α via straight lines. Since "height" is determined by slope, all crossings in the braid diagram are of the same type.[1] Since we employ periodic boundary conditions on the x variable, all of the braids are **closed**: left and right hand endpoints of strands are abstractly identified (perhaps by a nontrivial permutation of the strands). Denote by $\mathcal{D}_p^n$ the set of all n-strand period p discretized braids. For topological braids, a **singular braid** arises when any strands intersect. Since all of the braids we consider are lifts of graphs u, the only possible intersection is that which occurs when two strands are tangent in the projection. For a discretized braid $\mathbf{u}$, the singular braids are defined to be those braids at which anchor points on two different strands coincide in a non-transverse fashion (looking at neighboring points): specifically, $\Sigma = \Big\{\mathbf{u} \ : \ u_i^\alpha = u_i^\beta \text{ for some } i \text{ and } \alpha \neq \beta, \text{ and } (u_{i-1}^\alpha - u_{i-1}^\beta)(u_{i+1}^\alpha - u_{i+1}^\beta) \geq 0\Big\}$. The set Σ carves $\mathcal{D}_p^n$ into components: these are the discretized braid classes, denoted $[\mathbf{u}]$.

2.2. Discretizing the first equation in the standard way would yield a family of nearest-neighbor coupled equations of the form $\frac{d}{dt}u_i = u_{i-1} - 2u_i + u_{i+1} + \tilde{g}_i(u_i, u_{i+1})$ in which uniform parabolicity would manifest itself in terms of the derivatives of the right hand side with respect to the first and third variables. Instead of explicitly discretizing the PDE, we use the broadest possible category of nearest neighbor equations for which a comparison principle holds: these are related to the **monotone systems** of, e.g., [12, 17, 23] and others.

A **parabolic relation** of period p is a sequence of maps $\mathcal{R} = \{\mathcal{R}_i : \mathbb{R}^3 \to \mathbb{R}\}$, $i = 1...p$, such that $\partial_1 \mathcal{R}_i > 0$ and $\partial_3 \mathcal{R}_i \geq 0$ for every i. These include discretizations of uniform parabolic PDE's, as well as a variety of other discrete systems [21], including monotone twist maps [20]. The small amount of degeneracy permitted ($\partial_3 \mathcal{R}_i = 0$, *or* $\partial_1 \mathcal{R}_i = 0$) does not affect the manifestation of a comparison principle. Given a discretized braid $\mathbf{u} = \{u_i^\alpha\}$ and a parabolic relation $\mathcal{R}$, one evolves the braid according to the equation $\frac{d}{dt}(u_i^\alpha) = \mathcal{R}_i(u_{i-1}^\alpha, u_i^\alpha, u_{i+1}^\alpha)$. Stationary curves correspond to a braid $\mathbf{u}$ such that $\mathcal{R}_i(\mathbf{u}) = 0$ for all i. The parabolic relation $\mathcal{R}$ induces a flow on $\mathcal{D}_p^n$ which respects a braid-theoretic comparison principle.

[1] These are examples of **Legendrian** braids from contact geometry.

Lemma 1. *Let $\mathcal{R}$ be any parabolic relation and $\mathbf{u} \in \Sigma$ any singular braid. Then the flow line $\mathbf{u}(t)$ of $\mathcal{R}$ passing through $\mathbf{u} = \mathbf{u}(0)$ leaves a neighborhood of Σ in forward and backward time so as to strictly decrease the algebraic length of $\mathbf{u}(t)$ in the braid group as t passes through zero.*

2.3. Given the discretizations of the braid classes and the dynamics, one is left with a conveniently finite dimensional problem. For purposes of a forcing theory, we use relative braids. Given a period p braid $\mathbf{v}$, denote by $\mathcal{D}_p^n$ REL $\mathbf{v}$ the space of all n strand, period p discretized braids which have $\mathbf{v}$ as a sub-braid.

In this context, the simplest version of Conley's index can be defined for braid classes (see [10] for an introduction to the Conley index). To do so, it must be shown that the braid classes [**u** REL **v**] are **isolated** in the sense that (informally) no flow lines within [**u** REL **v**] are tangent to the boundary of this set. It follows from Lemma 1 that [**u** REL **v**] is isolated for the flow assuming that the braid class is **proper**, *i.e.*, no free strands of **u** can "collapse" onto **v** or onto each other. Furthermore, to ensure compactness, we assume that the braid class [**u** REL **v**] is bounded.

The **homotopy braid index** of [**u** REL **v**] is defined as the Conley index of the braid class, computed as follows. Choose any $\mathcal{R}$ which fixes **v**. Define $\mathcal{E}$ to be those braids on the boundary of [**u** REL **v**] along which evolution under the flow of $\mathcal{R}$ exits the braid class. The homotopy braid index is defined to be the pointed homotopy class[2] ([**u** REL **v**]) := ([**u** REL **v**]$/\mathcal{E}, \{\mathcal{E}\}$) . As this is simply the Conley index of the isolating block [**u** REL **v**] under the flow of $\mathcal{R}$, it is easy to show that **h**([**u** REL **v**]) is well-defined and independent of the choice of $\mathcal{R}$ (so long as it is parabolic and fixes **v**) as well as the choice of **v** within its braid class [**v**]. Although the homotopy type of a quotient of a braid class seems difficult to compute, the homology CH_*([**u** REL **v**]) is both efficacious and computable. [3]

2. A few theorems

3.1. The homotopy braid index has both topological and dynamical implications. The most important result about the index is the following invariance theorem:

Theorem 2 ([14]). *The homotopy braid index is an invariant of topological braid pairs.*

Otherwise said, any two discretizations of a topological braid pair have identical homotopy indices, regardless of the period of the discretization used. The proof of this theorem involves a singular perturbation argument applied to a "stabilization operator" on discretized braids. While the precise topological content of the index is as yet unclear, a duality operator on braids was discovered and analyzed in [14].

[2] We omit a few technicalities concerning the rare cases in which the discretized braids get "locked" because of too-coarse discretization: see [14] for details.

[3] The theorems about the index **h** were predicated by rigorous computer experiments of M. Allili. Recently the index of more complicated braids has been calculated using rigorous homology computations in collaboration with S. Day.

This has the pleasant corollary that, roughly speaking, adding a full twist to a braid class shifts the homotopy index up two dimensions (a dimension shift on the homology level; a double suspension on the homotopy level).

3.2. The dynamical consequences of the index are forcing results. A simple example: given any parabolic relation $\mathcal{R}$ which has as a stationary solution braid. Then, by adding free strands that yield a nontrivial braid index, there must be some invariant set for $\mathcal{R}$ within this braid class. At this point, one uses Morse theory ideas: if $\mathcal{R}$ is a gradient flow, then there must be a stationary solution of the form of the grey strand. If the flow is not of gradient type, then finer information can still detect stationary and/or periodic curves. By iterating the process of adding free strands and computing a nontrivial index, one can go quite far. The following forcing theorem (for gradient-type $\mathcal{R}$) is a very general forcing theorem:

Theorem 3 ([14]). *Let $\mathcal{R}$ be a parabolic recurrence relation which* (i) *is of gradient type, and* (ii) *is dissipative (roughly, that large solutions are "repelled" from infinity). If $\mathcal{R}$ fixes a discretized braid* $\mathbf{v}$ *which is not in the trivial braid class (that is, if it has any crossings whatsoever), then there are an infinite number of distinct braid classes for multiple periods which arise as stationary solutions of $\mathcal{R}$.*

3. Open problems

The dynamical implications of the braid Conley index are one direction of research. Here one can also think of extending the theory to arbitrary braids and their natural dynamics; Cauchy-Riemann equations. This is subject of current research. The topological issues however are just as interesting. To list a few:

(1) What is the relation between the braid Conley index and the algebraic properties of positive braids. Can the algebraic structure of the braid group be used to compute the index?
(2) The structure of the braid classes both discrete and continuous; for computing the index one often needs to decide how many connected components a discrete braid class has. One conjecture is that if the total number of discretization points is 1+ # of intersections in the braid diagram, then there is only one component?
(3) For dics diffeomorphisms the index can be used to find periodic points. In the area preserving case the index provides more information than Nielsen-Thurston theory. Are there situations in the general case where the index can also provide additional information?
(4) In [14] an extensive application of the braid Conley index to fourth order equations is given. It applies when such equations can be reduced; twist property. This reduction property is proved for a large class of equations. Numerical evidence suggests that this reduction is always possible, see [24]. Can the theory be extended to fourth order conservative equations in general?

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Mean field equations and Green functions on torus

Chang Shou Lin

In the talk, I want to consider the equation

$$\Delta u + 8\pi e^u = 8\pi\delta_0 \quad \text{on} \quad T, \tag{1}$$

where T is a torus, and δ_0 is the Dirac measure with the singularity at 0. For simplicity, the fundamental cell of T is denoted as $\{s\omega_1 + t\omega_2 \mid |s| \le \frac{1}{2}, |t| \le \frac{1}{2}\}$, where $\omega_1 = 1$ and $\omega_2 = a + ib$ with $b > 0$. By adding a constant to u, equation (1) can also be written as

$$\Delta u + 8\pi \left(\frac{e^u}{\int e^u} - \frac{1}{|T|} \right) = 8\pi(\delta_2 - \frac{1}{|T|}), \tag{2}$$

where $|T|$ is the area of torus. Usually, an equation similar to (2) is called a mean field equation. Clearly, equation (2) is invariant under adding a constant c to u. Hence, a solution is normalized by

$$\int_T u dx = 0 \tag{3}$$

If we let

$$u = v - 8\pi G(x), \tag{4}$$

where $G(x)$ is the Green function with the singularity at 0.

$$-\Delta G(x) = \delta_0 - \frac{1}{|T|}, \tag{5}$$

$$\int_T G(x) dx = 0.$$

Then $v(x)$ satisfies

$$\Delta v + 8\pi \left(\frac{e^{-8\pi G(x)}}{\int e^{-8\pi G(x)} e^v} - \frac{1}{|T|} \right) = 0 \quad \text{on} \quad T \tag{6}$$

It is obvious that equation (6) is the Euler-Lagrange equation of the nonlinear function J:

$$J(\phi) = \frac{1}{2} \int |\nabla \phi|^2 dx - 8\pi \log \int_T e^{-8\pi G} e^\phi dx. \tag{7}$$

We want to talk about motivations for studying equation (1). The first one comes from conformal geometry. The problem can generally be posed as the followings: Given N singular points $p_1, \dots, p_N$ and $-2 < \alpha_j$ for $j = 1, 2, \dots, N$, we want to find a metric $ds^2 = e^u |dx|^2$ such that the Gaussian curvature ds^2 is equal to 1 on $T \backslash \{p_1, \dots, p_N\}$, and

$$e^u = |z - p_j|^{\alpha_j} + O(1) \quad \text{for } z \text{ near } p_j.$$

The problem is equivalent to solving

$$\Delta u + \rho e^u = \sum_{j=1}^{N} 2\pi\alpha_j \delta_{p_j}, \tag{8}$$

where

$$\rho = \sum_{j=1}^{N} 2\pi\alpha_j.$$

Clearly, when $N = 1$ and $\alpha_j = 4$. Equation (8) is the same as equation (1). Of course, the conformal metric with conical singularity at $p_1, \ldots, p_N$ might not exist. For example, if $\alpha_j < 0$ for all j, then the equation (8) does not have a solution. In general, the existence for equation (8) is an interesting problem. For example, we can prove for the case $N = 1$, and $0 < \rho \neq 8\pi$, equation (8) does have a solution for any torus T. For a proof for this or more general situations, see [1,2,4,5].

The second motivations comes from the self-dual equation of the Chern-Simons model. Let $p_1, \ldots, p_N$ be vortex points of the equation, the equation can be written as a second order elliptic equation:

$$\Delta v + \frac{1}{\kappa^2} e^v (1 - e^v) = 4\pi \sum_{j=1}^{N} \delta_{p_j} \quad \text{on} \quad T. \tag{9}$$

For the Chern-Simons model, when $\kappa \to 0$, there are two types of solutions:

(1) $v \to 0$ uniformly on $T \backslash \{p_1, \ldots, p_n\}$ as $\kappa \to 0$,
(2) $v \to -\infty$ a.e as $\kappa \to 0$.

Solutions of first type resembles topological solutions in $\mathbb{R}^2$ and the other one resembles non-topological solutions in $\mathbb{R}^2$. For solutions of type (2), actually, there might have complicated phenomenon. For $N = 2$ and $p_1 = p_2 = 0$, Nolasco-Tarantello [6] obtained a sequence of solutions $v_k = G + w_k + c_k$, where c_k is a constant and $\int w_k = 0$, such that either

1. w_k uniformly converges to w in C^2 and w is a minimizer of the nonlinear equation J of (7).

or,

2. $w_k \to -\infty$ uniformly in $C^2_{loc}(T \backslash \{q\})$ for some point $q \neq 0$, and $\frac{e^{-8\pi G} e^{w_k}}{\int e^{-8\pi G} e^{w_k}}$ tends to the Dirac measure δ_q. In this case there exists no minimizer of $J(u)$.

In short, Nolasco-Tarantello obtained the following results.

Theorem 1. (Nolasco-Tarantello) ω_k *uniformly converges if and only if a minimizer of J exists.*

1. Main Theorems

Generally, the mean field equation can be written as

$$\Delta u + \rho \left(\frac{h(x) e^u}{\int h(x) e^u} - \frac{1}{|\Sigma|} \right) = 0 \quad \text{on} \quad \Sigma, \tag{10}$$

where Σ is a compact Riemann surface, and $h(x) \geq 0$. Here $h(x)$ can be allowed to have isolated zeros. As mentioned in Introduction, the zero of $h(x)$ comes from the prescribed singularities. At each zero x_0 of $h(x)$, we assume

$$h(x) = |x - x_0|^{2\alpha} \tilde{h}(x) \quad \text{for } x \text{ near } x_0,$$

where α is a positive integer and $\tilde{h}(x) > 0$ in a neighborhood of x_0. It is well-known that if $\rho \neq 8\pi m$, $m \in N$, then solutions of (10) is uniformly bounded, Thus the classic degree of Leray-Schauder can be well-defined. Denote it by d_ρ. Then we have

Theorem 2. *Suppose $h(x) > 0$ and $\rho \neq 8\pi m$. Let $\chi(\Sigma)$ be the Euler Characteristic of Σ. Then*

$$d_\rho = \begin{cases} 1 & \text{if } \rho < 8\pi \\ \dfrac{(-\chi(\Sigma)+1)\dots(-\chi(\Sigma)+m)}{m!} & \text{if } 8m\pi < \rho < 8(m+1)\pi \end{cases}$$

If $h(x)$ might have zeros somewhere, the index formulas can not be expressed so nicely. See [3]. In general, if $\rho \neq 8\pi m$, then the degree depends only on the topology of the underlying manifold Σ, but, if $\rho = 8m\pi$, then the geometry become important. For example, for equation (1), we can prove.

Theorem 3. *There exists a solutions for equation (1) if and only if the Green function has extra critical points other than three half-periods.*

Theorem 4. *There exists a minimizer for the nonlinear function J, if and only if there exists a global minimum points of G, which is not a half period.*

For equation (1), it is important to know the number of critical points of $G(x)$, and the problem of degeneracy or non-degeneracy for critical points of G is related to the bubbling analysis for equation (1). From the numberical computations, we propose the following two open problems:

Problem 1. the Green function has five critical points at most.

Problem 2. the Green function has five critical points if and only if all half-periods are non-degenerate saddle points.

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Nonlinear Elliptic Equations with Critical Hardy Potential

Michel Willem

We consider first the problem

$$(1)\qquad \begin{cases} -\Delta_p u - \lambda \dfrac{|u|^{p-2}}{|x|^p} u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \lim\limits_{|x|\to\infty} u(x) = 0 & \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is not necessarily bounded and $0 \in \Omega$. When Ω is bounded, this problem was considered by Azorero and Peral. We describe the lower semi-continuity approach due to P. Sintzoff (see [2]).

When $1 < p < N$ and

$$\lambda < \left(\frac{N-p}{p}\right)^p, f \in (\mathcal{D}_0^{1,p}(\Omega))',$$

there exists a least energy solution. Let us remark that, when $p \neq 2$, the natural functional is *not* convex and, for some $f \in \mathcal{D}(\Omega)$, there exists a non minimizing solution.

An open problem is to describe the multiplicity of the solutions of (1) e.g. when Ω is a ball.

We consider next the Hardy inequality for exterior domains :

$$(2)\qquad \|\frac{u}{\delta}\|^p_{L^p(\Omega)} \leq C_p(\Omega)\|\nabla u\|^p_{L^p(\Omega)}.$$

The exterior domain $\Omega \subset \mathbb{R}^N$ has a $\mathcal{C}^2$ compact boundary and

$$\delta(x) = \mathrm{dist}(x, \partial\Omega).$$

It is proved in [1], that (2) holds if and only if $1 < p \neq N$. (When Ω is bounded, it was proved by Marcus, Mizel and Pinchover that (2) holds for any $p > 1$.)

Let us define

$$\lambda_p(\Omega) = \inf\left\{\|\nabla u\|^p_{L^p(\Omega)} : u \in \mathcal{D}_0^{1,p}(\Omega), \|\frac{u}{\delta}\|_{L^p(\Omega)} = 1\right\}.$$

In [1] we prove that

$$\lambda_p(\Omega) < \min\left\{|\frac{N-p}{p}|^p, |\frac{p-1}{p}|^p\right\} \quad \Rightarrow \quad \lambda_p(\Omega) \text{ is achieved.}$$

An open problem is to prove (or disprove) the converse.

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Liouville-type theorems for superlinear parabolic problems

Pavol Quittner

We consider classical solutions of semilinear parabolic problems of the form

$$(1)\qquad \begin{cases} u_t - \Delta u = f(x,t,u,\nabla u), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where f has superlinear growth in u and Ω is an open set in $\mathbb{R}^N$. Under suitable additional assumptions on f, Ω and u_0, problem (1) possesses a unique solution defined on the maximal time interval $[0, T(u_0))$.

Let $\delta > 0$ and let $\|\cdot\|_\infty$ denote the norm in $L^\infty(\Omega)$. It turns out that a priori estimates of the form

$$(2)\qquad \|u(\cdot,t)\|_\infty \le C(\|u_0\|_\infty, \delta), \qquad 0 < t < T(u_0) - \delta,$$

play a very important role in the study of the asymptotic behavior of solutions of (1). Such estimates can be used in the study of positive and sign-changing equilibria (see [12], [1], [2] and the references therein), connecting orbits between equilibria (see [5], [1], [2]), nontrivial periodic solutions (see [12]), blow-up rate (see [6]), continuity of the maximal existence time $T(u_0)$ (see [11]), completeness of blow-up (see [13], [14], [9]) or optimal control problems governed by superlinear parabolic equations and systems (see [3]).

It is known that estimate (2) is not true in the critical and supercritical cases, in general (for example, if Ω is starshaped and $f = |u|^{p-1}u$, $p(N-2) \ge N+2$). On the other hand, estimate (2) was proved in [11] for a large class of superlinear functions $f = f(x,u)$ with subcritical growth. The proof in [11] was based on energy arguments. If energy methods cannot be used and the function f behaves like u^p ($p > 1$) for large u, then one can still use rescaling arguments (see [15], [14]). However, this approach is restricted to positive solutions and it also requires nonexistence of positive bounded solutions of the limiting equation

$$(3)\qquad u_t - \Delta u = u^p, \qquad x \in \mathbb{R}^N,\ t \in (-\infty, \infty),$$

(and the corresponding problem in a halfspace).

Problem (3) possesses (bounded radial) positive steady states if and only if $p(N-2) \ge N+2$. This suggests the following conjecture:

Conjecture 1. *Positive solutions of* (3) *exist if and only if* $p(N-2) \ge N+2$.

The nonexistence of bounded positive solutions of (3) follows from [4] provided $p(N-1)^2 < N(N+2)$. If $p(N-2) < N+2$ then the nonexistence of positive solutions of (3) satisfying additional requirements was proved in [8], [7] and [10]. In particular, the following theorem follows from [10].

Theorem 2. *Bounded positive radial solutions of* (3) *exist if and only if* $p(N-2) \ge N+2$.

In [9], a generalization of the following Liouville type theorem was proved and used in the proof of completeness of blow-up for problems with indefinite nonlinearities.

Theorem 3. *Let $p > 1$. Then the equation*

$$u_t - \Delta u = x_1 u^p, \qquad x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N,\ t \in (-\infty, \infty), \tag{4}$$

does not possess bounded positive solutions.

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The Morse complex for infinite dimensional manifolds

Pietro Majer
(joint work with Alberto Abbondandolo)

A useful approach to Morse theory is the so-called Morse complex approach. The idea consists in introducing the free Abelian group $M_k(f)$ generated by all the critical points x with Morse index $m(x) = k$ of a given Morse function $f : M \to \mathbb{R}$, and in defining a boundary operator $\partial_k : M_k(f) \to M_{k-1}(f)$ by counting the orbits in the intersections of unstable and stable manifolds of critical points, with respect to the negative gradient flow of f (notice that the intersection of the unstable manifold of x and the stable manifold of y has generically dimension $m(x) - m(y)$). When M is a compact manifold, the homology of this chain complex - the Morse complex of f - is isomorphic to the singular homology of M. This approach works also in the infinite dimensional setting considered by Palais and Smale in the sixties: a Morse function on an infinite dimensional Hilbert manifold, having critical points of finite Morse index, and satisfying the Palais-Smale compactness condition. When f is also bounded below, the homology of the Morse complex is again isomorphic to the singular homology of the ambient manifold. Otherwise, the Morse complex depends on the behavior of f at infinity, but its homotopy type is still considerably stable (for instance, two functions which have bounded uniform distance have homotopically equivalent Morse complexes). Actually, an appropriate level of generality consists of flows on Banach manifolds, having hyperbolic rest points of finite Morse index, admitting a Lyapunov function, and satisfying the analogue of the Palais-Smale condition (see [5]).

The Morse complex approach is very promising when critical points have infinite Morse index and co-index (critical points of this sort are invisible for classical infinite dimensional Morse theory). This fact was exploited by Floer, who applied this idea to a Cauchy-Riemann type PDE arising in Hamiltonian dynamics. One could try to apply this idea to the abstract negative gradient flow of a smooth functional: the hope is that, although the stable and unstable manifolds of critical points are infinite dimensional, their mutual intersections may be finite dimensional. This idea has little chances to work without additional structures: for instance, we have proved in [6] that if f is a smooth Morse function on a Hilbert manifold M, with critical points of infinite Morse index and co-index, then we can associate an arbitrary integer $a(x)$ to every critical point x, and find a metric g on M such that the unstable manifold of x and the stable manifold of y (with respect to the gradient flow determined by f and g) intersect transversally in a manifold of dimension $a(x) - a(y)$, for every pair of critical points x, y.

On a Hilbert space there is a natural class of functions which appear frequently in variational problems, namely those consisting of the sum of a non-degenerate quadratic form and of a nonlinear term with compact gradient. In this case, there is a well-defined notion of relative Morse index, and the Morse complex approach works (see [1]). The fact that the Hilbert space is naturally equipped with a splitting into two linear subspaces is essential here.

When trying to extend these ideas to functions defined on manifolds, the first problem is to determine those flows having stable and unstable manifolds intersecting in finite dimensional submanifolds. This question is reduced to the study of operators of the form $d/dt - A(t)$, where $A : \mathbb{R} \to \mathfrak{L}(H)$ is a path of bounded operators on the Hilbert space H, converging to hyperbolic operators for $t \to \pm\infty$. The study of the Fredholm property for such operators involves the notions of Fredholm pairs of linear subspaces, and of compact perturbation of a linear subspace (see [4] for a characterization of the Fredholm property for these operators, and for examples showing how many things can be wrong when H is infinite dimensional).

These results have allowed us to develop a complete Morse complex approach for quite a general class of gradient-like flows on a Hilbert manifold M (see [3]). The extra structure here is a fixed subbundle of the tangent bundle M, or more generally of an essential subbundle (that is, defined only up to compact perturbations). In order to deal with the question of finite dimensionality and orientability of intersections of unstable and stable manifolds, one has to understand the properties of some infinite dimensional Grassmannians, and of suitable determinant bundles over them (this aspect of the theory is extensively treated in [2]).

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Energy bounds for entire nodal solutions of autonomous elliptic equations via the moving plane method

Tobias Weth

We give a new lower bound for the energy of sign changing solutions of the pure critical exponent problem

$$(1) \qquad -\Delta u = |u|^{\frac{4}{N-2}}u, \qquad u \in D^{1,2}(\mathbb{R}^N), \qquad (N \geq 3).$$

Here $D^{1,2}(\mathbb{R}^N)$ denotes the completion of the space $\mathcal{C}_0^\infty(\mathbb{R}^N)$ of test functions with respect to the norm $\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2\, dx$, hence it is the largest space where the corresponding energy functional

$$u \mapsto \Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\, dx$$

is well defined. Problem (1) plays a crucial role in the study of problems in conformal geometry like the Yamabe problem and the prescribed scalar curvature problem. It also arises as a 'limiting problem' in the study of Palais-Smale sequences of the energy functional associated with the Dirichlet problem

$$(2) \qquad -\Delta u = \lambda u + |u|^{\frac{4}{N-2}} u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

on a smooth, bounded domain Ω with a parameter $\lambda \geq 0$. The set of *positive* solutions of (1) was completely classified and computed in celebrated papers by Aubin, Talenti, and Gidas-Ni-Nirenberg more than 25 years ago. This set forms an $N+1$-dimensional critical manifold of Φ in $D^{1,2}(\mathbb{R}^N)$ at the energy level $\frac{1}{N}S^{N/2}$, where S is the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. Much less understood is the set of *sign changing* solutions of (1). Ding [4] proved that there is a sequence $(u_n)_n$ of (conformally non-equivalent) sign changing solutions with $\Phi(u_n) \to \infty$. In fact, one can construct multiple sequences of this type, and the number of such sequences increases with the space dimension N, see [2, Section 3]. A well known and often used observation is the energy doubling of sign changing solutions of (1): Every sign changing solution $u \in D^{1,2}(\mathbb{R}^N)$ of (1) satisfies $\Phi(u) > \frac{2}{N}S^{N/2}$. Our following result states that this bound is not sharp.

Theorem 1. *There is an* $\varepsilon_0 > 0$ *such that* $\Phi(u) \geq \frac{2}{N}S^{N/2} + \varepsilon_0$ *for every sign changing solution* $u \in D^{1,2}(\mathbb{R}^N)$ *of (1).*

This improved bound is obtained by a contradiction argument. Assuming by contradiction that there is a sequence of sign changing solutions u_n of (1) such that $\Phi(u_n) \to \frac{2}{N}S^{N/2}$, we first transform this sequence into a convenient form, using the conformal invariance of (1). The transformed sequence consists of solutions having two bumps of opposite sign, and the distance of the bumps tends to infinity as $n \to \infty$. For large n, we then exclude the existence of such solutions by a variant of the moving plane method, thus obtaining a contradiction.
The new bound can be used to find new solutions for problem (2) with $\lambda = 0$ in some special domains Ω. This work is in preparation. For $\lambda = 0$, sign changing solutions of (2) have so far only been obtained for domains Ω with an involution symmetry, see [3] and the references therein.
The argument given above yields similar improved lower energy bounds for nodal solutions of other autonomous equations on $\mathbb{R}^N$. As an example we mention the nonlinear Schrödinger equation

$$-\Delta u + u = f(u), \qquad u \in H^1(\mathbb{R}^N)$$

with a superlinear and subcritical nonlinearity f.
An interesting different type of results concerning the set of sign changing solutions of (1) in dimension three is due to Bahri and Chanillo, see [1] and Professor Bahri's article in this report. They analyse the change in topology near sign changing critical points at infinity of the functional Φ.
We close this abstract by mentioning some open problems.

1. What is the optimal value of ε_0 ? Does there exist an energy minimizer within the set of sign changing solutions of (1) ?
2. Are there sign changing solutions of (1) which (modulo conformal transformations) do not have an $O(2)$-symmetry ?
3. Is every bounded sign changing solution of $-\Delta u = |u|^{\frac{4}{N-2}}u$ on $\mathbb{R}^N$ contained in the energy space $D^{1,2}(\mathbb{R}^N)$? This open problem has been pointed out by Professor Bahri.

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Symmetrization and Symmetry of Critical Points

Jean van Schaftingen

Let $\Omega \subset \mathbf{R}^N$ be a ball or an annulus, and consider the problem of finding a function $u : \Omega \to \mathbf{R}$ and $\lambda \in \mathbf{R}$ such that

$$(1) \qquad \begin{cases} -\Delta u = \lambda f(|x|, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

When $f(|x|, \cdot)$ is continuous and $|f(|x|, s)| \leq C(1 + |s|^p)$, with $1 \leq p < (N + 2)/(N - 2)$, the critical points of the functional

$$\varphi : \mathrm{H}^1_0(\Omega) \to \mathbf{R} : u \mapsto -\int_\Omega F(|x|, u)\, dx$$

(the function F is defined by $F(|x|, s) = \int_0^s f(|x|, \sigma)\, d\sigma$), restricted to the unit ball

$$\partial B(0, \mu) = \{u \in \mathrm{H}^1_0(\Omega) \ : \ \|\nabla u\|_2^2 = \mu\}$$

are solutions of (1).

When $f(|x|, -s) = -f(|x|, s)$ and $f(|x|, s)s > 0$ for every $x \in \Omega$ and for every $s \in \mathbf{R} \setminus \{0\}$, then for every $\mu > 0$, $\varphi|_{\partial B(0,\mu)}$ has infinitely many critical points [2]. An important tool of the proof is the Krasnoselskii genus, defined, for every closed set $A \subset \partial B(0, \mu)$ such that $-A = A$, by

$$\gamma(A) = \inf\{m \geq 0 \ : \ \text{there exists an odd map } h \in C(A, S^{m-1})\}.$$

If one defines

$$\Gamma_k = \{A \subset \partial B(0, \mu) \ : \ A \text{ is closed, } -A = A \text{ and } \gamma(A) \geq k\}$$

and

$$\beta_k = \inf_{A\in\Gamma_k} \sup_{u\in A} \varphi(u),$$

then the existence result can be restated more precisely as: For every $k \geq 1$, there is $u \in \partial B(0,\mu)$ such that $\varphi(u) = \beta_k$ and u_k is a critical point of $\varphi|_{\partial B(0,\mu)}$, and $\beta_k \to 0$ as $k \to \infty$.

Since the problem (1) is invariant under rotations, there is a natural question about the symmetry of the solutions whose existence has been proved. We prove that there exist solutions which share many properties of the solutions obtained previsously which are invariant under some isometries. In order to state the result, we need the notion of cap symmetrization (also called foliated Schwarz symmetrization [3]), defined as follows: If $u : \Omega \to \mathbf{R}$ is measurable and $e \in S^{N-1}$, then the cap symmetrization u^* of u with respect to e is the unique function such that for every $r > 0$ and $c \in \mathbf{R}$,

$$\partial B(0,r) \cap \{x \in \Omega \ : \ u^*(x) > c\}$$

is a geodesic ball in $\partial B(0,r)$ centered around re, and has the same $N-1$–dimensional Hausdorff measure as

$$\partial B(0,r) \cap \{x \in \Omega \ : \ u(x) > c\}.$$

Our main result is then: For every $1 \leq k \leq N$, there is $u_k \in \partial B(0,\mu)$ such that $\varphi(u_k) = \beta_k$ and u_k is a critical point of $\varphi|_{\partial B(0,\mu)}$, and $u_k^* = u_k$ [4]. If the solution u_k is not radially symmetric, then the level β_k is degenerate and $\beta_N = \beta_k$. Typically, our result should then only give one or two distinct critical levels with axially symmetric critical points.

The proof is based on the fact that for every sequence $(A_m)_{m\geq 1}$ in Γ_k such that $\sup_{u\in A_k} \varphi(u) \to \beta_k$, there exists a sequence $(u^m)_{m\geq 1}$ such that $u^m \in A_m$ and u^m converges to a critical point. In general, one can assume that $\gamma(A_m) = k$. Therefore, there exist an odd map $h_m \in C(A_m, S^{k-1}) \subset C(A_m, S^{N-1})$, and we could define A_m^* as the $\mathrm{H}^1(\Omega)$–closure of the cap symmetrizations of functions $u \in A_m$ with respect to $h_m(u)$. The set A_m^* is thus the closure of the image by an odd mapping of A_m. If this mapping was continuous, then one would have $\gamma(A_m^*) \geq \gamma(A_m) = k$. Unfortunately, the mapping $u \mapsto u^*$ is not continuous in $\mathrm{H}^1_0(\Omega)$ when $N \geq 3$ [1,4]. Moreover, $\|\nabla u^*\|_2 \leq \|\nabla u\|_2$, and there is no equality in general, so that $A_m^* \not\subset \partial B(0,\mu)$ in general.

This problem can be circumvented by approximating the cap symmetrization by continuous transformations. For a closed halfspace $H \subset \mathbf{R}^N$ with $\partial H \ni 0$, let σ_H denote the reflection with respect to ∂H, and let the polarization of $u : \Omega \to \mathbf{R}$ with respect to H be the function $u^H : \Omega \to \mathbf{R}$ defined by

$$u^H(x) = \begin{cases} \max\bigl(u(x), u(\sigma_H(x))\bigr) & \text{if } x \in \Omega \cap H, \\ \min\bigl(u(x), u(\sigma_H(x))\bigr) & \text{if } x \in \Omega \setminus H. \end{cases}$$

The polarizations are continuous transformations of $\mathrm{H}^1_0(\Omega)$. Moreover, $\|\nabla u^H\|^2 = \|\nabla u\|_2$ and $\varphi(u^H) = \varphi(u)$.

The symmetrizations can be used to approximate the cap symmetrizations: If $\cdot^*$ denotes the cap symmetrization with respect to $e \in S^{N-1}$, there exists a sequence of closed halfspaces $(H_n)_{n\geq 1}$ such that for every $u \in \mathrm{L}^p(\Omega)$,

$$u^{H_1 \dots H_n} \to u^*$$

in $\mathrm{L}^p(\Omega)$ as $n \to \infty$ [3,6]. Moreover, a sufficient condition on the sequence $(H_n)_{n\geq 1}$ can be provided so that

$$u^{H_1 \dots H_n} \to u^*$$

in $\mathrm{L}^p(\Omega)$ as $n \to \infty$ for every $u \in \mathrm{L}^p(\Omega)$ [4]. (This condition is satisfied by almost every sequence $(H_n)_{n\geq 1}$.) Using this condition, it is possible to construct a set $A_m^\#$ such that for every $v \in A_m^\#$, there is $u \in A_m$, $\ell \geq 1$ and $(H_i)_{1\leq i\leq \ell}$ such that $v = u^{H_1 \dots H_\ell}$ and $\|v - u^*\|_p \leq 1/m$, where $\cdot^*$ is the symmetrization with respect to $h_m(u)$. Therefore,

$$\gamma\big(\overline{A_m^\#}\big) \geq k,$$

and there is a critical point which is a limit of points in the sets $A_m^\#$. By construction this critical point is invariant under some cap symmetrization.

The essential assumptions for this result are that φ is defined on a $C^{1,1}$–submanifold of a Sobolev space on a radial domain, that φ satisfies the Palais-Smale condition at the level β_k and that $\varphi(u^H) \leq \varphi(u)$. The method is applicable to Neumann boundary conditions, and also works for cylindrical domains [4]. There are similar results for other minimax principles as the mountain pass Lemma and the linking Theorem [5].

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Elliptic Problems of Ambrosetti-Prodi Type

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(joint work with E.N.Dancer)

We consider the following problem of Ambrosetti-Prodi type:

$$\begin{cases} -\Delta u = g(u) - s\varphi_1(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $s > 0$ is a positive parameter, Ω is a smooth bounded domain in R^N, $\varphi_1(x) > 0$ is the eigenfunction of $-\Delta$ in Ω with Dirichlet boundary condition corresponding to the first eigenvalue λ_1, $\lim_{t\to+\infty} \frac{g(t)}{t} = \mu > \lambda_1$, $\lim_{t\to-\infty} \frac{g(t)}{t} = \nu < \lambda_1$. Here $\mu = +\infty$ and $\nu = -\infty$ are allowed.

In the pioneering paper [1], Ambrosetti and Prodi showed that if $0 < \nu < \lambda_1 < \mu < \lambda_2$, the second eigenvalue of $-\Delta$ in Ω with Dirichlet boundary condition, and $g(t)$ is convex, then (1) has exactly two solutions for $s > 0$ large enough. If $+\infty > \mu > \lambda_2$, it was proved by Hofer [3] and Solimini [5] that if s is large enough, then (1) has at least four solutions for large s.

The results in [1, 3, 5] suggests that (1) have more solutions if (ν, μ) contains more eigenvalues. In [4], Lazer and McKenna made a conjecture that if $\mu = +\infty$ and $g(t)$ does not grow too rapidly, then (1) has an unbounded number of solutions as $s \to +\infty$.

There was no result on the Lazer and McKenna conjecture in partial differential equation setting till recently. Breuer, McKenna and Plum [2], showed that (1) has at least four solutions if $g(t) = t^2$ and Ω is a rectangle in R^2.

We will show that for some typical nonlinearities, such as $g(t) = |t|^p$, $1 < p < (N+2)/(N-2)$ if $N \geq 3$, $1 < p < +\infty$ if $N = 2$, or $g(t) = t_+^p + \lambda t$, where $t_+ = t$ if $t \geq 0$, $t_+ = 0$ if $t < 0$, $\lambda \in (-\infty, \lambda_1)$, $N \geq 3$ and $1 < p < (N+2)/(N-2)$, the Lazer and McKenna conjecture is true. We prove this conjecture by constructing solutions with many sharp peaks near some maximum points of the first eigenfunction $\varphi_1(x)$ in Ω.

We are not able to prove that the Lazer and McKenna conjecture is true for $g(t) = t_+^p + \lambda t$ if $N = 2$. The main reason is that for $N \geq 3$ and $g(t) = t_+^p + \lambda t$, we use the solution of the following problem to build up a multipeak solution for (1):

$$\begin{cases} -\Delta u = (u-1)_+^p, \ u > 0 & \text{in } R^N, \\ u \in D^{1,2}(R^N). \end{cases} \tag{2}$$

But if $N = 2$, (2) has no solution.

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Morse lemma at infinity for the sign-changing Yamabe problem

Abbas Bahri

We consider a toy problem on S^3:

$$6u - 8\Delta u = u^5 \tag{2}$$

where the sign of u is not prescribed.

This problem ist known to have infinitely many solutions (W. Y. Ding) and infinitely many asymptots which are weighted combinations of these solutions properly scaled, translated and rotated.

We provide then a normal form for the functional associated to (1) near these asymptots and establish a formula for the difference of topology induced by suitable combinations of these asymptots.

This result uses the following conjecture:

There exists $c(p) > 0$ such that for all $u \in \mathbb{R}^p$, for all $a_1, \ldots, a_p \in \mathbb{R}^3$

$$\sup_k \left| u^t \frac{\partial A}{\partial a_i^k} u \right| + |Au|^2 \geq c(p) \sum \frac{u_i^2}{|a_i - a_j|^2}$$

where $A = (\alpha_{ij})_{i,j=1,\ldots,p}$ is given by $\alpha_{ij} = \begin{cases} 0 & i = j \\ |a_i - a_j|^{-1} & i \neq j \end{cases}$

We think that there are continuous forms of this inequality as the number of points tends to infinity.

Two remarkable consequences of this work are the following:

(1) An important piece of the difference of topology is provided by the "fiber"-bundle:

$$\begin{array}{l} \{u \in \mathbb{R}^p - \{0\} : u^t A u < 0\} \\ \downarrow \\ (S^3)^p_* \end{array}$$

where

$$(S^3)^p_* = \{(a_1, \ldots, a_p) : a_i \in S^3 \text{ such that } a_i \neq a_j \text{ for } i \neq j\}$$

(2) The exit set from the asymptots into the variational space does not depend on the precise form of the functions involved in the definition of each asymptot, but rather on the value of $u^t A u$ where u is the "direction vector" corresponding to the sign of these functions at infinity.

Open problems:

1. The topology of S

A formula for the difference of topology related to the asymptot $\sum \omega_i$ follows from our results. The formula is not totally obvious, but parts of it are already clear, as the exploration of the normal form of

$$P = \sum \bar{\omega}_i(\tilde{\tilde{a}}_j)\bar{\omega}_j^{\infty}\varepsilon_{ij} - \sum c_{ij}\varepsilon_{ij}^3$$

shows. A quantity very close to P is P_∞,

$$P_\infty = \sum \bar{\omega}_i^{\infty}\bar{\omega}_j^{\infty}\varepsilon_{ij} - \sum c_{ij}\varepsilon_{ij}^3.$$

Thinking of ε_{ij} as

$$\frac{1}{\sqrt{\lambda_i\lambda_j}|a_i - a_j|},$$

we get

$$P = \left(\cdots \frac{\bar{\omega}_j^{\infty}}{\sqrt{\lambda_j}} \cdots\right) A \begin{pmatrix} \vdots \\ \frac{\bar{\omega}_i^{\infty}}{\sqrt{\lambda_i}} \\ \vdots \end{pmatrix} - \sum c_{ij}\varepsilon_{ij}^3.$$

We thus see that an important piece for this difference of topology is

$$S = \{u \in \mathbb{R}^p - \{0\} s.t. (Au, u) \leq 0\}.$$

This seems not to be quite true at first glance, since $u = \begin{pmatrix} \vdots \\ \bar{\omega}_i^{\infty}/\sqrt{\lambda_i} \\ \vdots \end{pmatrix}$, so that the sign of $u_i = \frac{\bar{\omega}}{\sqrt{\lambda_i}}$ is prescribed. However, if we combine all the asymptots $\sum \pm\omega_i$, with all possible signs in front of the ω_i's, the differences of topology of each of this family of asymptots should "convex-combine" and give

$$\left(\left(S^3\right)_*^p \times \left(\mathbb{R}^P - \{0\}\right), S\right)$$

as one important piece of this difference of topology. Another piece will be provided by the zero sets of the ω_i's and a last piece is less explicit, more related to the relative position of $\nabla\bar{\omega}^{\infty}(a_i)$ with respect to $\nabla\bar{\omega}_j^{\infty}$.

It is an important open problem to explore the topology of S.

2. Non critical asymptots

A natural question follows from our program in 1. Assume that we have only "masses" $\bar{\omega}_i$ such that $\bar{\omega}_i^{\infty}$ is positive. Then, it is fairly obvious that such combinations of $\bar{\omega}_i$ do not build a genuine asymptot since ${}^t uAu$ is negative on them. The same observation holds if $\bar{\omega}_i^{\infty}$ are all negative. The result should hold as well when there are many more negative (or positive) contributions than contributions of the opposite sugn.

We want to understand the critical configurations as the number of points tends to $+\infty$ and we want to relate them to discrete as well as continuous geometrical problems on S^3.

3. Establish the discrete inequality and continuous forms of the discrete inequality

The discrete inequality $|Au|^2 + |^t u \frac{\partial A}{\partial x_i^?} u| \geq c \sum \frac{u_i^2}{|x_i - x_j|^2}$ is a key hypothesis in the proof of our Morse Lemma at infinity. The proof of this conjecture for $p = 3$ by Y. Xu is quite involved. One would like to prove this inequality for every p and to describe continuous forms of this inequality (as the number of points tends to $+\infty$).

4. Singular and Regular solutions of Yamabe-type problems

We have established in [1] a link between regular and singular solutions to Yamabe-type equations.

This link is complicated to state (Proposition 10 of [1]). In words, some object built part of singular solutions, part of regular solutions and their stable and unstable manifolds does not change under homotopies of the domain Ω_t (for the Dirichlet Yamabe problem in $\mathbb{R}^3$) for which each piece (regular or singular) changes. A similar phenomenon is described in Contact Form Geometry ([1]).

We believe that it is important to explore this link. For geometers, the set of singular solutions is partly described in terms of Configuration Spaces (of Ω). For Analysts, (see [2]), very early on, the existence of smooth solutions is derived using properties of the barycenter spaces (of Ω).

Clearly, there are composite objects, part configuration spaces, part barycenters which should be glued up in order to understand better each piece. This general idea underlies all our work since [2] and [3]. It is natural for us to explore the link more. A very difficult conjecture (Conjecture A) for Yamabe-type problems is for example stated in [1]. Any insight here could turn out to be important in the study of regular solutions for the Yamabe problem (under Direchlet boundary conditions) on contractible domains of $\mathbb{R}^3$.

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Brouwer Degree for Nodal Blow-up

Bernold Fiedler
(joint work with Hiroshi Matano)

Consider solutions $u = u(t, x)$ of scalar parabolic equations

$$u_t = u_{xx} + \lambda u + |u|^{p-1}u$$

on $0 < x < \pi$ with Dirichlet boundary conditions. Assume $p > 1$ and $\lambda > m^2$. For solution profiles $u(t, .)$ with exactly $m + 1$ intervals of strict monotonicity let $u_1, \ldots, u_m$ denote the m extrema at positions $0 < x_1 < \cdots < x_m < \pi$. We consider *ancient solutions* $u(t, .) \to 0$ for $t \to \infty$, defined globally in backwards time.

Theorem 1. *Choose* $u_1, \ldots, u_m \in \mathbb{R} \cup \{\pm\infty\}$*, not all finite, such that* $0 < u_1 > u_2 <> \ldots$ *. Then there exists an ancient solution with prescribed extrema* $u_1, \ldots, u_m$*, precisely, at blow-up time.*

Theorem 2. *Choose* $\sigma_0, \ldots, \sigma_m > 0$ *such that* $\sigma_0 + \sigma_2 + \cdots = \sigma_1 + \sigma_3 + \cdots = 1$*. Then there exists* $0 < \theta < \pi$ *and an ancient solution with extrema at positions* $x_1, \ldots, x_m$*, precisely, at blow-up time, such that* $x_0 = 0$*,* $x_{m+1} = \pi$ *and*

$$x_{j+1} - x_j = \begin{cases} \theta\sigma_j & \text{for } j = 0, 2, \ldots \text{even} \\ (\pi - \theta)\sigma_j & \text{for } j = 1, 3, \ldots \text{odd} \end{cases}$$

Extremal values satisfy $0 < u_1 > u_2 <> \ldots$ *but are not prescribed.*

Remarks.

(i) The proof uses elementary Brouwer degree for maps which encode extremal values u_j and positions x_j, respectively.

(ii) Theorem 1 loses track of positions x_j, and Theorem 2 loses track of extremal values u_j.

(iii) Analogous results hold for $0 > u_1 < u_2 >< \ldots$ and for nonstrict inequalities.

(iv) As a curiosity, for finite u_j, we note that there exist unique linear combinations of $\sin x, \ldots, \sin(mx)$ with prescribed u_j on σ_j, respectively. A similar "shape interpolation" result holds true for the first eigenfunctions of Sturm-Liouville problems of Dirichlet type.

Open Problem: Marry theorems 1 and 2 !

Critical exponents for uniformly elliptic extremal operators

Patricio Felmer

A cornerstone in the study of nonlinear elliptic partial differential equations is

$$(1) \qquad \Delta u + u^p = 0, \quad u \geq 0 \quad \text{in} \quad \mathbb{R}^N,$$

for which a complete description of the solutions depending on the exponent p is known. The main result is the existence of a number $p_N^* = (N + 2)/(N - 2)$, known as critical exponent, such that when $1 < p < p_N^*$ no non-trivial solution to equation 1 exists, when $p = p_N^*$ then, up to scaling, equation 1 possesses exactly one solution whose behavior at infinity is like $|x|^{-(N-2)}$ and when $p > p_N^*$ then equation 1 admits radial solutions with behavior at infinity like $|x|^{-\alpha}$, for $\alpha = 2/(p-1)$.

In the proof of these basic results various important tools has been adapted and developed, such as the celebrated Pohozaev identity, energy integrals, the moving planes technique based on the maximum principle, the Kelvin transform and Harnack inequalities. In this respect the work by Pohozaev [6], Caffarelli, Gidas and Spruck in [1], Gidas and Spruck [5] and Chen and Li [2] have been fundamental.

In the recent article [3], the author and Quaas consider a similar equation but replacing the Laplacian by a Pucci's maximal operator

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + u^p = 0, \quad u \geq 0 \quad \text{in} \quad \mathbb{R}^N, \tag{2}$$

$p > 1$ and $0 < \lambda \leq \Lambda$. We recall the definition of Pucci's maximal operator: consider the set $S^N_{\lambda,\Lambda}$ of symmetric matrices satisfying $\lambda I \leq A \leq \Lambda I$ and define

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \sup_{A \in S^N_{\lambda,\Lambda}} \operatorname{tr}(AM).$$

While the Pucci's maximal operator shares many properties with the Laplacian, like homogeneity, maximum principle and others, they divert from it in a fundamental manner. In fact the Pucci's maximal operator does not have divergence form and it does not possesses the equivalent of the Kelvin transform, preventing the use of many crucial tools in the analysis of the equation. Using different techniques in [3] the author and Quaas considered the problem restricted to the radially symmetric case and prove the existence of a critical exponent p^* playing the role of the critical exponent p_N^* for the Laplacian.

In the case of the operator $\mathcal{M}^+_{\lambda,\Lambda}$, the dimension like number

$$\tilde{N}_+ = \frac{\lambda}{\Lambda}(N-1) + 1 \tag{3}$$

plays an important role. In fact, it is proved in [3] that the critical exponent p^* satisfies

$$\max\{\frac{\tilde{N}_+}{\tilde{N}_+ - 2}, p_N^*\} < p^* < \frac{\tilde{N}_+ + 2}{\tilde{N}_+ - 2}.$$

The Pucci's extremal operators represents somehow the simplest version of a fully non-linear, autonomous uniformly elliptic operator. It is the purpose of this report to present results of critical exponents, in the case of radially symmetric solutions, for an extense class of extremal operators, extending and deepening the understanding started in [3].

We consider a set $\mathcal{A} \subset S^N_{\lambda,\Lambda}$ satisfying $PAP^{-1} \in \mathcal{A}$, for all $A \in \mathcal{A}$ and for all orthogonal matrix P. Then, we define the maximal operator

$$\mathcal{M}_{\mathcal{A}}(M) = \sup_{A \in \mathcal{A}} \operatorname{tr}(AM)$$

In [4] we consider the nonlinear equation

$$\mathcal{M}_{\mathcal{A}}(D^2u) + u^p = 0, \quad u \geq 0 \quad \text{in} \quad \mathbb{R}^N. \tag{4}$$

We prove the existence of a critical exponent $p^*_{\mathcal{A}}$ that determine the range of $p > 1$ for which we have existence or non-existence of radial solution to 4. More precisely, we prove the following

Theorem 1. *Let $\mathcal{M}_{\mathcal{A}}$ be a maximal operator as defined above. Then there are two dimension like numbers $N_\infty \leq N_0$ depending on the operator $\mathcal{M}_{\mathcal{A}}$. Assume that $N_\infty > 2$ then there is a critical exponent $p^*_{\mathcal{A}}$ such that*

$$\max\{\frac{N_\infty}{N_\infty - 2}, p_0\} < p^*_{\mathcal{A}} < p_\infty, \tag{5}$$

where

$$p_0 = \frac{N_0 + 2}{N_0 - 2} \qquad p_\infty = \frac{N_\infty + 2}{N_\infty - 2},$$

and such that:

i) If $1 < p < p^*_{\mathcal{A}}$ then there is no non-radial solution to 4.

ii) If $p = p^*_{\mathcal{A}}$ then there is a unique radial solution of 4 whose behavior at infinity is like $r^{-(N_\infty - 2)}$.

iii) If $p^*_{\mathcal{A}} < p$ then there is a unique radial solution to 4 whose behavior at infinity is like $r^{-\alpha}$.

In ii) and iii) uniqueness is meant up to scaling.

Remark 2. *The results discussed in this report are stated for maximal operators. Analogous results can be obtained for minimal operators.*

Theorem 1, i) may be seen as a Liouville type theorem for radially symmetric solutions. Liouville type theorems are the on the basis of existence results in bounded domains via degree theory. Actually, the success of this approach depends on a priori bounds for the positive solutions of the equation, and these a priori bounds are obtained by a blow-up technique and a Liouville type theorem.

In this direction we may state the following

Open Problem. Let $p^*_{\mathcal{A}}$ be the critical exponent found in Theorem 1.

1) Let $u \geq 0$ be a C^2 solution of

$$\mathcal{M}_{\mathcal{A}}(D^2u) + u^{p^*_{\mathcal{A}}} = 0, \quad \text{in} \quad \mathbb{R}^N$$

then u is radially symetric?

2) Let $u \geq 0$ be a C^2 solution of

$$\mathcal{M}_{\mathcal{A}}(D^2u) + u^p = 0, \quad \text{in} \quad \mathbb{R}^N$$

with $1 < p < p^*_{\mathcal{A}}$, is $u \equiv 0$?

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Sign-changing multi-bump solutions for NLS with steep potential wells

Kazunaga Tanaka

(joint work with Yohei Sato)

We consider the existence and multiplicity of solutions of the following nonlinear Schrödinger equations:

$$(P_\lambda) \qquad -\Delta u + (\lambda^2 a(x) + 1)u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N).$$

Here $p \in (1, \frac{N+2}{N-2})$ if $N \geq 3$, $p \in (1, \infty)$ if $N = 1, 2$ and $a(x) \in C(\mathbb{R}^N, \mathbb{R})$ is non-negative on $\mathbb{R}^N$. We consider multiplicity of solutions (including positive and sign-changing solutions) when the parameter λ is very large.

For $a(x)$ we assume

(a1) $a(x) \in C(\mathbb{R}^N, \mathbb{R})$, $a(x) \geq 0$ for all $x \in \mathbb{R}^N$ and the potential well $\Omega = \mathit{int}\ a^{-1}(0)$ is a non-empty bounded open set with smooth boundary $\partial\Omega$ and $a^{-1}(0) = \overline{\Omega}$, where $a^{-1}(0) = \{x \in \mathbb{R}^N;\ a(x) = 0\}$.
(a2) $0 < \liminf_{|x|\to\infty} a(x) \leq \sup_{x\in\mathbb{R}^N} a(x) < \infty$.

When λ is large, the potential well Ω plays important roles and the following Dirichlet problem appears as a limit of (P_λ):

$$(1) \qquad -\Delta u + u = |u|^{p-1}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

We remark that solutions of (P_λ) and (1) can be characterized as critical points of

$$(2) \qquad \Psi_\lambda(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + (\lambda^2 a(x) + 1)u^2) - \frac{1}{p+1}|u|^{p-1}\, dx,$$

$$(3) \qquad \Psi_{\Omega(u)} = \int_\Omega \frac{1}{2}(|\nabla u|^2 + u^2) - \frac{1}{p+1}|u|^{p+1}\, dx.$$

Bartsch and Wang [3] and Bartsch, Pankov and Wang [4] (see also [2]) studied such a situation firstly. Their assumptions on $a(x)$ and nonlinearity are more general and as a special case of their results we have

(1) There exists a least energy solution $u_\lambda(x)$ of (P_λ). Moreover $u_{\lambda_n}(x)$ converges strongly to a least energy solution of (3) after extracting a subsequence $\lambda_n \to \infty$.
(2) When $N \geq 3$ and $p \in (1, \frac{N+2}{N-2})$ is close to $\frac{N+2}{N-2}$, there exists at least $cat(\Omega)$ positive solutions of (P_λ) for large λ. Here $cat(\Omega)$ denotes Lusternik-Schnirelman category of Ω.
(3) For any $n \in \mathbb{N}$, there exists $\lambda(n) \geq 1$ such that (P_λ) has n pairs of (possibly sign-changing) solutions $\pm u_{1,\lambda}(x), \cdots, \pm u_{n,\lambda}(x)$ for $\lambda \geq \lambda(n)$. Moreover they converge to distinct solutions $\pm u_1(x), \cdots, \pm u_n(x)$ of (1) after extracting a subsequence $\lambda_k \to \infty$.

Here we remark that in [3], [4] they consider mainly the case where Ω is connected.

In this paper we consider the case where Ω consists of multiple connected components: $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_\ell$ and we consider the multiplicity of positive and sign-changing solutions for large λ.

We have studied the multiplicity of positive solutions in our previous paper [11] (also independently by Cao and Noussair [8]) and it is shown that for any choice of components $\Omega_{k_1}, \Omega_{k_2}, \cdots, \Omega_{k_m}$, there exists a positive solution $u_\lambda(x)$ of (P_λ) for large λ such that after extracting a subsequence $\lambda \to \infty$,

$$u_{\lambda_n}(x) \quad \to \quad \begin{cases} u_i(x) & \text{in}\, \Omega_i \ (i \in \{k_1, \cdots, k_m\}), \\ 0 & \text{in}\, \mathbb{R}^N \setminus (\Omega_{k_1} \cup \cdots \cup \Omega_{k_m} \end{cases}$$

strongly in $H^1(\mathbb{R}^N)$. Here $u_i(x)$ is a least energy solution of

$$-\Delta u + u = |u|^{p-1}u \quad \text{in}\, \Omega_i \qquad u = 0 \quad \text{on}\, \partial\Omega_i. \tag{4}$$

In particular, (P_λ) has at least $2^\ell - 1$ positive solutions for large λ.

It is natural to ask the existence of a sequence of solutions of (P_λ) converging to solutions of (4) in each Ω_i, which may not be least energy solutions. For the sake of simplicity, we assume that Ω consists of 2 components, that is,

$$\Omega = \Omega_1 \cup \Omega_2. \tag{5}$$

When $\Omega = \Omega_1 \cup \Omega_2$, we have two limit problems (4) which are corresponding to

$$\Psi_{\Omega_i}(u) = \int_{\Omega_i} \frac{1}{2}(|\nabla u|^2 + u^2) - \frac{1}{p+1}|u|^{p+1}\, dx :\ H_0^1(\Omega_i) \to \mathbb{R} \quad (i = 1, 2)$$

It is well-known that each functional has an unbounded sequences of critical points $(u_j^{(i)}(x))_{j=1}^\infty \subset H_0^1(\Omega_i)$ $(i = 1, 2)$. For a given pair $(u_{j_1}^{(1)}(x), u_{j_2}^{(2)}(x))$ we study the existence of a sequence of solutions $u_\lambda(x) \in H_0^1(\mathbb{R}^N)$ which converges to $(u_{j_1}^{(1)}(x), u_{j_2}^{(2)}(x))$ after extracting a subsequece. Here we give a partial answer to this problem. More precisely, we try to find a solution $u_\lambda(x) \in H_0^1(\mathbb{R}^N)$ which converges to $(u_1^{(1)}(x), u_{j_2}^{(2)}(x))$ as $\lambda_n \to \infty$, where $u_1^{(1)}(x)$ is a mountain pass solution of (4) in Ω_1 and $u_{j_2}^{(2)}(x)$ is a minimax solution of (4) in Ω_2.

To find an unbounded sequence of critical values of a functional $I(u) \in C^1(E, \mathbb{R})$ defined on an infinite dimensional Hilbert space E, the $\mathbb{Z}_2$-symmetry of $I(u)$, that is, $I(\pm u) = I(u)$ for all $u \in E$, plays an important role. We remark that $\Psi_\lambda(u) \in$

$C^1(H^1(\mathbb{R}^N),\mathbb{R})$ and a functional $\tilde{\Psi}(u_1,u_2)=\Psi_{\Omega_1}(u_1)+\Psi_{\Omega_2}(u_2)\in C^1(H_0^1(\Omega_1)\times H_0^1(\Omega_2),\mathbb{R})$ corresponding to (4) in $\Omega_1\cup\Omega_2$ have different symmetries; $\Psi_\lambda(u)$ is $\mathbb{Z}_2$-symmetric and $\tilde{\Psi}(u_1,u_2)$ is $(\mathbb{Z}_2)^2$-symmetric, that is,

$$\begin{aligned} \Psi_\lambda(su) &= \Psi_\lambda(u) \quad \text{for all}\, s\in\mathbb{Z}_2=\{-1,1\},\, u\in H^1(\mathbb{R}^N),\\ \tilde{\Psi}(s_1u_1,s_2u_2) &= \tilde{\Psi}(u_1,u_2) \quad \text{for all}\, s_1,s_2\in\{-1,1\},\\ &\qquad\qquad (u_1,u_2)\in H_0^1(\Omega_1)\times H_0^1(\Omega_2). \end{aligned}$$

Note that $\mathbb{Z}_2$-action on $\Psi_\lambda(u)$ is corresponding to the following action on $\tilde{\Psi}(u_1,u_2)$:

$$\tilde{\Psi}(su_1,su_2)=\tilde{\Psi}(u_1,u_2) \quad \text{for all}\, s\in\{-1,1\},\, (u_1,u_2)\in H_0^1(\Omega_1)\times H_0^1(\Omega_2)$$

and and there are no symmetries of $\Psi_\lambda(u)$ corresponding to the following $\mathbb{Z}_2$-symmetry of $\tilde{\Psi}(u_1,u_2)$:

$$\tilde{\Psi}(u_1,\pm u_2)=\tilde{\Psi}(u_1,u_2). \tag{6}$$

We also remark that solutions $(u_1^{(1)}(x),u_{j_2}^{(2)}(x))$ are obtained using group action (6). Thus to construct solutions $u_\lambda(x)$ converging to $(u_1^{(1)}(x),u_{j_2}^{(2)}(x))$, we need to develop a perturbation theory from symmetry and in this paper we use ideas from Ambrosetti [1], Bahri-Berestycki [5], Struwe [13] and Rabinowitz [12] (See also Bahri-Lions [6], Tanaka [14] and Bolle [7]). In these papers perturbation theories are developed for elliptic problem: $-\Delta u=|u|^{p-1}u+f(x)$ in a bounded domain Ω.

Theorem 1. *Assume (a1)–(a2) and (5). Then there exists a sequence of minimax values $b_k=\inf_{\sigma\in\Lambda_k}\max_{\theta\in S_+^{n(k)}}\Psi_{\Omega_2}(\sigma(\theta))$ for $\Psi_{\Omega_2}(u)$ whose corresponding critical points are connectable with mountain pass critical point for $\Psi_{\Omega_1}(u)$. More precisely, for any $k\in\mathbb{N}$ there exists $\lambda(k)\geq 1$ such that for any $\lambda\geq\lambda(k)$, (P_λ) has a solution $u_\lambda(x)$ such that*

1. *$\Psi_\lambda(u_\lambda)\to c_{mp}+b_k$ as $\lambda\to\infty$, where c_{mp} is a minimax value given by the mountain pass theorem.*
2. *For any given sequence $\lambda_n\to\infty$, we can extract a subsequence $\lambda_{n_\ell}\to\infty$ such that $u_{\lambda_{n_\ell}}(x)$ converges to a function $u(x)$ strongly in $H^1(\mathbb{R}^N)$. Moreover $u(x)$ satisfies (4) in $\Omega_1\cup\Omega_2$, $u=0$ in $\mathbb{R}^N\setminus(\Omega_1\cup\Omega_2)$ and $u>0$ in Ω_1.*
3. *Moreover $\Psi_{\Omega_1}(u|_{\Omega_1})=c_{mp}$, $\Psi_{\Omega_2}(u|_{\Omega_2})=b_k$.*

Remark 2. *(i) Minimax values b_k are defined using ideas from [1, 5, 12, 13].*
(ii) In the proof the discreteness of critical values due to Dancer [9, 10] plays an important role.
(ii) When $N=1$, we can show a stronger result. More precisely, any given pair $(u_{j_1}^{(1)},u_{j_2}^{(2)})$ of limit problems are connectable, that is, there exists a solution $u_\lambda(x)$ which converges to $u_{j_i}^{(i)}(x)$ in Ω_i $(i=1,2)$.

Next we deal with positive solutions. Our result is under the influence of [3].

Theorem 3. *Assume (a1)–(a2), (5) and* $N \geq 3$*. Then there exists a* $p_1 \in (1, \frac{N+2}{N-2})$ *and* $\lambda_1 \geq 1$ *such that for* $p \in (p_1, \frac{N+2}{N-2})$ *and* $\lambda \geq \lambda_1$*,* (P_λ) *possesses at least* $cat(\Omega_1) + cat(\Omega_2) + cat(\Omega_1 \times \Omega_2)$ *positive solutions.*

Remark 4. *(i) We remark that* $cat(\Omega_1 \cup \Omega_2) = cat(\Omega_1) + cat(\Omega_2)$ *and the argument of Bartsch and Wang [3] ensures the existence of at least* $cat(\Omega_1) + cat(\Omega_2)$ *positive solutions, which converges to a positive solution in one of the components and to 0 elsewhere after extracting a subsequence. Our Theorem 3 ensures additional* $cat(\Omega_1 \times \Omega_2)$ *positive solutions, which converges to positive solutions in both components* Ω_1*,* Ω_2*.*
(ii) We conjecture that (P_λ) *has at least* $cat(\Omega_1) + cat(\Omega_2) + cat(\Omega_1) \times cat(\Omega_2)$ *positive solutions for large* λ*.*
(iii) we can also show the existence of $cat(\Omega_1 \times \Omega_2)$ *sign-changing solutions which converge to positive solutions in* Ω_1 *and negative solutions in* Ω_2*.*

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On the Brezis-Nirenberg Problem

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(joint work with Richard Lavine)

Let $B_1(0)$ be the unit ball in $\mathbb{R}^N$, and consider the problem

$$(1) \qquad \begin{cases} -\,div(|x|^{-2a}\nabla u) = |x|^{-bp}u^{p-1} + \lambda |x|^{-2(a+1)+c}u \\ u(x) > 0 \text{ in } B_1(0) \\ u \in \mathcal{D}_a(B_1(0)) \end{cases}$$

The constants a, b, p, and N satisfy the conditions:

$$a < \frac{N-2}{2}, \quad a \leq b \leq a+1, \quad p = \frac{2N}{N-2(1+a-b)}, \quad c > 0.$$

When dealing with radial solutions, can allow: $a - \frac{N-2}{2} < b \leq a+1$. The space $\mathcal{D}_a(B_1(0))$ is the Hilbert space obtained by the completion of $C_0^\infty(B_1(0))$ with respect to the inner product

$$(u,v)_a = \int_{B_1(0)} |x|^{-2a} \nabla u \cdot \nabla v \, dx.$$

For $a = b = 0$ and $c = 2$, the problem (1) becomes

$$\begin{cases} -\Delta u = u^{p-1} + \lambda u \\ u(x) > 0 \text{ in } B_1(0) \\ u \in \mathcal{D}_0(B_1(0)) \end{cases} \tag{2}$$

with $p = \frac{2N}{N-2}$ being the critical Sobolev exponent. The problem (2) has been studied in [1] and the following intriguing fact was noticed.

Theorem 1. *Denote by $\lambda_1 = \lambda_1(N)$ the first eigenvalue in*

$$-\Delta\varphi = \lambda\varphi, \quad \varphi = 0 \text{ on } \partial B_1(0)$$

(a) for $N \geq 4$ problem (2) has solution if and only if $\lambda \in (0, \lambda_1)$, and
(b) for $N = 3$ problem (2) has solution if and only if $\lambda \in (\lambda_1/4, \lambda_1)$.

Due to the Gidas-Ni-Nirenberg result (see [5]) any solution of (2) is radial, i.e. $u(x) = u(|x|)$, and one may employ a Pohozaev type argument adjusted for ODE's in order to prove the nonexistence of solutions in the situation (b) of Theorem 1.

In this work we fill in a gap between previous existence and nonexistence results due to Nicolaescu [6] and, Chou and Geng [4]. We give the precise range of the parameter λ so that (1) has *radial* solution. We also give a new characterization of the lower bound for nonexistence, as well as a procedure for finding a sharp "Pohozaev factor" in the nonexistence argument. One should emphasize that the moving plane arguments of [5] break down in the weighted situation of the problem (1) and nonradial solutions should also be expected (see [2]). We state below the result and mention an alternative way of getting to the sharp Pohozaev identity.

Theorem 2. *For $\nu = \frac{N-2-2a}{c}$ let $Z_\pm(\nu)$ be the first positive zero of the Bessel function of the first kind $J_{\pm\nu}(x)$, and let*

$$\lambda_1 = \left(\frac{c}{2} Z_+(\nu)\right)^2, \quad \lambda^* = \left(\frac{c}{2} Z_-(\nu)\right)^2.$$

(a) For $1 \leq \nu$, problem (1) has a solution if and only if $\lambda \in (0, \lambda_1)$.
(b) For $1 > \nu$, $\lambda_1 > \lambda^* > 0$ and problem (1) has solution if and only if $\lambda \in (\lambda^*, \lambda_1)$.

By symmetry reduction, problem (1) becomes

$$\begin{cases} -u_{rr} - \dfrac{N-2a-1}{r} u_r = r^{2a-bp} u^{p-1} + \lambda r^{c-2} u \\ u(x) > 0 \text{ in } (0,1) \\ u \in \mathcal{D}_{a,R}(0,1) \end{cases} \tag{3}$$

Let $u(r) = r^{-\frac{N-2-2a}{2}} v(-\ln r)$ with $t = -\ln r$. Problem (3) transforms to

$$\begin{cases} -v_{tt} + \left(\dfrac{N-2-2a}{2}\right)^2 v = v^{p-1} + \lambda e^{-ct} v \\ v(t) > 0 \text{ in } (0,\infty), \quad v \in H_0^1(0,\infty) \end{cases} \tag{4}$$

By rescaling we can choose $N-2-2a = 1$ so that the equation (4) becomes

$$-v_{tt} + \frac{1}{4} v = v^{p-1} + \lambda e^{-ct} v$$

Multiplying by the "Pohozaev factor" $\phi v_t - \dfrac{\phi_t}{2} v$ for some differentiable ϕ. Rearranging terms:

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\phi v_t^2}{2} - \frac{\phi_t v_t v}{2} + v^2\left(\frac{\phi_{tt}}{4} - \frac{\phi}{8} + \frac{\lambda e^{-ct}\phi}{2}\right) + \frac{\phi}{p} v^p\right) \\ &= v^2\left(\frac{\phi_{ttt}}{4} - \frac{\phi_t}{4} + \lambda e^{-ct}\phi_t - \frac{\lambda c e^{-ct}\phi}{2}\right) + \frac{p+2}{2p}\phi_t v^p. \end{aligned}$$

We write ϕ as the product of two factors $\phi(t) = f(t)\cdot g(t)$ to get

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{2}W[v,f]W[v,g] + \frac{v^2}{4}\left[\left(f_{tt} - \frac{1}{4}f + \lambda e^{-ct} f\right) g \right.\right. \\ &\quad \left.\left. + \left(g_{tt} - \frac{1}{4}g + \lambda e^{-ct} g\right) f\right] + \frac{fg}{p} v^p\right\} \\ &= \frac{v^2}{4}\left[g\frac{d}{dt}\left(f_{tt} - \frac{1}{4}f + \lambda e^{-ct} f\right) + f\frac{d}{dt}\left(g_{tt} - \frac{1}{4}g + \lambda e^{-ct} g\right)\right. \\ &\quad \left. + 3g_t\left(f_{tt} - \frac{1}{4}f + \lambda e^{-ct} f\right) + 3f_t\left(g_{tt} - \frac{1}{4}g + \lambda e^{-ct} g\right)\right] \\ &\quad + \frac{p+2}{2p}(f_t g + f g_t) v^p. \end{aligned}$$

Here $W[v,f](t) = v(t)f_t(t) - v_t(t)f(t)$ is the Wronskian.

The above identity suggests to take the factors f and g as solutions of the linear equation

$$-f_{tt} + \frac{1}{4} f = \lambda e^{-ct} f \tag{5}$$

We then obtain the reduced identity

$$\frac{d}{dt}\left(W[v,f]W[v,g] + \frac{2}{p} v^p f g\right) = \frac{p+2}{p} v^p (fg)_t$$

which by integration from 0 to ∞ yields

$$-v_t^2(0) f(0) g(0) = \frac{p+2}{p}\int_0^\infty v^p (fg)_t \, dt$$

If the left hand side is nonpositive, while the right hand side is positive, we get the desired contradiction. These sign conditions reveal the sharp range for the parameter λ so that (4) has no solution.

There is an alternative way of obtaining the reduced identity. Namely, by combining equations (4) and (5) one gets

$$\frac{d}{dt}W[v,f] = v^{p-1}f$$

Similarly for a different eigenfunction g,

$$\frac{d}{dt}W[v,g] = v^{p-1}g$$

Combining by the Product Rule one gets

$$\frac{d}{dt}\left(W[v,f]W[v,g]\right) = v^{p-1}\left(fW[v,g] + gW[v,f]\right)$$

which is a slightly different form of the reduced identity.

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On the joint actions of diffusion and spatial heterogeneity on single and multiple species

Yuan Lou

We first study the effects of diffusion and spatial heterogeneity on the total population size of a single species, and consider

$$\mu\Delta\theta + \theta\big[m(x) - \theta\big] = 0 \qquad \text{in } \Omega \quad \theta > 0 \quad \text{in } \Omega, \tag{1.1a}$$

$$\frac{\partial\theta}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{1.1b}$$

where the migration rate μ is assumed to be a positive constant, the function $\theta = \theta(x,\mu)$ represents the density of the species at location x, and $m(x)$ denotes its local intrinsic growth rate. The habitat of the species Ω is a bounded region in R^N with smooth boundary $\partial\Omega$, and n is the outward unit normal vector on $\partial\Omega$.

We assume that

(A) $m(x)$ is non-constant, bounded and measurable, and $\int_\Omega m(x)\,dx > 0$.

For solutions of (1.1), the following results hold.

Theorem 1. *Suppose that assumption* (A) *holds.*

(a) *For every $\mu > 0$, the problem (1.1) has a unique positive solution $\theta(x,\mu)$ such that $\theta \in W^{2,p}(\Omega)$ for every $p \geq 1$.*
(b) *As $\mu \to 0+$, the solution $\theta(x,\mu) \to m_+(x)$ in $L^p(\Omega)$ for every $p \geq 1$; as $\mu \to \infty$, the solution $\theta(x,\mu) \to \frac{1}{|\Omega|}\int_\Omega m(x)\,dx$ in $W^{2,p}(\Omega)$ for every $p \geq 1$.*
(c) *If $m(x)$ is Hölder continuous in $\overline{\Omega}$, then $\theta \in C^2(\bar{\Omega})$. Moreover, $\theta(x,\mu) \to m_+(x)$ in $L^\infty(\Omega)$ as $\mu \to 0$, and $\theta(x,\mu) \to \frac{1}{|\Omega|}\int_\Omega m(x)\,dx$ in $C^2(\overline{\Omega})$ as $\mu \to \infty$.*

In view of Theorem 1, we introduce the function

$$F(\mu) \equiv \begin{cases} \displaystyle\int_\Omega m_+(x)\,dx, & \mu = 0, \\ \displaystyle\int_\Omega \theta(x,\mu)dx, & \mu > 0, \\ \displaystyle\int_\Omega m(x)\,dx, & \mu = \infty, \end{cases} \tag{1.2}$$

which can be interpreted as the total population size of the species.

Theorem 2. *[1] Suppose that assumption* (A) *holds.*

(a) *The function $F(\mu)$ satisfies $F(\mu) > F(\infty)$ for every $\mu \in (0,\infty)$.*
(b) *If $m(x) \geq 0$ in Ω, then for every $\mu \in (0,\infty)$, $F(\mu)$ satisfies*

$$F(0) = F(\infty) < F(\mu). \tag{1.3}$$

Part (a) of Theorem 2 implies that spatial heterogeneity increases the population size of species; Part (b) of Theorem 2 implies that when $m(x)$ is non-negative, the total population size is minimized at $\mu = 0$ and $\mu = \infty$, and maximized at some intermediate value μ^*.

If the function $m(x)$ changes sign, we have

Theorem 3. *[1] Suppose that $\Omega = (0,1)$, $m \in C^2[0,1]$, m changes sign, and $m(x) = 0$ has only nondegenerate roots in $[0,1]$. Then there exist positive constants μ_0 and c_0 such that $F(\mu) - F(0) \geq c_0\mu^{\frac{2}{3}}$ for every $\mu \in (0,\mu_0)$. Thus,*

$$F(\infty) < F(0) < \sup_{0<\mu<\infty} F(\mu). \tag{1.4}$$

Theorems 2 and 3 suggest that the total population size of species is usually maximized at some intermediate migration rate, and it turns out that this fact has interesting applications to multiple species in the context of ecological invasions. In this connection, we apply Theorems 2 and 3 to study the Lotka-Volterra competition-diffusion model

$$
(1.5) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \mu\Delta u + u\,[m(x) - u - bv] & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial v}{\partial t} = \nu\Delta v + v\,[m(x) - cu - v] & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 \qquad \text{on } \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases}
$$

where $u(x,t)$ and $v(x,t)$ represent the population densities of competing species 1 and 2 with respective migration rates μ and ν, the function $m(x)$ represents their common intrinsic growth rate, and b and c are inter-specific competition coefficients. We shall assume that μ, ν, b, and c are positive constants, the growth rate $m(x)$ satisfies (A), and u_0 and v_0 are non-negative functions that are not identically equal to zero.

For the last two decades there has been tremendous interest, by both mathematicians and ecologists, in two-species Lotka-Volterra competition models with spatially heterogeneous interactions.

When both μ and ν are sufficiently small [2] or both are sufficiently large, the function $m(x)$ is positive,and $0 < b, c < 1$, then (1.5) has a unique, globally asymptotic stable positive steady state. Hence, it seems reasonable to expect that for other ranges of migration rates, the dynamics of (1.5) still should be well behaved; e.g., the two competing species can coexist. However, as we shall see later, given any non-constant positive function $m(x)$, there exists a set of parameters $b, c \in (0,1)$ and $\mu, \nu > 0$ such that one of the semi-trivial steady states of (1.5) is the global attractor of (1.5). Therefore, the joint action of spatial heterogeneity and diffusion can drive one of the species to extinction.

We start by studying the joint effects of migration and spatial heterogeneity on the invasion of the species 2 when it is rare. The case of species 1 is similar. Mathematically, this is equivalent to studying the stability of the semi-trivial steady state $(\theta, 0)$ of (1.5), where $\theta = \theta(x,\mu)$ is the unique positive solution of (1.1). We are interested in the case $0 < c < 1$. As an application of Theorem 2, the stability of $(\theta, 0)$ when $c \in (0,1)$ can be described by

Theorem 4. *[1] If assumption* (A) *holds and $m(x)$ is non-negative, then there exists some constant $c_* = c_*(m,\Omega) \in (0,1)$ such that the following results hold.*

(a) *For every $c \in (0, c_*)$, the steady state $(\theta, 0)$ is unstable when $\mu > 0$ and $\nu > 0$.*
(b) *For every $c \in (c_*, 1)$, there exists $\bar\nu = \bar\nu(c, m, \Omega) > 0$ such that* (i) *for every $\nu \in (0, \overline{\nu})$, the steady state $(\theta, 0)$ is unstable when $\mu > 0$;* (ii) *for every $\nu > \bar\nu$, the steady state $(\theta, 0)$ changes stability at least twice as μ increases from 0 to ν.*

Remark 5. The most interesting case is when $c_* < c < 1$ and $\nu > \bar\nu$, where the followings hold.

(a) If $b > 1$, it is well known that without migration, species 2 always drives species 1 to extinction. However, Theorem 4 shows that with migration, for some ranges of migration rates, species 2 may fail to invade when rare.
(b) If $b < 1$, it is well known that, without migration, species 1 always coexists with species 2. Surprisingly, as shown in Theorem 6, for certain migration rates, species 2 is able to drive species 1 to extinction for arbitrary initial conditions.

For every $c > 0$, define

$$\Sigma_c = \big\{(\mu, \nu) \in (0, \infty) \times (0, \infty) : (\theta, 0) \text{ is linearly stable}\big\}. \tag{1.6}$$

By Theorem 4, the set Σ_c is non-empty for every $c \in (c_*, 1)$.

Theorem 6. *[1] If assumption* (A) *holds and $m(x)$ is non-negative, then for every $c \in (c_*, 1)$, there exists $b_* = b_*(c, \Omega, m) \in (0, 1]$ such that if $b \in (0, b_*)$ and $(\mu, \nu) \in \Sigma_c$, then $(\theta, 0)$ is globally asymptotically stable.*

We conjecture that Theorem 6 holds with $b_* = 1$.

For the case when $m(x)$ changes sign, the stability of $(\theta, 0)$ can be described similarly.

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