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Mini-Workshop: Operators on Spaces of Analytic Functions

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Abstract. The major topics discussed in this workshop were invariant subspaces of linear operators on Banach spaces of analytic functions, the ideal structure of H^∞, asymptotics for condition numbers of large matrices, and questions related to composition operators, frequently hypercyclic operators, subnormal operators and generalized Cesàro operators. A list of open problems raised at this workshop is also included.

Mathematics Subject Classification (2000): 46E15, 46E20, 46J15, 46J20, 47B15, 47B20, 47B32, 47B33, 47B37, 47B38.

Introduction by the Organisers

During the last twenty years there has been considerable activity and progress in the study of Banach spaces of analytic functions and their operators such as composition operators, multiplication operators and related natural classes of operators on these spaces. By use of the theory of analytic functions it is often possible to obtain very precise knowledge of the fine structure of these particular operators, and that has repercussions for our understanding of all bounded linear operators between Banach or Hilbert spaces. For example, the invariant subspace problem for separable Hilbert spaces can be reduced to the study of the invariant subspace structure of certain composition operators (E. Nordgren, P. Rosenthal and F. S. Wintrobe, 1987) or certain multiplication operators (C. Apostol, H. Bercovici, C. Foiaş and C. Pearcy, 1985).

Therefore, questions concerning invariant subspaces were one of the major topics in this mini–workshop. A. Atzmon considered in his first contribution scales of Banach spaces of analytic functions on which the lattice of invariant subspaces for the shift or the backward shift has no proper gaps. In his second talk he reported on recent progress concerning the translation invariant subspace problem on weighted ℓ^2 spaces over $\mathbb{Z}$. A. Aleman and S. Richter investigated relations between the index $\dim M/zM$ of an invariant subspace M for the operator of multiplication by the variable z on Hilbert spaces of analytic functions on the unit disc $\mathbb{D}$ and the boundary behavior of the functions it contains, as well as the connection between abstract properties of this operator and the boundary behavior of functions in such Hilbert spaces of analytic functions.

Questions related to composition operators were the second major subject of the meeting. W. Smith gave an equivalent formulation of the Brennan conjecture concerning the derivative of conformal maps of the unit disc in terms of compactness of certain weighted composition operators. Th. W. Gamelin related the essential spectrum of composition operators on uniform algebras to the notion of hyperbolic boundedness. E. Gallardo–Gutiérrez focused on the problem of characterizing boundedness or compactness of composition operators on Hardy spaces $H^p(\Omega)$ over simply connected domains Ω in the complex plane. In particular she presented a complete characterization of symbols inducing bounded and compact composition operators on H^p–spaces over Lavrientev domains ($1 \leq p < \infty$). C. Sundberg considered natural topologies on the set of composition operators on Banach spaces of analytic functions on the unit disc, for which all composition operators induced by analytic maps $\varphi : \mathbb{D} \to \mathbb{D}$ are bounded. M. Jury considered the C^* algebra C_Γ generated by the composition operators C_φ, $\varphi \in \Gamma$, Γ a non–elementary Fuchsian group, and obtained index theorems for certain sums of weighted composition operators. His proofs involve the use of methods from noncommutative geometry.

Spectral properties of generalized Cesàro operators and weighted averaging operators on Hardy and weighted Bergman spaces were considered by E. Albrecht.

N. Feldman introduced the idea of interpolating measures for subnormal operators S, a natural generalization of the notion of bounded point evaluations. In the case that $S = M_z$ on $P^2(\mu)$ is pure, he gave a characterization of interpolating measures. He also included some illuminating examples. W. T. Ross showed that, for a finite, compactly supported positive measure μ, the set of cyclic multiplication operators on $L^2(\mu)$ coincides with the set S_μ of all multiplication operators M_ϕ on $L^2(\mu)$, $\phi \in L_i^\infty(\mu)$, where $L_i^\infty(\mu)$ denotes the set of all μ–essentially bounded functions that are one–to–one on a set of full μ-measure. He also showed that S_μ has a common cyclic vector if and only if the continuous part of μ vanishes.

H. Jarchow discussed the problem of bounded extensions of bounded bilinear forms with an application to H^∞ and the disc algebra.

R. Mortini gave in his talk a complete characterization of all countably generated prime ideals in H^∞: A nonzero prime ideal I in H^∞ is countably generated if and only if either $I = \{f \in H^\infty;\, f(z_0) = 0\}$ for some $z_0 \in \mathbb{D}$ or if I is the ideal generated by the n-th roots of $S_\sigma(z) = \exp\Big(-\frac{1+\bar{\sigma}z}{1-\bar{\sigma}z}\Big)$ for some $\sigma \in \partial\mathbb{D}$.

N. Nikolski showed that the condition numbers of the worst $N \times N$ matrices A satisfying a Besov space $B^s_{p,q}$ functional calculus behave asymptotically as $N^s/\det(A)$ for $N \to \infty$. The proof depends on estimates of Besov analytic capacities of N points subsets of the unit disc.

S. Shimorin discussed a new type of area theorems for univalent functions and, as a consequence, obtained a new series of sharp integral inequalities. His work was motivated by recent progress on Brennan's conjecture.

For a scale of spaces of analytic functions in $\mathbb{D}$ including the Korenblum space, A. Borichev obtained quantitative uniqueness theorems of the Lyubarski–Seip type.

Motivated by Birkhoff's ergodic theorem, F. Bayart and S. Grivaux have recently introduced an interesting new concept in hypercyclicity, that of frequently hypercyclic operators. In her talk, S. Grivaux gave some sufficient criteria in terms of Banach space properties of the underlying space. K.-G. Grosse–Erdmann showed that, for a frequently hypercyclic operator T, the operator $T \oplus T$ is hypercyclic. He also proved that an operator T on the space of entire functions that commutes with the differentiation operator and that is not a multiple of the identity is frequently hypercyclic.

On Wednesday morning a problem session chaired by Th. W. Gamelin had been organized. Some of the problems discussed during that session are included at the end of this report. Further open questions were pointed out in many of the talks.

This mini–workshop was organized by Ernst Albrecht (Saarbrücken), Jean Esterle (Bordeaux), Raymond Mortini (Metz) and Stefan Richter (Knoxville). Unfortunately, Jean Esterle was unable to participate. All the participants were grateful for the hospitality and the stimulating atmosphere of the Forschungsinstitut Oberwolfach.

Mini-Workshop: Operators on Spaces of Analytic Functions

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Abstracts

Invariant subspaces for shifts and for backward shifts

Aharon Atzmon

Let X be an infinite dimensional complex Banach space and T a bounded linear operator on X. Denote by $Lat\ T$ the collection of all closed invariant subspaces of T. A natural question is whether for every two elements M_1, M_2 such that $M_1 \subset M_2$ and $\dim M_2/M_1 > 1$, there exists an element M in $Lat\ T$ such that $M_1 \subsetneq M \subsetneq M_2$. If the answer is positive, we say that $Lat\ T$ has no proper gaps.

We consider this problem for shifts and backward shifts on some classical Banach spaces of analytic functions on the unit disc $\mathbb{D}$.

For $1 \leq p < \infty$, and $\alpha \in \mathbb{R}$, we denote by D^p_α the Banach space of all analytic functions f on $\mathbb{D}$ for which the norm

$$\|f\| = \Big(\sum_{n=0}^{\infty} |\hat{f}(n)|^p (n+1)^\alpha\Big)^{1/p}$$

is finite, where $\hat{f}(n)$ denotes the n-th Taylor coefficient of f. For $1 \leq p < \infty$, and $\alpha > -1$, we denote by A^p_α the Banach space of all analytic functions f on $\mathbb{D}$ for which the norm

$$\|f\| = \Big(\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z)\Big)^{1/p}$$

is finite, where dA denotes the area measure. The spaces A^p_0 will be denoted by A^p.

The shift S and the backward shift B defined on analytic functions f on $\mathbb{D}$ by

$$\begin{aligned} Sf(z) =& zf(z), && z \in \mathbb{D}, \\ Bf(z) =& z^{-1}(f(z) - f(0)), && z \in \mathbb{D} \setminus \{0\}, \end{aligned}$$

are bounded linear operators on all these Banach spaces.

We prove the following results:

Theorem 1. *$Lat\, T$ has no proper gaps for the spaces D^p_α, $\alpha \geq 0$, $p = 1, 2$, and for $1 \leq p < \infty$, $\alpha > p - 1$.*

Theorem 2. *Let E be a Banach algebra of holomorphic functions on $\mathbb{D}$ in which the polynomials are dense, which is invariant under the shift S and the backward shift B. If for some $\alpha \geq 0$,*

$$\begin{aligned} \|z^n\| =& O(n^\alpha), \quad n \to \infty, \quad \textit{and} \\ \log^+ \|B^n\| =& O(n^{1/2}), \quad n \to \infty, \end{aligned}$$

then $Lat\, S$ has no proper gaps.

Remark. In this case the elements of $Lat\, S$ are closed ideals in E.

Theorem 3. *$Lat\, B$ has no proper gaps for the spaces A^2_α, $\alpha > -1$, the spaces A^p, $1 < p \leq 2$, and the spaces D^p_α, $1 < p < \infty$, $\alpha < -1$.*

Remarks. 1.) From the known results on the structure of S invariant subspaces for the spaces $H^p(\mathbb{D})$, $1 \le p < \infty$, and the structure of w^* closed ideals in $H^\infty(\mathbb{D})$, it follows that $Lat\, S$ has no proper gaps on $H^p(\mathbb{D})$ for $1 \le p < \infty$, and by duality the same holds true for $Lat\, B$ on these spaces.

2.) It is known that the problem whether $Lat\, S$ has no proper gaps on the Bergman space A^2 is equivalent to the invariant subspace problem on Hilbert space, and by duality, the same is true for the problem whether $Lat\, B$ has no proper gaps for the Dirichlet space D_1^2.

The proofs of Theorems 1–3 are given in [2] and are based on an invariant subspace theorem in [1].

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Hilbert spaces of entire functions and translation invariant subspaces

Aharon Atzmon

Let $\omega = (\omega_n)_{n\in\mathbb{Z}}$ be a sequence of positive numbers such that for some constant $c > 0$

$$c^{-1} \le \frac{\omega(n+1)}{\omega(n)} \le c, \qquad \forall n \in \mathbb{Z}, \tag{1}$$

and denote by $\ell^2_\omega(\mathbb{Z})$ the Hilbert space of all sequences $a = (a(n))_{n\in\mathbb{Z}}$ for which the norm

$$\|a\| = \Big(\sum_{n=-\infty}^{\infty} |a(n)|^2 \omega^2(n) \Big)^{1/2}$$

is finite. The assumption (1) implies that $\ell^2_\omega(\mathbb{Z})$ is translation invariant, and it is an open problem whether for every such ω this space has a nontrivial translation invariant subspace. The problem was reduced by Apostol [1] to the case that

$$\log \omega(n) = o(|n|), \qquad n \to \pm\infty.$$

The answer is known to be positive if $\sum_{n=-\infty}^{\infty} \frac{|\log \omega(n)|}{n^2+1} < \infty$ [2], when ω is non–increasing and $\omega(n) = 1$ for $n \ge 0$ [6], when $\omega(-n)\omega(n) = 1$, $\forall n \in \mathbb{Z}$, and the sequence $(\log \omega(n))_{n=0}^{\infty}$ is concave [5], and when ω is even [4]. Our new results are:

Theorem 1. *Assume that ω is a sequence which satisfies (1), $\omega(0) = 1$, the sequence $(\omega(n))_{n=0}^{\infty}$ is non–decreasing and $\sum_{n=1}^{\infty} n^{-3/2} \log \omega(n) < \infty$. If*

$$\limsup_{n\to\infty} n^{-1/2} \log \omega(-n) < 0$$

then $\ell^2_\omega(\mathbb{Z})$ has a nontrivial translation invariant subspace.

To state our second result, we introduce a notation. For a sequence ω that satisfies (1), we denote by ω^* the sequence on $\mathbb{Z}_+$ defined by

$$\log \omega^*(n) = \frac{n^{3/2}}{\pi} \sum_{j=1}^{\infty} \frac{\log \omega(j)}{j^{3/2}(n+j)}, \qquad n \in \mathbb{Z}_+ .$$

Theorem 2. *Let ω be a sequence that satisfies (1), the sequence $(\log \omega(n))_{n=0}^{\infty}$ is concave and non–negative, and one of the following two condition holds:*

(a) $\sum_{n=1}^{\infty} n^{-2} \log(n) < \infty$.

(b) $\log \omega(n+1) + \log \omega(n-1) - 2 \log \omega(n) = O(n^{-1})$, $n \to \infty$.

If $\sum_{n=1}^{\infty} n^{-3/2} \log(n) = \infty$, and $\omega(-n)\omega^*(n) = O(1)$, $n \to \infty$, then $\ell^2_\omega(\mathbb{Z})$ has a nontrivial translation invariant subspace.

The proofs of the theorems are based on sampling theorems for Hilbert spaces of entire functions of zero exponential type introduced in [3].

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Interpolating Measures for Subnormal Operators

Nathan S. Feldman

If $\mu \in M^+(K)$ is a positive regular Borel measure supported on a compact set K in the complex plane, then let $R^2(K,\mu)$ denote the closure of $\mathrm{Rat}(K)$, the rational functions with poles off K, in $L^2(\mu)$. If we define $S_{K,\mu} = M_z$ on $R^2(K,\mu)$, then $S_{K,\mu}$ is a typical rationally cyclic subnormal operator. When K is polynomially convex, then $R^2(K,\mu) = P^2(\mu)$, the closure of the analytic polynomials in $L^2(\mu)$, and $S_\mu := S_{K,\mu}$ will be a cyclic subnormal operator.

If $\lambda \in K$, then λ is a bounded point evaluation (b.p.e.) for $S_{K,\mu} = M_z$ on $R^2(K,\mu)$ if there is a constant $C > 0$ such that $|f(\lambda)| \leq C\|f\|_{L^2(\mu)}$ for all $f \in \mathrm{Rat}(K)$. This is equivalent to requiring that the densely defined linear operator $A : \mathrm{Rat}(K) \to \mathbb{C}$ given by $A(f) = f(\lambda)$ extends to an (onto) bounded linear operator $A : R^2(K,\mu) \to \mathbb{C}$ (the extension is also called A).

Thomson's Theorem [1] says that if $S_\mu = M_z$ on $P^2(\mu)$ is pure, then b.p.e.'s always exist for S_μ. However it is known (see [2]) that b.p.e.'s need not exist for

$R^2(K,\mu)$ spaces. We are looking to generalize the idea of a b.p.e. for a $R^2(K,\mu)$ space to the notion of an interpolating measure for any subnormal operator.

For a measure $\nu \in M^+(K)$, ν will be an *interpolating measure* for $S_{K,\mu} = M_z$ on $R^2(K,\mu)$ if the densely defined map $A : \mathrm{Rat}(K) \to L^2(\nu)$ defined by $A(f) = f$ extends to be an (into and) *onto* bounded linear operator $A : R^2(K,\mu) \to L^2(\nu)$.

Question 1. If K is a compact set in the complex plane and μ a measure on K, then does $S_{K,\mu} = M_z$ on $R^2(K,\mu)$ have an interpolating measure?

For an arbitrary operator S on a Hilbert space $\mathcal{H}$, a measure ν is said to be an *interpolating measure* for S if there exists an (into and) *onto* bounded linear operator $A : \mathcal{H} \to L^2(\nu)$ such that $AS = N_\nu A$, where $N_\nu = M_z$ on $L^2(\nu)$.

Question 2. If S is a subnormal operator, then does S have an interpolating measure? If not, which subnormal operators have interpolating measures?

Theorems. (a) *If $S_\mu = M_z$ on $P^2(\mu)$ is pure and G is the set of b.p.e.'s for S_μ, then a measure ν is an interpolating measure for S_μ if and only if ν is a discrete measure carried by G whose atoms form a $P^2(\mu)$ interpolating sequence.*

(b) *If $S = M_z$ on $H^2(G)$ where $G = \mathbb{D} \setminus [0,1]$, then Lebesgue measure on $[0,1]$ is an interpolating measure for S.*

(c) *If $S = M_{\overline{z}}$ on $L^2_a(\mathbb{D})^\perp$ is the dual of the Bergman operator, then for any compact set $K \subseteq \mathbb{D}$, $\nu =$ area measure on K is an interpolating measure for S.*

Question 3. Are there bounded regions G in $\mathbb{C}$ such that $S = M_z$ on the Bergman space $L^2_a(G)$ has a continuous interpolating measure that is supported on ∂G?

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Common cyclic vectors for normal operators

William T. Ross

(joint work with Warren Wogen)

It is well known that every bounded normal operator on a separable Hilbert space is unitarily equivalent to a multiplication operator $M_\phi : L^2(\mu) \to L^2(\mu)$, $M_\phi f = \phi f$, where μ is a positive, finite, compactly supported measure in the plane, and $\phi \in L^\infty(\mu)$, the μ-essentially bounded functions.

Using a beautiful theorem of Bram [4], along with the change of variables formula and the Stone-Weierstrass theorem, one can show [6] that the cyclic multiplication operators on $L^2(\mu)$ are

$$S_\mu := \{(M_\phi, L^2(\mu)) : \phi \in L^\infty_i(\mu)\}.$$

Here ‘cyclic’ means that there is an $f \in L^2(\mu)$ such that

$$\bigvee \{\phi^n f : n = 0, 1, \cdots\} = L^2(\mu),$$

and $L_i^\infty(\mu)$ are the μ-essentially bounded functions that are one-to-one on a set of full μ-measure. We ask the question: Does S_μ have a *common cyclic vector*? In other words, is there a *single* $f \in L^2(\mu)$ so that

$$\bigvee \{\phi^n f : n = 0, 1, \cdots\} = L^2(\mu)$$

for *every* $\phi \in L_i^\infty(\mu)$?

Such common cyclic vector problems have been explored before [3, 8] but usually for classes of operators of the form M_ϕ^*, where ϕ is a non-constant multiplier of some reproducing kernel Hilbert space of analytic functions on a planar domain. The main theorem from these papers is that this class of operators has a common cyclic vector and the vector takes the form

$$f = \sum_{j=1}^{\infty} c_j k_{\lambda_j},$$

where k_{λ_j} are the reproducing kernels for the Hilbert space.

Our main common cyclic vector theorem is the following:

Theorem 1. *Suppose $\mu = \mu_d + \mu_c$, is the decomposition of μ into its discrete part μ_d and its continuous part μ_c. Then S_μ has a common cyclic vector if and only if $\mu_c \equiv 0$.*

The proof that S_{μ_d} has a common cyclic vector, as well as some other results, was shown in [7] using Borel series. In [6], we show, using construction involving Szegö’s theorem, that S_m, where m is standard Lebesgue measure on the unit circle T, does not have a common cyclic vector. We then use this result to show, by using the theory of Lebesgue spaces [5], that S_{μ_c} (assuming $\mu_c \not\equiv 0$) does not have a common cyclic vector. The result now follows from the decomposition

$$(M_\phi, L^2(\mu)) \simeq (M_\phi, L^2(\mu_d)) \oplus (M_\phi, L^2(\mu_c)).$$

Since S_m does not have a common cyclic vector, one can ask as to whether or not there is some interesting subclass of S_m that *does* have a common cyclic vector. We have the following positive result:

Theorem 2. *Let $\mathcal{A}$ be the class of $\phi \in C^{1+\epsilon}(T)$ for some $\epsilon > 0$, such that ϕ is injective, except possibly for a finite number of points, and such that the derivative of ϕ never vanishes. Then $\mathcal{A}$ has a common cyclic vector.*

The proof of this theorem involves some estimates of harmonic measure and a generalization of Szegö’s theorem from [1, 2].

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Branching point area theorems for univalent functions

Serguei Shimorin

Area methods are a classical tool in the theory of univalent functions. Such topics as Grunsky, Goluzin, or Schiffer-Tammi inequalities are in fact different modifications of the Polynomial Area Theorem which in turn reduces to an appropriate application of the Green formula. Recent progress (due to Hedenmalm and the author) in estimating integral means of the derivatives of univalent functions (see [2], [1]) was based on the use of area inequalities combined with techniques of Bergman spaces. This is an inspiration for taking a new look on area methods in the theory of conformal mappings.

The classical polynomial area theorem for univalent functions can be formulated as follows: if ψ is a function univalent in the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$, and p is a Laurent polynomial

$$p(z) = \sum_{n=0}^{m} \frac{p_n}{z^n},$$

then the composition $p(\psi)$ written as a Laurent series converging for z near $\mathbb{T} = \partial\mathbb{D}$

$$p(\psi(z)) = \sum_{k=-\infty}^{+\infty} c_k z^k$$

satisfies

$$\big[p(\psi), p(\psi)\big] \le 0, \tag{1}$$

where $\big[\cdot,\cdot\big]$ is the indefinite inner product defined as

$$\big[g, g\big] = \sum_{k\in\mathbb{Z}} k|c_k|^2 \qquad \text{for} \qquad g(z) = \sum_{k\in\mathbb{Z}} c_k z^k. \tag{2}$$

It is well-known that the polynomial area theorem is equivalent to different modifications of the Grunsky inequalities and Goluzin inequalities which are usually more important for the applications than the area theorem itself. In fact, the derivation of Grunsky or Goluzin inequalities from the area theorem reduces to an *effective* (in appropriate sense, say, for rational functions p written in the form of sums of partial fractions) calculation of the positive and negative parts of the decomposition of $p(\psi)$ with respect to the above indefinite norm.

A new type of area theorems is obtained by considering branching point compositions with univalent functions. More precisely, we consider functions of the form

$$p(z) = (z - \mu_1)^{\theta_1} \dots (z - \mu_n)^{\theta_n} q(z), \tag{3}$$

where $\mu_1, \dots, \mu_n \in \psi(\mathbb{D})$, "branching multiplicities" θ_k are from the interval $(0, 1)$, and q are rational functions with poles in $\psi(\mathbb{D})$. The main point of the study is an analysis in the two-sided Dirichlet space on the unit circle supplied with the natural indefinite inner product. As a result, we obtain a new series of sharp integral inequalities. We discuss also branching point versions of Grunsky and Goluzin inequalities. In order to formulate them in an economical form, we introduce an abstract language of domination of kernel functions. This language gives a better understanding of even such classical topics as Goluzin-Lebedev inequalities for univalent functions.

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Asymptotics of Condition Numbers of Large Matrices

Nikolai Nikolski

How to bound the resolvent of a matrix or an operator in terms of its "spectral data"? For instance, for $N \times N$ matrices acting on $\mathbb{C}^N$ the question is, given a set Υ_N of $N \times N$ invertible matrices, how to find a function Φ_N such that

$$\|T^{-1}\| \leq \Phi_N(\delta)$$

for every $T \in \Upsilon_N$, where δ stands for the minimum modulus of the eigenvalues of T, $\delta = \min |\lambda_i(T)|$? how the best possible majorant Φ_N looks like? how it behaves as $N \longrightarrow \infty$? does it always exist?

It is well-known that, to be meaningful, these questions require a kind of normalization. In numerical analysis, the usual normalization is to replace $\|T^{-1}\|$ by the condition number $CN(T) = \|T\| \cdot \|T^{-1}\|$ and then look for an estimate for $CN(T)$ in terms of $\|T\|/\delta$. An equivalent approach is simply to include the normalization condition $\|T\| \leq 1$ into the definition of Υ.

Here, we use the second kind of normalization conditions and consider operators $T : X \longrightarrow X$ acting both on finite dimensional Banach/Hilbert space X, $\dim X = N < \infty$ ("$N \times N$ matrices") and on infinite dimensional ones. In order to have a more flexible classification of these operators than those given by the standard normalization $\|T\| \le 1$, we consider families Υ of operators obeying a functional calculus over a function space (algebra) A, i.e.

$$\|f(T)\| \le C\|f\|_A$$

for every polynomial f. Let A_C be the set of all such operators. Using this functional calculus classification for $N \times N$ matrices, we should consider as a true parameter for asymptotics of inverses or condition numbers not the dimension N but the degree of the minimal annihilating polynomial of T, $n = \deg(m_T) \le N$. That is why we can pass for free from $N \times N$ matrices to infinite dimensional algebraic operators of degree $\le n$ on an arbitrary Banach/Hilbert space X. Recall that T is algebraic if there exists a polynomial $p \ne 0$ such that $p(T) = 0$; we write the minimal annihilating polynomial in a monic form, $m_T(z) = m_\sigma(z) = \prod_{1 \le k \le n}(\lambda_k - z)$, where $\sigma = \{\lambda_1, \dots, \lambda_n\}$ is the spectrum $\sigma(T)$ of T (consisting of eigenvalues of T, with possible multiplicities as they occur in the minimal polynomial).

Given a function space A on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, a constant $C \ge 1$, and a family $\sigma = \{\lambda_1, \dots, \lambda_n\}$ in $\overline{\mathbb{D}} \backslash \{0\}$, we denote $\Upsilon(m_\sigma, A_C)$ the set of all algebraic operators T such that $T \in A_C$ and $m_\sigma(T) = 0$. In order to measure the size of inverses and condition numbers we use the following quantities

$$\varphi(m_\sigma, A_C) = \sup\{\|T^{-1}\| : T \in \Upsilon(m_\sigma, A_C)\},$$

$$\Phi_n(\Delta, A_C) = \sup\{\varphi(m_\sigma, A_C) : \sigma \subset \{z : 1/\Delta \le |z| \le 1\}, card(\sigma) \le n\} =$$

$$= \sup\{\|T^{-1}\| : T \in A_C,\ r(T^{-1}) \le \Delta,\ \deg(T) \le n\},$$

where $r(\cdot)$ stands for the spectral radius. If A is a Banach algebra, then the worst operator realizing the sup in $\varphi(m_\sigma, A_1)$ is the quotient operator $S/m_\sigma A$ of the shift operator $S : A \longrightarrow A$, $Sf = zf$ acting on $A/m_\sigma A$, or equivalently, its adjoint operator $S^*f = \frac{1}{z}(f - f(0))$ (the backward shift) acting on the subspace $(A/m_\sigma A)^* = K_{m_\sigma} \subset A^*$,

$$K_{m_\sigma} = span\Big(\frac{1}{1 - \overline{\lambda} z} : \lambda \in \sigma\Big),$$

with an obvious modification in the case of multiplicities in $\sigma = \{\lambda_1, \dots, \lambda_n\}$ (m times repeated λ gives the series $\dfrac{1}{(1 - \overline{\lambda} z)^j}$, $1 \le j \le m$).

Notice also that the standard normalization $\|T\| \le 1$ is equivalent to the condition $T \in A_1$, where $\|\cdot\|_A$ is the uniform norm $\|\cdot\|_\infty$ for the case of a Hilbert space X, $\|f\|_\infty = max_{|z| \le 1}|f(z)|$, and is the Wiener class norm $\|f\|_W = \sum_{k \ge 0} |\hat{f}(k)|$ for the case of an arbitrary Banach space X.

The case of all N-dimensional operators (matrices) acting on a Banach space, $T : X \longrightarrow X$, $\dim X = N$, was considered by J.J.Schäffer (1970) who, answering a question of B.L. van der Waerden, has showed that $\|T^{-1}\| \cdot |det(T)| \le k_N \|T\|^{N-1}$ for every invertible $N \times N$ matrix T, where $k_N \le \sqrt{eN}$. It follows that $\|T^{-1}\| \leq \sqrt{eN}(r(T^{-1}))^N$ for every contraction $\|T\| \leq 1$ on a N-dimensional Banach space X. E.Gluskin, M.Meyer, and A. Pajor (1994) gave another proof to Schäffer's result and disproved a Schäffer conjecture about the lower estimate showing by a probabilistic method that k_N is not bounded and moreover, $k_N \geq \frac{c_1}{\log\log(N)}\sqrt{\frac{N}{\log(N)}}$. The same paper contains a stronger counterexample by J. Bourgain giving $k_N \geq c_2\sqrt{\frac{N}{\log(N)}}$. Finally, H. Queffelec (1994), using a deterministic (number theory) approach, proved that Schäffer's inequality is sharp, i.e., $k_N \geq c_3\sqrt{N}$.

In our approach, considering an operator T in $\Upsilon(A_C, m_\sigma)$, we begin with an observation that $\|T^{-1}\| \leq C \cdot inf\|f\|_A$, where f runs over all solutions to the Bezout equation $zf + m_\sigma g = 1$. In the case when A is a Banach algebra, it leads to a two sided estimate $a \cdot cap_A(\sigma) \leq \varphi(m_\sigma, A_C) \leq b \cdot cap_A(\sigma)$, where $a, b > 0$ are constants and cap_A stands for the A-zero capacity (at $z = 0$)

$$cap_A(\sigma) = \inf\{\|f\|_A : f(0) = 1, f|\sigma = 0\}.$$

Therefore, in order to estimate $\varphi(m_\sigma, A_C)$, or to know asymptotics for $\Phi_n(\Delta, A_C)$ as $\Delta \longrightarrow \infty$ and/or $n \longrightarrow \infty$, we need to bound $cap_A(\sigma)$ or the following maximal capacities, an annular and a circular ones,

$$\kappa_n(\Delta, A) = \sup\{cap_A(\sigma) : \sigma \subset \{z : 1/\Delta \leq |z| \leq 1\}, card(\sigma) \leq n\},$$
$$\underline{\kappa}_n(\Delta, A) = \sup\{cap_A(\sigma) : \sigma \subset \{z : |z| = 1/\Delta\}, card(\sigma) \leq n\}.$$

This idea applying to the analytic Besov spaces $A = B^s_{p,q}$ ($s \geq 0$, $1 \leq p, q \leq \infty$) gives the following estimates

$$cap_{B^s_{p,q}}(\sigma) \leq c\frac{(card(\sigma))^s}{|m_\sigma(0)|}, \quad \kappa_n(\Delta, B^s_{p,q}) \leq c\Delta^n n^s$$

for $s > 0$, where $c = c(s, q)$, and

$$cap_{B^s_{p,q}}(\sigma) \leq c\frac{(\log(card(\sigma)))^{1/q}}{|m_\sigma(0)|}, \quad \kappa_n(\Delta, B^s_{p,q}) \leq c\Delta^n(\log(n))^{1/q}$$

for $s = 0$, where $c > 0$ is a numerical constant. It is also shown that for $s > 0$ these estimates are asymptotically sharp, namely

$$\frac{k}{2^{s+1}}n^s \leq \underline{\lim}_{\Delta\longrightarrow\infty} \frac{\underline{\kappa}_n(\Delta, A)}{\Delta^n} \leq \underline{\lim}_{\Delta\longrightarrow\infty} \frac{\kappa_n(\Delta, A)}{\Delta^n}.$$

The corresponding inequalities for the inverses and condition numbers are immediate consequences of these estimates.

Recall that Besov classes functional calculi are applied for the following operators.

(1) The set Υ of *Hilbert space* power bounded operators $\sup_{k\geq 0}\|T^k\| = a$. Here, V.Peller's $B^0_{\infty,1}$ calculus (1982) is applied, which implies $\|T^{-1}\| \leq const \cdot a^2\frac{\log(deg(T))}{|m_T(0)|}$.

(2) A similar bound holds for *Banach space* Tadmor-Ritt operators $T : X \longrightarrow X$, $\|R(\lambda, T)\| \leq C|\lambda - 1|^{-1}$ for $|\lambda| > 1$, by using a P.Vitse's $B^0_{\infty,1}$ functional calculus (2004).

(3) The set of Banach space Kreiss operators T defined by the resolvent estimate $\|R(\lambda, T)\| \leq C(|\lambda| - 1)^{-1}$ for $|\lambda| > 1$. In this case, we use P.Vitse's $B^1_{1,1}$ functional calculus (2003), which gives an asymptotics $\Delta^n n$ (up to constants do not depending on Δ and n) for the capacity $\kappa_n(\Delta, B^1_{1,1})$ and for inverses/condition numbers $\Phi_n(\Delta, (B^1_{1,1})_C)$, where as before $n = \deg(T)$, $\Delta > 1$. A higher growth of the resolvent $\|R(\lambda, T)\| \leq C(|\lambda| - 1)^{-s}$, $s > 1$, leads to a $B^s_{1,1}$ calculus (V. Peller; P. Vitse), and hence to the corresponding estimates for inverses and condition numbers in terms of the $B^s_{1,1}$ capacity (which is of the order of $\Delta^n n^s$).

(4) The set of *Hilbert space* operators T satisfying $C = \sup_{k\geq 1} \|T^k\|/k^\beta < \infty$, $\beta > 0$. By yet another Peller's theorem (1982), T obeys a $B^{2\beta}_{\infty,1}$ functional calculus. Therefore, $\|T^{-1}\| \leq c\dfrac{(deg(T))^{2\beta}}{|m_T(0)|}$, for every algebraic operator of this class. For $\beta < 1/2$, this is better than an estimate which follows from an $l^1_A(k^\beta)$-functional calculus (and, consequently, is true for all Banach space operators satisfying $\sup_{n\geq 1} \|T^k\|/k^\beta < \infty$). By the way, for $\beta > 1/2$, the obvious $l^1_A(k^\beta)$-functional calculus can not be improved even for operators on a Hilbert space: there exists a Hilberts space operator satisfying $\sup_{k\geq 1} \|T^k\|/k^\beta < \infty$ such that the norm $f \longmapsto \|f(T)\|$ is equivalent to $\|f\|_{l^1_A(k^\beta)}$ (N.Varopoulos, 1972).

We also consider yet another series of function spaces, namely,

$$l^q_A(w_k) = \Big\{ f = \sum_{k\geq 0} \hat{f}(k) z^k : \|f\|_{l^q_A(w_k)} = \Big(\sum_{k\geq 0} \big|\hat{f}(k)\big|^q w_k^q \Big)^{1/q} \Big\},$$

where $w_k > 0$ such that $\lim_k w_k^{1/k} = 1$, $0 < \inf_k \frac{w_{k+1}}{w_k} \leq \sup_k \frac{w_{k+1}}{w_k} < \infty$ and $1 \leq q \leq \infty$. It is shown that

$$cap_{l^q_A(w_k)}(\sigma) \leq \frac{\gamma_q(card(\sigma))}{|m_\sigma(0)|},$$

where $\gamma_q(n)$ is defined in terms of the so-called Lagrange transform of the weight sequence $(w_k)_{k\geq 0}$; in particular, for $w_k = k^\beta$, $k \geq 1$ ($w_0 = 1$), $\gamma_q(n) \leq n^\beta$ for $q \geq 2$ and $\gamma_q(n) \leq an^{\beta + \frac{1}{q} - \frac{1}{2}} + b$ for $1 \leq q < 2$, where $a, b > 0$ are constant depending only on q and β. As a corollary one can get an estimate $\big\|T^{-1}\big\| \leq \dfrac{bCn^{\beta+(1/2)}}{|m_T(0)|}$, where $b > 0$ is a constant depending on (w_k), for every Banach space operator T with $\deg(T) \leq n$ and $C = \sup_{k\geq 1} \|T^k\|/k^\beta < \infty$.

Previous results can be viewed as bounding $\|1/z\|_{A/m_\sigma A} \big/ \|1/z\|_{H^\infty/m_\sigma H^\infty}$. We also briefly consider the problem of comparison of the norms $a \longmapsto \|a\|_{A/m_\sigma A}$, where A is a function Banach algebra on $\mathbb{D}$, and $a \longmapsto \|a\|_{H^\infty/m_\sigma H^\infty}$ for an arbitrary function a on σ. This is a kind of the Nevanlinna-Pick interpolation problem for a function algebra A. In fact, we consider three function algebras only, namely, $B^1_{1,1}$, W, $B^0_{\infty,1}$, and obtain estimates for the quantity

$$k_n(A) = \sup\Big\{\frac{\|a\|_{A/m_\sigma A}}{\|a\|_{H^\infty/m_\sigma H^\infty}} : a \in A, a \neq 0, \sigma \subset \mathbb{D}, card(\sigma) \leq n\Big\}.$$

We show that for all three algebras k_n is equivalent to n despite the fact that these (strictly) embedded algebras $B^1_{1,1} \subset W \subset B^0_{\infty,1} \subset H^\infty \bigcap C(\overline{\mathbb{D}})$ are quite different. For instance, $\|z^n\|_{B^1_{1,1}}$ is equivalent to n as $n \longrightarrow \infty$, but for three other algebras $\|z^n\|$ is bounded; the constants describing the asymptotics of the worst norm $\|T^{-1}\|$ as $T \in A_C, n = \deg(T) \longrightarrow \infty$, also behave differently for all three algebras (namely, as it follows from the preceding results, they are of the order of $\log(n+1)$, $\sqrt{n}$, and n, respectively).

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Uniqueness theorems for Korenblum type spaces

Alexander Borichev
(joint work with Yuri Lyubarskii)

Given a topological space X of analytic functions in the unit disc $\mathbb{D}$ and a class $\mathcal{E}$ of subsets E of $\mathbb{D}$, we call a non-decreasing positive function $M : [0,1) \to (0,\infty)$ a minorant for the pair $(X,\mathcal{E})$ and write $M \in \mathcal{M}(X,\mathcal{E})$ if

$$\begin{aligned} f \in X, &\qquad E \in \mathcal{E}, \\ \log|f(z)| \leq -M(|z|), &\qquad z \in E, \end{aligned} \tag{1}$$

imply that $f = 0$.

Clearly, $\mathcal{M}(X,\mathcal{E}) \neq \emptyset$ implies that $\mathcal{E} \subset \mathcal{U}(X)$, where $\mathcal{U}(X)$ is the family of the uniqueness subsets E for the space X: $E \in \mathcal{U}(X)$ if and only if

$$f \in X, \quad f\big|E = 0 \quad \Longrightarrow \quad f = 0.$$

Suppose that $H^\infty \subset X \subset A(\lambda)$, for some λ, where

$$A(\lambda) = \big\{f \in \mathrm{Hol}(\mathbb{D}) : \sup_{z\in\mathbb{D}} |f(z)|/\lambda(|z|) < \infty\big\}.$$

Then a simple argument shows that the class $\mathcal{M}(X,\mathcal{U}(X))$ is empty. The reason is that the class $\mathcal{U}(X)$ contains subsets $E \subset \mathbb{D}$ which are not massive enough: E may be the union of clusters E_j of nearby points in such a way that the estimate (1) on $x \in E_j$ implies a similar estimate (with M replaced by $M/2$) on the whole E_j. That is why we need to consider only elements in the family $\mathcal{U}(X)$ which are sufficiently separated. In [3], the authors deal with the case $X = H^\infty$, and consider the class $\mathcal{SU}(H^\infty)$ of hyperbolically separated subsets E of $\mathbb{D}$ that are uniqueness subsets for H^∞. They prove that

$$M \in \mathcal{M}(H^\infty, \mathcal{SU}(H^\infty)) \iff \int_0 \frac{dt}{tM(1-t)} < \infty.$$

Here we work with the scale of spaces

$$\mathcal{A}_s^r = \big\{f \in \mathrm{Hol}(\mathbb{D}) : \log|f(z)| \le r\log^s \frac{1}{1-|z|} + c_f\big\}, \qquad r,s>0,$$

$$\mathcal{A}_s = \bigcup_{r<\infty} \mathcal{A}_s^r.$$

We have $H^\infty \subset \mathcal{A}_s \subset \mathcal{A}_1 \subset \mathcal{A}_t$, $0<s<1<t$, where $\mathcal{A}_1$ is the so called Korenblum space,

$$\mathcal{A}_1 = \big\{f \in \mathrm{Hol}(\mathbb{D}) : |f(z)| \le \frac{c_f}{(1-|z|)^{c'_f}}\big\}.$$

The uniqueness subsets for $\mathcal{A}_s$ are described by Korenblum [2] (1975, $s=1$) and Seip [5] (1995, $s>0$). For $0<s<1$, we define $\mathcal{SU}(\mathcal{A}_s)$ as the class of hyperbolically separated subsets E of $\mathbb{D}$ that are uniqueness subsets for $\mathcal{A}_s$.

Theorem 1. *For regular M, $0<s<1$,*

$$M \in \mathcal{M}(\mathcal{A}_s, \mathcal{SU}(\mathcal{A}_s)) \iff \int_0 \frac{dt}{tM(1-t)} < \infty.$$

For $s=1$, no hyperbolically separated subset of $\mathbb{D}$ belongs to $\mathcal{U}(\mathcal{A}_1)$. The results of Korenblum and Seip do not give a complete description of $\mathcal{U}(\mathcal{A}_1^r)$, $r<\infty$. However, there is only a small gap between necessary conditions and sufficient conditions. In particular, it is known that every $\mathcal{U}(\mathcal{A}_1^r)$ contains hyperbolically separated subsets. We define $\mathcal{SU}(\mathcal{A}_1)$ as the class of $E \subset \mathbb{D}$ such that for every r there exists a hyperbolically separated subset E_r of E such that $E_r \in \mathcal{U}(\mathcal{A}_1^r)$.

Theorem 2. *For regular M,*

$$M \in \mathcal{M}(\mathcal{A}_1, \mathcal{SU}(\mathcal{A}_1)) \iff \int_0 \frac{dt}{tM(1-t)} < \infty.$$

For $s>1$, we introduce

$$\rho_s(z) = (1-|z|)\Big(\log\frac{1}{1-|z|}\Big)^{(1-s)/2},$$

and say that E is s-separated if for some $\varepsilon>0$,

$$|\lambda-\mu| \ge \varepsilon\rho_s(\lambda), \qquad \lambda,\mu \in E, \quad \lambda \ne \mu.$$

We define $\mathcal{SU}(\mathcal{A}_s)$, $s>1$, as the class of $E \subset \mathbb{D}$ such that for every r there exists an s-separated subset E_r of E such that $E_r \in \mathcal{U}(\mathcal{A}_s^r)$.

Theorem 3. *For regular M, $s>1$,*

$$M \in \mathcal{M}(\mathcal{A}_s, \mathcal{SU}(\mathcal{A}_s)) \iff \int_0 \Big(\log\frac{1}{t}\Big)^{s-1}\frac{dt}{tM(1-t)} < \infty.$$

Remarks. 1. In Theorems 1–3, when the integrals diverge, we can find $E \in \mathcal{SU}(\mathcal{A}_s)$ and $f \in H^\infty \setminus \{0\}$ satisfying the estimate (1).

2. Our result should be compared to that by Pau and Thomas [4] concerning $\mathcal{M}(H^\infty, \mathcal{E})$, for some special classes $\mathcal{E} \subset \mathcal{SU}(H^\infty)$.

3. By duality, using a method of Havinson [5], we can deduce from Theorem 2 a result on approximation by simple fractions with restrictions on coefficients in the space $C_A^\infty = C^\infty(\mathbb{T}) \cap H^\infty$.

Question. How to get analogous results for the Bergman space (no description of uniqueness subsets is known yet), for the spaces $\mathcal{A}_s^r$, $0 < s < 1$?

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Ergodic linear measure-preserving transformations of Banach spaces

Sophie Grivaux

(joint work with Frédéric Bayart)

The topic of this talk is the study of the dynamics of bounded linear operators on a separable complex Banach space X. The results presented here are taken from [3]. We concentrate on the measure-theoretic point of view: under which conditions on X and $T \in \mathcal{B}(X)$ can T be viewed as an ergodic measure-preserving transformation of $(X, \mathcal{B}, m)$, where m is a probability measure on X? We undertake this study with an aim at frequent hypercyclicity questions. The notion of frequent hypercyclicity was introduced in [1]:

Definition 1. An operator T on X is said to be *frequently hypercyclic* provided there exists a vector x such that for every non empty open subset U of X, the set of integers n such that $T^n x$ belongs to U has positive lower density. In this case, x is called a *frequently hypercyclic vector* for T.

This is a more restrictive notion than the classical hypercyclicity (see for instance [6]), the main difference being the lack of Baire Category methods in the study of frequent hypercyclicity theory. This lead us to develop in [1] the measure-theoretic method we alluded to above, which was inspired by the works of Flytzanis ([5]) and Bourdon and Shapiro ([4]). A major role in such questions is played by the eigenvectors associated to eigenvalues of T of modulus 1:

Definition 2. Let T be a bounded operator on X and σ a probability measure on the unit circle $\mathbb{T} = \{\lambda \in \mathbb{C} \;;\; |\lambda| = 1\}$. We say that T has a *σ-spanning set of unimodular eigenvectors* if for every σ-measurable subset A of $\mathbb{T}$ with $\sigma(A) = 1$, the eigenspaces $\ker(T - \lambda I)$, $\lambda \in A$, span a dense subspace of X. If the measure

σ can be chosen to be continuous (i.e. $\sigma(\{\lambda\}) = 0$ for every $\lambda \in \mathbb{T}$), then we say that T has a *perfectly spanning set of unimodular eigenvectors.*

The starting point of our investigation here is the following result of [1]: let T be a bounded operator on a separable Hilbert space H. If T has a perfectly spanning set of unimodular eigenvectors, then T is frequently hypercyclic, because it admits an invariant gaussian measure with respect to which T is weak-mixing. We obtain extensions of this result to the Banach space setting. The possibility of such extensions depends on two features of the pair (X, T): the geometry of X (its type, Fourier-type, whether it contains c_0 or not... and the possibility to parametrize the unimodular eigenvector fields in a regular way: if the unimodular eigenvectors are σ-spanning, there exists a countable family of functions $E_i : (\mathbb{T}, \mathcal{B}, \sigma) \to X$ such that $||E_i(\lambda)|| \leq 1$ for $i \geq 1$ and $\overline{\text{sp}}[E_i(\lambda)\ ,\ i \geq 1] = \ker(T - \lambda)$ for every $\lambda \in \mathbb{T}$ ([2]). "Regularity" means that the functions E_i can be chosen to be Lipschitz, or α-Hölderian, etc...

Theorem 3. *Under one of the following assumptions, T admits an invariant non-degenerate gaussian measure with respect to which it is weak-mixing. In particular T is frequently hypercyclic.*

(1) *if X has type 2 (no regularity assumption on the E_i's)*
(2) *if X is arbitrary and the E_i's are Lipschitz (no assumption on the space)*
(3) *if X does not contain a copy of c_0 and the E_i's are α-Hölderian for some $\alpha > \frac{1}{2}$ (mixed assumptions)*
(4) *if X is a p-convex and p'-concave Banach lattice for some $1 < p \leq 2$ and the E_i's are α-Hölderian for some $\alpha > \frac{1}{2} - \frac{1}{p'}$: for instance for the spaces $L^p(\mu)$, $1 < p \leq 2$.*

We do not know if these assumptions are really necessary: if $T \in \mathcal{B}(X)$ has a perfectly spanning set of unimodular eigenvectors, is it always true that T admits a non-degenerate gaussian measure with respect to which it is weak-mixing? Is it always frequently hypercyclic?

We also investigate the converse assertion, which is known to be true in the Hilbert space case ([5]):

Theorem 4. *Let X be a space of cotype 2, and $T \in \mathcal{B}(X)$. Suppose that there exists a non-degenerate invariant gaussian measure m for T. Then the unimodular eigenvectors of T span a dense subspace of X. If T is ergodic (or weak-mixing, which is the same in this context) with respect to m, then the unimodular eigenvectors of T are perfectly spanning. This does not remain true on general Banach spaces.*

We conclude by recalling the main open question in hypercyclicity, which motivated this study (T is weak-mixing on $(X, \mathcal{B}, m)$ if and only if $T \times T$ is ergodic on $(X \times X, \mathcal{B} \otimes \mathcal{B}, m \otimes m)$): if T is hypercyclic on X, is $T \oplus T$ hypercyclic on the direct sum $X \oplus X$?

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On frequent hypercyclicity

Karl-Goswin Grosse-Erdmann

(joint work with Antonio Bonilla and Alfredo Peris)

Motivated by Birkhoff's ergodic theorem, F. Bayart and S. Grivaux have recently introduced an interesting new concept in hypercyclicity, that of frequently hypercyclic operators.

We recall that a vector x in a topological vector space X is called *hypercyclic* for a (continuous and linear) operator T on X if its orbit $\{x, Tx, T^2x, \ldots\}$ is dense in X, that is, if the orbit meets every non-empty open subset U of X; see [6]. Now, Bayart and Grivaux call x *frequently hypercyclic* for T if its orbit meets every non-empty open subset U of X 'often' in the sense that each set $\{n \in \mathbb{N} : T^n x \in U\}$ has positive lower density. The operator T is called *frequently hypercyclic* if it possesses a frequently hypercyclic vector. This new notion has been thoroughly investigated by Bayart and Grivaux [1], [2].

We report here on joint work with Antonio Bonilla [3], [4] and with Alfredo Peris [7]; we address some problems posed by Bayart and Grivaux, and we present new examples of frequently hypercyclic operators on spaces of analytic functions.

In the sequel, we let X denote an F-space, that is, a complete and metrizable topological vector space, and we assume that X is separable.

A. Question 4.8 of [2] asks if every frequently hypercyclic operator T on X has the property that the operator $T \oplus T$, acting on $X \times X$, is hypercyclic. This question was motivated by the 'Great open problem in hypercyclicity', due to D. Herrero, that asks if the same property holds for *all* hypercyclic operators.

Theorem 1 ([7]). *For any frequently hypercyclic operator T on X the operator $T \oplus T$ is hypercyclic.*

The proof uses, in an essential way, a theorem of Erdös and Sárközy by which the difference set $A - A$ of any set A of positive lower density has bounded gaps.

B. Weighted backward shift operators on the sequence space $\ell^p, 1 \leq p < \infty$, are defined by $T(x_n) = (w_{n+1}x_{n+1})$, where $w = (w_n)$ is a positive bounded sequence.

In [2] it is shown that $\sum_{n=1}^{\infty} \frac{1}{(w_1 \ldots w_n)^p} < \infty$ is a sufficient condition for frequent hypercyclicity of T, while the following condition (C) is shown to be necessary: there exists a sequence (n_k) of positive lower density such that

$$\sum_{k=1}^{\infty} \frac{1}{(w_1 \ldots w_{n_k})^p} < \infty.$$

In Conjecture 2.10 of [2] it is suggested that condition (C), in fact, characterizes frequent hypercyclicity of T. This is not the case.

Example ([7]). There exists a weighted backward shift operator that satisfies condition (C) but that is not frequently hypercyclic.

The characterization of frequently hypercyclic weighted backward shifts remains an open problem.

C. It is well known that the set of hypercyclic vectors of a hypercyclic operator is always residual. In contrast, Bayart and Grivaux [2] have shown that there exist frequently hypercyclic operators for which the set of frequently hypercyclic vectors is non-residual. In fact, we have the following.

Theorem 2 ([3]). *Under weak conditions (see Theorems* 6.1 *and* 6.5 *of* [3]*) on a frequently hypercyclic operator T its set of frequently hypercyclic vectors is of first category in X.*

As a consequence, all known frequently hypercyclic operators T have this property; it is an open problem if a set of frequently hypercyclic vectors can be of second category.

We also study another notion of 'bigness'. It is an immediate consequence of the residuality of the set of hypercyclic vectors that every element in the underlying space can be written as a sum of two hypercyclic vectors. For frequent hypercyclicity the situation is more complicated.

Examples ([3]). There are frequently hypercyclic operators for which every vector is the sum of two frequently hypercyclic vectors, while there are other frequently hypercyclic operators for which this is not the case.

D. One of the most remarkable results in hypercyclicity, due to Godefroy and Shapiro [5], states that every operator on the space $H(\mathbb{C})$ of entire functions that commutes with the differentiation operator is hypercyclic, unless it is a multiple of the identity. This contains, in particular, MacLane's result that the differentiation operator itself is hypercyclic, and Birkhoff's result that the translation operator $f \mapsto f(\cdot + 1)$ is hypercyclic. We improve the result of Godefroy and Shapiro to frequent hypercyclicity.

Theorem 3 ([4]). *Let T be an operator on $H(\mathbb{C})$ that commutes with the differentiation operator and that is not a scalar multiple of the identity. Then T is frequently hypercyclic.*

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Brennan's conjecture for weighted composition operators

Wayne Smith

Brennan's conjecture concerns integrability of the derivative of a conformal map τ of the unit disk $\mathbb{D}$. The conjecture is that, for all such τ,

$$\int_{\mathbb{D}} (1/|\tau'|)^p dA < \infty$$

holds for $-2/3 < p < 2$. Simple examples show this range can not be extended. The conjecture was formulated and established for $-2/3 < p < 1 + \delta$, where $\delta > 0$ is small, by J. E. Brennan in [1]. The range for which the conjecture is known has recently been extended to $-2/3 < p \leq 1.782$ by S. Shimorin [2].

We show that Brennan's conjecture is equivalent to a statement about weighted composition operators. Let τ be as above and let ϕ be an analytic self-map of $\mathbb{D}$. Define, for f analytic on $\mathbb{D}$,

$$(A_{\varphi,p} f)(z) = \left(\frac{\tau'(\phi(z))}{\tau'(z)} \right)^p f(\phi(z)).$$

There are always choices of ϕ that make $A_{\varphi,p}$ *bounded* on the Bergman space $L^2_a(\mathbb{D})$. We are interested in the set of p for which there is a choice of ϕ (depending on τ) that makes $A_{\varphi,p}$ *compact* on $L^2_a(\mathbb{D})$. We show that this happens if and only if $(1/\tau')^p \in L^2_a(\mathbb{D})$; see [3]. Thus Brennan's conjecture is equivalent to such a choice of ϕ existing for the range $-1/3 < p < 1$, and this is known for $-1/3 < p \leq .891$.

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Composition Operators on Uniform Algebras

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(joint work with P. Galindo, M. Lindström)

Let A and B be uniform algebras, with spectrum M_A and M_B respectively. A unital homomorphism from A to B can be viewed as a composition operator $C_\phi : A \to B$, where $\phi : M_B \to M_A$ is the restriction of the dual of the homomorphism to M_B. We are interested in comparing composition operators from A to B in the essential operator norm. We are also interested in the essential spectrum of composition operators on A (unital endomorphisms of A), and in relating the essential spectrum to the notion of hyperbolic boundedness introduced in [GGL1].

The pseudohyperbolic metric $\rho(z,w) = |z-w/|1-\bar{z}w|$ on the open unit disk in the complex plane induces a pseudohyperbolic metric ρ_A on M_A by $\rho_A(x,y) = \sup \rho(f(x), f(y))$, where the supremum is taken over all $f \in A$ satisfying $||f|| < 1$. The Gleason parts of A are the open pseudohyperbolic balls of radius 1 in M_A. (See [Kö].)

A subset E of M_A is hyperbolically bounded if it is contained in a finite number of pseudohyperbolic balls of radius strictly less that 1.

Theorem 1. *Let A be a uniform algebra, and let ϕ be a self-map of M_A that induces a composition operator C_ϕ on A. The following statements are equivalent.*
(i) There is a decomposition of M_A into disjoint clopen subsets $E_1, \ldots, E_m$ such that the iterates of ϕ converge uniformly on each E_j in the pseudohyperbolic metric to an attracting cycle in E_j for ϕ.
(ii) There is $n \geq 1$ such that the nth iterate $\phi^n(M_A)$ under ϕ is a hyperbolically bounded subset of M_A.
(iii) The essential spectral radius of C_ϕ is strictly less than 1, that is, the essential spectrum of C_ϕ does not meet the unit circle.

The equivalence of (i) and (ii) is proved in [GGL1], and the equivalence with (iii) is proved in [GGL2]. These results have a long history (see [Ka], [Kö], [Kl], [Zh]), and they are closely related to those of P. Gorkin and R. Mortini [GM]. Recent work of J. Feinstein and H. Kamowitz [FK] establishes the equivalence of (i) and (iii) for unital semi-simple Banach algebras.

Two composition operators C_ϕ and C_ψ from A to B are in the same norm vicinity if $||C_\phi - C_\psi|| < 2$. This occurs if and only if there is $r < 1$ such that $\rho_A(\phi(y), \psi(y)) \leq r$ for all $y \in M_B$. Norm vicinities are clopen in the uniform operator norm.

Two composition operators C_ϕ and C_ψ from A to B are in the same hyperbolic vicinity if there is a subset Y of M_B that is norming for B, a subset E of Y, and an $r < 1$ such that $\rho_A(\phi(y), \psi(y)) \leq r$ for all $y \in E$, and $\phi(Y \backslash E$ and $\psi(Y \backslash E$ are hyperbolically bounded in M_A. The following two theorems are proved in [GGL2].

Theorem 2. *Let A and B be uniform algebras, and let ϕ and ψ be maps from M_B to M_A that induce a composition operators C_ϕ and C_ψ from A to B. Then C_ϕ and C_ψ belong to the same hyperbolic vicinity if and only if there is a subalgebra* *of A of finite (linear) codimension such that the restrictions of C_ϕ and C_ψ to the subalgebra belong to the same norm vicinity.*

Theorem 3. *Hyperbolic vicinities are clopen with respect to the essential operator norm.*

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Composition operators on Hardy spaces of a simply connected domain

Eva A. Gallardo-Gutiérrez

(joint work with María J. González and Artur Nicolau)

Let Ω be a simply connected domain properly contained in the complex plane $\mathbb{C}$ with locally rectifiable boundary $\partial\Omega$. Let τ be a Riemmann map that takes the open unit disc $\mathbb{D}$ onto Ω. For $1 \le p < \infty$, the Hardy space $\mathcal{H}^p(\Omega)$ consists of holomorphic functions F on Ω such that the norm

$$\|F\|_p^p = \frac{1}{2\pi} \sup_{0<r<1} \int_{\tau(\{|z|=r\})} |F(w)|^p \, |dw|$$

is finite. Here, $|dw|$ denotes the arc-length measure on $\partial\Omega$. Although this norm depends on the choice of the Riemann map, it is not hard to see that any other Riemann map induces an equivalent norm on $\mathcal{H}^p(\Omega)$(see [1]).

If Φ is a holomorphic map on Ω, that takes Ω into itself, then the equation

$$C_\Phi F = F \circ \Phi$$

defines a composition operator C_Φ on the space $\mathcal{H}(\Omega)$ of all holomorphic functions on Ω. In case that Ω is the open unit disc $\mathbb{D}$, Littlewood [2] proved in 1925 that any composition operator C_Φ is bounded on any Hardy space $\mathcal{H}^p(\mathbb{D})$; a result known as *Littlewood Subordination Principle*. In the eighties, Shapiro [4] characterized the compactness of C_Φ on $\mathcal{H}^p(\mathbb{D})$ in terms of the *Nevanlinna counting function* for Φ.

Recently, Shapiro and Smith [5] have shown that the geometry of the domain Ω plays an important role in the boundedness and compactness of C_Φ on $\mathcal{H}^p(\Omega)$. In particular, they prove that the condition of boundedness for the derivative of the Riemann map τ and its reciprocal actually characterizes the domains Ω for which every composition operator is bounded in $\mathcal{H}^p(\Omega)$. Moreover, $\mathcal{H}^p(\Omega)$ supports a compact composition operator if and only if $\partial\Omega$ has finite one-dimensional Hausdorff measure.

The aim of our work is to relate the geometry of the domain Ω to the fact that boundedness (respectively compactness) of C_Φ on $\mathcal{H}^p(\Omega)$ can be characterized in terms of a Nevanlinna type condition for Φ in Ω. If $\delta(z, \partial\Omega)$ denotes the distance from z to the boundary of Ω, we consider the function $\widetilde{N}_{\Phi,\Omega}$ associated to Φ in Ω by

$$\widetilde{N}_{\Phi,\Omega}(w) = \begin{cases} \displaystyle\sum_{z\in\Phi^{-1}\{w\}} \delta(z,\partial\Omega) & \text{if} \quad w \in \Phi(\Omega) \\ 0 & \text{if} \quad w \notin \Phi(\Omega). \end{cases}$$

The function $\widetilde{N}_{\Phi,\mathbb{D}}$ is closely related to the classical *Nevanlinna counting function*, and both Littlewood's Subordination Principle an Shapiro's Compactness Theorem can be restated in terms of $\widetilde{N}_{\Phi,\mathbb{D}}$.

The advantage of considering the function $\widetilde{N}_{\Phi,\Omega}$ is that precisely the geometry of the domain Ω plays a fundamental role to determine what symbols Φ induce bounded and compact composition operators on $\mathcal{H}^p(\Omega)$. Roughly speaking, we show that whenever the boundary of Ω is, in some sense, quasi smooth, bounded and compact composition operators on $\mathcal{H}^p(\Omega)$, $1 \le p < \infty$ are completely characterized with a condition similar to that one in the already known spaces $\mathcal{H}^p(\mathbb{D})$.

In particular, we show that for any simply connected domain Ω, under the extra hypotheses that Φ is a finitely-valent symbol, the *Littlewood type inequality*

$$\widetilde{N}_{\Phi,\Omega}(w) \lesssim \delta(w,\partial\Omega), \qquad (w \in \Omega)$$

is necessary for C_Φ to be bounded in $\mathcal{H}^p(\Omega)$, for any $1 \le p < \infty$. Moreover, we also prove that the corresponding "little-oh" condition is necessary for the compactness of C_Φ on $\mathcal{H}^p(\Omega)$, $1 \le p < \infty$. Nevertheless, *Littlewood type inequality* does not suffice for characterizing boundedness of composition operators C_Φ, even induced by univalent symbols.

On the other hand, we show that *Littlewood type inequality* is sufficient, without any extra assumption on the valence of the symbol Φ, if we impose a geometrical condition on the domain: $\partial\Omega$ is a Lavrentiev curve. On the contrary, it is no longer necessary if we drop the extra assumption on Φ of being finitely-valent. The key point is a link between the geometry of the underlying domain Ω and the symbol inducing the composition operator.

Finally, we relate both facts, characterizing those symbols inducing bounded and compact composition operators on $\mathcal{H}^p(\Omega)$, whenever Ω is a Lavrentiev domain.

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Comparing topologies on the space of composition operators

Carl Sundberg
(joint work with Eero Saksman)

Consider the set of analytic composition operators

$$\mathcal{C} = \{C_\phi\,;\,\phi : \mathbb{D} \to \mathbb{D} \text{ analytic}\}$$

where $\mathbb{D}$ is the unit disk in the complex plane. If X is a Banach space of analytic functions in $\mathbb{D}$ for which all $C_\phi \in \mathcal{C}$ act boundedly on X then we consider the topology τ_X naturally induced on $\mathcal{C}$ by considering it as a subspace of $\mathcal{B}(X)$. We consider among others the cases where X is the Hardy space H^2, the standard weighted Bergman space

$$A^2_\alpha = \left\{f \text{ analytic in } \mathbb{D}\,;\,\|f\|^2_{A^2_\alpha} = \frac{\alpha+1}{\pi}\int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty\right\}$$

for $\alpha > -1$, the Hardy space H^∞, and the Bloch space B. Some of our main results:

(i) τ_{H^2} is strictly finer than $\tau_{A^2_\alpha}$.
(ii) τ_{H^∞} is strictly finer than τ_{H^2}.
(iii) For all $\alpha, \beta > -1$, $\tau_{A^2_\alpha} = \tau_{A^2_\beta}$.
(iv) τ_{H^2} and τ_B are not comparable.

Our results have obvious relevance to the study of isolation in $\mathcal{C}$ in the various topologies τ_X, see e.g. [1, 2, 3, 4, 6]. Our results also 'explain' known relationships concerning compactness, e.g. if $C_\phi - C_\psi$ acts as a compact operator on H^2 then it also acts as a compact operator on A^2_α [5].

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Analytic contractions, nontangential limits, and the index of invariant subspaces

Stefan Richter
(joint work with Alexandru Aleman, Carl Sundberg)

Surprising results of Apostol, Bercovici, Foias, and Pearcy from the mid 1980s implied that the structure of the invariant subspace lattice of the Bergman shift differs drastically from the Beurling lattice of invariant subspaces of the unilateral shift acting on the Hardy space H^2, [4]. In fact, a solution to the invariant subspace problem would be found if one could decide whether or not the Bergman lattice has proper gaps. The constructions of Apostol, Bercovici, Foias and Pearcy use dual algebra techniques and they are valid for many contractive multiplication operators M_ζ acting on Hilbert spaces of analytic functions such that the powers M_ζ^n converge to zero in the strong operator topology. For the Bergman space a function theoretic approach to these results was given by Hedenmalm [7], also see [8]. The main point is that the Bergman lattice contains invariant subspaces $\mathcal{M}$ of arbitrary index, i.e. $\dim \mathcal{M}/\zeta\mathcal{M}$ can be any positive integer or even be infinite. Hedenmalm used interpolating and sampling sequences to construct invariant subspaces of high index. On the other hand there has been some indication in the literature that invariant subspaces containing functions that are sufficiently regular near the boundary of the unit disc always must have index 1, [9, 10, 1].

In this work we investigate the connection between nontangential limiting behaviour of functions in a Hilbert space of analytic functions and the structure of the invariant subspace lattice of the multiplication operator M_ζ. In fact, by analyzing the set where ratios of functions in the space have nontangential limits we find a sufficient condition for the existence of invariant subspaces with high index that in many cases is also necessary.

Let $\mathcal{H}$ be a Hilbert space of analytic functions on the open unit disc $\mathbb{D}$ such that the operator M_ζ of multiplication with the identity function ζ defines a contraction operator. Let k denote the reproducing kernel of $\mathcal{H}$ and let g be a nonzero function in $\mathcal{H}$. Then for every $\lambda \in \mathbb{D}$ and $f \in \mathcal{H}$ one easily checks the fundamental inequality

$$\left|\frac{f}{g}(\lambda)\right|^2 \leq \left((1-|\lambda|^2)||\frac{f}{1-\overline{\lambda}\zeta}||^2\right)\left((1-|\lambda|^2)\frac{||k_\lambda||^2}{|g(\lambda)|^2}\right).$$

By use of a unitary dilation of M_ζ one shows that the first factor of the right hand side of the inequality is bounded by a positive harmonic function. Hence the meromorphic function f/g is in the Hardy space $H^2(\Omega)$ for every subregion $\Omega \subseteq \mathbb{D}$

where the second factor of the right hand side is bounded. We are thus lead to define

$$\Delta(\mathcal{H}) = \{z \in \partial\mathbb{D} : \text{ nt-}\limsup_{\lambda\to z}(1-|\lambda|^2)\frac{||k_\lambda||^2}{|g(\lambda)|^2} < \infty\},$$

where we have used the prefix "nt" to abbreviate the word nontangential. By the above remarks together with some classical results of Privalov and Caratheodory it is not hard to prove that the definition of $\Delta(\mathcal{H})$ is independent of $g \neq 0$ (up to sets of linear Lebesgue measure 0), and that f/g has nontangential limits a.e. on $\Delta(\mathcal{H})$. In fact, $\Delta(\mathcal{H})$ is the largest set with this property:

Theorem 1. *If $\mathcal{H}$ be a Hilbert space of analytic functions on $\mathbb{D}$ such that $||M_\zeta|| \leq 1$, then up to sets of measure 0 the set $\Delta(\mathcal{H})$ is the largest subset of $\partial\mathbb{D}$ such that for every $f, g \in \mathcal{H}$, $g \neq 0$, the meromorphic function f/g has nontangential limits a.e. on $\Delta(\mathcal{H})$. We say that $\mathcal{H}$ admits nontangential limits on $\Delta(\mathcal{H})$.*

For our applications of Theorem 1 we need to have a description of $\Delta(\mathcal{H})$ in terms of the norm on $\mathcal{H}$ instead of the reproducing kernel. For this reason we associate the set $\Sigma(\mathcal{H})$ with $\mathcal{H}$. It is the smallest set that carries the spectral measure of the unitary summand of the minimal co-isometric extension of M_ζ. More precisely, if $S^* \oplus R$ denotes the minimal co-isometric extension of the contraction M_ζ, R unitary, S^* a backward shift of some multiplicity, then it turns out that the spectral measure of R is absolutely continuous, and we set $\Sigma(\mathcal{H}) = \{z \in \partial\mathbb{D} : w(z) > 0\}$, where $w \in L^1(\partial\mathbb{D})$ is chosen so that $w(z)|dz|$ is a scalar valued spectral measure for R. In the Sz. Nagy Foias theory of contractions R has been called the *-residual part of the minimal unitary dilation of M_ζ, and $\Sigma(\mathcal{H})$ has also been denoted by $\Sigma_*(M_\zeta)$, see [5].

It is easy to see that $\Sigma(\mathcal{H})$ has Lebesgue measure 0, if and only if $||\zeta^n f|| \to 0$ for all $f \in \mathcal{H}$ as $n \to \infty$. Furthermore, it turns out that one always has $\Delta(\mathcal{H}) \subseteq \Sigma(\mathcal{H})$ a.e..

Theorem 2. *Let $\mathcal{H}$ be a Hilbert space of analytic functions on $\mathbb{D}$ such that $||M_\zeta|| \leq 1$ and $\dim \mathcal{H}/\zeta\mathcal{H} = 1$.*

If there is a $c > 0$ such that for all $f \in \mathcal{H}$ and all $\lambda \in \mathbb{D}$ we have

$$\left\|\frac{\zeta - \lambda}{1-\overline{\lambda}\zeta}f\right\| \geq c||f||,$$

then $\Delta(\mathcal{H}) = \Sigma(\mathcal{H})$ a.e. and the following four conditions are equivalent:
(1) there exists an invariant subspace $\mathcal{M}$ of M_ζ of index > 1,
(2) $|\Delta(\mathcal{H})| = 0$, i.e. $\mathcal{H}$ does not admit nontangential limits on any set of positive measure,
(3) $\|\zeta^n f\| \to 0$ for some $f \in \mathcal{H}$, $f \neq 0$,
(4) $\|\zeta^n f\| \to 0$ for all $f \in \mathcal{H}$, i.e. $|\Sigma(\mathcal{H})| = 0$.

For $i = 1, 2, 3$ the implications $(i+1) \Rightarrow (i)$ follow under the weaker hypothesis that the constant c in the theorem may depend on the point $\lambda \in \mathbb{D}$. In fact, it is trivial that (4) implies (3). The implication (3) $\Rightarrow$ (2) easily follows from a classical theorem of Khinchin and Kolmogorov (see Theorems A3 and A4 of [6]). The proof that (2) implies (1) uses results from the theory of dual algebras together with the information gained from Theorem 1. The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are false in the larger generality that we just alluded to, but at this point we do not know whether conditions (1) and (2) are equivalent in this more general setting. For so-called analytic $P^2(\mu)$-spaces conditions (1)-(4) are equivalent, this has been proven in [2].

The proofs of Theorems 1 and 2 along with further details and results will appear in [3].

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Some examples concerning boundary behavior and index in Hilbert spaces of analytic functions

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(joint work with Stefan Richter, Carl Sundberg)

By a Hilbert space of analytic functions on the unit disc $\mathbb{D}$ we mean, as usual, a Hilbert space H continuously contained in the Fréchet space of all analytic functions in $\mathbb{D}$. We shall assume throughout that the operator of multiplication by the independent variable $(M_z)f(z) = zf(z)$ is bounded on H, its spectrum equals the closed unit disc and that

$$\dim H/(M_z - \lambda)H = 1 \tag{1}$$

for all $\lambda \in \mathbb{D}$. Clearly, by Fredholm theory this number is actually independent of the choice of $\lambda \in \mathbb{D}$. If $M \subset H$ is a closed invariant subspace for M_z then the restriction $M_z|M$ will share the above properties except possibly the relation (1). In fact, in the mid '80s Apostol, Bercovici, Foiaş and Pearcy proved that there are natural examples of such spaces H and M_z-invariant subspaces $M \subset H$ such that the index of M defined by

$$\text{ind}M = \text{dim}M/zM$$

can equal any prescribed positive integer or even ∞. Spaces of *Bergman type* are a particular class of spaces where their results apply. These are Hilbert spaces of analytic functions as above where the norm is given by integration against a positive measure carried by $\mathbb{D}$, i.e.

$$\|f\|^2 = \int_{\mathbb{D}} |f|^2 d\mu.$$

In contrast to this situation, it is well known that every nonzero invariant subspace of the Hardy space or the standard weighted Dirichlet spaces has index one. Recall that these spaces are contained in the usual Nevanlinna class, while spaces of Bergman type always contain functions of unbounded characteristic. This leads to the natural question whether invariant subspaces of Hilbert spaces of analytic functions contained in the Nevanlinna class always have index one? The answer to this question is negative and it follows from more recent work of Esterle [1]. There is however a quite subtle connection between boundary behavior and index which occurs in the context of spaces of Bergman type.

Theorem 1. *Let M be a closed invariant subspace of the Bergman-type space H such that M contains a dense subset S with the property that there exists a set E of positive Lebesgue measure on the unit circle such that whenever $f, g \in S,\ f, g \neq 0$ the meromorphic function f/g has nontangential limits a.e. on E. Then M has index one.*

There are a number of results from the 90's which seem to indicate that for the standard Bergman space the same conclusion should hold under weaker assumption. For example, the following question appeared to be reasonable:

Does the conclusion of the theorem still hold if M contains one nonzero function which has nontangential limits on a set of positive measure on the unit circle?

In [2] we have shown that the answer is negative in general. The result is a consequence of our study of majorization functions for invariant subspaces of the Bergman spaces. On the other hand, we show that the answer is affirmative when the set in question has closed subsets with positive measure and finite entropy.

To motivate the second problem, let us return for a moment to the standard Hardy and Bergman spaces. On H^2 the operator M_z is an isometry while on any space of Bergman type, M_z is a contraction and $M_z^n \to 0$ in the strong operator topology. Do these operator-theoretic properties of M_z on a Hilbert space of analytic functions influence the boundary behavior of the functions in the space and the index of invariant subspaces? Surprisingly, an affirmative answer can be obtained under the (mild) regularity assumption that

$$\inf\{\|(z-\lambda)(1-\bar{\lambda}z)^{-1}f\|/\|f\| : \ \lambda \in \mathbb{D}, f \in H, f \neq 0\} > 0.$$

This is the content of the main results in [3] and we shall omit the details here. What we want to point out is that without regularity assumptions, the above question has at its turn a negative answer even for operators that are in some sense close to isometries.

Example 2. There exists a Hilbert space of analytic functions on $\mathbb{D}$ with the following properties:
(i) The selfcommutator $M_z^* M_z - M_z M_z^*$ has rank 2,
(ii)For every $f \in H \setminus \{0\}$ we have $\liminf_n \|z^n f\| > 0$,
(iii) H contains functions that have no nontangential limits on any set of positive measure on the unit circle,
(iv) There are invariant subspaces for M_z on H of arbitrary index.

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Hahn-Banach Extension of Bilinear Forms

Hans Jarchow

Given two Banach spaces X and Y, let $\mathcal{L}^2(X \times Y)$ be the Banach space of all (bounded) bilinear forms on $X \times Y$, where X and Y are given Banach spaces. We say that (X, Y) has the **bilinear extension property (BEP)** if, no matter how we choose Banach spaces $\widetilde{X}$ and $\widetilde{Y}$ containing X, resp. Y, as subspaces, each $\beta \in \mathcal{L}^2(X \times Y)$ is the restriction of some $\widetilde{\beta} \in \mathcal{L}^2(\widetilde{X} \times \widetilde{Y})$. Our aim is to figure out under which conditions a pair of Banach spaces has *BEP*.

With each $\beta \in \mathcal{L}^2(X \times Y)$ we associate the operator $u_\beta \in \mathcal{L}(X, Y^*)$ defined by $\langle u_\beta(x), y \rangle = \beta(x, y)$ (for $x \in X$ and $y \in Y$). The resulting map $\mathcal{L}^2(X \times Y) \to \mathcal{L}(X, Y^*) : \beta \mapsto u_\beta$ is an isometric isomorphism. This trivial observation links our topic with operator ideals. In the new setting, tools like factorization and trace duality are available.

We use standard notation and facts of Banach space theory. The ideals we are going to work with can be derived by means of certain standard procedures from the classical Banach ideals Π_p of p-summing operators, $\mathcal{I}_p$ of p-integral operators and Γ_p of p-factorable operators ($1 \le p \le \infty$). The procedures are the formation of compositions $\mathcal{A} \circ \mathcal{B}$, of quotients $\mathcal{A}^{-1} \circ \mathcal{B}$ and $\mathcal{A} \circ \mathcal{B}^{-1}$ of quasi-Banach ideals $\mathcal{A}$ and $\mathcal{B}$, and the formation of the dual ideal $\mathcal{A}^d$ and the trace dual $\mathcal{A}^*$ of a given quasi-Banach ideal $\mathcal{A}$. Details can be found in [7], [4], and [3].

Among the best known relations are $\Pi_p^* = \mathcal{I}_{p^*}$, $\mathcal{I}_p^* = \Pi_{p^*}$, $\Gamma_p^* = \Pi_p^d \circ \Pi_{p^*}$, and $[\Pi_p^d \circ \Pi_{p^*}]^* = \Gamma_p$. These relations are even isometric with respect to canonical norms on the involved ideals, but we skip such details.

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We shall use the short hand notation $X \in \mathcal{A}$ to indicate that the identity operator of a given Banach space X belongs to $\mathcal{A}(X,X)$, where $\mathcal{A}$ is a quasi-Banach ideal.

The ideal Γ_2^* will be important. It can be represented in various ways. First, $\Gamma_2^*(X,Y)$ consists of all operators $u : X \to Y$ which admit a factorization $u : X \stackrel{w}{\to} H \stackrel{v}{\to} Y$ where v^* and w belong to Π_2. This amounts to a factorization $k_Y u : X \to L^\infty(\mu) \stackrel{w}{\longrightarrow} L^1(\mu) \to Y^{**}$ where μ is a probability measure and $k_Y : Y \hookrightarrow Y^{**}$ is the canonical embedding. In addition, Γ_2^* coincides with the quotient ideal $\Gamma_2^{-1} \circ \mathcal{I}_1 \circ \Gamma_2^{-1}$ ([7], 17.5.2).

The following is a straightforward consequence of the extension property of $L^\infty(\mu)$ (for μ finite, say), see e.g. [4]. By $\otimes_\pi$ we refer to the formation of projective tensor products.

Proposition 1. *Let X and Y be Banach spaces. The following statements are equivalent:*

(i) *(X,Y) has BEP.*
(ii) *$X \otimes_\pi Y$ is a subspace of $\widetilde{X} \otimes_\pi \widetilde{Y}$ whenever X is a subspace of $\widetilde{X}$ and Y is a subspace of $\widetilde{Y}$.*
(iii) *$\mathcal{L}(X,Y^*) = \Gamma_2^*(X,Y^*)$.*

A Banach space operator is **approximable** if it can be approximated, uniformly on compact sets, by finite rank operators. By (iii) of the preceding proposition, every operator $X \to Y^*$ is approximable whenever (X,Y) has *BEP*.

Corollary 2. . *If (X,X^*) has BEP, then* $\dim\ X < \infty$.

In fact, the canonical map $k_X : X \hookrightarrow X^{**}$ belongs to Γ_2^* iff $\dim\ X < \infty$.

Application of trace duality to Proposition 1.(iii) yields:

Proposition 3. *If (X,Y) has BEP, then $\Gamma_2(Y^*,X) = \mathcal{I}_1(Y^*,X)$.*

The converse is true if X or Y^* has the metric approximation property. But it is not clear what happens in the in general case.

The preceding result has an interesting consequence. Recall that a Banach space X is said to **verify GT** (‘**Grothendieck's Theorem**’) if $\mathcal{L}(X,\ell^2) = \Pi_1(X,\ell^2)$; cf. [8]. In such a case, we will also write $X \in GT$, and we say that X is a GT-space, etc.

Proposition 4. *If (X,Y) has BEP, then X^* and Y^* verify GT.*

In fact, combine Dvoretzky's Theorem on spherical sections and the fact that the ideal Π_1 of 1-summing operators is the injective hull of $\mathcal{I}_1$ to get that every operator $Y^* \to \ell^2$ is 1-summing, that is, Y^* verifies GT. By symmetry, $X^* \in GT$.

It is not known if GT spaces necessarily do have cotype 2. Let us write $Z \in GT \wedge C_2$ if Z is a cotype 2 space verifying GT.

Under suitable additional assumptions, Proposition 3 has a converse. From Corollary 1 we get:

Proposition 5. *Suppose that X^* and Y^* are in $GT \wedge C_2$. (X, Y) has the BEP iff every operator $X \to Y^*$ is approximable.*

It suffices to require that X^* and Y^* satisfy GT, have cotype 2 and that one of them embeds into a Banach space having cotype 2 and the approximation property.

Another case where the converse of Proposition 4 holds occurs if a certain weak form of lattice structure is available. Recall that a Banach space is said to have the property $g\ell_2$ if every 1-summing operator from that space into ℓ^2 factors through an L^1-space. $g\ell_2$ is a self-dual property, and it is shared by every Banach lattice. The terminology refers to the paper [5] by Y. Gordon and D.R. Lewis. We shall write $X \in GT \wedge g\ell_2$ if the Banach space X verifies both, GT and $g\ell_2$. It is easy to see that this happens iff $\mathcal{L}(X, \ell^2) = \Gamma_1(X, \ell^2)$. Compare also with [4], 17.11 and 17.12.

Proposition 6. *For every Banach space X the following are equivalent:*

(i) *X^* verifies GT.*
(ii) *(X, Y) has BEP, for every $\mathcal{L}^\infty$- space Y.*
(iii) *(X, Y) has BEP, for every Banach space such that $Y^* \in GT \wedge g\ell_2$.*

Since (X, Y) has *BEP* iff (Y, X) does, we may also state that for Y^* to verify GT it is necessary and sufficient that (X, Y) has *BEP* whenever $X^* \in GT \wedge g\ell_2$.

Moreover:

Proposition 7. *If $X^* \in GT \wedge g\ell_2$ and $Y^* \in GT$, then (X, Y) has BEP.*

In fact, every operator $u : X \to Y^*$ is now in $\Gamma_2^{-1} \circ (\Pi_1 \circ \Gamma_\infty) \circ \Gamma_2^{-1} = \Gamma_2^{-1} \circ \mathcal{I}_1 \circ \Gamma_2^{-1} = \Gamma_2^*$.

Actually, we can do better. We use $\otimes_\varepsilon$ to signalize the formation of injective tensor products. It is obvious that (X, Y) has *BEP* if the Banach spaces X and Y satisfy $X \otimes_\pi Y = X \otimes_\varepsilon Y$. It is a deep result of G. Pisier that every Banach space of cotype 2 is contained in an infinite dimensional Banach space P such that $P \otimes_\pi P = P \otimes_\varepsilon P$ and both, P and P^* verify $GT \wedge C_2$; see [8]. In combination with Proposition 6, this leads to

Corollary 8. *Suppose that $X^* \in GT \wedge g\ell_2$. Then $\mathcal{L}(X, Z) = \Pi_2(X, Z)$ for every cotype 2 space Z.*

The announced improvement of Proposition 6 is now part of

Proposition 9. *The following statements on a Banach space X are equivalent.*

(i) *If $Y^* \in GT \wedge C_2$, then (X, Y) has BEP.*
(ii) *Every operator from X into any cotype 2 space is 2-summing.*

But even in this case, we do not know whether X^* must have cotype 2.

By results of J. Bourgain [1], [2], Proposition 7 applies if $X = Y$ equals H^∞, or the disk algebra A. Recall that A and H^∞ do not have the property $g\ell_2$ ([6]). Also, Proposition 7 applies if we take $X = Y$ to be a subspace of $\mathcal{C}(K)$ such that $\mathcal{C}(K)/X$ is reflexive; see [4], Ch.15. It is well-known that $\mathcal{C}(K)/X$ even has $g\ell_2$ in such a case. Of course, we can also choose X from the first and Y from the second group of examples, etc.

We conclude by another easy consequence of Pisier's results:

Proposition 10. *If X and Y are cotype 2 spaces. Then (X, Y) has BEP iff $X \otimes_\varepsilon Y = X \otimes_\pi Y$.*

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C*-algebras generated by composition operators

Michael T. Jury

A theorem of Coburn [1] computes the C*-algebra generated by the unilateral shift on the Hardy space H^2, and shows that this C*-algebra contains the compact operators $\mathcal{K}$ and computes the quotient $C^*(S)/\mathcal{K} \cong C(\partial\mathbb{D})$. Motivated by this result, and the success of C*-algebra methods in the study of Toeplitz operators generally, it is natural to investigate the C*-algebras generated by families of composition operators acting on spaces of analytic functions, and in particular to ask whether it is possible to extract single-operator information (such as Fredholmness and the Fredholm index) from C*- or K-theory invariants. Here we consider the C*-algebra $\mathcal{C}_\Gamma$ generated by the set of composition operators $\{C_\gamma : \gamma \in \Gamma\}$, acting on the Hardy space H^2, where Γ is a non-elementary discrete group of Möbius transformations of the unit disk $\mathbb{D}$ (i.e. a *Fuchsian group.*) Our main theorem is the following [2]:

Theorem 1. *If Γ is a non-elementary Fuchsian group, then the C*-algebra $\mathcal{C}_\Gamma$ contains the unilateral shift S, and hence the algebra of compact operators $\mathcal{K}$. Moreover, there is an exact sequence*

$$0 \to \mathcal{K} \to \mathcal{C}_\Gamma \to C(\partial\mathbb{D}) \times \Gamma \to 0$$

With this exact sequence in hand, we obtain an element of the *Ext* group $Ext(C(\partial\mathbb{D}) \times \Gamma, \mathcal{K})$. It is then natural to ask which *Ext*-class is obtained, and for a topological formula for the pairing of this class with the K-theory group $K_1(C(\partial\mathbb{D}) \times \Gamma)$. Such a formula, together with a method for determining the spectrum of an element of $C(\partial\mathbb{D}) \times \Gamma$, would amount to an index formula for elements of $\mathcal{C}_\Gamma$ analogous to the winding number formula for the Fredholm index of Toeplitz operators on the circle. In this setting, however, the spectral problem seems intractable, and standard topological techniques are unavailable for the computation of the index pairing since the relevant topological space would be $\Gamma\backslash S^1$, which is pathological. However in some special cases, though the spectral problem is still open, we can make some progress on the computation of the index pairing. In particular, by appeal to techniques of noncommutative geometry, in particular a computation of the Chern-Connes character of the *Ext*-class arising of the extension and the pairing of (periodic) cyclic cohomology with K-theory, we obtain the following result:

Theorem 2. *Let Γ be as above, and moreover assume Γ is cocompact and torsion-free. If $f = \sum f_\gamma[\gamma]$ is invertible in $C(\partial\mathbb{D}) \times \Gamma$ and $f^{-1} = \sum g_\gamma[\gamma]$, then the operator $T = \sum T_{f_\gamma} U_\gamma \in \mathcal{C}_\Gamma$ is Fredholm, and*

$$ind(T) = \sum_{\gamma\in\Gamma} \frac{-1}{2\pi i} \int_{\partial\mathbb{D}} g_{\gamma^{-1}}(\gamma^{-1}(z)) df_\gamma(z)$$

In the case where Γ is elementary (e.g. finite cyclic) we can replace $\mathcal{C}_\Gamma$ with the C*-algebra generated by the operators C_γ and the unilateral shift, and since the spectral theory of the crossed product $C(\partial\mathbb{D}) \times \mathbb{Z}_n$ is easily worked out, we get the following index theorem:

Theorem 3. *Let $\gamma(z) = \lambda z$ where λ is a primitive n^{th} root of unity. Then the operator*

$$\sum_{j=0}^{n-1} T_{f_j} C_\gamma^j$$

is Fredholm if and only it the determinant

$$h(z) = \begin{vmatrix} f_0(z) & f_1(z) & \cdots & f_{n-1}(z) \\ f_{n-1}(\lambda z) & f_0(\lambda z) & \cdots & f_{n-2}(\lambda z) \\ \vdots & & & \vdots \\ f_1(\lambda^{n-1}z) & \cdots & \cdots & f_0(\lambda^{n-1}z) \end{vmatrix}$$

is nowhere vanishing on $\partial\mathbb{D}$, in which case the Fredholm index of T is $-1/n$ times the winding number of h around the origin.

The problem of computing C*-algebras generated by composition operators is still very much open. Recent work of Kriete, MacCluer and Moorhouse [3] considers the case of the C*-algebra generated by the shift and a single non-automorphic linear-fractional composition operator; in this case the quotient $C^*(S, C_\varphi)/\mathcal{K}$ can be computed explicitly and is a type I C*-algebra. On the other hand, the *Ext* class obtained in Theorem 1 above coincides (at least up to scalar multiplies) with the class recently constructed by J. Lott [4], and it may be possible to obtain more satisfactory index results by considering this class in the context of that construction.

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Spectral properties of generalized Cesàro operators

Ernst Albrecht
(joint work with Len Miller, Michael M. Neuman, Vivien G. Miller)

The spectral picture of the classical Cesàro operators C_0, C_1 and C_∞ on $\ell^2(\mathbb{N}_0)$, $L^2(0,\infty)$ and $L^2(0,1)$, given by

$$C_0(a) := \Big(\frac{1}{n+1}\sum_{j=0}^{n} a_j\Big)_{n=0}^{\infty}, \qquad a = (a_n)_{n=0}^{\infty} \in \ell^2(\mathbb{N}_0),$$

$$(C_k f)(t) := \frac{1}{t}\int_0^t f(s)ds\,, \qquad f \in L^2(0,k),\ k = 1,\infty,$$

has been computed by Brown, Halmos and Shields in [4]. In [2, 3], Aleman and Siskakis initiated the study of integral operators of the type S_g, where g is holomorphic on the unit disc $\mathbb{D}$ and

$$(S_g f)(z) := \frac{1}{z}\int_0^z f(\zeta)g(\zeta)d\zeta\,,$$

on Hardy and weighted Bergman spaces. For $g(\zeta) \equiv (1-\zeta)^{-1}$ the operator $C := S_g$ (on the Hardy space H^2) is unitarily equivalent to C_0. The operator C is known to be subnormal on H^2 [6] and, for $1 < p < \infty$, $-1 < \alpha < \infty$, it is subdecomposable on the Hardy spaces H^p and the weighted Bergman spaces $L_a^{p,\alpha}$ on the disc [9, 5]. See [7] for the theory of subdecomposable operators.

Various special cases have been considered by Young [10, 11] for the Hilbert spaces H^2 and L_a^2. In [1] we show the following generalization of a result of Young to the non Hilbert space situation.

Theorem 1. *Suppose that $g(z) = \sum_{j=1}^{m} \frac{a_j}{1-b_j z} + h(z)$ where b_j, $1 \le j \le m$, are distinct points on the unit circle $\partial\mathbb{D}$, $|a_j| > 0$ for each j, and $h \in H^\infty(\mathbb{D})$. Then for every $\alpha \ge -1$ and $p > 1$,*

(i) $S_g \in \mathcal{L}(L_a^{p,\alpha})$ *has point spectrum*

$$\sigma_p(S_g|_{L_a^{p,\alpha}}) = \left\{ \frac{g(0)}{n} : \Re\left(\frac{a_j}{g(0)}\right) < \frac{2+\alpha}{np},\ 1 \le j \le m \right\},$$

and each eigenvalue of S_g *is simple.*

(ii) *If* $h(0) = 0$*, then* $\sigma_p(S_g|_{L_a^{p,\alpha}})$ *is finite.*

(iii) $\sigma(S_g|_{L_a^{p,\alpha}}) = \sigma_p(S_g|_{L_a^{p,\alpha}}) \cup \bigcup_{j=1}^m \overline{D}_j$*, where* $D_j = D\left(\frac{p\,a_j}{2(2+\alpha)}\right)$*, for each* j*,* $1 \le j \le m$*.*

(iv) S_g *has essential spectrum* $\sigma_e(S_g) = \bigcup_{j=1}^m \partial D_j$*. Moreover, if* $\lambda \in \rho_e(S_\varphi)$ *then* $\operatorname{ind}(\lambda - S_g) = -\sum_{j=1}^m \chi_{D_j}(\lambda)$*.*

For further (in particular local) spectral properties of such operators we refer to [8].

It is also shown, that for $g(\zeta) \equiv \frac{1+\zeta}{1-\zeta}$ the operator S_g is hyponormal on H^2, which gives a partial answer to a question of Aleman and Siskakis [2].

In joint work (in progress) with V. G. Miller we investigate local spectral properties of operators generalizing the continuous Cesàro operators C_1 and C_∞ and of certain generalized averaging operators on H^p.

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Countably generated prime ideals in H^∞

Raymond Mortini
(joint work with Pamela Gorkin)

We report on joint work with Pamela Gorkin [3] on the structure of prime ideals in the algebra H^∞ of bounded analytic functions in the open unit disk. In 1984, Gorkin [2] and Mortini [5], [6], independently confirmed a conjecture of J. Kelleher [4] and F. Forelli [1] by showing that a nonzero prime ideal in H^∞ is finitely generated if and only if it is a maximal ideal of the form $M(z_0) = \{f \in H^\infty : f(z_0) = 0\}$ for some $z_0 \in \mathbb{D}$. These maximal ideals actually are principal ideals; they are generated by the single function $z - z_0$. In Mortini's thesis [5] an example of a non-maximal, countably generated prime ideal is given: it is the ideal $I = I(S, S^{1/2}, S^{1/3}, \dots)$ generated by the n-th roots of the atomic inner function $S(z) = \exp\left(-\frac{1+z}{1-z}\right)$. This was the first explicit example of a non-maximal prime ideal in H^∞. Are there any other countably generated prime ideals in H^∞, apart from those given by inner rotations of the function S? For $|\sigma| = 1$, let $S_\sigma(z) = S(\overline{\sigma} z)$ be the atomic inner function with singularity at the point σ. In [3] we confirm the conjecture [8] that a nonzero prime ideal I in H^∞ is countably generated if and only if either $I = M(z_0)$ for some $z_0 \in \mathbb{D}$ or if $I = I(S_\sigma, S_\sigma^{1/2}, S_\sigma^{1/3}, \dots)$ for some $\sigma \in \mathbb{C}$ with $|\sigma| = 1$. The proof uses maximal ideal space techniques and is based on some factorization theorems for Blaschke products and on Suarez's result [9] that H^∞ is a separating algebra. It is pointed out that the situation in $H^\infty + C$ is different: here there exist no countably generated prime ideals at all. We also recall that in the disk-algebra $A(\mathbb{D})$, or more generally, in any analytic trace $A = \mathscr{C} \cap H^\infty$ of a C^*-algebra $\mathscr{C}$ with $C \subseteq \mathscr{C} \subseteq QC$, a prime ideal P is countably generated if and only if $P = (0)$ or $M(z_0) = \{f \in A : f(z_0) = 0\}$ for some $z_0 \in \mathbb{D}$ (see [7]).

In the final part we draw attention to a not so well-known result that in a commutative unital Banach algebra any countably generated *closed* ideal actually is finitely generated. We present Udo Klein's elegant Baire category argument to prove that assertion.

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List of open problems

The following list contains some of the problems which were posed during the problem session of the workshop.

1. Aharon Atzmon: Two problems

1.1. A problem about closed ideals in H^∞.

Problem 1. *Let I_1, I_2 be closed ideals in H^∞ (of the unit disc), such that $I_1 \subset I_2$ and $\dim I_2/I_1 > 1$. Does there exist a closed ideal J of H^∞ such that $I_1 \subsetneq J \subsetneq I_2$?*

From the known results on the structure of w^* closed ideals in H^∞, it follows that the answer is positive if we assume that the ideals I_1 and I_2 are w^* closed, and in this case one can choose J to be also w^* closed. Using the results in [1], one can show that the same holds true if we assume only that I_1 is w^* closed. It is also shown in [1], that either we have a positive solution for the pair of ideals I_1, I_2, or there exists a point λ in the unit circle such that $(z-\lambda)I_2 \subset I_1$. This implies that one can always find a closed invariant subspace J of H^∞ that satisfies $I_1 \subsetneq J \subsetneq I_2$.

1.2. **A problem about invariant subspaces of the backward shift on the Bergman spaces A^p.** For $1 \le p < \infty$, let A^p denote the Bergman space of all holomorphic functions on the unit disc $\mathbb{D}$ which are in $L^p(\mathbb{D})$. It is well known that this is a closed subspace of $L^p(\mathbb{D})$ on which the backward shift is a bounded operator.

Problem 2. *Assume that $1 \le p < \infty$, and that M_1, M_2 are two invariant subspaces for the backward shift on A^p such that $M_1 \subset M_2$ and $\dim M_2/M_1 > 1$. Does there exist an invariant subspace M of B such that $M_1 \subsetneq M \subsetneq M_2$?*

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2. Stefan Richter: Questions concerning the boundary behaviour of functions in the space H^2_d

For $d \geq 1$ the space H^2_d is a space of analytic functions on B_d, the open unit ball of $\mathbb{C}^d$. It is defined by the reproducing kernel $k_\lambda(z) = \frac{1}{1-\langle z,\lambda\rangle}$, where $\langle z, \lambda\rangle = \sum_{i=1}^n z_i\overline{\lambda}_i$ for $\lambda, z \in B_d$. If we use multi-index notation, then one checks that for $f(z) = \sum_{j\in\mathbb{N}_0^d} \hat{f}(j)z^j$ the norm on H^2_d is given by $||f||^2 = \sum_{j\in\mathbb{N}_0^d} |\hat{f}(j)|^2 \frac{j!}{|j|!}$. H^2_d is contained in $H^2(\partial B_d)$, the Hardy space of the ball, and the multiplier algebra $M(H^2_d)$ is contained in $H^\infty(B_d)$. For $d > 1$ these inclusions are proper.

The space H^2_d and the operator tuple $M_z = (M_{z_1}, \ldots, M_{z_d})$ is important for the operator theory of $d-$contractions, i.e. tuples $T = (T_1, \ldots, T_d)$ of commuting Hilbert space operators that satisfy $|| \sum_{i=1}^d T_i x_i||^2 \leq \sum_{i=1}^n ||x_i||^2$. For example, Drury proved the following von Neumann-type inequality:

$$||p(T)|| \leq ||p||_{M(H^2_d)}$$

for all $d-$contractions T and polynomials $p(z) = p(z_1, \ldots, z_d)$. Here $|| \cdot ||_{M(H^2_d)}$ denotes the multiplier norm on H^2_d, see [1]. More recently, H^2_d has appeared in the work of many authors. There are now a dilation theorem and a commutant lifting theorem for $d-$contractions, results on Nevanlinna-Pick interpolation for $M(H^2_d)$, and Beurling-type theorems for the invariant subspaces of M_z.

In order to refine these recent results it would be good to know more about the function theory of H^2_d. For example, since $H^2_d \subseteq H^2(\partial B_d)$ it follows that functions in H^2_d for almost every $z \in \partial B_d$ have limits if z is approached from within a so-called Koranyi region, [3]. For $d > 1$ one should be able to say more. For the first question we think of the one variable classical Dirichlet space as it is contained in H^2.

Problem 1. *Work out the details of a capacity theory for H^2_d, and prove that functions in H^2_d have nontangential limits everywhere on ∂B_d except perhaps a set of "capacity 0".*

Let M_0 be the set of representing measures for the ball algebra, i.e. all those measures $\mu \geq 0$ that are supported in ∂B_d and satisfy $\int f d\mu = f(0)$ for every f analytic in B_d and continuous on $\overline{B}_d$. In [2], Lemma 2.2, it was shown that for every $f \in H^2_d$ and $\lambda \in B_d$ one has $|f(\lambda)|^2 \leq 2\mathrm{Re}\langle fk_\lambda, f\rangle - ||f||^2$, i.e. $|f|^2$ has a pluriharmonic majorant, and it follows that H^2_d is contained in the closure of the polynomials in Lumer's Hardy space, [3]. Furthermore, if for $r < 1$ we write $f_r(z) = f(rz)$, then we obtain $\int |f_r|^2 d\mu \leq ||f||^2$ for every $f \in H^2_d$ and every $\mu \in M_0$. One deduces that for every $\mu \in M_0$ f_r converges in $L^2(\mu)$ as $r \to 1^-$. This suggests

Problem 2. *If $f \in H^2_d$, then does f have radial limits outside of sets that are "totally null" (see* [3]*), i.e. is there a set E with $\mu(E) = 0$ for every $\mu \in M_0$ and such that $f(rz)$ converges for every $z \in \partial B_d \setminus E$ as $r \to 1^-$?*

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3. Wayne Smith: Distribution of mass by a Bergman function

Let L_a^1 be the Bergman space of functions analytic on the unit disk $\mathbb{D}$ with norm $\|f\|_1 = \int_{\mathbb{D}} |f| dA$. For $\varepsilon > 0$, let $\Sigma_\varepsilon = \{z : |\arg z| < \varepsilon\}$.

Conjecture. *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $f \in L_a^1$ and $f(0) = 0$, then*

$$\int_{f^{-1}(\Sigma_\varepsilon)} |f| dA > \delta \|f\|_1.$$

This conjecture has applications to the theory of extremal dilatations of quasi-conformal mappings; see [1]. An equivalent formulation first appeared in [2] where it was established when $\varepsilon \geq \frac{\pi}{2} - \eta$, some $\eta > 0$. In [1] the conjecture was shown to hold for all *univalent* $f \in L_a^1$ and all $\varepsilon > 0$. The conjecture fails when formulated for L_a^p, any $p > 1$; see [1].

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4. Raymond Mortini: Problems concerning the Corona property and interpolating Blaschke Products

Problem 1 Let $\Omega \subseteq \mathbb{C}$ be a domain such that $A = H^\infty(\Omega)$ contains a nonconstant function. Then A is a point separating uniform algebra. It is still unknown whether Ω has the corona property, that is whether Ω is dense in the maximal ideal space (or spectrum) $M(A)$ of A. Equivalently:

Let $f_1, \ldots, f_N \in A$. Suppose that $\sum_{j=1}^N |f_j| \geq \delta > 0$ in Ω. Does there exist $g_j \in A$ such that $1 = \sum_{j=1}^N g_j f_j$?

For the unit disk $\mathbb{D}$ the answer is yes and that was proven by Carleson in his famous Corona Theorem. It is also known that each finitely connected domain supporting non-constant bounded analytic functions has the corona property. A class of infinitely connected domains for which the answer is known, are the so called Denjoy domains. These are domains of the form $\Omega = \mathbb{C} \setminus K$, where $K \subseteq \mathbb{R}$ is a compact subset of the reals. Astonishingly, it is also not known whether the Corona Theorem is true for the domains $\Omega = \mathbb{C} \setminus (K \times S)$, where K and S are compact subsets of the reals. A quite general class of domains for which the answer is yes are bounded domains for which the diameters of the components of $\mathbb{C} \setminus \Omega$ are bounded away from zero. See [1] and [2] for additional informations.

Let us also mention that in several complex variables it is still not known whether the unit ball $\mathbb{B}^n$ and the polydisk $\mathbb{D}^n$ have the corona property.

Problem 2 One of the important open questions in the theory of bounded analytic functions in the unit disk is the following problem of J.B. Garnett and P.W. Jones:

Can every Blaschke product be uniformly approximated by interpolating Blaschke products?

It is easy to see that the class $\mathcal{F}$ of finite products of interpolating Blaschke products has this property. In [6] it was shown that any Blaschke product having its zeros in a cone does belong to the closure of $\mathcal{F}$.

Earlier, in 1996, Garnett and Nicolau [3] showed that the linear hull of the set of interpolating Blaschke products is dense in $H^\infty(\mathbb{D})$. Then Oyma [8] and O'Neill [7] proved that every function of norm less than $1/27$ lies in the closed convex hull $\mathcal{H}$ of the interpolating Blaschke products. It is not known whether the whole unit ball of H^∞ lies in $\mathcal{H}$.

Good candidates for counter–examples in the Garnett–Jones approximation problem could be the so called *universal Blaschke products* (see [4]); these are inner functions B such that for a given sequence (z_n) tending to the boundary of $\mathbb{D}$ the set $\{B(\frac{z+z_n}{1+\overline{z}_n z}) : n \in \mathbb{N}\}$ is locally uniformly dense in the closed unit ball $\mathscr{B}$ of H^∞. They have the property that for any $f \in \mathscr{B}$ there exists $m \in M(H^\infty)$ such that $B \circ L_m = f$; here L_m is the Hoffman map of $\mathbb{D}$ onto the Gleason part $P(m)$ of m. On the other hand, interpolating Blaschke products b have the property that for every m, $b \circ L_m = b^* F$, where b^* is an another interpolating Blaschke product (respectively a unimodular constant) and F an invertible outer function bounded away from zero by a fixed constant depending only on the uniform separation constant of b (see [5]). Thus, in some sense, these universal functions seem to be "far away" from interpolating Blaschke products.

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5. Alexandru Aleman: Cyclic multiplicity of spaces of analytic functions

By a Hilbert space of analytic functions on the unit disc $\mathbb{D}$ we mean a Hilbert space H continuously contained in the Fréchet space of all analytic functions in $\mathbb{D}$ such that the operator of multiplication by the independent variable $(M_z)f(z) = zf(z)$ is bounded on H, its spectrum equals the closed unit disc and that

$$\dim H/(M_z - \lambda)H = 1\,, \qquad z \in \mathbb{D}.$$

Recall that a set $S \subset H$ generates H if the linear span of the polynomial multiples of the functions in S is dense in H. If S is singleton $\{x\}$, we say that $M_z|H$ is cyclic and that x is a cyclic vector for this operator. The above condition on the range of $M_z - \lambda$ is in general necessary but not sufficient for cyclicity. Atzmon (Proc. Amer. Math. Soc. 129 (2001), no. 7, 1963–1967) produced the first example of a Banach space of analytic functions (actually an ideal of the well known Wiener algebra) that satisfies this condition but is not cyclic, in fact it cannot be generated by any finite set. Examples of Hilbert spaces of analytic functions where M_z is not cyclic appear in some unpublished work of Sundberg and Aleman and Sundberg. These are essentially based on the construction of zero sequences for weighted Bergman spaces which contain subsequences that are not zero-sequences for the same space. It is not known whether these spaces are generated by a finite set or not. More generally, the problem is to find examples of Hilbert spaces of analytic functions on the unit disc as above, that are not finitely generated.

6. Carl Sundberg: When is the difference of two composition operators on H^2 compact?

Let $\phi, \psi : \mathbb{D} \to \mathbb{D}$ be analytic, where $\mathbb{D}$ is the unit disk in the complex plane, and denote by $C_\phi, C\psi$ the respective composition operators induced on the Hardy space H^2. We ask for a characterization of pairs ϕ, ψ for which $C_\phi - C_\psi$ is compact.

The question was completely answered by Jennifer Moorhouse [1] in the case when H^2 is replaced by any of the standard weighted Bergman spaces A^2_α. Her answer can be stated in the following way: $C_\phi - C_\psi$ is compact on A^2_α iff "the reproducing kernels think that $C^*_\phi - C^*_\psi$ is compact", i.e. iff $\left\|(C^*_\phi - C^*_\psi)\frac{k_\lambda}{\|k_\lambda\|}\right\| \to 0$ as $|\lambda| \to 1$, where k_λ is the reproducing kernel for A^2_α at λ.

On the other hand it is known that the composition operator C_ϕ is compact on H^2 iff "the reproducing kernel thinks it is", i.e. iff $\left\|C_\phi \frac{k_\lambda}{\|k_\lambda\|}\right\| \to 0$ as $|\lambda| \to 1$, where now of course k_λ denotes the reproducing kernel for H^2 at λ ([4]). We are led to conjecture that this statement is true for differences of composition operators on H^2 as well:

Conjecture. *Suppose ϕ, ψ are such that $\left\|(C_\phi - C_\psi)\frac{k_\lambda}{\|k_\lambda\|}\right\| \to 0$ as $|\lambda| \to 1$. Then $C_\phi - C_\psi$ is compact.*

In addition to the papers already mentioned we list some others where this question is considered.

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