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# Partielle Differentialgleichungen

Organised by
Tom Ilmanen (Zürich)
Reiner Schätzle (Tübingen)
Neil Trudinger (Canberra)

July 24th – July 30th, 2005

Abstract. The workshop dealt with partial differential equations in geometry and technical applications. The main topics were the combination of nonlinear partial differential equations and geometric measure theory, conformal invariance and the Willmore functional, and regularity of free boundaries.

*Mathematics Subject Classification (2000):* 35 J 60, 35 J 35, 58 J 05, 53 A 30, 49 Q 15.

## Introduction by the Organisers

The workshop *Partial differential equations*, organised by Tom Ilmanen (ETH Zürich), Reiner Schätzle (Universität Tübingen) and Neil Trudinger (Australian National University Canberra) was held July 24-30, 2005. This meeting was well attended by 46 participants, including 6 females, with broad geographic reprensentation. The program consisted of 15 talks and 9 shorter contributions and left sufficient time for discussions.

New results combining partial differential equations and geometric problems were presented in the area of minimal surfaces, free boundaries and singular limits, for example the construction of branched minimal surfaces, the regularity of free boundaries in the wake of the monotonicity formula of Weiss and a proof of a conjecture of De Giorgi.

A major part of the leading experts of partial differential equations with conformal invariance attended the workshop. Here new results were presented in conformal geometry, for the Yamabe problem, the Paneitz operator and the Willmore functional.

## Workshop: Partielle Differentialgleichungen

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# Abstracts

## A PDE Construction of a Class of Stable Branched Minimal Immersions

Leon Simon
(joint work with Neshan Wickramasekera)

This talk described recent joint work [SW05] with Neshan Wickramasekera, whose papers [Wic04a], [Wic04b] establish a rather complete regularity and compactness theory for stable branched minimal immersions of dimension $n$ in $\mathbb{R}^{n+1}$ near points of density less than 3. One of the main theorems discussed here (Theorem 2) can be viewed as a confirmation of the fact that there is a rather rich class of such stable branched minimal immersions in $\mathbb{R}^{n+1}$.

The case $n = 2$ is also of interest, although if $n = 2$ other techniques for generating branched minimal immersions with isolated branch points are available—for example modifications of the method [CHS84] can be used to prove quite general existence theorems in case $n = 2$ which complement the result for symmetric data proved in Theorem 2.

The approach here is to first look for a function $u_0(x, y) = u_0(re^{i\theta}, y)$ on the cylinder $D \times \mathbb{R}^{n-2}$ ($D = \{x \in \mathbb{R}^2 : |x| \leq 1\}$) which has prescribed bounded smooth boundary data $\varphi_0(e^{i\theta}, y)$ on $\partial D \times \mathbb{R}^{n-2}$ and which is a stationary point (i.e. a solution of the Euler-Lagrange equation) for the functional

$$\mathcal{F}_0(v) = \int_\Omega 4r^2 \sqrt{1 + (4r^2)^{-1}|D_x v|^2 + |D_y v|^2}\, dxdy.$$

Observe that this functional maps to the non-parametric area functional under the transformation $T : (re^{i\theta}, y) \mapsto (r^2 e^{2i\theta}, y)$, and so $u(re^{i\theta}, y) = u_0(r^{1/2}e^{i\theta/2}, y)$ will be a two-valued function with graph $G$ which is a minimal hypersurface, and $G$ will have branch points along the $(n-2)$-dimensional $C^1$ submanifold $\{(0, y, u_0(0, y)) : y \in \mathbb{R}^{n-2}\}$, assuming that $u_0(re^{i\theta}, y)$ is $C^1$ across the singular axis $r = 0$.

To get started we observe that the degenerate functional $\mathcal{F}_0$ can be approximated by the non-degenerate functionals $\mathcal{F}_\delta$ of the form

$$\mathcal{F}_\delta(v) = \int_\Omega 4r_\delta^2 \sqrt{1 + (4r_\delta^2)^{-1}|D_x v|^2 + |D_y v|^2}\, dxdy,$$

where, for $\delta \in (0, \frac{1}{2})$, $r_\delta$ is a smooth function of the variables $x = (x_1, x_2)$ with $r_\delta \equiv r$ for $r \geq \delta$ and $\delta \geq r_\delta \geq \delta/2$ for $r \in [0, \delta)$. As discussed in [SW05], standard quasilinear elliptic PDE methods then guarantee the existence of uniformly bounded (independent of $\delta$) smooth solutions $u_\delta(re^{i\theta}, y)$ with boundary data $\varphi_0$. The key point then is to discuss what happens near $r = 0$ as $\delta \downarrow 0$. In general a discontinuity occurs at $r = 0$ as we let $\delta \downarrow 0$. On the other hand $u_0 = \lim_{\delta_j \downarrow 0} u_{\delta_j}$ (for a suitable sequence $\delta_j \downarrow 0$) is a smooth function on $(D \setminus \{0\}) \times \mathbb{R}^{n-2}$ by standard quasilinear elliptic estimates away from the singular axis $\{0\} \times \mathbb{R}^{n-2}$. By a similar approximation procedure we can dispense with the requirement that the boundary data $\varphi_0$ of $u_0$ is smooth and instead take $\varphi_0$ to be an arbitrary bounded continuous function.

The first main result gives a precise analysis of the structure of $G = \operatorname{graph} u$ above a point $(0, y_0)$ where $u_0$ is discontinuous. Here $\overline{G}$ denotes closure of $G$ as a subset of $\mathbb{R}^{n+1}$.

**Theorem 1.** *Suppose $u_0$ as above is discontinuous at some point $(0, y_0) \in \{0\} \times \mathbb{R}^{n-2}$, and $\rho_0 \in (0, \frac{1}{4}]$. Then there is a $\rho_1 \in (0, \rho_0]$ and a point $(0, y_1, t_1) \in B_{\rho_0}(0, y_0) \times \mathbb{R}$ such that $B_{\rho_1}(0, y_1, t_1) \cap (\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R}) \subset \overline{G}$, $G$ (as an $n$-dimensional integer multiplicity varifold in $\mathbb{R}^{n+1}$) has a unique tangent cone $\mathbb{C}$ at $(0, y_1, t_1)$ of the form*

$$\mathbb{C} = |H_1| + |H_2|,$$

*where $H_1, H_2$ are distinct $n$-dimensional half-spaces meeting at angle $\neq \pi$ along the common boundary $\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R}$, $|H_j|$ is the multiplicity 1 varifold corresponding to $H_j$, and*

$$\overline{G} \cap B_{\rho_1}(0, y_1, t_1) = L_1 \cup L_2,$$

*where each $L_j$ is an embedded $C^\infty$ manifold-with-boundary, with boundary $\partial L_j$ (taken in the open ball $B_{\rho_1}(0, y_1, t_1)$) given by $\partial L_j = B_{\rho_1}(0, y_1, t_1) \cap (\{0\} \times \mathbb{R}^{n-2} \times \mathbb{R})$, $L_j$ has the tangent half-space $H_j$ at the point $(0, y_1, t_1)$, and $(L_1 \setminus \partial L_1) \cap (L_2 \setminus \partial L_2) = \emptyset$.*

The proof uses results for stable minimal hypersurfaces from [SS81], Allard's boundary regularity theorem [All75], and modifications of classical quasilinear elliptic PDE theory, in particular the gradient estimates of [BDM69], [Tru72], [Sim76]. A direct corollary of the above is the following general existence theorem for symmetric boundary data.

**Theorem 2.** *If the boundary data $\varphi_0$ satisfies the $\mathbb{Z}_k$ symmetry condition $\varphi_0 \circ S_k = \varphi_0$ for some odd $k \geq 3$, where $S_k(re^{i\theta}, y) = (re^{i(\theta + 2\pi/k)}, y)$, then the solution $u_0$ above can be selected so that $u_0 \circ S_k = u_0$, and then $u(re^{i\theta}, y) = u_0(r^{1/2} e^{i\theta/2}, y)$ is a $C^{1,\alpha}(D \times \mathbb{R}^{n-2}) \cap C^0(\overline{D} \times \mathbb{R}^{n-2})$ two-valued function such that $u \circ S_k = u$ and for each $\sigma \in [\frac{1}{2}, 1)$*

$$\sup_{0<|x|<\sigma,\, y \in \mathbb{R}^{n-2}} |x|^{-\alpha} |D_x u(x, y)| \leq C,$$

*where $\alpha = \alpha(k, n, \sigma, M) \in (0, 1/2)$ and $C = C(k, n, \sigma, M) > 0$, with $M$ any upper bound for $\sup_{\partial \mathcal{C}} |\varphi_0|$. In particular, the closure of $G$ in $D \times \mathbb{R}^{n-2} \times \mathbb{R}$ is a $C^{1,\alpha}$ stable branched immersion given explicitly by the covering map*

$$\Phi(re^{i\theta}, y) = (re^{i\theta}, y, u_0(r^{1/2} e^{i\theta/2}, y))$$

*which is $4\pi$-periodic in the $\theta$ variable and which has boundary values*

$$(e^{i\theta}, y, \varphi_0(e^{i\theta/2}, y))$$

*at $r = 1$.*

Theorem 2 evidently follows directly from Theorem 1 because the construction of $u_0$ (described in [SW05]) guarantees that $u_0$ can be selected so that $u_0 \circ S_k = u_0$, and $u$ (as a two-valued function) inherits the same symmetry, so $\overline{S}_k(G) = G$, where $\overline{S}_k(e^{i\theta}, y, t) = (e^{i(\theta+2\pi/k)}, y, t)$, and hence $\overline{S}_k(H_1 \cup H_2) = H_1 \cup H_2$ because $H_1 \cup H_2$ is a tangent cone of $G$. But this is of course impossible because $\overline{S}_k(H_1)$ is a union of $k$ distinct half-spaces and $k \geq 3$.

There is also a version of Theorems 1,2 for the $q$-valued case when $u(re^{i\theta}, y) = u_0(r^{1/q}e^{i\theta/q}, y)$, where $u_0$ is a stationary point of the functional

$$\mathcal{F}(v) = \int_\Omega (qr^{q-1})^2 \sqrt{1 + (qr^{q-1})^{-2}|D_x v|^2 + |D_y v|^2}\, dx dy.$$

We refer to [SW05] for the precise statements.

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## References


[All75] W. Allard, *On the first variation of a varifold—boundary behavior*, Annals of Math. **101** (1975), 418–446.

[BDM69] E. Bombieri, E. De Giorgi, and M. Miranda, *Una maggiorazzione a priori relativa alle ipersuperfici minimali non parametriche*, Arch. Rat. Mech. Anal. **32** (1969), 255–267.

[CHS84] L. Caffarelli, R. Hardt, and L. Simon, *Minimal surfaces with isolated singularities*, Manuscripta Math. **48** (1984), 1–18.

[Sim76] L. Simon, *Interior gradient bounds for non-uniformly elliptic equations*, Indiana Univ. Math. J. **25** (1976), 821–855.

[SS81] R. Schoen and L. Simon, *Regularity of stable minimal hypersurfaces*, Comm. Pure and Appl. Math **34** (1981), 741–797.

[SW05] L. Simon and N. Wickramasekera, *Stable branched minimal immersions with prescribed boundary*, Preprint (2005).

[Tru72] N. Trudinger, *A new proof of the interior gradient bound for the minimal surface equation in n dimensions*, Proc. Nat. Acad. Sci. U.S.A. **69** (1972), 821–823.

[Wic04a] N. Wickramasekera, *A rigidity theorem for stable minimal hypercones*, To appear in J. Diff. Geom. (2004).

[Wic04b] ———, *A regularity and compactness theory for immersed stable minimal hypersurfaces of multiplicity at most 2*, Preprint (2004).


## An estimate for conformal surfaces with bounds on the Willmore energy

Ernst Kuwert

(joint work with Reiner Schätzle)

The Willmore energy of an immersed surface $f : \Sigma \to \mathbb{R}^n$ is the integral

$$W(f) = \frac{1}{4} \int_\Sigma |H|^2 \, d\mu_g,$$

where $\mu_g$ is the measure associated to the induced metric $g$, and $H$ is the mean curvature vector. Here we consider only closed surfaces, and denote by $\beta_{n,p}$ the infimum of the Willmore energy in the class of oriented surfaces of genus $p \in \mathbb{N}_0$. It is well-known that $\beta_{n,0} = 4\pi$, the value being attained only by round spheres. The existence result of L. Simon implies $\beta_{n,p} > 4\pi$, whereas comparison surfaces show that $\beta_{n,p} < 8\pi$, see [1]. By a result of Li and Yau, immersions with $W(f) < 8\pi$ will in fact be embeddings, and we will work in this range. The Willmore conjecture says that $\beta_{3,1} = 2\pi^2$, and a proposed proof can be found in [3].

The main geometric feature of the Willmore functional is its invariance with respect to the Möbius group of $\mathbb{R}^n$, i.e. we have

$$0 \notin f(\Sigma),\, \hat{f} = \frac{f}{|f|^2} \quad \Rightarrow \quad W(\hat{f}) = W(f).$$

In the following, we assume $p \geq 1$ and denote by $g_0$ the unique constant curvature metric conformal to $g$ with $\mu_{g_0}(\Sigma) = \mu_g(\Sigma)$. Writing $g = e^{2u} g_0$ we have

$$-\Delta_g u + K_{g_0} e^{-2u} = K_g.$$

**Theorem 1.** *Let $f : \Sigma \to \mathbb{R}^n$ be an immersion of a closed, oriented surface of genus $p \geq 1$, where $n \leq 4$. Suppose that $W(f) \leq \omega_{n,p} - \delta$ for some $\delta > 0$, where $\omega_{3,1} = 8\pi$ and in general $\omega_{n,p} \leq 8\pi$ satisfies the following additional restrictions:*

(1) $\omega_{n,p} - 4\pi \leq \sum_{i=1}^{k} (\beta_{n,p_i} - 4\pi)$ *for any partition $p = p_1 + \ldots + p_k$ with $1 \leq p_i < k$.*
(2) $\omega_{4,p} \leq \beta_{4,p} + \frac{8\pi}{3}$.

*Then, after applying a suitable Möbius transformation, we have the estimate*

$$\|u\|_{L^\infty(\Sigma)} \leq C(p, \delta).$$

**Remarks**:

(a) If $\beta_{n,p} \geq 6\pi$ for all $p$ (which is expected), then $\omega_{n,p} = 8\pi$.
(b) After dilating to achieve unit area and also translating, one obtains the estimate
$$\|f\|_{W^{2,2}(\Sigma, g_0)} \leq C(p, \delta, g_0).$$
(c) For $p = 1$, one can show that the set of conformal structures obtained from surfaces as in the theorem is a bounded subset of the moduli space.

For the proof of the theorem, we need the almost graphical decomposition from [1] for surfaces with small second fundamental form in $L^2$. The second main ingredient is the theory of complete conformal immersions with second fundamental form in $L^2$ due to S. Müller and V. Sverak [2].

## References

[1] L. Simon, *Existence of surfaces minimizing the Willmorre functional*, Comm. Anal. Geom. **1** (1993), 281–326.
[2] S. Müller, V. Sverak, *On surfaces of finite total curvature*, J. Differential Geom. **42** (1995), 229–258.
[3] M. Schmidt, *A proof of the Willmore conjecture*, math.DG/0203224 (2002).

## Singular sets of vectorial elliptic problems

Giuseppe Mingione
(joint work with Frank Duzaar, Erlangen & Jan Kristensen, Oxford)

Let me consider an elliptic system of the type

$$\text{div } a(x,u,Du) = 0 \; , \tag{1}$$

where the vector field $a : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ is assumed to be $C^1$ with respect to the last variable, and satisfying the following growth, ellipticity, and Hölder continuity assumptions:

$$\begin{cases} |a(x,v,z)| \le L(1+|z|^{p-1}) \\ \nu(1+|z|^2)^{\frac{p-2}{2}}|\lambda|^2 \le \langle a_z(x,v,z)\lambda,\lambda\rangle \le L(1+|z|^2)^{\frac{p-2}{2}}|\lambda|^2 \\ |a(x,u,z)-a(y,v,z)| \le L\omega(|x-y|+|u-v|)(1+|z|^{p-1}) \; . \end{cases} \tag{2}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded open subset, $n \ge 2$, $N \ge 1$, $x$, $y \in \Omega$, $u$, $v \in \mathbb{R}^N$ and $z$, $\lambda \in \mathbb{R}^{nN}$, while $\omega\colon \mathbb{R}^+ \to (0,1)$ is a bounded, concave modulus of continuity such that $\omega(s) \le s^\alpha$, for $\alpha \in (0,1)$. Moreover $0 < \nu < L$, and $p > 1$. Partial $C^{1,\alpha}$-regularity of solutions takes place, in the sense that if $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ is a weak solution to (1), then there exists an open subset $\Omega_u \subset \Omega$ such that $|\Omega \setminus \Omega_u| = 0$, and $Du \in C^{0,\alpha}_{\text{loc}}(\Omega_u,\mathbb{R}^{nN})$. See [1] and related references.

Now, let me consider an integral functional of the Calculus of Variations

$$\mathcal{F}(v) := \int_\Omega F(x,v,Dv)\,dx \; . \tag{3}$$

Here $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ is an integrand, which I assume to be $C^2$ with respect to the last variable, and satisfying

$$\begin{cases} \nu|z|^p \le F(x,v,z) \le L(1+|z|^p) \\ \nu(1+|z|^2)^{\frac{p-2}{2}}|\lambda|^2 \le \langle F_{zz}(x,v,z)\lambda,\lambda\rangle \le L(1+|z|^2)^{\frac{p-2}{2}}|\lambda|^2 \\ |F(x,u,z)-F(y,v,z)| \le L\omega(|x-y|+|u-v|)(1+|z|^p) \end{cases} \tag{4}$$

with a similar notation as for (2). Once again, if $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$, then there exists an open subset $\Omega_u \subset \Omega$ such that $|\Omega \setminus \Omega_u| = 0$, and $Du \in C^{0,\alpha/2}_{\text{loc}}(\Omega_u,\mathbb{R}^{nN})$.

Moreover, for both minima and solutions, the so called higher integrability holds: there exists *an explicitly computable* number $s \equiv s(L/\nu) > p$, such that

$$Du \in L^s_{\text{loc}}(\Omega,\mathbb{R}^{nN}) \; . \tag{5}$$

In a rougher form, such regularity results are known since the beginning of the eighties. A natural problem is now proving that the singular set of $u$

$$\Sigma_u := \Omega \setminus \Omega_u \; ,$$

actually has Hausdorff dimension $\dim_{\mathcal{H}}(\Sigma_u)$ strictly less than $n$; see [3], Question (a), page 117. As for systems, the following results are essentially contained in [5, 6], taking into account the later improvements in [2, 4].

**Theorem 1.** *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution to the system* (1)*, under the assumptions* (2)*. Then*
(R1): *In general*

$$\dim_{\mathcal{H}}(\Sigma_u) \leq n - \min\{2\alpha, s - p\} .$$

(R2): *If either $n \leq p + 2$, or if $u$ is already locally Hölder continuous in $\Omega$, then*

$$\dim_{\mathcal{H}}(\Sigma_u) \leq n - 2\alpha .$$

(R3): *If $a(x, u, Du) \equiv a(x, Du)$, then the estimate in* (R2) *holds again.*
(R4): *In the case $p = 2$ all the previous inequalities become strict and, as for the cases treated in* (R2)-(R3)*, it follows that $\mathcal{H}^{n-2\alpha}(\Sigma_u) = 0$.*

The number $s$ in (R1) is the higher integrability exponent from (5). (R3) tells us that the possibility of "reducing" the dimension of the singular sets is determined in a quantitative way by the regularity of $a(x, z)$ with respect to $x$. This helps explaining (R1)-(R2): when dealing with a complete vector field of the type $a(x, u, z)$, the system (1) can be view as div $b(x, Du) = 0$, where $b(x, z) \equiv a(x, u(x), z)$. At this point the Hölder continuity of $x \to b(x, z)$ is lost, since $u(x)$ may exhibit high irregularity. Nevertheless, it is possible to use (5) to bound the oscillations of $u(x)$, and getting (R1). Accordingly, in the low dimensional case $n \leq p + 2$, it is possible to prove that $u$ is a-priori more regular on large subsets, and (R2) follows. The proof rests on a combined application of Gehring's lemma, interpolation methods, and difference quotients technique: at the end $Du$ is in a suitable Fractional Sobolev space. In turn, this implies the estimate on the singular set via abstract measure theoretical arguments.

As for minima, the following results have been proven in [4]:

**Theorem 2.** *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be local minimizer of the functional $\mathcal{F}$, under the assumptions* (4)*. Then*
(R5): *In general*

$$\dim_{\mathcal{H}}(\Sigma_u) \leq n - \min\{\alpha, s - p\} .$$

(R6): *If either $n \leq p + 2$, or if $u$ is already locally Hölder continuous in $\Omega$, then*

$$\dim_{\mathcal{H}}(\Sigma_u) \leq n - \alpha .$$

(R7): *If $F(x, v, z) \equiv f(x, z) + g(x, v)$, and $g$ is a bounded function, then the estimate in* (R6) *holds again.*
(R8): *In the case $p = 2$ all the previous inequalities become strict and, as for the cases treated in* (R6)-(R7)*, it follows that $\mathcal{H}^{n-\alpha}(\Sigma_u) = 0$.*

Here we again prove that $Du$ is in a suitable Fractional Sobolev space, but the implementation is this time completely different. Indeed, due to the Hölder continuity assumption in the second variable $(4)_3$, the functional $\mathcal{F}$ does not posses the related Euler-Lagrange system in general, and it is not possible to use any test function technique. On the contrary, in [4] we introduce a new,"variational difference quotient method", based on the minimality of $u$, and a delicate iteration/interpolation procedure in the setting of Fractional Sobolev spaces. Moreover, certain *ad hoc* Calderón-Zygmund type estimates for general non-linear elliptic systems must be derived and used. The result of Theorem 2 was not known even for $C^\infty$ integrands $F(x,v,z)$, see the Introduction of [4].

Let me finally consider the following Dirichlet problem:

$$(6)\qquad \begin{cases} -\text{div } a(x,u,Du) = 0 & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \\ u_0 \in C^{1,\alpha}(\overline{\Omega},\mathbb{R}^N)\ , \quad \partial\Omega \text{ is } C^{1,\alpha} \text{ regular .} & \end{cases}$$

It is known that even if $\partial\Omega$ and $u_0$ are such regular, there exist in general boundary irregular points, but, surprisingly enough, *the existence of even one regular boundary point for Du was not known.* In [2] such a gap is partially filled, by extending at the boundary the estimates from Theorem 1. We have

**Theorem 3.** *Let $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ be a weak solution to* (6)*, under the assumptions* (2)*, with $\alpha > 1/2$. Moreover, assume that either $n \leq p+2$, or $a(x,u,Du) \equiv a(x,Du)$. Then almost every boundary point $x \in \partial\Omega$, in the sense of the usual surface measure, is a regular point, i.e. the gradient is $C^{0,\alpha}$-regular in a neighborhood of $x$, relative to $\overline{\Omega}$. Moreover, when $p = 2$, we can allow the borderline case $\alpha = 1/2$.*

In [2] we introduce a new, interpolation-like technique for treating fractional difference quotients, working without testing the system, but via a direct comparison method. I feel that such a technique may be of its own interest.

## References


[1] F. Duzaar & J. F. Grotowski: *Optimal interior partial regularity for nonlinear elliptic systems: the method of A-harmonic approximation.* manuscripta math. **103** (2000), 267-298.

[2] F. Duzaar & J. Kristensen & G. Mingione: *The existence of regular boundary points for non-linear elliptic systems.* Preprint Dip. Mat. Univ. Parma. 2005.

[3] M. Giaquinta, M.: *Introduction to regularity theory for nonlinear elliptic systems.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1993.

[4] J. Kristensen & G. Mingione: *The singular set of minima of integral functionals.* Arch. Ration. Mech. Anal. To appear.

[5] G. Mingione: *The singular set of solutions to non-differentiable elliptic systems.* Arch. Ration. Mech. Anal. **166** (2003), 287–301.

[6] G. Mingione: *Bounds for the singular set of solutions to non linear elliptic systems.* Calc. Var. Partial Differential Equations **18** (2003), 373–400.


## Convexity in Carnot groups

Valentino Magnani

We give an account of recent results and open questions related to the notion of convexity in Carnot groups. A Carnot group $\mathbb{G}$ is a connected, simply connected graded nilpotent Lie group equipped with a system of left invariant horizontal vector fields $X_1, X_2, \dots, X_m$, spanning the first layer $V_1$ of the Lie algebra and satisfying the Lie bracket generating condition, [8]. These vector fields give the horizontal directions at each point of the space and define the so-called Carnot-Carathéodory distance, [9].

Let $\Omega$ be an open subset of $\mathbb{G}$. A function $u : \Omega \longrightarrow \mathbb{R}$ is H-convex if its restriction $t \longrightarrow u(x\exp(tX))$ is convex with respect to $t$, where $X \in V_1$, $x \in \Omega$ and $\gamma(t) = x\exp(tX)$ is the unique integral curve of $X$ passing through $x$, namely, a horizontal line. This notion has been proposed by Caffarelli and Cabré and studied by Danielli, Garofalo and Nhieu, [5], see also [12]. Recall that horizontal lines constitute a special subset of (sub-Riemannian) geodesics. However, extending convexity of $u$ to all geodesics of the group would yield a trivial notion, [13].

A first regularity property shows that H-convex functions which are locally bounded above are locally Lipschitz with respect to the Carnot-Carathéodory distance, [14]. Balogh and Rickly have shown that H-convex functions in the Heisenberg group are automatically locally bounded above, [2]. Recently, Rickly has shown that measurable H-convex functions are locally bounded above in any Carnot group and that the measurability assumption can be removed if the step is not greater than two, [17]. A detailed study of H-convex functions and H-convex sets in Carnot groups can be found in [16]. Here we mention that it is still not clear whether H-convex functions are locally bounded above in arbitrary Carnot groups.

The following estimates for continuous H-convex functions have been achieved in [5],

$$\sup_{y \in B_{\xi,r}} |u(y)| \leq C \fint_{B_{\xi,\lambda r}} |u(y)|\, dy \tag{1}$$

$$\|\nabla_H u\|_{L^\infty(B_{\xi,r})} \leq \frac{C}{r} \fint_{B_{\xi,\lambda r}} |u(y)|\, dy. \tag{2}$$

where $\lambda > 1$ is a fixed number, $C > 0$ depends on the group and $\nabla_H u = (X_1 u, \dots, X_m u)$.

Convexity in Carnot groups can be also introduced in the viscosity sense, according to the following definition by Lu, Manfredi and Stroffolini, [12]. An upper semicontinuous function $u : \Omega \longrightarrow \mathbb{R}$ is said to be v-convex if for every $x \in \Omega$ and every $\varphi \in C^2(\Omega)$ being greater than or equal to $u$ in a neighbourhood of $x$ and such that $u(x) = \varphi(x)$, we have $\nabla^2_H \varphi(x) \geq 0$. The horizontal Hessian $\nabla^2_H \varphi(x)$ has elements $\frac{1}{2}\big(X_i X_j + X_j X_i\big)\big(\varphi(x)\big)$, for every $i, j = 1, \dots, m$. Through comparison with subelliptic cones, whose existence and uniqueness in the Heisenberg group is provided by results of Bieske, [3], estimates (1) and (2) for v-convex functions have been obtained in [12]. By recent results of Wang, [22], these estimates have been further extended to Carnot groups, [11].

A natural question concerns the equality between v-convexity and H-convexity. A first positive answer has been achieved in the Heisenberg group, [2], then different proofs have been given in arbitrary Carnot groups, [11], [14], [16], [21].

Precisely, an upper semicontinuous function is v-convex if and only if it is H-convex.

Concerning second order regularity results, a natural question is extending the classical Aleksandrov-Busemann-Feller differentiability theorem to H-convex functions. A way to reach this result is showing that the second order distributional derivatives $X_iX_ju$, $i,j=1,\dots m$, of an H-convex function $u$ are measures, namely $u \in BV_H^2(\Omega)$. In fact, as it is shown by Ambrosio and the author [1], if $u \in BV_H^2(\Omega)$, then for a.e. $x \in \Omega$ there exists a unique polynomial $P_{[x]}$ of homogeneous degree less than or equal to two satisfying

$$\frac{1}{r^2}\fint_{B_{x,r}} |u - P_{[x]}| \longrightarrow 0.$$

Here $P_{[x]}$ is the second order approximate differential of $u$ at $x$. By a standard method, [7], it can be shown that functions in $BV_H^2(\Omega)$, satisfying (1) and (2) have a.e. pointwise second order differential. Then an important issue is studying whether H-convex functions belong to $BV_H^2(\Omega)$. The H-convexity easily implies that the symmetrized second order derivatives $(X_iX_ju + X_jX_iu)/2$ are measures, then proving that $X_iX_ju$ are measures is equivalent to showing that so are $[X_i,X_j]u$ and we arrive at the following problem:

(3) *Is it true that $[X_i,X_j]u$ are measures when $u$ is an H-convex function?*

This is an open question in arbitrary Carnot groups. A positive answer in Heisenberg groups has been given by Gutiérrez and Montanari, [10], and its extension to step two Carnot groups has been established by Danielli, Garofalo, Nhieu and Tournier, [6]. Trudinger has achieved a further extension to free divergence Hörmander vector fields of step two, [20]. The interesting feature of this approach is in finding a suitable subelliptic nonlinear operator satisfying a monotonicity property. In the Euclidean case, Trudinger and Wang obtained this property for $k$-Hessian operators applied to $k$-convex functions, [18], [19]. For a real symmetric matrix $A$ we define

$$F_k(A) = \sum_{1\le i_1<i_2<\dots<i_k\le n} M_{i_1i_2i_3\dots i_k}(A)$$

where $M_{i_1i_2i_3\dots i_k}(A)$ are the $k$-minors on the diagonal of the matrix. A function $u \in C^2(\Omega)$ is $k$-convex if $F_j[u] := F_j(\nabla^2u) \ge 0$ for every $j=1,\dots,k$. The monotonicity theorem, as proved in [18], shows that functions $u,v \in C^2(\Omega)\cap C(\overline{\Omega})$ satisfying $u \le v$ on $\Omega$, $u=v$ on $\partial\Omega$ and such that $u+v$ is 2-convex yield the inequality

$$\int_\Omega F_k[v] \le \int_\Omega F_k[u].$$

The proof follows immediately from both integration by parts and the free divergence formula $\sum_{j=1}^n \left(\partial_{x_j}\,\partial_{r_{ij}}F_k\right)(\nabla^2u)=0$. In Heisenberg groups the corresponding operator satisfying a monotonicity theorem has been found in [10] and it has the form

$$\mathcal{F}_2[u] = F_2(\nabla_H^2u) + 12n(\partial_tu)^2.$$

Then its form has been extended to two step Carnot groups in [6], obtaining

$$\mathcal{F}_2[u] = F_2(\nabla_H^2 u) + \frac{3}{4} \sum_{1 \le i < j \le m} \Big([X_i, X_j]u\Big)^2.$$

As observed in [20], these operators possess a free divergence formula. In fact, defining $G_2(A) = F_2(A) + \frac{1}{2}\sum_{i<j}(a_{ij} - a_{ji})^2$ and noting that $\mathcal{F}_2[u] = G_2(X^2 u)$, where $X^2 u = (X_i X_j u)_{ij}$ is the nonsymmetrized horizontal Hessian, one finds

$$\sum_{j=1}^{m} X_j \left( (\partial_{r_{ij}} G_2)(X^2 u) \right) = 0,$$

then the monotonicity theorem easily follows for more general free divergence, two step Hörmander vector fields, [20]. As a consequence of this theorem, the estimate

$$\int_{\Omega'} F_2(\nabla_H^2 u) + \frac{3}{4} \sum_{1 \le i < j \le m} \Big([X_i, X_j]u\Big)^2 \le C \left( \int_{\Omega} |u| \right)^2 \tag{4}$$

can be achieved for a function $u \in C^2(\Omega)$, satisfying $F_j(\nabla_H^2 u) \ge 0$ for $j = 1, 2$, where $\Omega'$ is compactly contained in $\Omega$ and $C$ depends on $\operatorname{dist}(\Omega', \partial\Omega)$. Now, to establish that $[X_i, X_j]u \in L^2_{loc}(\Omega)$ when $u$ is H-convex, we introduce the following definition. According to [20], we say that a function $u \in C^2(\Omega)$ is k-convex with respect to the vector fields $X_j$ (or simply k-convex) if $F_j(\nabla_H^2 u) \ge 0$ for any $j = 1, \dots, k$. The larger class of locally summable k-convex functions is obtained by closure of $C^2$ smooth k-convex functions with respect to $L^1_{loc}$-convergence. Hence, estimate (4) shows that locally summable 2-convex functions satisfy $[X_i, X_j]u \in L^2_{loc}(\Omega)$. Now we notice that in Carnot groups a function $u \in C^2(\Omega)$ is H-convex if and only if $\nabla_H^2 u \ge 0$, therefore by a suitable smooth convolution it can be seen that the class of locally summable m-convex functions coincides with that of locally Lipschitz H-convex functions. As a result, in step two Carnot groups any H-convex function $u$ has the property $[X_i, X_j]u \in L^2_{loc}(\Omega)$.

Now, it would be desirable having a characterization of the $L^1_{loc}$-limits of $k$-convex functions analogous to the case $k = m$. Here it is helpful the following distributional characterization of H-convex functions in step two Carnot groups. A Radon measure $\mu$ such that $\nabla_H^2 \mu \ge 0$ is defined by an $L^1_{loc}$-limit of smooth $H$-convex functions, [14]. The problem of extending this characterization to higher step Carnot groups relies on the validity of the key identity

$$X_i X_j \theta(x) = X_i X_j \theta(x^{-1}) \tag{5}$$

for mollifiers $\theta$ such that $\theta(x) = \theta(x^{-1})$. Presently, this identity holds in step two Carnot groups, whereas it is not known in groups of higher step. The same approach of [14] and equality (5) easily imply that locally summable k-convex functions can be characterized in distributional sense, as in Lemma 2.2 of [19].

Second order differentiability can be extended to k-convex functions. In fact, among the gradient estimates obtained in [20] for k-convex functions, it is shown that

$$\sup_{\Omega'} \frac{|u(x) - u(y)|}{d(x,y)^\alpha} \leq C \, \|u\|_{L^1(\Omega)} \, ,$$

under the condition $k > (Q-1)m/(Q+m-2)$. As a consequence, arguing as in [4], from the fact that $[X_i, X_j]u \in L^2_{loc}(\Omega)$ and the approximate second order differentiability of functions in $BV^2_H(\Omega)$, the classical Aleksandrov-Busemann-Feller's theorem extends to k-convex functions in step two Carnot groups, when $k > (Q-1)m/(Q+m-2)$, [20].

## References


[1] L.Ambrosio, V.Magnani, *Weak differentiability of BV functions on stratified groups*, Math. Z., **245**, 123-153, (2003)

[2] Z.Balogh, M.Rickly, *Regularity of convex functions on Heisenberg groups*, Ann. Scuola Norm. Sup. Sci. (5), **2**, n.4, 847-868, (2003)

[3] T.Bieske, *On ∞-harmonic functions on the Heisenberg group*, Comm. in Partial Differential Equations, **27**, n.3-4, 727-762, (2002)

[4] N.Chaudhuri, N.S.Trudinger, *An Aleksandrov type theorem for k-convex functions*. Bull. Aus. Math. Soc. **71**, 305-314, (2005)

[5] D.Danielli, N.Garofalo, D.M. Nhieu, *Notions of convexity in Carnot groups*, Comm. Anal. Geom., **11**, n.2, 263-341, (2003)

[6] D.Danielli, N.Garofalo, D.M. Nhieu, F.Tournier *The theorem of Busemann-Feller-Alexandrov in Carnot groups*, Comm. Anal. Geom. **12**, n.4, 853-886, (2004)

[7] L.C.Evans, R.F.Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, (1992)

[8] G.B.Folland, E.M. Stein, *Hardy Spaces on Homogeneous groups*, Princeton University Press, 1982

[9] M.Gromov, *Carnot-Carathéodory spaces seen from within*, Subriemannian Geometry, Progress in Mathematics, **144**. ed. by A.Bellaiche and J.Risler, Birkhauser Verlag, Basel, 1996.

[10] C.E. Gutiérrez, A. Montanari, *On the second order derivatives of convex functions on the Heisenberg group*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **3**, n.2, 349-366, (2004)

[11] P.Juutinen, G.Lu, J.Manfredi, B.Stroffolini, *Convex functions in Carnot groups*, preprint (2005)

[12] G.Lu, J.Manfredi, B.Stroffolini, *Convex functions on the Heisenberg group*, Calc. Var., **19**, n.1, 1-22, (2004)

[13] R.Monti, M.Rickly, *Geodetically convex sets in the Heisenberg group*, J. Convex Anal. **12**, n.1, 187-196, (2005)

[14] V.Magnani, *Lipschitz continuity, Aleksandrov theorem and characterizations for H-convex functions*, preprint (2003), to appear on Math. Ann.

[15] P.Pansu, *Une inégalité isoperimetrique sur le groupe de Heisenberg*, C.R. Acad. Sc. Paris, **295**, Série I, 127-130, (1982)

[16] M.Rickly, *On questions of existence and regularity related to notions of convexity in Carnot groups*, Ph.D. thesis (2005)

[17] M.Rickly, *First order regularity of convex functions on Carnot groups*, preprint (2005)

[18] N.Trudinger, X.J.Wang, *Hessian measures. I.* Topol. Methods Nonlinear Anal. **10**, n.2, 225-239, (1997)

[19] N.Trudinger, X.J.Wang, *Hessian measures. II.* Ann. of Math. (2) **150**, n.2, 579-604, (1999)

[20] N.Trudinger, *On Hessian measures for non-commuting vector fields*, ArXiv preprint: math.AP/0503695, (2005)

[21] C.Wang, *Viscosity convex functions on Carnot groups*, **133**, n.4, 1247-1253, (2005)

[22] C.Wang, *The Euler Equations of absolutely minimizing Lipschitz extensions for vector fields satisfying Hörmander condition*, preprint (2005)

## Expanding and contracting surfaces

Oliver C. Schnürer

**Abstract.** We show that convex surfaces become round under expanding and contracting curvature flows.

We consider families of convex embedded surfaces $M_t$ in $\mathbb{R}^3$ that evolve according to

$$\dot{X} = -F\nu,$$

where $X$ is the embedding vector, $\nu$ the outer unit normal, and the normal velocity $F$ is a symmetric function of the principal curvatures $\lambda_1$ and $\lambda_2$. We will assume that all convex surfaces considered here are closed and embedded into $\mathbb{R}^3$.

### 1. Contracting Flows

For normal velocities of homogeneity one, i. e. for normal velocities positive homogeneous of degree one in the principal curvatures, there are several results that show convergence to a round point for convex hypersurfaces, see e. g. [4] for the mean curvature flow, $F = H = \lambda_1 + \ldots + \lambda_n$, or [1]. A surface is said to converge to a round point, if it converges to a point and, after appropriate rescaling, to a round sphere.

Ben Andrews showed that convex surfaces converge to round points under Gauß curvature flow [2].

We generalize this result to other normal velocities of homogeneity larger than one [5].

**Theorem 1.1.** *Any family of closed strictly convex surfaces $M_t$, flowing according to*

$$\dot{X} = -|A|^2\nu,$$

*converges to a round point in finite time.*

The proof is based on the following observation.

**Theorem 1.2.** *For any family $M_t$ of closed strictly convex surfaces, flowing according to $\dot{X} = -|A|^2\nu$,*

$$\max_{M_t} \frac{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2}{\lambda_1\lambda_2}$$

*is non-increasing in time.*

Note especially that this quantity is not scaling invariant.

## 2. Expanding Flows

For normal velocities of homogeneity minus one, starshaped hypersurfaces converge to round spheres at infinity [3], i. e. they converge to infinity and, after appropriate rescaling, to round spheres.

Our result for expanding surfaces addresses the inverse Gauß curvature flow [6].

**Theorem 2.3.** *Any family of closed strictly convex surfaces $M_t$, flowing according to*

$$\dot{X} = \frac{1}{K}\nu,$$

*converges to a round sphere at infinity in finite time.*

Once again, the main ingredient of the proof is a monotone quantity.

**Theorem 2.4.** *For surfaces $M_t$ solving the inverse Gauß curvature flow,*

$$\max_{M_t} \frac{(\lambda_1 - \lambda_2)^2}{{\lambda_1}^2 {\lambda_2}^2}$$

*is non-increasing in time.*

## 3. Monotone Quantities

The monotonicity of the quantities mentioned above is a direct consequence of the maximum principle.

In the following table, we have a few other normal velocities $F$ and quantities $w$ such that $\max_{M_t} w$ is monotone during the respective flows. In order to find these quantities, we used an algorithm that checks for symmetric rational functions of the principal curvatures that

- $w \geq 0$ with equality at umbilic points,
- umbilic points are the only critical points,
- $w$ has an appropriate scaling behavior,
- the evolution equation of $w$ is such that we can apply the maximum principle to prove monotonicity.

## References


[1] Ben Andrews, *Pinching estimates and motion of hypersurfaces by curvature functions*, `arXiv:math.DG/0402311`.

[2] Ben Andrews, *Gauss curvature flow: the fate of the rolling stones*, Invent. Math. **138** (1999), no. 1, 151–161.

[3] Claus Gerhardt, *Flow of nonconvex hypersurfaces into spheres*, J. Differential Geom. **32** (1990), no. 1, 299–314.

[4] Gerhard Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom. **20** (1984), no. 1, 237–266.

[5] Oliver C. Schnürer, *Surfaces contracting by $|A|^2$*, 2004, `arXiv:math.DG/0409388`.

[6] Oliver C. Schnürer, *Surfaces expanding by the inverse Gauß curvature flow*, 2004, `arXiv:math.DG/0412297`.

Table 1

| | |
|---|---|
| $H^2$ | $\dfrac{(\lambda_1+\lambda_2)^3(\lambda_1-\lambda_2)^2}{\left({\lambda_1}^2+{\lambda_2}^2\right)\lambda_1\lambda_2}$ |
| $H^3$ | $\dfrac{\left({\lambda_1}^2+\lambda_1\lambda_2+{\lambda_2}^2\right)(\lambda_1+\lambda_2)^2(\lambda_1-\lambda_2)^2}{\left({\lambda_1}^2-\lambda_1\lambda_2+{\lambda_2}^2\right)\lambda_1\lambda_2}$ |
| $H^4$ | $\dfrac{\left({\lambda_1}^2+\lambda_1\lambda_2+{\lambda_2}^2\right)(\lambda_1+\lambda_2)^6(\lambda_1-\lambda_2)^2}{{\lambda_1}^2{\lambda_2}^2}$ |
| $\lvert A\rvert^2+\beta H^2$, $0\le\beta\le 5$ | $\dfrac{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{\lambda_1\lambda_2}$ |
| $\operatorname{tr} A^3$ | $\dfrac{\left(3{\lambda_1}^2+2\lambda_1\lambda_2+3{\lambda_2}^2\right)(\lambda_1-\lambda_2)^2}{\lambda_1\lambda_2}$ |
| $\operatorname{tr}\lvert A\rvert^\alpha$, $\alpha=2,\,4,\,5,\,6$ | $\dfrac{\left({\lambda_1}^{\alpha-2}+{\lambda_2}^{\alpha-2}\right)(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{\lambda_1\lambda_2}$ |
| $H\lvert A\rvert^2$ | $\dfrac{(\lambda_1+\lambda_2)^2(\lambda_1-\lambda_2)^2}{\lambda_1\lambda_2}$ |
| $\lvert A\rvert^4$ | $\dfrac{\left({\lambda_1}^4+2{\lambda_1}^3\lambda_2+4{\lambda_1}^2{\lambda_2}^2+2\lambda_1{\lambda_2}^3+{\lambda_2}^4\right)(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)\lambda_1\lambda_2}$ |

| | |
|---|---|
| $-\dfrac{H^2}{K^2}$ | $\dfrac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)\lambda_1\lambda_2}$ |
| $-\dfrac{\lvert A\rvert^2}{K^2}$ | $\dfrac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)\lambda_1\lambda_2}$ |
| $-\dfrac{H^3}{K^3}$ | $\dfrac{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{\left({\lambda_1}^2+{\lambda_2}^2\right)\left({\lambda_1}^2+\lambda_1\lambda_2+{\lambda_2}^2\right){\lambda_1}^3{\lambda_2}^3}$ |

## Shock Reflection and Free Boundary Problems for Degenerate Elliptic Equations

Mikhail Feldman
(joint work with Gui-Qiang Chen)

When a plane shock hits a wedge head on, it experiences a reflection and then a self-similar reflected shock moves outward as the original shock moves forward in time. Experimental, computational, and asymptotic analysis have shown that various patterns of reflected shocks may occur, including regular and Mach reflection [1, 5, 6, 7, 8, 10]. However, there has been no rigorous mathematical result on the global existence and structural stability of shock reflection, especially for potential flow which has widely been used in aerodynamics. Such problems involve several difficulties in the analysis of nonlinear partial differential equations including equations of elliptic-hyperbolic mixed type, free boundary problems, degenerate ellipticity along the sonic line, and corner singularities.

In this work we prove existence of a shock reflection solution for potential flow, in case when a wedge angle is close to $\frac{\pi}{2}$. Potential flow equation for self-similar solutions, in self-similar variables $(\xi, \eta) = (\frac{x}{t}, \frac{y}{t})$, is

$$\operatorname{div}\left(\rho(|D\varphi|^2, \varphi)D\varphi\right) + 2\rho(|D\varphi|^2, \varphi) = 0,$$

with

$$\rho(|D\varphi|^2, \varphi) = H(\rho_0^{\gamma-1} - \varphi - \frac{1}{2}|D\varphi|^2),$$

where $\varphi(\xi, \eta)$ is the pseudo-velocity potential, $H(s) = s^{1/(\gamma-1)}$, and $\gamma > 1, \rho_0 > 0$ are constants. Equation is elliptic-hyperbolic mixed, which is elliptic if and only if

$$|D\varphi| < \sqrt{\frac{2(\gamma-1)}{\gamma+1}(\rho_0^{\gamma-1} - \varphi)}.$$

For a regular reflection solution, the hyperbolic part of the solution can be computed explicitly. Thus unknown is the ellipticity region $\Omega^+$, and the solution $\varphi$ in $\Omega^+$. The boundary of $\Omega^+$ consists of the shock curve, location of which is unknown, and the known part, which includes the sonic line. Ellipticity is expected to degenerate along the sonic line. We reformulate the problem as a free boundary problem for a modified equation, which is elliptic, with ellipticity degenerating at the sonic line. Free boundary condition is derived from the Rankine-Hugoniot condition on the shock curve. We solve this free boundary problem, using an iteration scheme of [2, 3]. On each step, an "iteration free boundary" is given, and we solve a boundary value problem for the degenerate elliptic equation. In particular, we obtain $C^{1,1}$ estimates of the solution near the sonic line, at which ellipticity degenerates. These estimates use the nonlinear structure of elliptic degeneracy, and a scaling technique similar to [4, 9]. A fixed point $\varphi$ of the iteration procedure is a solution of the free boundary problem for the modified equation. Then we prove, by a careful gradient estimate, that $\varphi$ satisfies the original potential flow equation.

## References


[1] G. Ben-Dor, *Shock Wave Reflection Phenomena*, Springer-Verlag: New York, 1991.
[2] G.-Q. Chen and M. Feldman, Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type, *J. Amer. Math. Soc.* **16** (2003), 461–494.
[3] G.-Q. Chen and M. Feldman, Steady transonic shocks and free boundary problems in infinite cylinders for the Euler equations, *Comm. Pure Appl. Math.* **57** (2004), 310–356.
[4] P. Daskalopoulos and R. Hamilton, The free boundary in the Gauss curvature flow with flat sides. *J. Reine Angew. Math.* **510** (1999), 187–227.
[5] J. Glimm and A. Majda, *Multidimensional Hyperbolic Problems and Computations*, Springer-Verlag: New York, 1991.
[6] J. K. Hunter, Transverse diffraction of nonlinear waves and singular rays, *SIAM J. Appl. Math.* **48** (1988), 1–37.
[7] J. Hunter and J. B. Keller, Weak shock diffraction, *Wave Motion,* **6** (1984), 79–89.
[8] J. B. Keller and A. A. Blank, Diffraction and reflection of pulses by wedges and corners, *Comm. Pure Appl. Math.* **4** (1951), 75–94.
[9] F.H. Lin and L. Wang A class of fully nonlinear elliptic equations with singularity at the boundary. *J. Geom. Anal.* **8** (1998), no. 4, 583–598.
[10] C. S. Morawetz, Potential theory for regular and Mach reflection of a shock at a wedge. *Comm. Pure Appl. Math.*, **47** (1994), 593–624.


## Concentration phenomena for Liouville's equation

Michael Struwe

Let $\Omega$ be a bounded domain of $\mathbb{R}^4$ and let $u_k$ be solutions to the equation

$$\Delta^2 u_k = V_k e^{4u_k} \text{ in } \Omega, \tag{1}$$

where $V_k \to 6$ uniformly in $\Omega$ as $k \to \infty$ and with

$$\int_\Omega V_k e^{4u_k}\,dx \le \Lambda < \infty \tag{2}$$

uniformly in $k$. Geometrically, the solutions $u_k$ to (1) correspond to conformal metrics $g_k = e^{2u_k} g_{\mathbb{R}^4}$ on $\Omega$ having $Q$-curvature $Q_k = V_k/2$.

Equation (1) is the fourth order analogue of Liouville's equation

$$-\Delta u_k = V_k e^{2u_k} \text{ in } \Omega \subset \mathbb{R}^2. \tag{3}$$

with $V_k \to 1$ uniformly in $\Omega$ as $k \to \infty$. Thus, for problem (1), (2) we may expect similar results to hold as have been obtained by Brezis-Merle [2], Li-Shafrir [7] in the two-dimensional case, where a striking quantization is observed due to the fact that all solutions $u$ of the limit equation

$$-\Delta u = e^{2u} \text{ on } \mathbb{R}^2,\ \int_{\mathbb{R}^2} e^{2u}\,dx < \infty, \tag{4}$$

giving rise to conformal metrics $g = e^{2u} g_{\mathbb{R}^2}$ on $\mathbb{R}^2$ with Gauss curvature $K \equiv 1$, by a result of Chen-Li [4] up to scaling are given by the function $u(x) = log(\frac{2}{1+|x|^2})$ which is induced by stereographic projection of the standard sphere.

In contrast to the two-dimensional case, in 4 dimensions there is a much greater abundance of solutions to the corresponding limit equation

$$\Delta^2 u = 6e^{4u} \text{ on } \mathbb{R}^4 \tag{5}$$

than can be obtained by pull-back of the spherical metric on $S^4$. In fact, by a result of Chang-Chen [3] for any $\alpha \in ]0, 16\pi^2]$ there exists a solution $u_\alpha$ of (5) of total volume $\int_{\mathbb{R}^4} 6e^{4u_\alpha}\, dx = \alpha$. Only the solutions that achieve the maximal volume $\int_{\mathbb{R}^4} 6e^{4u}\, dx = 16\pi^2$ up to scaling again are of the form $u(x) = log(\frac{2}{1+|x|^2})$. In particular, there are solutions $u_k$ to (1) with $V_k \equiv 6$ such that $u_k(0) \to \infty$ while $u_k(x) \to -\infty$ for all $x \neq 0$ and $\int_\Omega V_k e^{4u_k}\, dx \to 0$ as $k \to \infty$.

There is a further complication in the four-dimensional case, illustrated by the following simple example. Consider the sequence $(v_k)$ on $\mathbb{R}^4$, defined by letting $v_k(x) = w_k(|x^1|)$, where $w_k'''' = 6e^{4w_k}$ on $[0, \infty[$ with $w_k(0) = w_k'(0) = w_k'''(0) = 0$, and $w_k''(0) = -k$, $k \in \mathbb{N}$. Given $\Lambda > 0$, we can find a sequence of radii $R_k \to \infty$ as $k \to \infty$ such that $\int_{B_{R_k}(0)} 6e^{4v_k}\, dx = \Lambda$. After scaling $u_k(x) = v_k(R_k x) + \log R_k$ we then obtain a sequence of solutions $u_k$ to equation (5) on $\Omega = B_1(0)$ such that $u_k(x) \to \infty$ for all $x \in S_0 = \{x \in \Omega;\, x^1 = 0\}$ and $u_k(x) \to -\infty$ away from $S_0$ as $k \to \infty$. Scaling back, from $(u_k)$ we reobtain the normalized functions $v_k$ which however fail to converge to a solution of the limit problem (5) and develop an interior layer on the hypersurface $\{x \in \mathbb{R}^4;\, x^1 = 0\}$, instead.

The following concentration-compactness result in [1], obtained jointly with Adimurthi and Frederic Robert, therefore seems best possible.

**Theorem 1.** *Let $\Omega$ be a bounded domain of $\mathbb{R}^4$ and let $(u_k)_{k\in\mathbb{N}}$ be a sequence of solutions to (1), (2) as above.*

*Then either i) a subsequence $(u_k)$ is relatively compact in $C^{3,\alpha}_{loc}(\Omega)$, or ii) there exist a subsequence $(u_k)$ and a closed nowhere dense set $S_0$ of vanishing measure and at most finitely many points $x^{(i)} \in \Omega$, $1 \le i \le I \le C\Lambda$, such that, letting*

$$S = S_0 \cup \{x^{(i)};\, 1 \le i \le I\},$$

*we have $u_k \to -\infty$ uniformly locally away from $S$ as $k \to \infty$. Moreover, there is a sequence of numbers $\beta_k \to \infty$ such that*

$$\frac{u_k}{\beta_k} \to \varphi \text{ in } C^{3,\alpha}_{loc}(\Omega \setminus S),$$

*where $\varphi \in C^4(\Omega \setminus \{x^{(i)};\, 1 \le i \le I\})$ is such that*

$$\Delta^2 \varphi = 0, \varphi \le 0, \varphi \not\equiv 0,$$

*and*

$$S_0 = \{x \in \Omega \setminus \{x^{(i)};\, 1 \le i \le I\};\, \varphi(x) = 0\}.$$

*Finally, near any point $x_0 \in S$ where $\sup_{B_r(x_0)} u_k \to \infty$ for every $r > 0$ as $k \to \infty$, in particular, near any concentration point $x^{(i)}$, there exist points $x_k \to x_0$ and suitable radii $r_k \to 0$ such that after scaling we have*

$$v_k(x) = u_k(x_k + r_k x) + \log r_k \le 0 \le \log 2 + v_k(0) \text{ on } B_L(0) \tag{6}$$

*for any $L > 0$ and sufficiently large $k$. As $k \to \infty$ then either a subsequence $v_k \to v$ in $C^{3,\alpha}_{loc}(\mathbb{R}^4)$, where $v$ solves the limit equation (5), or there holds $v_k \to -\infty$ almost everywhere and there is a sequence of numbers $\gamma_k \to \infty$ such that a subsequence*

$$\frac{v_k}{\gamma_k} \to \psi \text{ in } C^{3,\alpha}_{loc}(\mathbb{R}^4),$$

*where $\psi \leq 0$ is a non-constant quadratic polynomial.*

We regard Theorem 1 as a first step towards a more complete description of the possible concentration behavior of sequences of solutions to problem (1), (2).

Considering (1) as a system of second order equations for $u_k$ and $\Delta u_k$, respectively, it is possible to obtain some partial results in this regard from the observation that (2) provides uniform integral bounds for $\Delta u_k$ up to a remainder given by a harmonic function. The latter component may be controlled if one imposes, for instance, Navier boundary conditions $u_k = \Delta u_k = 0$ on $\partial\Omega$. In fact, in this case J. Wei [14] has shown (in the notation of Theorem 1) that $S_0 = \emptyset$ and that at any concentration point $x^{(i)}$ the rescaled functions $v_k \to v$ in $C^{3,\alpha}_{loc}(\mathbb{R}^4)$, where $v$ is the profile induced by stereographic projection.

As shown by Robert [12], the same result holds if for some open subset $\emptyset \neq \omega \subset \Omega$ we have the a-priori bounds

$$||(\Delta u_k)^-||_{L^1(\Omega)} \leq C, ||(\Delta u_k)^+||_{L^1(\omega)} \leq C,$$

for all $k \in \mathbb{N}$, where $s^\pm = \pm max\{0, \pm s\}$. Also in the radially symmetric case there is a complete description of the possible concentration patterns; see [12].

In the geometric context similar results hold for the related problem of describing the possible concentration behavior of solutions to the equation of prescribed $Q$-curvature on a closed 4-manifold $M$. Here the bi-Laplacian in equation (1) is replaced by the Paneitz-Branson operator and $V_k$ again may be interpreted as being proportional to the $Q$-curvature of the metric $g_k = e^{2u_k} g_M$. In the case when $M = S^4$, Malchiodi-Struwe [10] have shown that any such sequence $(g_k)$ of metrics when $V_k$ converges uniformly to some smooth limit $V_0 > 0$ either is relatively compact or blows up at a single concentration-point where a round spherical metric forms after rescaling. Further compactness results and references can be found in [9].

Related results on compactness issues for fourth order equations can be found in Hebey-Robert-Wen [6] and Robert [11].

## References


[1] Adimurthi, F. Robert, M Struwe: *Concentration phenomena for Liouville's equation in dimension four*, preprint 2005.

[2] H. Brézis, F. Merle: *Uniform estimates and blow-up behaviour for solutions of $-\Delta u = V(x)e^u$ in two dimensions*, Comm. Partial Differential Equations 16 (1991), 1223-1253.

[3] S.-Y. A. Chang, W. Chen: *A note on a class of higher order conformally covariant equations*, Discrete Continuous Dynamical Systems 7 (2001), 275-281.

[4] W. Chen, C. Li: *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. 63 (1991), 615-623.

[5] O. Druet, F. Robert: *Bubbling phenomena for fourth-order four-dimensional PDEs with exponential growth*, Proc. Amer. Math. Soc., to appear.
[6] E. Hebey, F. Robert, Y. Wen: *Compactness and global estimates for a fourth order equation of critical Sobolev growth arising from conformal geometry*, Comm. Contemp. Math., to appear.
[7] Y. Li, I. Shafrir: *Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two*, Indiana Univ. Math. J. 43 (1994), 1255-1270.
[8] C. S. Lin.: *A classification of solutions of a conformally invariant fourth order equation in* $\mathbb{R}^n$, Comm. Math. Helv. 73 (1998), 206-231.
[9] A. Malchiodi: *Compactness of solutions to some geometric fourth-order equations*, preprint 2005.
[10] A. Malchiodi, M. Struwe: *The Q-curvature flow on $S^4$*, preprint 2004.
[11] F. Robert: *Positive solutions for a fourth order equation invariant under isometries*, Proc. Amer. Math. Soc. 131 (2003), 1423-1431.
[12] F. Robert: *Quantization issues for fourth order equations with critical exponential growth*, preprint 2005.
[13] F. Robert, M Struwe: *Asymptotic profile for a fourth order PDE with critical exponential growth in dimension four*, Adv. Nonlinear Stud. 4 (2004), no. 4, 397-415.
[14] J. Wei: *Asymptotic behavior of a nonlinear fourth order eigenvalue problem*, Comm. Partial Differential Equations 21 (1996), no. 9-10, 1451-1467.


## A fully nonlinear version of the Yamabe problem

Matthew J. Gursky

In this talk we discuss the problem of prescribing symmetric functions of the eigenvalues of the Ricci curvature for a conformal metric.

If $A = \frac{1}{n-2}\big(\mathrm{Ric} - \frac{1}{2(n-1)}\mathrm{R}\, g\big)$ denote the Weyl-Schouten tensor (Ric = Ricci, R = scalar curv.); then given a conformal metric $g_u = e^{-2u}g$ we have

$$A_u = A + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g.$$

The model problem we describe in detail is prescribing the elementary symmetric polynomians of the eigenvalues of $A_u$; i.e. given $f \in C^\infty$, we want to find $g_u = e^{-2u}g$ s.t.

$$(*) \qquad \sigma_k^{1/k}\Big(\lambda\Big(A + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g\Big)\Big) = fe^{-2u}.$$

The talk is organized as follows: (I) ellipticity, (II) variational properties, (III) estimates, (IV) existence results.

**Theorem 1** (with J. Viaclovsky)**.** *If $k > \frac{n}{2}$, $M^n \neq S^n$, $(M^n, g)$ "admissible", there always exist solutions of $(*)$. Moreover the solution space is compact.*

## Critical regularity for elliptic equations from Littlewood-Paley theory

Denis Labutin

We prove a general theorem on local regularity improvement for critical linear elliptic equations. The main point and value of the theorem is that it allows to derive the full ($C^\infty_{\mathrm{loc}}$-) regularity for critical nonlinear equations and systems. The theorem states:

**Theorem 1.** *Suppose* $u \in W^{s,p}(B)$, $V \in L^{n/(\alpha-\beta-\gamma)}(B)$, *where* $\alpha, \beta, \gamma \geq 0$, $\alpha > \beta + \gamma$, *and*

$$Lu + P(V(x)Qu) = 0.$$

*Here* $L, P, Q \in \Psi^{\alpha,\beta,\gamma}$ *are* $\Psi do$, $L$ *is elliptic. Assume that* $\alpha - \beta \geq s \geq \gamma$ *and* $\gamma > s - \frac{n}{p} > \alpha - \beta - n$.

*Then* $u \in W^{s,p+\varepsilon}\left(B_{1/2}\right)$ *for some* $\varepsilon = \varepsilon(\alpha, \beta, \gamma, n, p)$.

Despite the theorem allows to deduce the full regularity of many critical nonlinear problems, it is *not* an a-priori estimate. Namely, it is claimed that

$$\|u\|_{W^{s,p+\varepsilon}}\left(B_{1/2}\right) \leq C(u),$$

where $C$ depends on the profile of $u$, not just $\|u\|_{W^{1,p}}$. The meaning of the theorem is that in critical problems the regularity is improved without any assumptions on the structure of the nonlinearity, if the nonlinearity is in $L^p$ with $\infty > p > 1$.

## Analytic aspects of the Toda system

Guofang Wang

(joint work with J. Jost, C.-S. Lin)

In this talk, we address the analytic aspects of the 2-dimensional (open) Toda system (Toda lattice) for $SU(N+1)$

$$-\Delta u_i = \sum_{j=1}^{N} a_{ij} e^{u_j}, \tag{1}$$

for $i = 1, 2, \cdots, N$, where $K = (a_{ij})_{N\times N}$ is the Cartan matrix for $SU(N+1)$ given by

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

The Toda system is a very natural generalization of the Liouville equation

$$-\Delta u = 2e^u. \tag{2}$$

Both equations are completely integrable and important in integrable systems theory. The Liouville equation and the Toda system arise naturally from many mathematical problems and many physical models. In Chern-Simons theories, the Liouville equation is closely related to Abelian models, while the Toda system is related to non-Abelian models. See for instance the books [5] and [10] and the references therein.

We want to generalize all analytic results for the Liouville equation to the Toda system. However, many known results for the Liouville equation heavily depend on the maximum principle, which is not valid for the Toda system. On the other hand, solutions of the Toda stsyem are closely related to geometric objects, flat connections and holomorphic curves into $\mathbb{C}P^n$. With these geometric intepretations, we can overcome the mentioned difficulty-without maximum principle.

First of all, we established a Moser-Trudinger type inequality for the Toda system in [6].

**Thereom 1.** Let $\Sigma$ be a closed surface with area 1. Define a functional $\Phi^N : (H^1(\Sigma))^N \to \mathbb{R}$ by

$$\Phi_M(u) = \frac{1}{2}\sum_{i,j=1}^{N}\int_\Sigma a_{ij}(\nabla u_i \nabla u_j + 2M_i u_j) - \sum_{i=1}^{N} M_i \log\int_\Sigma \exp(\sum_{j=1}^{N} a_{ij}u_j).$$

Then, the functional has a lower bound if and only if

$$M_j \le 4\pi, \quad \text{for } j = 1, 2, \cdots, N. \tag{3}$$

Then, we classified all entire solutions of the Toda system with finite energy, which are obstruction of the compactness of the solutions. These are usually called “bubbles”.

**Thereom 2.** Any $C^2$ solution $u = (u_1, u_2, \cdots, u_N)$ of (1) in $\mathbb{R}^2$ with $\int_{\mathbb{R}^2} e^{u_j} < \infty$ for any $j = 1, 2, \cdots, N$ comes from a rational curve into $\mathbb{C}P^N$.

When $N = 1$, it is the classifaction result of Chen-Li in [1]. Any rational curve in $\mathbb{C}P^N$ can be transformed to

$$\phi_0(z) = [1, z, \cdots, \sqrt{\binom{N}{k}}z^k, \cdots, z^N],$$

by a holomorphic isometry, which is an element of $PSL(N+1,\mathbb{C})$. Hence the space of solutions of (1) with energy is equivalent to $PSL(N+1,\mathbb{C})/PSU(N+1)$. The dimension of the solution space is $N^2 + 2N$. Theorem 2 can be restated in a geometric way as follows:

**Thereom 3.** Any totally unramified holomorphic map $\phi$ from $\mathbb{C}$ to $\mathbb{C}P^N$ satisfying a finite energy condition can be compactified to a rational curve.

Very recently, we gave in [8] a more precise asymptotic behavior of "blow-up". For simplicity, we only consider the case $N = 2$. To describe it, we assume that there is a sequence of solutions $u^k = (u_1^k, u_2^k)$ of

$$\left\{ \begin{array}{rcl} -\Delta u_1^k &=& 2e^{u_1^k} - e^{u_2^k} \\ -\Delta u_2^k &=& 2e^{u_2^k} - e^{u_1^k} \end{array} \right. \quad \text{in } B_2. \tag{4}$$

Here $B_r$ denotes a disk of radius $r$ and center 0. Suppose that $u^k$ bubbles off, i.e., $\max_{x\in B_2}\{u_1^k, u_2^k\} \to \infty$ as $k \to \infty$. More precisely we assume that

(a) 0 is the only blow-up point of $u^k$.
(b) $\max_{\partial B_2} u_i^k - \min_{\partial B_2} u_i^k \le c$ for $i = 1, 2$.
(c) $\int_{B_2} e^{u_i^k} dx \le c$ for $i = 1, 2$ and any $k$.

Assume that $\lambda^k = \lambda_1^k := \max_{B_2} u_1^k \ge \max_{B_2} u_2^k =: \lambda_2^k$. Let $x_k$ be the maximum point of $u_1^k$. Set $v^k = (v_1^k, v_2^k)$ by

$$v_i^k(x) = u_i(\epsilon_k x + x^k) - \lambda^k \quad \text{for} \quad i = 1, 2,$$

where $\epsilon_k = e^{-\frac{1}{2}\lambda_k}$. Since we only consider the case that the bubble is a solution of the Toda system, we may further assume that

(d) $v_2^k(0)$ is bounded from below.

Then, there exists a solution $v^0 = (v_1^0, v_2^0)$ of the Toda system

$$\left\{ \begin{array}{l} -\Delta v_1^0 = 2e^{v_1^0} - e^{v_2^0}, \\ -\Delta v_2^0 = 2e^{v_2^0} - e^{v_1^0}, \end{array} \right. \quad \text{in } \mathbb{R}^2$$

such that $v_i^k - v_i^0$ converges to zero in $C^2_{loc}(\mathbb{R}^2)$.

Now we state our result.

**Thereom 4.** Let $u^k = (u_1^k, u_2^k)$ be a sequence of solutions to (4). Suppose that that $(a)$-$(d)$ hold. Then there exist two constants $r_0 > 0$ and $c > 0$ independent of $k$, such that

$$|u_i^k(x) - \lambda^k - v_i^0(\epsilon_k^{-1}(x - x^k))| < c \quad \text{in} \quad B_{r_0} \tag{5}$$

for $i = 1, 2$.

When $N = 1$, this was proved in [9] by using the method of moving planes. The Harnack inequality for solutions follows from Theorem 5. We also obtained various existence results in [8].

**Thereom 5.** Let $\Sigma$ be a Riemann surface with Gaussian curvature $K$ and $N = 2$. Suppose that

$$\min\{\Delta \log h_1(x), \Delta \log h_2(x)\} + 4\pi - 2K(x) > 0. \tag{6}$$

Then there is a solution $u = (u_1, u_2)$ of

$$\begin{aligned} -\Delta u_1 &= 2\rho_1 \left( \frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1}} - 1 \right) - \rho_2 \left( \frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2}} - 1 \right) \\ -\Delta u_2 &= 2\rho_2 \left( \frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2}} - 1 \right) - \rho_1 \left( \frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1}} - 1 \right), \end{aligned} \tag{7}$$

for $\rho_1 \in (0, 4\pi]$ and $\rho_2 \in (0, 4\pi]$.

When $N = 1$, this is a result of [4]. See also [2] and [3].

## References


[1] Chen, W. X and Li, C., *Classification of solutions of some nonlinear elliptic equations,* Duke Math. J., **63** (1991) 615–622

[2] Chen, C. C. and Lin, C.-S., *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces,* Comm. Pure Appl. Math., **55** (2002), 728–771.

[3] Chen, C. C. and Lin, C.-S., *Topological degree for a mean field equation on Riemann surfaces,* Comm. Pure Appl. Math. **56** (2003) 1667–1727.

[4] Ding, W., Jost, J., Li, J. and Wang, G., *The differential equation $\Delta u = 8\pi - 8\pi h e^u$ on a compact Riemann surface,* Asian J. Math. **1** (1997), 230–248.

[5] Dunne, G. *Self-dual Chern-Simons Theories.* Lecture Notes in Physics, Vol. **m36**, Springer-Verlag, Berlin, 1995.

[6] Jost, J.; Wang G. Analytic aspects of the Toda system: II. An application to the relativistic $SU(3)$ Chern-Simons model. Comm. Pure Appl. Math. **54** (2001) 1289–1319.

[7] Jost, J.; Wang G. Classification of solutions of a Toda system in $\mathbb{R}^2$. Int. Math. Res. Not., **2002** (2002), 277–290.

[8] Jost, J.; Lin, C.-S.; Wang G. Analytic Aspects of the Toda System: II. Bubbling behavior and existence of solutions, to appear in: Comm. Pure Appl. Math. (2005)

[9] Li, Y. Y., *Harnack type inequality: the method of moving planes,* Comm. Math. Phys. **200** (1999), 421–444.

[10] Yang, Y. Solitons in field theory and nonlinear analysis, Springer-Verlag, New York, 2001.


## Forth order approximation of harmonic maps from surfaces

Tobias Lamm

In a recent research project [4] we study a regularization of the Dirichlet energy for maps from a smooth and compact Riemannian surface $(M, g)$ into a smooth and compact Riemannian manifold $(N, h)$, which we assume to be isometrically embedded into some $\mathbb{R}^l$. More precisely we consider the functional

$$E_\varepsilon(u) := \int_M (|Du|^2 + \varepsilon |\Delta u|^2), \tag{1}$$

for every $\varepsilon > 0$, where $\Delta$ is the Laplace-Beltrami operator of $(M, g)$. One of the reasons why it is interesting to study $E_\varepsilon$ is that for every $\varepsilon > 0$, $E_\varepsilon$ satisfies the Palais-Smale condition and critical points of $E_\varepsilon$ are smooth. Because of these facts we study sequences $u_\varepsilon$ $(\varepsilon \to 0)$ of critical points of $E_\varepsilon$ with uniformly bounded energy. Due to this uniform bound we already know that there exists a subsequence $\varepsilon_k \to 0$ such that $u_{\varepsilon_k} \rightharpoonup u_0$ weakly in $W^{1,2}(M, N)$. After proving the so called $\delta_0$-estimates and using the result of Sacks & Uhlenbeck [11] on the removability of point singularities of harmonic maps with finite energy, we are actually able to show that $u_0$ is a smooth harmonic map. Because of the phenomenon of bubbling it is well known that the above mentioned weak convergence can not be improved to yield strong convergence in general. What we can show is that there exist at most finitely many points $x_1, \dots, x_p \in M$ such that $u_{\varepsilon_k} \to u_0$ in $C^m_{\text{loc}}(M \backslash \{x_1, \dots, x_p\}, N)$, for all $m \in \mathbb{N}$, and, by performing a blow-up at $x_i$, $i \in \{1, \dots, p\}$, we show that there exists a non-trivial harmonic two-sphere $\omega^i$, a sequence of points $x^i_k \in M$, $x^i_k \to x_i$, and a sequence of radii $t^i_k \in \mathbb{R}_+$, $t^i_k \to 0$, such that $u_{\varepsilon_k}(x^i_k + t^i_k \cdot) \to \omega^i$ in $C^m_{\text{loc}}(\mathbb{R}^2, N)$, where here and in the following we always identify $\omega^i : S^2 \to N$ and $\omega^i : \mathbb{R}^2 \to N$ with the help of the stereographic projection. By iterating this, one can actually find all non-trivial harmonic two-spheres which separate at $x_i$. The most difficult task, in this situation, is to show that there is no energy-loss during this process (this is the so called energy identity). This means that in the limit the energy $E_{\varepsilon_k}(u_{\varepsilon_k})$ is given as the sum of the Dirichlet energy of the weak limit $u_0$ and the Dirichlet energies of the non-trivial harmonic two-spheres, which we obtain by rescaling.

**Theorem 1.** *Let $(M^2, g)$ be a smooth, compact Riemannian surface without boundary and let $N = S^{l-1} \hookrightarrow \mathbb{R}^l$ be the standard sphere. Moreover let $u_\varepsilon \in C^\infty(M, N)$ ($\varepsilon \to 0$) be a sequence of critical points of $E_\varepsilon$ with uniformly bounded energy. Then there exists a sequence $\varepsilon_k \to 0$ and at most finitely many points $x_1, \dots, x_p \in M$ such that $u_{\varepsilon_k} \to u_0$ weakly in $W^{1,2}(M, N)$ and in $C^m_{\text{loc}}(M \backslash \{x_1, \dots, x_p\}, N)$, for all $m \in \mathbb{N}$, where $u_0 : M \to N$ is a smooth harmonic map.*
*By performing a blow-up at each $x_i$, $1 \le i \le p$, one gets that there exist at most finitely many non-trivial smooth harmonic maps $\omega^{i,j} : S^2 \to N$, $1 \le j \le j_i$, sequences of points $x^{i,j}_k \in M$, $x^{i,j}_k \to x_i$, and sequences of radii $t^{i,j}_k \in \mathbb{R}_+$, $t^{i,j}_k \to 0$, such that*

(2)
$$\max\{\frac{t^{i,j}_k}{t^{i,j'}_k}, \frac{t^{i,j'}_k}{t^{i,j}_k}, \frac{\text{dist}(x^{i,j}_k, x^{i,j'}_k)}{t^{i,j}_k + t^{i,j'}_k}\} \to \infty, \quad \forall \;\; 1 \le i \le p, \;\; 1 \le j, j' \le j_i, \;\; j \ne j',$$

$$\frac{\varepsilon_k}{(t^{i,j}_k)^2} \to 0 \quad \forall \;\; 1 \le i \le p, \;\; 1 \le j \le j_i, \tag{3}$$

$$\lim\nolimits_{k\to\infty} E_{\varepsilon_k}(u_{\varepsilon_k}) = E_0(u_0) + \sum_{i=1}^{p} \sum_{j=1}^{j_i} E_0(\omega^{i,j}) \tag{4}$$

*and*

$$\int_{B_{R_0}(x_i)} (|Dw^{j_i}_k|^2 + \varepsilon_k |\Delta w^{j_i}_k|^2) \to \int_{B_{R_0}(x_i)} |Du_0|^2, \quad \forall 1 \le i \le p, \tag{5}$$

*where $w^{j_i}_k = u_{\varepsilon_k} - \sum_{j=1}^{j_i} (\omega^{i,j}(\frac{\cdot - x^{i,j}_k}{t^{i,j}_k}) - \omega^{i,j}(\infty))$,*

$$0 < R_0 < \frac{1}{2} \min\{\text{inj}(M), \min\{\text{dist}(x_i, x_j) | 1 \le i \ne j \le p\}\}$$

*is some number and $\infty$ is identified with the north pole of $S^2$ by stereographic projection.*

In the existing literature there are quite a few different approaches for the regularization of the Dirichlet energy for maps from surfaces, or the approximation of harmonic maps from surfaces. Here we mention a few of them.
Sacks & Uhlenbeck [11] studied a $p$-harmonic approximation and used it to prove the existence of minimal immersions of two-spheres. The statement corresponding to (4) above, for the p-harmonic approximation was shown to be true for general target manifolds $N$ by Chen & Tian [1] if one consideres minimizing sequences. To our knowledge the case of an arbitrary sequence of critical points of the $p$-approximation is still open.
Struwe [12] considered the harmonic map heat flow (and also Palais-Smale sequences for the Dirichlet energy with tension field converging to zero in $L^2$) and proved (4) in this case with $\geq$ instead of $=$. Later, the energy identity was established by Qing [10], Ding & Tian [2], Wang [13] and Lin & Wang [7].
Jost [3] considered a mountain-pass sequence and proved the energy identity in this situation.
Parker [9] proved the energy identity for sequences of harmonic maps and showed that the energy identity is wrong for general Palais-Smale sequences.
Lin & Rivière [5], [6] proved the energy identity for sequences of stationary harmonic maps in higher dimensions.
Recently, Lin & Wang [8] studied a Ginzburg-Landau type approximation of harmonic maps and showed the energy identity and no-neck property for this approximation.

The main step in the proof of Theorem 1 consists of showing the energy identity in the presence of one bubble. This is done by using the special structure of the target and by deriving estimates in Lorentz spaces for sequences of critical points of the functional $E_\varepsilon$.

## References

[1] J. Chen and G. Tian. Compactification of moduli space of harmonic mappings. *Comm. Math. Helv.*, 74:201–237, 1999.
[2] W. Y. Ding and G. Tian. Energy identity for a class of approximate harmonic maps from surfaces. *Comm. Anal. Geom.*, 3:543–554, 1995.
[3] J. Jost. *Two-dimensional geometric variational problems.* John Wiley and Sons, Chichester, 1991.
[4] T. Lamm. Fourth order approximation of harmonic maps from surfaces. Preprint Universität Freiburg.
[5] F. Lin. Gradient estimates and blow-up analysis for stationary harmonic maps. *Ann. of Math.*, 149:785–829, 1999.
[6] F. Lin and T. Rivière. Energy quantization for harmonic maps. *Duke Math. J.*, 111:177–193, 2002.
[7] F. Lin and C. Wang. Energy identity of harmonic map flows from surfaces at finite singular time. *Calc. Var. Partial Differ. Equ.*, 6:369–380, 1998.
[8] F. Lin and C. Wang. Harmonic and quasi-harmonic spheres II. *Comm. Anal. Geom.*, 10:341–375, 2002.
[9] T. Parker. Bubble tree convergence for harmonic maps. *J. Differential Geom.*, 44:595–633, 1996.

[10] J. Qing. On singularities of the heat flow for harmonic maps from surfaces into spheres. *Comm. Anal. Geom.*, 3:297–315, 1995.
[11] J. Sacks and K. Uhlenbeck. The existence of minimal immersions of 2-spheres. *Annals of Math.*, 113:1–24, 1981.
[12] M. Struwe. On the evolution of harmonic mappings of Riemannian surfaces. *Comm. Math. Helv.*, 60:558–581, 1985.
[13] C. Wang. Bubble phenomena of certain Palais-Smale sequences from surfaces to general targets. *Houston J. Math.*, 22:559–590, 1996.

## Singular perturbation limit of phase separation problem

Yoshihiro Tonegawa

There are many physical models in materials sciences where two different phases (say, phase A and B) co-exist with some separating interface between them. Traditionally, interface is a hypersurface with the obvious notion of area, normal vector field, curvatures, etc. However, it is sometimes desirable to consider a "diffused interface", where the traditional interface is replaced by a region of small thickness. To describe it, one introduces a scalar function $u$ which is suitably smooth, and $u(x) \approx 1\,(\text{or} - 1)$ indicates position $x$ is occupied by phase A (alternatively B), and region where $u \approx 0$ may be considered as the diffused separating interface. The reason to adopt such formulation is partly practical: numerically, it is easier to handle scalar functions than parametrized hypersurfaces. Often, it is also easy to incorporate other interacting fields such as flow fields in this formulation. One can heuristically introduce the notion of 'normal vector field' and 'curvature' for the diffused interface as well. On the other hand, it is an interesting mathematical challenge to check that these heuristics are in fact correct. Here, we briefly report the recent advances on the understanding of the model when the surface tension is the dominant force for the separation of the two phases, or alternatively, when the dominant energy comes from the area of the interfaces.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary. For a scalar function $u$ defined on $\Omega$, define

$$E_\varepsilon(u) = \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon}\,\mathrm{d}x,$$

where $\varepsilon > 0$ is a small parameter. With a suitable side condition such as volume constraint $\int_\Omega u\,\mathrm{d}x = m$, minimize $E_\varepsilon$ among all $u$ satisfying the integral condition. One can establish the existence of a minimizer $u_\varepsilon$ by the direct method of minimization. One can expect that $u_\varepsilon$ has the feature that most of $u_\varepsilon(x)$ is close to $\pm 1$, and there is a transition region with thickness of $O(\varepsilon)$, and the energy $E_\varepsilon(u_\varepsilon)$ corresponds to the area of the transition. As $\varepsilon \to 0$, the transition of $u_\varepsilon$ is expected to approach to an area minimizing hypersurface with the given volume fraction constraint, so in particular, it should be a constant mean curvature hypersurface. This has been proved in the 80's in the framework of $\Gamma$-convergence as in [6, 11] and others. We cite the related two time-dependent problems, one is the Allen-Cahn (AC) equation

$$\varepsilon\frac{\partial u_\varepsilon}{\partial t} = \varepsilon\Delta u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon}$$

and the other is the Cahn-Hilliard (CH) equation

$$\frac{\partial u_\varepsilon}{\partial t} = \Delta(-\varepsilon\Delta u_\varepsilon + \frac{W'(u_\varepsilon)}{\varepsilon}).$$

With suitable initial and boundary condtions, one has the following energy equalities, where we write $v_\varepsilon = -\varepsilon\Delta u_\varepsilon + \frac{W'(u_\varepsilon)}{\varepsilon}$.

$$\frac{d}{dt}E_\varepsilon(u) = -\frac{1}{\varepsilon}\int (f_\varepsilon)^2 \text{ for AC}, \qquad = -\int |\nabla f_\varepsilon|^2 \text{ for CH.}$$

It is proved under various assumptions that interface regions of the Allen-Cahn equation converge to the mean curvature flow when we take the limit $\varepsilon \to 0$. In the setting of varifold, Ilmanen [5] proved that the limit interface is a mean curvature flow in the sense of Brakke [3]. The integrality of the limit is proved [12], so the Allen-Cahn limit has all the measure-theoretic properties satisfied by the solution constructed initially by Brakke. One may also consider the following questions, naturally motivated by the AC and CH as well as Geometric Measure Theory: given a sequence of functions $\{u_\varepsilon\}$ with uniform energy bounds $E_\varepsilon(u_\varepsilon) \leq C$ and either (1) $\frac{1}{\varepsilon}||f_\varepsilon||^2_{L^2} \leq C$, or (2) $||f_\varepsilon||_{W^{1,p}} \leq C$, characterize the (subsequential) limit measure $\lim_{\varepsilon\to 0} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon}\,\mathrm{d}x$. The condition (1) is motivated by AC and it is expected that the bound should serve as an $L^2$ mean curvature bound. The condtion (2) is motivated by CH, and is suitably interpreted as a Sobolev mean curvature bound. With (2) and $p > \frac{n}{2}$, I proved that the limit measure is an integral varifold with bounded $L^q$ mean cuvature, and the condition $p > \frac{n}{2}$ gives $q > n-1$ [4, 13, 14]. The related result for the sharp interface case is discussed in [9]. Under the assumption (1), recently (a) with the radially symmetric assumption [2] (b) with the monotone assumption in one direction and $n = 3$ [7] and finally (3) with no assumption $n = 3$ [8], it is proved that the limit interface is an integral varifold with bounded $L^2$ mean curvature. The important ingradient for the proof is to establish the monotonicity formula for the properly scaled energy, which is analogous to those appearing in the theory of varifold [1]. As an effort to understand the energy $E_\varepsilon$, I also studied the property of the limit measure of stable critical points [15] and showed that the limit measure is a stationary integral varifold with generalized $L^2$ second fundamental form $\mathcal{A}$. The stability of the measure is expressed in terms of $\mathcal{A}$, just like smooth minimal surfaces. Even though we do not know in general that the support of the limit measure is smooth, we also can show that the Schoen's inequality for stable minimal hypersurfaces hold in terms of $\mathcal{A}$. Such inequality was the essential tool for the regularity theory of stable minimal hypersurfaces [10]. For $n = 2$, the support of the limit measure is a disjoint union of lines. For $n = 3$, tangent cones at every points are single planes with possible multiplicities. It is an interesting open problem to show that the support of limit measure is smooth for $n \leq 7$, which we expect it to be true.

## References


[1] W. Allard, *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972), 417–491.
[2] G. Bellettini, L. Mugnai, *On the approximation of the elastica functional in radial symmetry*, Calc. Var. **24** (2005), 1–20.
[3] K. Brakke, *The motion of a surface by its mean curvature*, Princeton University Press, Princeton, N.J., (1978).
[4] J.E. Hutchinson, Y. Tonegawa, *Convergence of phase interfaces in the van der Waals - Cahn - Hilliard theory*, Calc. Var. **10** (2000), 49–84.
[5] T. Ilmanen, *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, J. Differential Geom. **38** (1993), 417–461.
[6] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Rational Mech. Anal. **98** (1987), 123–142.
[7] R. Moser, *A higher order asymptotic problem related to phase transitions*, preprint.
[8] M. Roger, R. Schätzle, *On a modified conjecture of De Giorgi*, in preparation.
[9] R. Schätzle, *Hypersurfaces with mean curvature given by an ambient Sobolev function*, J. Diff. Geom. **58** (2001), 371–420.
[10] R. Schoen, L. Simon, *Regularity of stable minimal hypersurfaces*, Comm. Pure Appl. Math. **34** (1981), 741–797.
[11] Sternberg, P. *The effect of a singular perturbation on nonconvex variational problems*, Arch. Rational Mech. Anal. **101** (1988), 209–260.
[12] Y. Tonegawa, *Integrality of varifolds in the singular limit of reaction-diffusion equations*, Hiroshima Math. J. **33** (2003), 323–341.
[13] Y. Tonegawa, *Phase field model with a variable chemical potential*, Pro. Royal Soc. Edinburgh, **132A** (2002), 993–1019.
[14] Y. Tonegawa, *Diffused interface with the chemical potential in the Sobolev space*, to appear in Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5).
[15] Y. Tonegawa, *On stable critical points for a singular perturbation problem*, Comm. Analysis and Geometry **13** (2005), 439–459.


## Two phase free boundary regularity problem for Harmonic measure and Poisson kernel


Tatiana Toro

(joint work with Carlos Kenig)


Several approaches have been used to study 2-phase free boundary regularity problems. Motivated by applications to the flow of liquids in models of jets and cavities, Alt, Caffarelli and Friedman consider the functional

$$J(v) = \int_B \left( |\nabla v|^2 + Q_+(x)\chi_{\{v>0\}} + Q_-(x)\chi_{\{v<0\}} \right) dx$$

where $B$ is an open set in $\mathbb{R}^n$, $Q_\pm$ are smooth functions in $B$ such that $Q_+ - Q_- \geq c > 0$, and $v \in \mathcal{K}$ and

$$\mathcal{K} = \{v \in L^1_{loc}(B), \nabla v \in L^2(B), v = u_0 \in \partial B\} \text{ where } u_0 \in L^1_{loc}(B), \nabla u_0 \in L^2(B).$$

**Theorem 1.** [1] *There exists $u \in \mathcal{K}$ such that $J(u) = \inf_{v \in K} J(v)$. The function $u$ satisfies*

$$\Delta u = 0 \text{ in } B \cap \partial\{u > 0\} \text{ and } (u_\nu^+)^2 - (u_\nu^-)^2 = Q_x - Q_- \quad \text{on } \partial\{u > 0\}.$$

*Moreover*

- *If $n = 2$ the free boundary $\partial\{u > 0\}$ is smooth.*
- *If $n \geq 3$ there exists a closed set $S \subset \partial\{u > 0\}$ such that $\mathcal{H}^{n-1}(S) = 0$ and $\partial\{u > 0\}\backslash S$ is smooth.*

The main tool in the proof of this theorem is the following monotonicity formula. Let $x_0 \in \partial\{u > 0\}$ and $r_0 > 0$ such $B(x_0, r_0) \subset B$ then for $0 < r < r_0$ the quantity

$$\frac{1}{r^4}\int_{B(x_0,r)} \frac{|\nabla u^+|}{|x - x_0|^{n-2}}\,dx \int_{B(x_0,r)} \frac{|\nabla u^-|}{|x - x_0|^{n-2}}\,dx$$

increases with $r$.

**Theorem 2.** [6] *If $n = 3$ and $u$ is as above then $\partial\{u > 0\}$ is smooth.*

**Theorem 3.** [11] *If $n \geq 4$, and $u$ is as above then the reduced boundary $\partial^*\{u > 0\}$ is smooth and $\Sigma = \partial\{u > 0\}\backslash\partial^*\{u > 0\}$ has Hausdorff dimension at most $n - 3$.*

The main additional tool in the proof of the last 2 theorems was Weiss' monotonicity formula (see [10], [11]) which states that for $0 < s < r < r_0$ and $x_0$ as above if $Q_\pm = \lambda_\pm$ where $\lambda_\pm$ are constants and if

$$\phi(r) = \frac{1}{r^n}\int_{B(x_0,r)} \left(|\nabla u|^2 + \lambda_+\chi_{\{v>0\}} + \lambda_-\chi_{\{v<0\}}\right)\,dx - \frac{1}{r^{n+1}}\int_{\partial B(x_0,r)} u^2$$

then

$$\phi(r) - \phi(s) = \int_s^r t^{-n}\int_{\partial B(x_0,r)} 2\left(\nabla u \cdot \frac{x - x_0}{|x - x_0|} - \frac{u}{r}\right)^2 d\mathcal{H}^{n-1}dt.$$

This monotonicity formula yields that the blow up limits of minimizers are homogeneous functions of degree 1.

There exist similarities between this problem and the area minimizing problem in codimension one. Hence a big open question in this area is whether the singular set $\Sigma$ above has Hausdorff dimension at most $n - 7$.

The free boundary regularity problem for harmonic measure and Poisson kernel lies between this problem and the one addressed by Bishop, Carleson, Garnett, Jones and Makarov in a series of papers in the late eighties (see [3], [4], [5]). They proved that for a domain $\Omega \subset \mathbb{R}^2$ if $\omega_+$ (resp. $\omega_-$) denotes the harmonic measure of $\Omega$ (resp. $\Omega^c$) with fixed pole then if $\omega_+$ and $\omega_-$ are mutually absolutely continuous then $\partial\Omega$ contains a Lipschitz piece.

Along these lines we prove the following results (see [8]).

**Theorem 4.** *Let $\Omega \subset \mathbb{R}^n$ be a $\delta$-Reifenberg flat domain for $\delta > 0$ small enough depending only on $n$. Let $\omega_\pm$ be as above. Assume that $h = \frac{d\omega_-}{d\omega_+}$ satisfies $\log h \in \mathrm{VMO}(d\omega_+)$. Then $\Omega$ is a Reifenberg flat domain with vanishing constant. In particular if $\mathcal{H}^n(\partial\Omega) < \infty$ then $\partial\Omega$ is rectifiable.*

**Theorem 5.** *Let $\Omega \subset \mathbb{R}^n$ be a 2-sided chord arc domain. Assume that $\log h_\pm \in \mathrm{VMO}(d\sigma)$, where $h_\pm = \frac{d\omega_\pm}{d\sigma}$. Then $\Omega$ is locally Reifenberg flat with vanishing constant. Furthermore $\overrightarrow{n} \in \mathrm{VMO}(d\sigma)$, where $\overrightarrow{n}$ denotes the measure theoretic normal to $\partial\Omega$.*

**Corollary 1.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded 2-sided chord arc domain. Assume that $\log h_+ \in C^{k,\alpha}(\partial\Omega)$ for some $k \geq 0$ and $\log h_- \in \mathrm{VMO}(d\sigma)$, then $\Omega$ is a $C^{k+1,\alpha}$ domain.*

Under the above hypothesis $\log h_+ \in \mathrm{VMO}(d\sigma)$ and by Theorem 5, $\Omega$ is locally Reifenberg flat with vanishing constant. Hence by Alt and Caffarelli's result in the case $k = 0$ or by Kinderlehrer and Nirenberg's work in the case $k \geq 1$ we conclude that $\Omega$ is a $C^{k+1,\beta}$ domain (see [2] and [9]). In fact we can take $\beta = \alpha$ by [7].

### References


[1] W. Alt, L. Caffarelli & A. Friedman, *Variational problems with two phases and their free boundaries*, Trans. Amer. Math. Society **282** (1984), 431–461.

[2] W. Alt & L. Caffarelli, *Existence and Regularity for a minimum problem with free boundary*, J. Reine Angew. Math. **325** (1981), 105–144.

[3] C. Bishop, *A characterization of Poissonian domains*, Ark. Mat. **29** (1991), 1–24.

[4] C. Bishop, L. Carleson, J. Garnett & P. Jones, *Harmonic measure supported on curves*, Pacific J. Math **138** (1989), 233–236.

[5] C. Bishop & P. Jones, *Harmonic measure and arc length*, Ann. of Math. **132** (1990), 511–547.

[6] L. Caffarelli, D. Jerison & C.Kenig *Global energy minimizers for free boundary problems and full regularity in three dimensions*, Noncompact problems at the intersection of geometry, analysis, and topology, Contemp. Math. Amer. Math. Soc. **350** (2004), 83–97.

[7] D. Jerison, *Regularity of the Poisson Kernel and Free Boundary Problems*, Colloquium Mathematicum, **60–61**, (1990), 547–567.

[8] C. Kenig & T. Toro *Free boundary regularity below the continuous threshold: 2-phase problems* to appear in J. Reine Angew. Math.

[9] D. Kinderlehrer & L. Nirenberg, *Regularity in free boundary problems*, Ann. Scuola Norm. Sup. Pisa **4** (1977), 373-391.

[10] G. Weiss *Partial regularity for the minimum problem with free boundary*, Journal of Geom. Anal. **9**,(1999), 317–326.

[11] G. Weiss *Partial regularity for weak solutions of an elliptic free boundary problem*, Comm. Partial Diff. Equations. **23**,(1998), 439–455.


## On a modified conjecture of De Giorgi


Matthias Röger

(joint work with Reiner Schätzle)


We study the $\Gamma$-convergence of functionals arising in the Van der Waals-Cahn-Hilliard theory. The corresponding limit functional is given as the sum of the area and the Willmore functional. The problem under investigation was proposed as modification of a conjecture of De Giorgi and partial results were obtained by several authors. We prove here the modified conjecture in dimensions $n = 2, 3$.

Let a set $\Omega \subset \mathbb{R}^n$, a standard double well potential $W(t) := (1 - t^2)^2$ be given and define for $\varepsilon > 0$ functionals $\mathcal{F}_\varepsilon : L^1(\Omega) \to \mathbb{R}$,

$$\mathcal{F}_\varepsilon(u) := \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\Big) d\mathcal{L}^n + \int_\Omega \frac{1}{\varepsilon}\Big( - \varepsilon\Delta u + \frac{1}{\varepsilon}W'(u)\Big)^2 d\mathcal{L}^n$$

if $u \in L^1(\Omega) \cap W^{2,2}(\Omega)$ and $\mathcal{F}_\varepsilon(u) := \infty$ if $u \in L^1(\Omega) \setminus W^{2,2}(\Omega)$.
Further put $\sigma := \int_{-1}^{1} \sqrt{2W}$ , and for $\mathcal{X} = 2\mathcal{X}_E - 1$ with $E \subseteq \Omega$ and $\partial E \cap \Omega \in C^2$ define

$$\mathcal{F}(\mathcal{X}) := \sigma \mathcal{H}^{n-1}(\partial E \cap \Omega) + \sigma \int_{\partial E \cap \Omega} |\vec{\mathbf{H}}_{\partial E}|^2 d\mathcal{H}^{n-1}.$$

Our goal is to prove in small space dimensions a modification of a conjecture of De Giorgi (see [4]), as stated in the following theorem.

**Theorem 1** (Modified De Giorgi Conjecture)**.** *Let $n = 2, 3$. For any $\mathcal{X} = 2\mathcal{X}_E - 1$ with $E \subset \Omega$, $\partial E \cap \Omega \in C^2$,*

$$\Gamma(L^1(\Omega)) - \lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\mathcal{X}) \quad = \quad \mathcal{F}(\mathcal{X})$$

*holds.*

Compared to the original conjecture of De Giorgi the structure of the approximate functionals $\mathcal{F}_\varepsilon$ is different in the choice of the double well potential and, more importantly, in the the second term of $\mathcal{F}_\varepsilon$, where instead of the 'energy density' $\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)$ the factor $\frac{1}{\varepsilon}$ appears.

The $\Gamma$-convergence of the first part of the functionals $\mathcal{F}_\varepsilon$ to the first term of $\mathcal{F}$, which is basically the area functional, was already proved by Modica and Mortola [7] see also [8]. The second part of $\mathcal{F}$ is up to a constant identical to the Willmore functional.

The modified De Giorgi conjecture as stated above has attracted much attention over the years. Bellettini and Paolini [2] (see also [1]) have proved the *limsup-estimate* necessary for the Gamma-convergence. Loreti and March considered in [6] the gradient flows corresponding to the functionals $\mathcal{F}_\varepsilon$, $\mathcal{F}$ and proved the convergence as $\varepsilon \to 0$ by formal asymptotic expansions.

The *liminf-estimate* turns out to be the difficult part in the proof of the Modified De Giorgi Conjecture and only recently partial results were obtained. In [1] Bellettini and Mugnai proved the Gamma-convergence for rotationally symmetric data in $\mathbb{R}^2$ and Moser proved in [9] the liminf-estimate in three space dimensions if the data are monotone in one direction. The lower semi-continuity of $\mathcal{F}$, which is a necessary condition for $\mathcal{F}$ being a $\Gamma$-limit, follows from a result of Schätzle in [10], where the lower semi-continuity of the Willmore functional under weak convergence of currents is shown.

To prove the liminf estimate in space dimensions $n = 2, 3$ for general data we combine the approach of Hutchinson and Tonegawa in [5], [11] with arguments used by Chen in [3]. We consider

$$\mathcal{X} = 2\mathcal{X}_E - 1 \quad \text{with } \partial E \cap \Omega \text{ in } C^2,$$
$$u_\varepsilon \in W^{2,2}(\Omega), \qquad u_\varepsilon \to \mathcal{X} \quad \text{in } L^1(\Omega)$$

and define *energy measures* $\mu_\varepsilon$ and *discrepancy measures* $\xi_\varepsilon$,

$$\mu_\varepsilon := \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big)\mathcal{L}^n, \qquad \xi_\varepsilon := \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}W(u_\varepsilon)\Big)\mathcal{L}^n.$$

The main step in the proof of lower semi-continuity is to deduce that $\mu_\varepsilon$ converges to a rectifiable varifold $\mu$ with $\sigma^{-1}\mu$ having integral multiplicity and with weak mean curvature $\vec{H}_\mu \in L^2(\mu)$ satisfying the liminf estimate

$$\int_\Omega |\vec{H}_\mu|^2 \, d\mu \;\leq\; \liminf_{\varepsilon\to 0} \int_\Omega \frac{1}{\varepsilon}\Big( -\varepsilon\Delta u + \frac{1}{\varepsilon}W'(u)\Big)^2 d\mathcal{L}^n.$$

The major challenge in this part is the control of the discrepancy measures, which is much more delicate here as in [5], [11] and which requires a careful analysis and some additional arguments.

From a result in [10] relating the mean curvature of the limit varifold to the local geometry given by $\partial E$ it follows that

$$\vec{H}_{\partial E} \quad = \quad \vec{H}_\mu$$

holds $\mathcal{H}^{n-1}$-almost everywhere on $\partial E$. In addition from [7] we obtain

$$\mathcal{H}^{n-1}\lfloor \partial E \leq \sigma^{-1}\mu$$

and we arrive at the *liminf estimate*

$$\mathcal{F}(\mathcal{X}) \quad \leq \quad \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon).$$

## References


[1] G. Bellettini, L. Mugnai, *On the approximation of the elastica functional in radial symmetry*, Calculus of Variations and Partial Differential Equations, **online first** (2004).

[2] G. Bellettini, M. Paolini, *Approssimazione variazionale di funzionali con curvatura*, Seminario di Analisi Matematica, Dipartimento di Matematica dell'Università di Bologna, Tecnoprint Bologna, (1993) pp. 87-97.

[3] Chen, X., *Global asymptotic limit of solutions of the Cahn-Hilliard equation*, Journal of Differential Geometry, **44**, No. 2, (1996) pp. 262-311.

[4] E. de Giorgi, *Some remarks on* $\Gamma-$*convergence and least squares methods*, Composite Media and Homogenization Theory (G. Dal Maso and G.F. Dell'Antonio, eds.), Progress in Nonlinear Differential Equations and their Applications, Vol. 5, Birkhäuser, (1991) pp. 135-142.

[5] J. Hutchinson, Y. Tonegawa, *Convergence of Phase Interfaces in the van der Waals-Cahn-Hilliard Theory*, Calculus of Variations and Partial Differential Equations, **10**, no. 1, (2000) pp. 49-84.

[6] P. Loreti, R. March, *Propagation of fronts in a nonlinear fourth order equation*, European Journal of Applied Mathematics, **11**, (2000) pp. 203-213.

[7] L. Modica, S. Mortola, *Un esempio di* $\Gamma$*-convergenza*, Bollettino della Unione Matematica Italiana, **14B**, (1977) pp. 285-299.

[8] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion*, Archive for Rational Mechanics and Analysis, **98**, no. 2, (1987) pp. 123-142.

[9] R. Moser, *A higher order asymptotic problem related to phase transitions*, preprint (2004).

[10] R. Schätzle, *Lower semicontinuity of the Willmore functional for currents*, (2004) submitted.

[11] Y. Tonegawa, *Phase field model with a variable chemical potential*, Proceedings of the Royal Society of Edinburgh, Sect. A, **132**, no. 4, (2002) pp. 993-1019.

**Multiplicity results for the prescribed $Q$-curvature**

Mohameden Ould Ahmedou

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On a riemannian 4-manifold, $(M, g)$, the **Paneitz-Operator** is defined by:

$$P_g^4\varphi = \Delta_g^2\varphi - div_g\left(\frac{2}{3}R_g g - 2Ric_g\right)d\varphi,$$

where $R_g$ and $Ric_g$ resp. the sclar and Ricci curvature.
This operator enjoys many nice properties, in particular it is conformally invariant that is

$$P_{g_w} = e^{-4w}P_g \quad \text{for } g_w = e^{2w}g.$$

It is a natural extension of the Laplace-Beltrami Operator on surfaces and gives rise to a fourth order conformal invariant: the so-called $Q$**-Curvature** defined by:

$$Q_g = \frac{1}{12}(-\Delta_g R_g + R_g^2 - 3\,|Ric_g|) \quad .$$

The $Q$-curvature transforms in a very nice way under conformal change of metric, namely if $g' = e^{2w}g$, then $Q_g$ and $Q_{g'}$ are related by the following relation:

$$P_g^4 w + 2Q_g = 2Q_{g'}e^{4w}2Q_g \quad .$$

The Paneitz-Operator for manifolds of dimension $\geq 4$, has been obtained by T. Branson, it is defined defined by:

$$P_g^n u = \Delta_g^2 u - div_g\left(a_n R_g g + b_n Ric_g\right)du + \frac{n-4}{2}Q_g^n u,$$

where $Q_g^n = \frac{-1}{2(n-1)}\Delta_g R_g + c_n R_g^2 - \frac{2}{(n-2)^2}\left|Ric_g\right|^2$.
The Paneitz operator has been subject of intensive studies, see please the lectures notes of A. Chang [Chang 05].
The Paneitz operator $P_g^n, n \geq 5$ is also conformally invariant, indeed if $\tilde{g} = \varphi^{\frac{4}{n-4}}g$ then for all $\psi \in C^\infty(M)$ there holds:

$$P_g^n(\psi\varphi) = \varphi^{\frac{n+4}{n-4}}P_{\tilde{g}}^n(\psi).$$

In particular for $\psi = 1$ we have:

$$P_g^n(\varphi) = \frac{n-4}{2}Q_{\tilde{g}}^n\varphi^{\frac{n+4}{n-4}}. \tag{1}$$

In view of equation (1) a natural question arises: Can we prescribe the conformal invariant $Q$? That is for a given function $K$ defined on $(M, g)$, does there exist a metric $\tilde{g} \in [g]$ with $Q_{\tilde{g}} = K$?
Setting $\tilde{g} := u^{\frac{4}{n-4}}g$ with $u > 0$ the problem is equivalent to solving the following equation:

$$P_g^n(\varphi) = \frac{n-4}{2}K\varphi^{\frac{n+4}{n-4}} \tag{2}$$

The equation (2) has a variational structure, the solutions are critical points of the Euler-Lagrange functional. The space of variation is the Sobolev space $H^2(M)$. However due to the noncompactness of the embedding $H^2(M)$ to $L^{\frac{2n}{n-4}}(M)$, the Euler functional does not satisfy the "Palais-Smale condition" which leads to the failure of the classical existence mechanisms of the variational theory.
In this talk we consider the case where the manifold $M$ is the standard sphere $(S^n, g_c)$, where due to Kazdan-Warner type obstructions, the above question amounts to find conditions on $K$ to insure existence of solution to the above equation.
In previous works in collaboration with Z. Djadli and A. Malchiodi, [Djadli-Malchiodi-OuldAhmedou1], [Djadli-Malchiodi-OuldAhmedou2], we gave an Euler-Hopf type criterium to insure existence of solutions on 5 and 6 dimensional spheres and in this talk we report on new progess concerning the multiplicity of solutions. More precisely we adress the question of providing a lower bound on the numbers of metrics with prescribed $Q$-curvature. Such a lower bound has to be found in connection with the existence of *critical point at infinity* for the associated Euler Lagrange functional. Actually the existence of such *noncompact ends* explains the main difficulty in dealing with the problem, however it turns out that such noncompact orbits of the gradient can be treated as usual critical point once a *Morse Lemma at infinity* is performed as A. Bahri and J.M. Coron [Bahri-Coron 1991] did for the scalar curvature problem. In particular their topological contribution to the level sets of the functional can be computed. In this work we prove that, under generic conditions conditions on $K$ that this *topology at infinity* is a lower bound for the number of metrics in the conformal class of $g_c$ having prescribed $Q$-curvature. Such an inequality between the number of solutions and the topology induced by the noncompacts orbits of the gradient flow can be seen as a generalization of the well known *Morse inequalities* to this noncompact framework.

## References

[Bahri-Coron 1991] A. Bahri and J. M. Coron, The scalar curvature problem on the standard three dimensional spheres *J. Funct. Anal.* **95** (1991), 106-172.

[Chang 05] Sun-Yung A. Chang, Nonlinear elliptic equations in conformal geometry, *Nachdiplom lectures in Mathematics, ETH, Zurich*, Springer-Birkhäuser (2005).

[Djadli-Malchiodi-OuldAhmedou1] Z. Djadli, A. Malchiodi and M. Ould Ahmedou Prescribing a fourth order conformal invariant on the standard sphere, Part II : blow up analysis and applications, *Ann. Scuola Norm. Sup. Pisa cl. Sci.* **5** (2002), 387-434

[Djadli-Malchiodi-OuldAhmedou2] Z. Djadli, A. Malchiodi and M. Ould Ahmedou Prescribing a fourth order conformal invariant on the standard sphere, Part I : a perturbation result, *Communications in Contemporary mathematics* **4** (2002), p. 375-408.

**Convergence of equilibria of thin elastic films**

Stefan Müller
(joint work with Maximilian Schultz)

The relation between three-dimensional nonlinear elasticity and theories for lower-dimensional objects such as rods, beams, membranes, plates and shells has been an outstanding question since the very beginning of the research in elasticity. In fact there is a large variety of lower-dimensional theories. They are usually obtained by making certain (strong) apriori assumptions on the form of the solutions of the full three-dimensional problem and hence their rigorous range of validity is typically unclear. As highlighted already in the work of Fritz John, a key point is the geometric nonlinearity in elasticity, i.e the invariance of the elastic energy under rotations. In particular thin elastic objects can undergo large rotations even under small loads and this prevents any analysis based on a naive linearization.

The first rigorous results were only obtained in the early 90's using a variational approach which guarantees convergence of minimizers to a suitable limit problem. In this talk we discuss the convergence of (possibly non-minimizing) stationary points of the elastic energy functional. To set the stage let us first review the variational setting. Consider a cylindrical domain $\Omega_h = S \times (-h/2, h/2)$, where $S$ is a bounded subset of $\mathbb{R}^2$ with Lipschitz boundary. To a deformation $v : \Omega_h \to \mathbb{R}^3$ we associate the elastic energy (per unit height)

$$E^h = \frac{1}{h} \int_{\Omega_h} W(\nabla v)\, dz. \tag{1}$$

We assume that the stored-energy density function $W$ satisfies the following conditions:

$$W(RF) = W(F) \quad \forall R \in SO(3) \qquad \text{(frame indifference)}, \tag{2}$$

$$W = 0 \quad \text{on } SO(3), \tag{3}$$

$$W(F) \geq c \operatorname{dist}^2(F, SO(3)), \quad c > 0, \tag{4}$$

$$W \text{ is } C^2 \text{ in a neighbourhood of } SO(3). \tag{5}$$

Here $SO(3)$ denotes the group of proper rotations. The frame indifference implies that there exists a function $\tilde{W}$ defined on symmetric matrices such that $W(\nabla v) = \tilde{W}((\nabla v)^T \nabla v)$, i.e. the elastic energy depends only on the pull-back metric of $v$.

To discuss the limiting behaviour as $h \to 0$ it is convenient to rescale to a fixed domain $\Omega = S \times (-1/2, 1/2)$ by the change of variables $x = (z_1, z_2, hz_3)$ and $y(x) = v(z)$. With the notation

$$\nabla_h y = (\partial_1 y, \partial_2 y, \frac{1}{h}\partial_3 y) = (\nabla' y, \frac{1}{h}\partial_3 y) \tag{6}$$

we have

$$E^h(v) = I^h(y) = \int_\Omega W(\nabla_h y)\, dx. \tag{7}$$

The variational approach leads to a hierarchy of limiting theories depending on the scaling of $I^h$. More precisely we have for $h \to 0$ in the sense of Gamma-convergence

$$\frac{1}{h^\beta} I^h \xrightarrow{\Gamma} I_\beta. \tag{8}$$

This implies, roughly speaking, that minimizers of $I^h$ (subject to suitable boundary conditions or body forces) converge to minimizers of $I_\beta$, provided $I^h$ evaluated on the minimizers is bounded by $Ch^\beta$. Gamma-convergence was first established by LeDret and Raoult for $\beta = 0$ [5], then for all $\beta \geq 2$ in [3, 4] (see also [8, 9] for results for $\beta = 2$ under additional conditions). For $0 < \beta < 5/3$ convergence was recently obtained by Conti and Maggi [2], see also [1]. The exponent $\beta = 5/3$ is conjectured to be relevant for the crumpling of elastic sheets [6, 10, 2].

Here we focus on the case $\beta = 2$ which leads to Kirchhoff's geometrically nonlinear bending theory. For the limit problem the natural class of admissible functions is given by $W^{2,2}$ isometric immersions from $S$ to $\mathbb{R}^3$, i.e.,

$$\mathcal{A} := \left\{ y \in W^{2,2}(\Omega, \mathbb{R}^3) : \partial_3 y = 0, (\nabla' y)^T \nabla' y = Id \right\}. \tag{9}$$

The limiting energy functional is

$$I_2(\bar{y}) = \begin{cases} \frac{1}{24} \int_S Q_2(A)\, dx_1 dx_2, & \text{if } y \in \mathcal{A}, \\ +\infty, & \text{else.} \end{cases} \tag{10}$$

Here $A$ is the second fundamental form and $Q_2$ is a quadratic form which can be computed from the linearization $\partial^2 W/\partial^2 F(Id)$ of the 3d energy at the identity. If $W = \frac{1}{2}\text{dist}^2(F, SO(3))$ then simply $Q_2(A) = |A|^2$.

In this talk we consider convergence of equilibria for the case $\beta = 2$. Instead of considering the reduction from 3d to 2d we focus on the simpler limit from 2d to 1d. Thus we start from a thin strip

$$\Omega_h = (0, L) \times (-h/2, h/2) \tag{11}$$

and after the rescaling $(z_1, z_2) = (x_1, hx_2)$, $\nabla_h = (\partial_1, \frac{1}{h}\partial_2)$ we consider the functional

$$J^h(y) = \int_\Omega W(\nabla_h y) - h^2 g(x_1) \cdot y\, dx. \tag{12}$$

The corresponding Gamma-limit is given by

$$J_2(\bar{y}) = \int_0^L \frac{1}{24} E\kappa^2 - g \cdot \bar{y}\, dx_1, \tag{13}$$

where

$$\bar{y} : (0, L) \to \mathbb{R}^2, \quad \bar{y}' = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \kappa = \theta', \tag{14}$$

and where $J_2$ takes the value $+\infty$ if $\bar{y}$ is not of the above form (here we took the liberty to identify maps on $\Omega$ which are independent of $y_2$ with maps on $(0, L)$). It is convenient to fix one endpoint by requiring $\bar{y}(0) = 0$. Integrating the linear term by parts we easily see that the Euler-Lagrange equation corresponding to the limit functional is given by

$$-\frac{1}{12}E\theta'' + \tilde{g}\cdot\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} = 0, \quad \tilde{g}(x_1) = \int_L^{x_1} g(\xi)\,d\xi. \tag{15}$$

**Theorem 1.** *Assume that (2 – 5) hold, that the energy $W$ is differentiable and the derivative $DW$ is globally Lipschitz. Let $y^{(h)}$ be a sequence of stationary points of $J^h$ (subject to the boundary condition $y^{(h)}(0,x_2) = (0,hx_2)$ at $x_1 = 0$ and to natural boundary conditions on the remaining boundaries). Assume that*

$$\int_\Omega W(\nabla_h y^{(h)}) \le Ch^2. \tag{16}$$

*Then*

$$y^{(h)} \to \bar{y} \quad \text{in } W^{1,2}(\Omega;\mathbb{R}^2), \tag{17}$$

$$\partial_2\bar{y} = 0, \quad \partial_1\bar{y} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \tag{18}$$

*and $\theta$ solves (15).*

Remarks. 1. An easy application of the Poincaré inequality shows that the estimate (16) holds automatically for minimizers.
2. Mielke [7] used a centre manifold approach to compare solutions in a thin strip to a 1d problem. His approach gives a comparison already for finite $h$, but it requires that the nonlinear strain $(\nabla_h y)^T\nabla_h y$ is close to the identity in $L^\infty$ (and applied forces $g$ cannot be easily included).

The proof uses in particular the quantitative rigidity estimate in [3] and a compensated compactness argument.

## References


[1] S. Conti *Habilitation thesis*, University of Leipzig, 2003.

[2] S. Conti and F. Maggi, in preparation.

[3] G. Friesecke, R.D. James and S. Müller *A theorem on geometric rigidity and the derivation of nonlinear plate theory from three dimensional elasticity*, Comm. Pure Appl. Math. **55** (2002), 1461–1506.

[4] G. Friesecke, R.D. James and S. Müller, *A hierarchy of plate models, derived from nonlinear elasticity by Gamma-convergence*, Preprint MPI-MIS **7/ 2005**, to appear in: Arch. Ration. Mech. Anal.

[5] H. LeDret and A. Raoult, *The nonlinear membrane model as a variational limit of nonlinear three-dimensional elasticity*, J. Math. Pure Appl. **73** (1995), 549–578.

[6] A.E. Lobkovsky, S. Gentges, Hao Li, D. Morse and T. Witten, *Scaling properties of stretching ridges in a crumpled elastic sheet*, Science **270** (1995), 1482–1485.

[7] A. Mielke *On Saint Venant's problem for an elastic strip*, Proc. Roy. Soc. Edinburgh Sect. A **110** (1988), 161–181.

[8] O. Pantz, *Une justification partielle du modèle de plaque en flexion par Γ-convergence*, C.R. Acad. Sci. Paris Sér. I Math. **332** (2001), 587–592.

[9] O. Pantz, *On the justification of the nonlinear inextensional plate model*, Arch. Ration. Mech. Anal. **167** (2003), 179–209.

[10] S.C. Venkataramani, *Lower bounds for the energy in a crumpled elastic sheet- a minimal ridge*, Nonlinearity **17** (2004), 301–312.

## Beltrami Systems and the Hilbert-Smith Conjecture

Gaven J. Martin

### 1. Introduction

We show how uniqueness for solutions to the $n$–dimensional Beltrami systems (the governing equations for the theories of non-linear elasticity and conformal geometry) is obtained from our solution to the Hilbert-Smith conjecture on topological transformation groups in the uniformly elliptic setting.

To being with consider two material bodies $\Omega$ and $\Omega'$. Each has a microstructure. Such microstructures define maps ($G$ on $\Omega$ and $H$ on $\Omega'$) from the domain into $S(n)$, the space of $n \times n$ positive definite matrices of determinant equal to 1. Thus at each point of $\Omega$ and $\Omega'$ we have an oriented lattice structure, but no canonical scale. The mappings $G$ and $H$ are only assumed bounded and measurable. Bounded means there are no infinitely thin microcrystals and gives us ellipticity. In most interesting applications the microstructure will not be continuous as it will jump along some interface and these interfaces may be of fractal nature and occur at all scales.

We consider the problem of deforming the elastic body $(\Omega, G)$ into $(\Omega, H)$. Such a deformation will be an orientation preserving homeomorphism (by the principle of interpenatrability of matter) $f : \Omega \to \Omega'$ taking the microstructure of $\Omega$ to that of $\Omega'$. The first order PDEs describing this are the Beltrami systems,

$$Df^t(x)H(f(x))Df(x) = J(x,f)^{2/n}G(x), \qquad \text{a.e. } \Omega \tag{1}$$

Here $Df(x)$ is the differential matrix of $f$ and $J(x,f)$ the Jacobian determinant. These equations, although first order, are highly nonlinear and with only measurable coefficients. To fix ideas, we study solutions only in the natural Sobolev class $W^{1,n}_{loc}(\Omega,\Omega')$ of functions whose first order derivatives are locally $L^n$ integrable.

Surprisingly, in two dimensions, when written in complex notation, the equations (1) are equivalent to the linear Beltrami equation,

$$f_{\overline{z}}(z) = \mu(z)f_z(z), \qquad \text{a.e. } \Omega \tag{2}$$

for some bounded measurable Beltrami coefficient $\mu$ with $\|\mu\|_\infty = k < 1$. This equation is intimately connected with conformal geometry, Teichmülller spaces and so forth. Existence, uniqueness and optimal regularity are all completely understood, and considerable progress has been made on the degenerate elliptic case $\|\mu\|_\infty = 1$. See [1] for all of this.

In higher dimensions nothing is known concerning existence beyond the classical results of Weyl-Schouten from the 1920's where $G$ and $H$ are smooth and the associated Weyl-Schouten tensor vanishes. Rather more is known about regularity and higher integrability, and this is also explained in [1]. Optimal regularity has not been proven, but is conjectured to depend precisely upon the $L^p$–norms of the spin operator acting on forms.

## 2. Uniqueness

Suppose we have two homeomorphic solutions $g, h : \Omega \to \Omega'$ solving (1). Higher integrability and some regularity results quickly imply that the map $f = g^{-1} \circ h : \Omega \to \Omega$ lies in $W^{1,n}_{loc}(\Omega, \Omega)$ and satisfies the PDE

$$Df^t(x)G(f(x))Df(x) = J(x,f)^{2/n}G(x), \qquad \text{a.e. } \Omega \tag{3}$$

Moreover, the space $\Gamma$ of all homeomorphic $W^{1,n}_{loc}$ solutions to (3) forms a locally compact topological transformation group acting effectively on $\Omega$. This group is actually the conformal group with respect to the measurable conformal structure induced by $G$. Here ellipticity proves local compactness since the conformal group is uniformly quasiconformal.

In this situation it is natural to ask if the group $\Gamma$ is a Lie group. In fact this was, more or less, Hilbert's 5th problem [3]. The 5th problem has been solved under the assumption that $\Gamma$ is locally euclidean by Gleason and Montgomery/Zippin [3]. The general problem (when $\Gamma$ is assumed to act on a locally euclidean space) is now known as the Hilbert-Smith conjecture. We announced the solution to this conjecture in the case that $\Gamma$ is quasiconformal in [2] using a key result of Yang [4] which bounds (from below) the Hausdorff dimension of any invariant metric as well as the structure theory which reduces the problem to the case of the $p$-adics. Control on the dimension of invariant metrics is provided by the ellipticity of the PDEs (3) via known distortion results from the theory of quasiconformal mappings. Thus we conclude our group is a Lie group.

The point here is that if $g = h$ on a nonempty set $X \subset \Omega$, then $f = identity$ on $X$ and so $\Gamma_0 = \overline{\langle f \rangle}$ will be a compact Lie group. Unlike the $p$-adics, nontrivial compact Lie groups have nontrivial elements of finite order. However any element of finite order in $\Gamma_0$ is the identity on $X$ and so from Smith theory, $X$ must locally look like a subset of a homology manifold of even co-dimension. In particular, $X$ cannot be of dimension $n-1$ or open. Returning to our original problem we find we have proven

**Theorem 1.** *Suppose $g, h \in W^{1,n}_{loc}(\Omega, \Omega')$ are solutions to (1) and $g = h$ on a set $X$ of topological dimension at least $n-1$. Then $g \equiv h$. This is best possible.*

That this is best possible follows since $g = identity$ and $h$ a rotation agree on the fixed point set of $h$, satisfy (1) with $G \equiv I_n$, and which could be a codimension 2 sphere (of Hausdorff dimension as close to $n$ as we might like if we accept a worse $G$).

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## References

[1] T. Iwaniec and G. J. Martin, *Geometric Function Theory and Nonlinear Analysis*, Oxford University Press, 2001.

[2] G. J. Martin, *The Hilbert–Smith conjecture for quasiconformal actions*, Electron. Res. Announc. Amer. Math. Soc., **5**, (1999), 66–70.

[3] D. Montgomery and L. Zippin, *Topological Transformation groups*, Interscience, New York, 1955.
[4] C-T. Yang, *p–adic transformation groups*, Michigan Math. J., **7**, (1960), 201–218.

## Mean curvature and minimal surfaces in CR manifolds

Andrea Malchiodi
(joint work with Jih-Hsin Cheng, Jenn-Fang Hwang and Paul Yang)

Let $M$ be a three dimensional manifold. A *contact structure* $\xi$ on $M$ is a completely non-integrable two-dimensional distribution, while a *contact form* $\Theta$ is a non-zero 1-form on $M$ which annihilates $\xi$. We will always assume $\Theta$ to be *oriented*, namely that $d\Theta(u,v) > 0$ if $(u,v)$ is an oriented basis of $\xi$. The *Reeb vector field* associated to $\Theta$ is the unique vector field $T$ such that $\Theta(T) = 1$ and such that $d\Theta(T,\cdot) = 0$.

A *CR structure* compatible with $\xi$ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -Id$. We assume that also $J$ is oriented, namely that for every non-zero vector field $X$, the couple $(X, JX)$ is an oriented basis of $\xi$.

A *CR manifold* (or pseudo-hermitian) is a manifold endowed with a $CR$ structure and with a global contact form $\Theta$. This gives rise to a natural volume form

$$V(\Omega) = \int_\Omega \Theta \wedge d\Theta$$

and to a metric on $\xi$ called *Levi form*

$$L_\Theta(v,w) = d\Theta(v, Jw).$$

We now recall the definition of the Tanaka-Webster connection and the associated curvature. Let $e_1$ be a field in $\xi$ with unit length, namely such that $L_\Theta(e_1,e_1) = 1$, and let $e_2 = Je_1$, so that $(e_1,e_2)$ is an oriented basis of $\xi$. Let $\{\Theta, e^1, e^2\}$ be the triple of forms dual to $\{T, e_1, e_2\}$. Then we have the *structure equations*

$$d\Theta = 2e^1 \wedge e^2; \tag{S1}$$

$$de^1 = -e^2 \wedge \omega \mod \Theta; \qquad de^2 = e^1 \wedge \omega \mod \Theta. \tag{S2}$$

The *Tanaka-Webster connection* is defined by

$$\nabla^{p.h.} e_1 = \omega \otimes e_2, \qquad \nabla^{p.h.} e_2 = -\omega \otimes e_1,$$

while the *Tanaka-Webster curvature* is given by

$$d\omega(e_1,e_2) = -2W.$$

Given a function $f$ and a vector field $V$ tangent to $\xi$ we define the *subgradient* of $f$ and the *subdivergence* of $V$ as

$$\nabla_b f = (e_1 f)e_1 + (e_2 f)e_2; \qquad div_b V = L_\Theta(\nabla^{p.h.}_{e_1} V, e_1) + L_\Theta(\nabla^{p.h.}_{e_2} V.e_2).$$

We have also the *sublaplacian* of $f$ which is given by

$$\Delta_b f = div_b(\nabla_b f)$$

For the Heisenberg group $H^1$ we have the standard choices

$$\hat{e}_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad \hat{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad \hat{T} = \frac{\partial}{\partial z}, \quad \hat{\Theta} = x\,dy - y\,dx + dz.$$

Now, given a regular surface $\Sigma \subseteq M^3$, we define a notion of *(pseudo)-mean curvature.* If $p \in \Sigma$ and if $T_p\Sigma \neq \xi(p)$, we define $e_1(p)$ as the unique (up to the sign) unit vector belonging to $T_p\Sigma \cap \xi(p)$, and $e_2(p) = J(p)e_1(p)$. Then we define the (p)-mean curvature $H$ in three equivalent ways

- **(1):** *as a second variation of the volume*: if $\Sigma$ is the boundary of an open set $\Omega$, then for a variation of $\Omega$ (or of $\Sigma$) in the direction $fe_2$ we have

$$\delta_{fe_2} V(\Omega) = \int_\Sigma f\Theta \wedge e^1; \qquad \delta_{fe_2}\int_\Sigma \Theta \wedge e^1 = -\int_\Sigma fH\Theta \wedge e^1$$

- **(2):** *viewing $\Sigma$ as a level set*: if $\Sigma = \{\psi = 0\}$, then

$$H = -div_b\left(\frac{\nabla_b\psi}{|\nabla_b\psi|}\right)$$

- **(3):** *using the Tanaka-Webster connection*: similarly to the curvature of a curve in the plane

$$\nabla^{p.h.}_{e_1} e_1 = He_2.$$

For the Heisenberg group the first two definitions coincide with those in [CDG], [DGN], [Pau]. Moreover, the *area* element $\Theta \wedge e^1$ coincides with the three dimensional Hausdorff measure of $\Sigma$, considered in [B] and in [FSS].

*Graphs in the Heisenberg group.* Let $u : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, and let $\Sigma$ be the graph of $u$

$$\Sigma = \left\{(x, y, u(x, y)) \mid (x, y) \in \mathbb{R}^2\right\}.$$

Recall that $e_1 \in T\Sigma \cap \xi$, and for a graph it is given by

$$e_1 = \frac{1}{D}\left[-(u_y + x)\begin{pmatrix}1\\0\\y\end{pmatrix} + (u_x - y)\begin{pmatrix}0\\1\\-x\end{pmatrix}\right],$$

where

$$D = \left[(u_x - y)^2 + (u_y + x)^2\right]^{\frac{1}{2}}.$$

One also finds that

$$H = \frac{1}{D^3}\{(u_y + x)^2 u_{xx} - 2(u_y + x)(u_x - y)u_{xy} + (u_x - y)^2 u_{yy}\},$$

so the equation $H \equiv 0$ is

$$(*) \qquad (u_y + x)^2 u_{xx} - 2(u_y + x)(u_x - y)u_{xy} + (u_x - y)^2 u_{yy} = 0$$

We have then the following classification result.

**Theorem A.** ([CHMY], see also [CH] and [GP] for some extensions) *The only entire $C^2$ smooth solutions to the equation $(*)$ are of the form*

(1.1) $u = ax + by + c$ *(a plane with a,b,c being real constants);*

(1.2) $u = -abx^2 + (a^2 - b^2)xy + aby^2 + g(-bx + ay)$

*($a, b$ being real constants such that $a^2 + b^2 = 1$ and $g \in C^2$).*

The main ingredient for proving Theorem A is the analysis of the *singular points* of $\Sigma$ (or of $u$), which are given by

$$S(u) = \left\{(x, y) \in \mathbb{R}^2 \ : \ u_x - y = u_y + x = 0\right\}.$$

For a minimal graph we have the following characterization of the singular points.

**Proposition** *Let $\Omega$ be a domain in the $xy-$plane. Let $u \in C^2(\Omega)$ be a solution of $(*)$. Let $p_0$ be a singular point of $u$. Then either $p_0$ is isolated in $S(u)$ or there exists a small neighborhood of $p_0$ which intersects with $S(u)$ in exactly a $C^1$ smooth curve through $p_0$.*

The analysis of the singular points can also be employed to study surfaces with bounded (p)-mean curvature in general 3-dimensional CR manifolds. We have indeed the following result.

**Theorem B.** *Let $M$ be a pseudohermitian 3-manifold. Let $\Sigma$ be a closed, connected surface, $C^2$ smoothly immersed in $M$ with bounded p-mean curvature. Then the genus of $\Sigma$ is less than or equal to 1. In particular, there are no constant p-mean curvature or p-minimal surfaces $\Sigma$ of genus greater than one in $M$.*

## References


[B] Balogh, Z., *Size of characteristic sets and functions with prescribed gradient*, J. reine angew. Math., 564 (2003) 63-83.

[CDG] Capogna, L., Danielli, D., Garofalo, N., *The geometric Sobolev embedding fir vector fields and the isoperimetric inequality*, Comm. Anal. Geom., 2 (1994) 203-215.

[CH] Cheng, J.-H. and Hwang, J.-F., *Properly embedded and immersed minimal surfaces in the Heisenberg group*, Bull. Aus. Math. Soc., 70 (2004) 507-520.

[CHMY] Jih-Hsin Cheng, Jenn-Fang Hwang, Andrea Malchiodi, Paul Yang: Minimal surfaces in pseudohermitian geometry and the Bernstein problem in the Heisenberg group, Annali Scuola Norm. Sup. Pisa, 1 (2005), 129-177.

[DGN] Danielli, D., Garofalo, N. and Nhieu, D-M., *Minimal surfaces, surfaces of constant mean curvature and isoperimetry in Carnot groups*, preprint, 2001.

[FSS] Franchi, B., Serapioni, R., and Serra Cassano, F., *Rectifiability and perimeter in the Heisenberg group*, Math. Ann. 321, (2001) 479-531.

[GP] Garofalo, N. and Pauls, S., *The Bernstein problem in the Heisenberg group*, preprint, 2004.

[Pau] Pauls, S. D., *Minimal surfaces in the Heisenberg group*, Geometric Dedicata, 104 (2004) 201-231.

**Regularity of free boundaries in parabolic obstacle type problems**

D. Apushkinskaya, Saarbrücken
(joint work with H. Shahgholian, Stockholm and N. Uraltseva, St. Petersburg)

This talk is inspired by a recent joint work with H. Shahgholian and N. Uraltseva (see [1]-[3]), and it concerns the regularity properties of a free boundary in a neighborhood of the fixed boundary of a domain for a parabolic obstacle problem with zero constraint.

For parabolic equations the simplest obstacle problem can be formulated as follows: let $\mathbb{D}$ be a domain in $\mathbb{R}^n$, $\mathbb{Q} = \mathbb{D}\times]0,T[$,

$$\mathbb{K} = \{w \in H^1(\mathbb{Q}) : \quad w \geq 0 \quad \text{a.e. in } \mathbb{Q}, \quad w = \phi \quad \text{on } \partial'\mathbb{Q}\},$$

where $\phi$ be a nonnegative function defined on the parabolic boundary $\partial'\mathbb{Q}$ of the cylinder $\mathbb{Q}$. It is required to find a function $u \in \mathbb{K}$ such that

$$\int_{\mathbb{D}} \partial_t u(w-u)dx + \int_{\mathbb{D}} DuD(w-u)dx + \int_{\mathbb{D}} (w-u)dx \geq 0$$

a.e. in $t \in ]0,T[$, and for all $w \in \mathbb{K}$.

It is known that if $u$ is a solution of this problem, then, in the sense of distributions, $u$ satisfies the equation

$$\Delta u - \partial_t u = \chi_\Omega \quad \text{in } \mathbb{Q},$$

where $\Omega = \{(x,t) \in \mathbb{Q} : u(x,t) > 0\}$, and $\chi_\Omega$ is the characteristic function of the set $\Omega$. The set $\Omega = \Omega(u)$ is called the *noncoincidence set*, while the set $\Lambda(u) = \{(x,t) : u(x,t) = |Du(x,t)| = 0\}$ is the *coincidence set* for the solution $u$; $\Gamma(u) = \partial\Omega \cap \Lambda(u)$ is the *free boundary.* The possibility must not be ruled out that the free boundary $\Gamma(u)$ and the fixed boundary $\partial'\mathbb{Q}$ meet at points where $\phi = 0$. Therefore, the points of contact may exist.

The regularity of the free boundary (far from $\partial'\mathbb{Q}$) for this problem has been investigated earlier only in the special case of the Stefan problem, where the boundary and initial conditions guarantee the additional property $\partial_t u \geq 0$; see [4]. It should be emphasized that results of [1]-[3] enable us to avoid any assumptions on the time-derivative of solutions. Results for an elliptic problem related to our ones can be found in [7]. It should be mentioned also a recent work [6] where a parabolic free boundary problem without presence of the contact points was considered. Results for an elliptic free boundary problem without presence of the contact points were obtained in [5]. Note that in [5]-[7] the more general free boundary problems were treated, without the assumption about the nonnegativity of the solution.

Our main result says that the boundary of the noncoincidence set $\Omega$ is Lischitz continuous near the part of the lateral surface of $\mathbb{Q}$ where the solution is equal to zero. In particular, this implies that, locally, inside $\mathbb{Q}$ and near that part, the free boundary is the graph of a $C^{1,\alpha}$-function.

**Remark.** Unfortunately, $C^{1,\alpha}$-regularity of the free boundary may fail to occur at the points of contact between the free boundary and the fixed boundary. The counterexample, showing that in the $t$-direction the free boundary $\partial\{u > 0\}$ may intersect the fixed boundary transversally, can be found in [3].

Our arguments are based on the blow-up technique, in combination with various monotonicity formulas, and on the result of the paper [2] concerning the global solutions of the parabolic obstacle problem with zero constraint (i.e., the solutions in the entire half-space $\{(x,t) \in \mathbb{R}^{n+1} : x_1 > 0\}$. It should be emphasized that our arguments do not require any additional assumptions on the free boundary.

## References

[1] D.E. Apushkinskaya, H. Shahgholian, N.N. Uraltseva, *Boundary estimates for solutions to the parabolic free boundary problem*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **271** (2000), 39–55.

[2] D.E. Apushkinskaya, H. Shahgholian, N.N. Uraltseva, *On the global solutions of the parabolic obstacle problem*, Algebra i Analiz **14** (2002), no. 1, 3–25 [Russian]; English transl., St. Petersburg Math. J. **14** (2003), no. 1, 1–17.

[3] D.E. Apushkinskaya, H. Shahgholian, N.N. Uraltseva, *Lipschitz property of the free boundary in the parabolic obstacle problem*, Algebra i Analiz **15** (2003), no. 3, 3–25 [Russian]; English transl., St. Petersburg Math. J. **15** (2004), no. 3, 375–391.

[4] L.A. Caffarelli, *The regularity of free boundaries in higher dimensions*, Acta Math. **139** (1977), 155–184.

[5] L.A. Caffarelli, L. Karp, H. Shahgholian, *Regularity of a free boundary with application to the Pompeiu problem*, Annals of Math. **151** (2000), 269–292.

[6] L.A. Caffarelli, A. Petrosyan, H. Shahgholian, *Regularity of a free boundary in parabolic potential theory*, J. Amer. Math. Soc. **17** (2004), no. 4, 827–869.

[7] H. Shahgholian, N. Uraltseva, *Regularity of a free boundary near contact points with the fixed boundary*, Duke Math. J. **116** (2003), no. 1, 1–34.

## Regularity of stable branched minimal hypersurfaces

Neshan Wickramasekera

The goal of the work presented in this talk is to understand the local structure, including the nature of singularities, of stable minimal hypersurfaces of a Euclidean space (or more generally, of a smooth Riemannian manifold) of arbitrary dimension. In [SS81], R. Schoen and L. Simon developed a partial regularity theory for $n$-dimensional stable minimal hyperpsurfaces assuming the (regular parts of the) hypersurfaces are embedded and their singularities have locally finite $(n-2)$-dimensional Hausdorff measure. Embeddedness guarantees that all tangent cones to the hypersurface are multiplicity 1 cones. i.e. multiple "sheets" cannot come together when passing to the weak limit of a sequence of geometric rescalings of the hypersurface at any given (singular) point. Note however that stable minimal hypersurfaces need not be embedded as is demonstrated by the simple example of a pair of transversely intersecting hyperplanes. Once the embeddedness hypothesis is removed, higher multiplicity and branching becomes a central issue. The simplest such instance is when the multiplicity is 2, and the basic goal in that case is to understand the nature of the hypersurface near a singular (i.e. non-immersed) point (a branch point) at which the hypersurface has a multiplicity 2 hyperplane as one of its tangent cones. This has recently been accomplished in [Wic]. The main regularity theorem of [Wic] (Theorem 1 below) explains fully the asymptotic nature of a stable minimal hypersurface near a branch point with a multiplicity 2 tangent plane, assuming that the singular set has finite $(n-2)$-dimensional Hausdorff measure. (The singular set consists only of "genuine" singularities, which include branch points if any exist. Thus, the points of self-intersection, where the hypersurface is immersed, are considered regular.) In stating this theorem and subsequently, we use the following notation: $B^n_\rho(0)$ denotes the open ball of the $n$-dimensional Euclidean space with radius $\rho$ and center the origin. $\mathcal{H}^n$ denotes the $n$-dimensional Hausdorff measure.

**Theorem 1.** *For each $\delta \in (0,1)$, there exists a number $\epsilon \in (0,1)$, depending only on $n$ and $\delta$, such that the following is true. If $M$ is an orientable immersed stable minimal hypersurface of $B^{n+1}_2(0)$, with $\mathcal{H}^{n-2}(\operatorname{sing} M) < \infty$, $0 \in \overline{M}$, $\frac{\mathcal{H}^n(M)}{\omega_n 2^n} \leq 3-\delta$ and $\int_{M\cap(B^n_1(0)\times\mathbf{R})} |x^{n+1}|^2 \leq \epsilon$, then $M_1 \cap (B^n_{1/2}(0) \times \mathbf{R}) = \operatorname{graph} u$ where $M_1$ is the connected component of $\overline{M} \cap (B^n_1(0) \times \mathbf{R})$ containing the origin, $u$ is either a single valued or a 2-valued $C^{1,\alpha}$ function on $B^n_{1/2}(0)$ satisfying*

$$\|u\|_{C^{1,\alpha}(B^n_{1/2}(0))} \leq C \left( \int_{M\cap(B^n_1(0)\times\mathbf{R})} |x^{n+1}|^2 \right)^{1/2}.$$

*Here the constants $C$ and $\alpha \in (0,1)$ depend only on $n$ and $\delta$.*

See the Appendix of [Wic] for the definition of the $C^{1,\alpha}$ norm of $u$ when $u$ is a 2-valued function.

This theorem rules out, for instance, the possibility of having a sequence of "necks" connecting two sheets and converging to a branch point of the hypersurface.

As an important corollary of this theorem, a compactness theorem (Theorem 2 below) for a class of singular stable minimal hypersurfaces is obtained, which in particular confirms the expectation that a branched stable minimal hypersurface cannot be approximated by a sequence of regular stable, minimal immersions.

**Theorem 2.** *Let $\delta \in (0,1)$. There exists $\sigma = \sigma(n,\delta) \in (0,1/2)$ such that the following is true. Suppose $M_k$ is a sequence of orientable stable minimal hypersurface immersed in $B^{n+1}_2(0)$ with $\mathcal{H}^{n-2}(\operatorname{sing} M_k) = 0$ for each $k$ and $\limsup_{k\to\infty} \frac{\mathcal{H}^n(M_k)}{\omega_n 2^n} \leq 3-\delta$. Then there exists a stationary varifold $V$ of $B^{n+1}_2(0)$ and a closed subset $S$ of $\operatorname{spt}\|V\| \cap B^{n+1}_\sigma(0)$ with $S = \emptyset$ if $2 \leq n \leq 6$, $S$ discrete if $n = 7$ and $\mathcal{H}^{n-7+\gamma}(S) = 0$ for every $\gamma > 0$ if $n \geq 8$ such that after passing to a subsequence, which we again denote $\{k\}$, $M_k \to V$ as varifolds and $(\operatorname{spt}\|V\| \setminus S) \cap B^{n+1}_\sigma(0)$ is an orientable immersed, smooth, stable minimal hypersurface of $B^{n+1}_\sigma(0)$.*

In low dimensions, the "smallness of excess" hypothesis of Theorem 1 can be dropped provided we assume that the mass ratio is sufficiently close to 2. Precisely, we have the following:

**Theorem 3.** *There exist fixed constants* $\epsilon \in (0,1)$, $\sigma \in (0,1)$, $C \in (0,\infty)$ *and* $\alpha \in (0,1)$ *such that the following holds. If* $2 \leq n \leq 6$, $M$ *is an orientable immersed stable minimal hypersurface of* $B_2^{n+1}(0)$ *with* $\mathcal{H}^{n-2}(\text{sing}\, M) < \infty$, $0 \in \overline{M}$ *and* $\frac{\mathcal{H}^n(M)}{\omega_n 2^n} \leq 2+\epsilon$, *then for some orthogonal rotation* $q$ *of* $\mathbf{R}^{n+1}$, *either* $q\,M_1 \cap (B_\sigma^n(0) \times \mathbf{R}) = \text{graph}\, u$ *where* $u$ *is either a single valued or a 2-valued* $C^{1,\alpha}$ *function on* $B_\sigma^n(0)$ *with*

$$\|u\|_{C^{1,\alpha}(B_\sigma^n(0))} \leq C \left( \int_{q\,M \cap (B_1^n(0)\times\mathbf{R})} |x^{n+1}|^2 \right)^{1/2}$$

*or there exists a pair of transverse hyperplanes* $P^{(1)}, P^{(2)}$ *of* $\mathbf{R}^{n+1}$ *such that* $q\,M_1 \cap (B_\sigma^n(0) \times \mathbf{R}) = \text{graph}\,(p^{(1)} + u^{(1)}) \cup \text{graph}\,(p^{(2)} + u^{(2)})$, *where* $P^{(1)} = \text{graph}\, p^{(1)}$, $P^{(2)} = \text{graph}\, p^{(2)}$, $u^{(i)} \in C^{1,\alpha}\,(P^{(i)} \cap (B_\sigma^n(0) \times \mathbf{R}); \mathbf{R})$ *with*

$$\|u^{(i)}\|_{C^{1,\alpha}(P^{(i)} \cap (B_\sigma^n(0)\times\mathbf{R}))} \leq C \left( \int_{M \cap (B_1^n(0)\times\mathbf{R})} \text{dist}^2\,(x, P) \right)^{1/2}$$

*for* $i = 1,2$. *Here* $M_1$ *denotes the connected component of* $\overline{M} \cap (B_1^n(0) \times \mathbf{R})$ *containing the origin.*

At a key stage of the proof of Theorem 1, a type of harmonic approximation is used, where it is shown that whenever the $L^2$-height excess relative to a (multiplicity 2) hyperplane of a stable minimal hypersurface is small in a cylinder, the hypersurface in a smaller cylinder is well approximated by the graph of a "2-valued harmonic" function. F. J. Almgren Jr. used multivalued harmonic functions in his work on area minimizing currents (of arbitrary dimension and co-dimension), in [Alm83], where harmonic meant Dirichlet energy minimizing. Under the weaker hypothesis of stability, the two-valued approximating functions do not satisfy this energy minimizing property. However, codimension 1 setting gives them a lot more structure, and a sufficiently detailed, asymptotic description of the functions is obtained in [Wic].

The work in [Wic] uses methods and results due to L. Simon [Sim93]; R. Hardt and L. Simon [HS79]; R. Schoen and L. Simon [SS81]; F. J. Almgren Jr. [Alm83] and the author [Wic04] at a number of crucial points. This work in fact should be viewed as a generalization of the results of [Wic04]. To prove that a stable hypersurface, when it is weakly close to a multiplicity 2 hyperplane, is well approximated by the graph of a 2-valued harmonic function of the type aforementioned, a blow-up argument is used, where sequences of stable hypersurfaces are blown up off (multiplicity 2) hyperplanes to produce the 2-valued harmonic functions. This blow up procedure is based on the approximate graphical decomposition of the hypersurface as in [SS81]. The next key step is to understand the asymptotics of these 2-valued approximating functions. The approach taken in analyzing these functions is to use geometric arguments, aimed at proving excess decay estimates for the graphs of the functions. To investigate the local regularity properties of these functions at points where their graphs blow up to transversely intersecting pairs of hyperplanes, and also to prove global decay estimates when the center point is a branch point of the function, variants of powerful techniques developed by Simon [Sim93] and Hardt and Simon [HS79] are used. In particular, a crucial ingredient is an estimate for the radial derivatives of the blow-up limit due to Hardt and Simon [HS79].

Another important technical ingredient of the analysis of the approximating 2-valued harmonic functions is the monotonicity of a frequency function, an idea used first in a geometric setting by Almgren [Alm83]. Both the frequency function directly associated with the two-valued function as well as the one associated with the single valued function obtained by taking the difference between the two values of the two-valued function are used in [Wic]. Either frequency function, for any given center point, is monotonically non-decreasing as a function of the radius. Thus, in particular, the points of the domain of the two-valued function may be classified according to the values assumed by the limit of the frequency function associated with the difference function. In a classical setting, e.g. if the function were (single valued) harmonic, this limit is equal to the vanishing order of the function at the point in question. In the setting of [Wic], it conveys analogous information, which may be regarded as the order of contact between the "two sheets" of the graph of the 2-valued function, (although admittedly at a branch point one does not have a useful notion of two sheets) and it reveals the local geometric picture of the graph; i.e. whether the graph locally consists of two disjoint harmonic disks, or of two self intersecting harmonic disks or whether it is branched. Furthermore, the rate of decay of the graph of the two valued function to its (unique) multiplicity 2 tangent plane at a branch point depends only on $n$. Said differently, there exists a fixed frequency gap, depending only on $n$, implying that the order of contact at a branch point cannot be arbitrarily close to 1.

As further corollaries of Theorem 1 in low dimensions, the following pointwise curvature estimate (Theorem 4) and the Bernstain type theorem (Theroem 5) are obtained. For smooth stable minimal hypersurfaces of dimension up to 5, these results were previously obtained (by entirely different methods) under weaker assumption of arbitrary mass bound by R. Schoen, L. Simon and S.-T. Yau [SSY75].

**Theorem 4.** *Let $\delta \in (0,1)$. There exist positive numbers $\Gamma$ and $\sigma$ depending only on $\delta$ such that if $2 \leq n \leq 6$ and $M$ is an immersed, stable minimal hypersurface of $B_2^{n+1}(0)$ with second fundamental form $A$, satisfying $\mathcal{H}^{n-2}(\operatorname{sing} M) = 0$ and $\frac{\mathcal{H}^n(M)}{\omega_n 2^n} \leq 3 - \delta$, then $\operatorname{sing} M \cap B_\sigma^{n+1}(0) = \emptyset$ and*

$$\sup_{M \cap B_\sigma^{n+1}(0)} |A| \leq \Gamma.$$

**Theorem 5.** *Let $\delta \in (0,1)$. Suppose $2 \leq n \leq 6$, $M$ is a complete, non-compact stable minimal hypersurface of $\mathbf{R}^{n+1}$ satisfying $\frac{\mathcal{H}^n\,(M \cap B_R^{n+1}(0))}{\omega_n R^n} \leq 3 - \delta$ for all $R > 0$. Then $M$ must be a union of at most 2 affine hyperplanes.*

The existence of a rich class of stable branched minimal hypersurfaces of the type considered in [Wic] has recently been established by L. Simon and the author [SW].

## References


[Alm83] F. J. Almgren Jr. *Almgren's big regularity paper: Q-valued functions minimizing Dirichlet's integral and the regularity of area minimizing rectifiable currents up to codimension 2.* World Scientific (2000).

[HS79] R. Hardt, L. Simon. *Boundary regularity and embedded solutions for the oriented Plateau problem.* Ann. of Math. **110** (1979), 436–486.

[SS81] R. Schoen, L. Simon. *Regularity of stable minimal hypersurfaces.* Comm. Pure and Appl. Math. **34** (1981), 742–797.

[SSY75] R. Schoen, L. Simon, S.-T. Yau. *Curvature estimates for minimal hypersurfaces.* Acta Math. **134** (1975), 276–288.

[Sim93] L. Simon. *Cylindrical tangent cones and the singular set of minimal submanifolds.* Journal of Differential Geometry **38** (1993), 585-652.

[SW] L. Simon, N. Wickramasekera. *Stable branched minimal immersions with prescribed boundary.* Preprint.

[Wic04] N. Wickramasekera. *A rigidity theorem for stable minimal hypercones.* Journal of Differential Geometry **68**, (2004), 433-514.

[Wic] N. Wickramasekera. *A regularity and compactness theory for immersed stable minimal hypersurfaces of multiplicity at most 2.* Preprint.


## $L^p$-estimates for steady compressible fluids


Jens Frehse

(joint work with Sonja Goj and Mark Steinhauer)


We consider the equations for steady compressible fluids in $\mathbb{R}^3$

$$\operatorname{div}(\varrho u) = 0, \qquad \int \varrho \mathrm{d}x = M, \quad \varrho \geq 0,$$

$$-\mu\Delta u - (\mu+1)\nabla \operatorname{div} u + (u \cdot \nabla u)\varrho = -\nabla \varrho^\gamma + f\varrho + g,$$

$u$ = velocity field, $\varrho$ = density.

Existence of solutions is known in important cases for $\gamma > \frac{3}{2}$. For $\gamma < \frac{3}{2}$ estimates of the convective term and $\varrho^\gamma$ are needed. The authors contribute to this question by proving a uniform (local) $L^q$-estimate in the case

$$5/4 \leq \gamma \leq 5/3.$$

It states that

$$\varrho u \nabla u \quad \text{and} \quad \varrho \in L_{\text{loc}}^{\frac{6\gamma}{5+2\gamma}}$$

this covers the case of "air", where $\gamma = 7/5 (< 3/2)$.

## Conformally Invariant differential equations on 4-manifolds

Paul Yang
(joint work with Alice Chang)

In recent years there has been a lot of work on the Q-curvature equations, and many people have asked about the geometric meaning of such equations. In this talk we describe a program utilizing some of the recent work on the Q-curvature equations and related $\sigma_k$ equations to study a class of conformal structures in dimension four, thus providing some motivation for the study of a family of nonlinear equations in conformal geometry. Recall the decomposition of the Riemann tensor in dimension four:

$$Rm = W \oplus (1/2)A \owedge g,$$

where $W$ is the Weyl tensor and $A = Rc - \frac{R}{6}g$ is the Schouten tensor, $Rc$ is the Ricci tensor and $R$ the scalar curvature of the metric $g$. Recall that the Weyl tensor transform by scaling under a conformal change in metric $\bar{g} = e^{2w}g$, thus the Schouten tensor contains all the derivative information in the conformal factor.

The well known conformal Laplacian $L = -6\Delta + R$ transforms under the conformal change of metric $\bar{g} = u^2 g$ by: $\bar{L}\phi = u^3 L(u\phi)$ and it gives rise to the scalar curvature equation: $Lu = \bar{R}u^3$. The existence theory is completed by the work of Yamabe ([Y]), Trudinger ([T]), Aubin ([A]) and Schoen ([S]).

The fourth order Paneitz operator in dimension four is given by $P = \Delta^2 + \delta\{(2/3)Rg - 2Rc\}d$ where $\delta$ is the divergence, and $d$ is the deRham differential. Under the conformal change in metric $\bar{g} = e^{2w}g$, the Paneitz operator transforms by $\bar{P} = e^{-4w}P$, and it computes the Q-curvature of the conformal metric $\bar{g}$ by

$$Pw + 2Q = 2\bar{Q}e^{4w}, \quad Q = \frac{1}{12}\{-\Delta R + 6\sigma_2(A)\}.$$

In general dimensions, the existence of analogue of the Paneitz operator and the corresponding $Q$-curvature equations has been worked out by Graham-Jenne-Mason-Sparling ([GJMS]) using the ambient metric construction of Fefferman and Graham ([FG]), and by Branson ([B]) using representation.

In dimension four the Q-curvature equation is closely connected with the Chern-Gauss-Bonnet formula for a compact oriented manifold in dimension four:

$$8\pi^2\chi = \int |W|^2 + \sigma_2(A)dV.$$

This result may be extended to domains on the standard 4-sphere:

**Theorem 1.** *([CQY1]) Let $g = e^{2w}g_0$ be a complete conformal metric defined on $\Omega \subset S^4$ satisfying the following curvature bounds: $R \geq c_1 > 0, |\nabla R| \leq c_2, Rc \geq -c_3$ and $\int_\Omega |Q|dV < \infty$ then $\Omega = S^4 \setminus \{q_1, \ldots, q_N\}$ and $8\pi^2\chi(\Omega) = \int_\Omega QdV + \sum_1^N I_k$, where $I_k$ are isoperimetric constants attached to the end at $q_k$.*

It follows from the tranformation rules, the positivity of the operators $L$ and $P$ are conformally invariant. The following is a conformally invariant set of critera:

**Theorem 2.** *([G]) On a compact 4-manifold $(M, g)$, if the conformal Laplacian is positive, and $\int \sigma_2(A)dV > 0$ then the Paneitz operator is positive, and $\int \sigma_2(A)dV \le 16\pi^2$, where equality holds only for the standard 4-sphere.*

The positivity of the operators $L$ and $P$ gives strong control of the topology and geometry of the 4-manifold in question. We say the metric $g$ belongs to the positive k-th cone (denoted by $g \in \Gamma_k^+$) if $\sigma_i(A_g) > 0$ for $i = 1, 2, \dots, k$.

**Theorem 3.** *([CGY1]) Under the same assumptions as Theorem 2, there exists a conformal metric $\bar{g} \in [g]$ and $g \in \Gamma_2^+$; and except for the standard 4-sphere, given any smooth positive function $f$, there is a conformal metric $\bar{g}$ so that $\sigma_2(\bar{A}) = f$.*

in It is also natural to consider 4-manifolds with boundary, then it is important to find natural boundary conditions under which the foregoing considerations may be extended. This work is being carried out in the forthcoming thesis of Sophie Chen. In a different direction, the solvability of the remaining $\sigma_k$ equations for $k = 3, 4$ is a special case of the recent work of Gursky and Viaclovsky:

**Theorem 4.** *([GV]) If $g \in \Gamma_k^+$, and $(n/2) < k$ then there exists a conformal metric $\bar{g}$ for which $\sigma_k(\bar{A}) = 1$.*

It is a simple algebraic fact that in dimension four, the condition $A \in \Gamma_2^+$ implies the positivity of the Ricci tensor. Consequently the conformal structures in dimension four with positive operators $L$ and $P$ have finite fundamental group. Thus up to a finite covering, we are considering simply connected 4-manifolds. The important work of Donaldson and Freedman implies that such manifolds are necessarily homeomorphic to two countable series of simply connected 4-manifolds: (i) $S^4$, and $k(CP^2)\#l(\bar{C}P^2)$, where $0 \le k < 5l + 4$; (ii) $k(S^2 \times S^2)$. The simplest one in this list is the 4-sphere and we have the following conformal sphere theorem:

**Theorem 5.** *([CGY2]) Assume $(M^4, g)$ has positive $L$ operator and that*

$$\int \sigma_2(A)dV \ge \int |W|^2 dV > 0,$$

*then $M$ is diffeomorphic to $S^4$, $RP^4$ or $CP^2$ and the inequality is sharp, that is the standard conformal structure on $CP^2$ realized the equality.*

Thus it is natural to ask for a given 4-manifold whether it is possible to find a conformal structure minimizing the quantity $\int |W|^2 dV$. This is a difficult question. The critical metrics for the functional $\int |W|^2 dV$ are called Bach flat metrics, and the Euler equation is the vanishing of the Bach tensor: $B_{ij} = \nabla_k \nabla^l W_{kijl} + (1/2) R^{kl} W_{kijl} = 0$. This class of metrics include the Einstein metrics, the conformally flat metrics, the self-dual or anti-self-dual metrics and Kahler metrics of constant scalar curvature. There is recent advance on the structure of Bach flat metrics by Tian and Viaclovsky:

**Theorem 6.** *([TV]) For a complete Bach flat 4-manifold satisfying a uniform Sobolev constant, and the curvature decay condition $|Rm(x)| = o(|r(x)^{-2}|)$, then the volume of geodesic balls have an upper bound:* $\mathrm{Vol}(B_r) \le Cr^4$. *Consequently $X$ is an ALE space.*

This previous result allows the construction of a bubble tree to describe the possible degeneration of a sequence of Bach flat Yamabe metrics. As a consequence, it is possible to formulate a finiteness result for the diffeomorphic classes of four dimensional conformal structures with positive operators $L$ and $P$:

**Theorem 7.** *([CQY2]) Let $\mathcal{A}$ denote the class of compact Bach flat 4-manifolds satisfying the following: (i) $Y(M,g) \geq c_1 > 0$, (ii) $\int |W|^2 dV \leq c_2$, (iii) $\int \sigma_2(A) dV \geq c_3 > 0$, then $\mathcal{A}$ contains only finite number of distinct diffeomorphism classes.*

It is reasonable to conjecture that there are no Bach-flat conformal structures in dimension four with positive operators $L$ and $P$ other than the standard 4-sphere satisfying the condition $16\pi^2 - \epsilon \leq \int \sigma_2(A) dV \leq 16\pi^2$ and it would be of interest to determine such a constant. The following is a first step in this direction:

**Theorem 8.** *([CQY2]) There exists a positive $\epsilon$ so that the assertion above holds for conformal structures belonging to the class $\mathcal{A}$.*

## References


[A] T. Aubin *Equations differentielles non lineaires et probleme de Yamabe concernant la courbure scalaire* J. Math. Pures Appl, **55** (1976), 269-296.

[B] T. Branson, *Differential operators canonically associated to a conformal structure* Math. Scand. **57** 1985,293–345.

[CGY1] S.Y.A. Chang, M.J. Gursky and P. Yang, *An Equation of Monge-Ampere type in conformal geometry, and four-manifolds of positive Ricci curvature* Annals of Math., **155** (2002), 709-787.

[CGY2] S.Y.A. Chang, M.J. Gursky and P. Yang, *A conformally invariant sphere theorem in four dimensions*, Pub. IHES **98** (2004), 105-143.

[CQY1] S.Y.A. Chang, J. Qing and P. Yang, *Compactification of a class of conformally flat 4-manifolds* Invent. Math. **142** (2000), 65-93.

[CQY2] S.Y.A. Chang, J. Qing and P. Yang, *On a conformal gap and finiteness theorem for a class of four manifolds*, preprint 2005.

[FG] C. Fefferman and C.R. Graham *Conformal invariants*, Soc. Math. de France, Asterisque, hors serie **1985**, 95-116.

[G] M. Gursky *The principal eigenvalue of a conformally invariant differential operator , with an application to semilinear elliptic PDE* Comm. Math. Phys. 207 **1999**, 131-142.

[GJMS] C. R. Graham, R. Jenne, L. Mason and G. Sparling *Conformally invariant powers of the Laplacian, I: existence* J. London. Math. Soc. (2) 46, **1992**, 557-565.

[GV] M. Gursky and J. Viaclovsky *Prescribing symmetric functions of the eigenvalues of the Ricci tensor*, preprint 2004.

[S] R. Schoen *Conformal deformation of a Riemannian metric to constant scalar curvature* J. Diff. Geom., **20**, (1984), pp 479-495.

[TV] G. Tian and J. Viaclovksy *Bach-flat asymptotically locally Euclidean metrics*, Invent. Math., **160** , 2005, 357-415.

[T] N. Trudinger *Remarks concerning the conformal deformation of Riemannian structure on compact manifolds* Ann. Scuo. Norm, Sup. Pisa, **3** (1968), pp 265-274.

[Y] H. Yamabe *On a deformation of Riemannian structures on compact manifolds* Osaka Math. J., **12** (1960), pp 21-37.

## Pure unrectifiability and universal singular sets in Tonelli's classical variational problem

Bernd Kirchheim
(joint work with M. Csörnyei, T. O'Neil, D. Preiss, S. Winter)

Given a Lagrangian

$$L : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$$

we consider the minimisation problem

$$E(u) = \int_a^b L(x, u(x), u'(x))\, dx \to \min \text{ over } u(a) = A, u(b) = B, u \in W^{1,1}.$$

Important assumptions that are often imposed on the Lagrangian are

- ($\omega$)-*superlinearity:* $L(x, u, p) \geq \omega(p)$ ; $\omega'' > 0$, $\omega(0) = \omega'(0) = 0$ and $\omega(x)/|x| \to +\infty$ as $|x| \to \infty$.
- *(LS)* $L \in C^\infty$ ($L \in C^3$ classically needed assumption)
- *(LC)* $L(x, u, \cdot)$ is convex $\forall x, u$.

One of the first results in the modern calculus of variations establishes the existence of minimisers under the assumptions superlinearity and (LC). Beside this, before 1923 Tonelli also proved his striking partial regularity result: superlinearity, (LS) and strict (LC) ensure that for the minimiser $u$ its derivative $u' : [a, b] \to [-\infty, +\infty]$ is continuous. In particular $S_u = \{x : |u'(x)| = \infty\}$ is closed (and of measure zero) and it can be shown that $u$ is smooth on $[a, b] \setminus S_u$.

The question, whether $S_u$ can indeed be nonempty was answered in the positive by Lavrienteff in 1926 and later Ball-Mizel ([BM]) and Davie ([D]), who showed that any closed set of measure zero is the singular set $S_u$ for a suitably chosen but nice Lagrangian $L$. See also [BGH] for more historical information.

Due to the superlinear growth, points $x$ with $|u'(x)| = \infty$ cost lots of energy $E$ and can, therefore, only occur $(x, u)$ is situated on the ground of a "steep valley" (with vertical direction) in the $(x, u)$ energy landscape. The natural question arises, how many such locations can be at all created, even we do not require them to belong to a single minimiser.

**Definition 1** (J.M.Ball) The universal singular set of the Lagrangian $L$ is the union of the two sets $USS^{\pm}(L)$ defined as

$$\{(x_0, u_0) : \exists (a, A, b, B) \exists u \text{ minim. for } E : u(x_0) = u_0, u'(x_0) = \pm\infty\}$$

Ball-Nadirashvili (see [BN]) established in 1993 that $USS(L)$ is a set of first category, in 1994 Sychev ([S]) showed it is of planar measure zero, but no examples of infinite length were known. Here we answer the question of its Hausdorff dimension by giving an essentially precise geometric characterization. We need the following definitions

**Definition 2** A Borel set $E \subset \mathbb{R}^2$ is purely unrectifiable if $H^1(E \cap \text{im}(\phi)) = 0$ for any lipschitz curve $\phi : \mathbb{R} \to \mathbb{R}^2$. We say that a lipschitz curve $\phi$ is a "non-climbing graph" if $\phi'(t) \neq 0$ and $\arg(\phi'(t)) \in [-\pi/2, \pi/2)$ a.e.

Note that, in particular, PU-sets can be of Hausdorff dimension two, although Fubini's theorem clearly implies that they are of planar measure zero. Much more information on this kind of sets can be found in [ACP]. Armed with these definitions, we obtain

**Theorem 1** Let $L$ have some superlinear growth and satisfy (LC). Then

$$H^1(USS^+(L) \cap \mathrm{im}(\phi)) = 0$$

for every non-climbing graph $\phi$.

**Theorem 2** If any compact purely unrectifiable set $S$ and any superlinearity $\omega$ are given then a Lagrangian $L$ with $\omega$-growth, (LC) and (LS) can be found with $S \subset USS^+(L)$.

It turns out that the difference between Theorem 1 and 2 can not be ignored, i.e. universal singular sets are purely unrectifiable only when tested by a suitably adapted class of curves.

**Theorem 3** Given any fixed superlinearity $\omega$ one can find a 1-rectifiable set of positive 1-measure $E$ (essentially the graph of a Cantor function over its singular points) and a Lagrangian $L$ of $\omega$-growth and satisfying (LC) and (LS) do exist such that $S \subset USS^+(L)$.

Finally, there is also a characterization of pure unrectifiablity (with respect to all curves $\phi$) provided by

**Theorem 1$^*$** If $S$ is fixed and if for any $\omega$ we find a Lagrangian $L$ with $\omega$-growth and $S \subset USS^+(L)$ then $S$ is purely unrectifiable.

## References

[ACP] G. Alberti, M. Csörnyei, D. Preiss, *Structure of null sets in the plane and applications*, Proceedings of the European Congress of Mathematics 2004, EMS Publishing House, 3 – 22.

[BM] J. M. Ball, V. Mizel, *One-dimensional variational problems whose minimizers do not satsify the Euler-Lagrange equation*, Arch. Rational Mech. Anal. **90** (1985), 325–388.

[BN] J. M. Ball, N. S. Nadirashvili, *Universal singular sets for the one-dimensional variational problems*, Calc. Var. PDE **1**/4 (1993), 429–438.

[BGH] G. Buttazo, M. Giaquinta, S. Hildebrandt, *One-dimensional variational problems. An introduction*, OUP 1998.

[D] A M. Davie, *Singular minimisers in the calculus of variations in one dimension*, Arch. Rational Mech. Anal. **101** (1988), 161–177.

[S] M. Sychev, *The Lebesgue measure of a singular set in the simplest problem in the calculus of variations*, Siberian Math. J. **35** (1994), 1220–1233.

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