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Analysis and Quantum Theory

Organised by
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September 18th – September 24th, 2005

Abstract. The MFO-Workshop 0538 *Analysis and Quantum Theory* focussed on problems in mathematical physics, especially those connected to quantum field theory. Several theoretical physicists introduced a mostly mathematically oriented audience to topics in theoretical physics such as perturbative quantum field theory and fractional quantum Hall effect, thus bridging the gap between theoretical physics and mathematics. Apart from these reviews, a selection of research reports on recent results in mathematical physics was presented.

Mathematics Subject Classification (2000): 81Txx.

Introduction by the Organisers

The MFO-Workshop 0538 *Analysis and Quantum Theory*, organized by V. Bach (U Mainz), J. Derezinski (U Warsaw), and J. P. Solovej (U Copenhagen), was held from September 18 through September 24, 2005. The workshop's general theme was mathematical physics, it especially focussed on

- an introduction to the Epstein-Glaser Method in quantum field theory by T. Hurth (CERN Geneva),
- an introduction to perturbative renormalization in quantum electrodynamics by E. Remiddi (U. Bolgna),
- an introduction to the theory of the fractional quantum Hall effect by J. Fröhlich (ETH Zürich),
- a review of the use of the Witten Laplacian in classical statistical mechanics by B. Helffer (U. Paris-Orsay),
- a review of the Bogolubov-Dirac-Fock approximation in relativistic quantum mechanics and quantum field theory by E. Séré (U Paris-Dauphine).

Besides these lectures of more tutorial and overview character, several seminars on current research results were delivered; see the abstracts below. The purpose of this workshops was twofold.

- The tutorial lectures were meant to bridge the gap between theoretical physics and mathematics. For example, even the notion of what is *perturbative* is different in these communities and a inexhaustive source of misunderstandings between mathematicians and physicists. Indeed, the lectures by T. Hurth and E. Remiddi were intended to present notions of *perturbative quantum field theory* to an, in majority mathematically oriented, audience.
- The seminars on current research were meant to communicate the state of the art on more specialized topics in the mathematical description of quantum mechanics and quantum field theory.

Besides the lectures present during daytime, an evening discussion on open problems and future developments took place on September 26. This discussion was chaired by J. Fröhlich who emphazised the importance that mathematical physics addresses *physically relevant* problems. This criterion in competition with the criterion of *feasability of its solution* should be the guiding principles of our choice of research topics. There was a general agreement of the audience to this statement.

It is a pleasure to thank the director G.-M. Greuel, the administrative staff, and the kitchen staff of MFO, as well as, the state of Baden-Würtemberg (the main source of funding of MFO), for their work which made this workshop possible.

Workshop: Analysis and Quantum Theory

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Abstracts

Convergence of the Unitary Evolution in the Van Hove scaling in the Friedrichs Model

Wojciech De Roeck
(joint work with Jan Derezinski)

Let $\mathcal{H} = \mathcal{H}_{\rm S} \bigoplus \mathcal{H}_{\rm R}$, where $\mathcal{H}_{\rm S}$ is a finite-dimensional and $\mathcal{H}_{\rm R}$ an infinite-dimensional Hilbert space. We will write $1_{\rm S}$ and $1_{\rm R}$ for the respective projections on these Hilbert spaces. Fix a self-adjoint $H_{\rm R}$ on $\mathcal{H}_{\rm R}$ and a self-adjoint $H_{\rm S}$ on $\mathcal{H}_{\rm S}$.
Let the free Hamiltonian H_0 on $\mathcal{H}$ be given as

$$H_0 = H_{\rm S} \oplus H_{\rm R}$$

Let $V \in \mathcal{B}(\mathcal{H}_{\rm S}, \mathcal{H}_{\rm R})$ and by a slight abuse of notation we will denote by V the coresponding operator on $\mathcal{H}$.
Fix $\lambda > 0$ and let the interacting Friedrichs Hamiltonian be

$$H_\lambda = H_0 + \lambda(V + V^*) \tag{1}$$

Under certain assumptions on V, we have the following result, due to Davies [2]

Theorem .1.

$$\lim_{\lambda \downarrow 0} 1_{\rm S} {\rm e}^{+{\rm i}\lambda^{-2} t H_0} {\rm e}^{-{\rm i}\lambda^{-2} t H_\lambda} 1_{\rm S} = {\rm e}^{-{\rm i}t\Gamma} \tag{2}$$

where $-{\rm i}\Gamma$ *(which depends on* $H_{\rm S}, H_{\rm R}, V$*) is a dissipative operator in* $\mathcal{H}_{\rm S}$*:*

$$-{\rm i}\Gamma + {\rm i}\Gamma^* \leq 0 \tag{3}$$

Hence ${\rm e}^{-{\rm i}t\Gamma}$ is a semigroup.
This result is well-known in much greater generality, see [3] for an extensive review.

Our aim is to generalize this result.
We define an asymptotic space $\mathcal{L} = \mathcal{L}_{\rm S} \oplus \mathcal{L}_{\rm R}$ with a unitary identification

$$J_\lambda : \mathcal{L} \to \mathcal{H} \tag{4}$$

such that $J_\lambda 1_{\rm S} = 1_{\rm S}$. ($J_\lambda$ affects only $\mathcal{H}_{\rm R}$).
We construct a unitary dilation $\mathcal{U}_t$ of the semigroup ${\rm e}^{-{\rm i}t\Gamma}$ on $\mathcal{L}$:

1. $\mathcal{U}_t$ is a unitary group.

$$\mathcal{U}_t \mathcal{U}_s = \mathcal{U}_{t+s} \qquad t, s \in \mathbb{R} \tag{5}$$

2. $\mathcal{U}_t$ dilates ${\rm e}^{-{\rm i}t\Gamma}$

$$1_{\rm S} \mathcal{U}_t 1_{\rm S} = {\rm e}^{-{\rm i}t\Gamma} \qquad t \in \mathbb{R}^+ \tag{6}$$

Now, finally, we claim that our particular dilation (dilations are not unique) can be obtained directly from the original evolution ${\rm e}^{-{\rm i}\lambda^{-2} t H_\lambda}$, when renormalized by a certain (very simple) unitary evolution ${\rm e}^{{\rm i}\lambda^{-2} t F}$ where F is a self-adjoint operator on $\mathcal{L}$:

Theorem .2.

$$\mathrm{s}-\lim_{\lambda\downarrow 0} \mathrm{e}^{\mathrm{i}\lambda^{-2}tF} J_\lambda^* \mathrm{e}^{-\mathrm{i}\lambda^{-2}H_\lambda} J_\lambda^* = \mathcal{U}_t \tag{7}$$

where $\mathrm{s}-\lim$ *denotes the strong operator limit*

Similar results have been obtained already in [1]. Naturally, this procedure is not restricted to the Friedrichs model. Presently we are investigating in what generality these results can be stated and how they can be used to extract physical information (e.g. about fluctuations).

Acknowledgment
Part of the work was done during a visit of both authors to the Erwin Schrödinger Institute (ESI) in Vienna, in the framework of the 2005 program "Open Quantum systems". We thank the ESI for hospitality and support.

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Lieb–Thirring type inequalities and Gagliardo–Nirenberg inequalities for systems

Jean Dolbeault
(joint work with Patricio Felmer, Michael Loss, and Eric Paturel)

Let V be a nonnegative potential such that the corresponding Schrödinger operator on $\mathbb{R}^d$ has an unbounded sequence of eigenvalues $(\lambda_i(V))_{i\in\mathbb{N}^*}$. If $\gamma > d/2$, then

$$\sum_{i\in\mathbb{N}^*} [\lambda_i(V)]^{-\gamma} \le \mathcal{C}(\gamma) \int_{\mathbb{R}^d} V^{\frac{d}{2}-\gamma}\, dx \tag{1}$$

for some optimal and explicit constant $\mathcal{C}(\gamma)$. Such an inequality is interesting for studying, *e.g.*, the stability of mixed states with occupation numbers.

The infimum of $\lambda_1(V)^\gamma \cdot \int_{\mathbb{R}^d} V^{\frac{d}{2}-\gamma}\, dx$ on all possible potentials V gives a lower bound for $[\mathcal{C}(\gamma)]^{-1}$, but it is also related to the optimal constant of a subfamily of Gagliardo-Nirenberg inequalities. This explains explains how Lieb-Thirring inequality type inequalities can be seen as generalizations of the Gagliardo-Nirenberg inequalities for systems of functions, with occupation numbers taken into account.

Inequality (1) can be generalized to

$$\sum_{i\in\mathbb{N}^*} F(\lambda_i(V)) = \mathrm{Tr}\left[F\left(-\Delta + V\right)\right] \le \int_{\mathbb{R}^d} G(V(x))\, dx \tag{2}$$

where F and G are appropriately related. As a special case corresponding to $F(s) = e^{-s}$, (2) is equivalent to an optimal euclidean logarithmic Sobolev inequality, which can be written as

$$\int_{\mathbb{R}^d} \rho \log \rho \, dx + \frac{d}{2} \log(4\pi) \int_{\mathbb{R}^d} \rho \, dx \leq \sum_{i \in \mathbb{N}^*} \nu_i \log \nu_i + \sum_{i \in \mathbb{N}^*} \nu_i \int_{\mathbb{R}^d} |\nabla \psi_i|^2 \, dx$$

where $\rho = \sum_{i \in \mathbb{N}^*} \nu_i |\psi_i|^2$, $(\nu_i)_{i \in \mathbb{N}^*}$ is any nonnegative sequence of occupation numbers and $(\psi_i)_{i \in \mathbb{N}^*}$ is any sequence of $L^2(\mathbb{R}^d)$ orthonormal functions.

Such inequalities provide interesting a *a priori* estimates for dynamical stability results, which are now investigated in a joint project with P. Felmer and E. Paturel, in the spirit of what has been done in kinetic theory, see for instance [1, 2, 6]. Another promising area of research is the rigorous justification of diffusive limits for quantum systems, following ideas which have recently been developed in [3, 4, 5]. Some aspects of this question are currently being investigated in a joint project with J. Mayorga and F. Méhats.

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Haag-Ruelle scattering theory in presence of massless particles

Wojciech Dybalski

The purpose of this talk is to report on recent advances in the scattering theory of relativistic particles. The framework used is local quantum physics [1].

The asymptotic interpretation of quantum field theory has been a subject of study for more than five decades: In the pioneering work of Haag and Ruelle [2, 3] it was assumed that the mass of the particle in question is isolated from the rest of the spectrum. The first author to study embedded masses (still in the realm of purely massive theories) was Herbst [4], who based his analysis on a certain regularity assumption. An alternative to this approach, based on a physically motivated stability requirement, was given by Buchholz and Fredenhagen [5]. While this latter analysis is meaningful only for purely massive theories, it turns out that the

approach of Herbst can be extended to the situation when massless particles are present.

Such an extension was presented in [6] making use of the methods from the scattering theory of massless particles [7, 8] and more recent results in harmonic analysis of local operators [9]. A model physical example, covered by the generalized theory, are stable atoms from the point of view of QED. (Charged particles are excluded by the sharp mass assumption [10]) The remaining open problem is to substitute the technical assumption of Herbst by some physically motivated criterion.

As mentioned above, the description of collisions of electrically charged particles lies outside the scope of the Haag-Ruelle theory. A natural language to study such collisions, within the realm of local quantum physics, is provided by the concept of particle weights [11]. The pertinent questions of existence and interpretation of particle weights will be the subject of future investigations.

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The representability problem for the two-body density matrix of a many-fermion system

Gero Friesecke

The representability problem consists in finding an intrinsic characterization of the range of the map from N-fermion wavefunctions to 2-body reduced density matrices. The problem was originally studied by C.N.Yang [7] in connection with superconductivity and by A.J.Coleman [2] in connection with quantum chemistry, where it is receiving renewed interest due to the fact that available partial results are beginning to be successfully exploited in numerical computations (see e.g. [8]).

In this note I give the mathematical formulation of the problem, explain the connection with fermion condensation and computational quantum chemistry, and present recent progress [5] in the case of four particles, including an intriguing application to computational complexity reduction. Namely, for a certain class of pair interactions, the associated linear quantum eigenvalue equation with $O(K^4)$ unknowns turns out to be equivalent to a system of two nonlinear quantum eigenvalue equations with only $O(K^2)$ unknowns.

1. Mathematical formulation

Assume for simplicity that the fermionic particles under consideration are electrons. Then the single-particle state space is the Hilbert space $\mathcal{H}_1 = L^2(V)$ with $V = \mathbb{R}^3 \times \mathbb{Z}_2$ (the $\mathbb{Z}_2$ corresponding to spin). The N-particle state space is the antisymmetrized tensor product

$$\mathcal{H}_N = \mathcal{H}_1 \wedge ... \wedge \mathcal{H}_1 = \\ \{\Psi \in L^2(V^N) \,|\, \Psi(.., x_i, .., x_j, ..) = -\Psi(.., x_j, .., x_i, ..) \text{ for all } i \neq j\}.$$

For an N-fermion state $\Psi \in A_N$, and $1 \leq p \leq N$, its p-body reduced density matrix is a linear operator on p-body space, $\widehat{\gamma}^p_\Psi : \mathcal{H}_p \to \mathcal{H}_p$, which is defined as follows: introduce the integral kernel

$$\gamma^p_\Psi(x_1, .., x_p, x'_1, .., x'_p) := \\ \binom{N}{p} \int_{V^{N-p}} \Psi(x_1, .., x_p, x_{p+1}, .., x_N)\overline{\Psi(x'_1, .., x'_p, x_{p+1}, .., x_N)} dx_{p+1}..dx_N$$

and set

$$(\widehat{\gamma}^p_\Psi \varphi)(x_1, .., x_p) := \int_{V^p} \gamma^p_\Psi(x_1, .., x_p, x'_1, .., x'_p)\, \varphi(x'_1, .., x'_p) dx_1..dx_p.$$

It is easy to show from the definition that $\widehat{\gamma}^p_\Psi$ is compact, selfadjoint, nonnegative, trace class, and has trace $\binom{N}{p}$(=number of p-tuples in the system). (The latter comes from the widely but not universally used normalization factor in the definition.) In the sequel we follow the convention to denote the one-body and two-body density matrix by $\widehat{\gamma}^1 =: \gamma$ and $\widehat{\gamma}^2 =: \Gamma$. We can now state the

***N*-representability problem for the 2-body density matrix:** Characterize the set of operators $\tilde{\Gamma} : \mathcal{H}_2 \to \mathcal{H}_2$ which are 'N-representable', i.e. for which there exists a $\Psi \in A_N$ such that $\Gamma_\Psi = \tilde{\Gamma}$.

From a mathematical point of view, there is nothing special about $p = 2$ and one can ask the question for arbitrary p. For why this is less interesting physically see Section 3 below.

Even when the underlying one-body Hilbert space $\mathcal{H}_1$ is of low finite dimension, the problem is highly nontrivial and poorly understood.

Example (taken from [5]) Let $N = 4$, $\dim \mathcal{H}_1 = 6$. Then in terms of an orthonormal basis $|1\rangle, .., |6\rangle$ of $\mathcal{H}_1$, $\mathcal{H}_2 = \mathcal{H}_1 \wedge \mathcal{H}_1 = Span\{|ij\rangle\}_{1 \leq i < j \leq 6}$, where $|ij\rangle$ is the antisymmetrized tensor product of $|i\rangle$ and $|j\rangle$. In particular, $\mathcal{H}_2$ is 15-dimensional. The space of selfadjoint linear operators $\tilde{\Gamma} : \mathcal{H}_2 \to \mathcal{H}_2$ is then the 105-dimensional

space of hermitean 15×15 matrices. It can be shown [5] that the first of the following two matrices is 4-representable but the second is not. (The ordering of the basis $|ij\rangle$ used to represent operators by matrices is the alphabetical ordering.)

$$\tilde{\Gamma}_1 = \frac{1}{3}\begin{pmatrix}
2 & & & & & & & & & & & & & & 1 \\
 & 1 & & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & & \\
 & & & 1 & & & & & & & & & & & \\
 & & & & 1 & & & & & & & & & & \\
 & & & & & 1 & & & & & & & & & \\
 & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & & & & & & \\
 & & & & & & & & 1 & & & & & & \\
1 & & & & & & & & & 2 & & & & & 1 \\
 & & & & & & & & & & 1 & & & & \\
 & & & & & & & & & & & 1 & & & \\
 & & & & & & & & & & & & 1 & & \\
 & & & & & & & & & & & & & 1 & \\
1 & & & & & & & & & 1 & & & & & 2
\end{pmatrix}$$

$$\tilde{\Gamma}_2 = \frac{1}{3}\begin{pmatrix}
1 & & & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & & \\
 & & & 1 & & & & & & & & & & & \\
 & & & & 1 & & & & & & & & & & \\
 & & & & & 1 & & & & & & & & & \\
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 & & & & & & & & 1 & & & & & & \\
 & & & & & & & & & 1 & & & & & \\
 & & & & & & & & & & 1 & & & & \\
 & & & & & & & & & & & 2 & 1 & 1 & \\
 & & & & & & & & & & & 1 & 2 & 1 & \\
 & & & & & & & & & & & 1 & 1 & 2 &
\end{pmatrix}$$

For a recent review of known necessary conditions for representability see [3]. Nontrivial conditions not covered in [3] include Bach's inequality [1] and Erdahl's three-index condition [8]. The representability problem has not even been solved in the simpler case of the one-body density matrix. In the latter case, the convex hull of the representable set is known [6]; from the table of possible ranks, which is also known [4] and reveals various forbidden values, one sees however that the representable set is not convex.

2. Connection with fermion condensation

The following theorem was proved by C.N.Yang [7](see also [2]).

Theorem For all $\Psi \in A_N$, $\lambda_{max}(\Gamma_\Psi) \leq N/2$ if N is even and $\leq (N-1)/2$ if N is odd, both bounds being sharp.

(We remark in passing that representability does not depend on the eigenvalues of Γ alone – see the two matrices above which only differ by a re-ordering of the basis. The set of representable matrices is only invariant under *some* base changes in pair state – those induced by a one-body base change.)

To understand the relevance to fermion condensation, start from the identity

$$\langle\chi|\Gamma_\Psi|\chi\rangle = \langle\Psi|a^+(\chi)a(\chi)|\Psi\rangle \text{ for all } \chi \in A_2,\ \Psi \in A_N,$$

where $a(\chi) : \mathcal{H}_N \to \mathcal{H}_{N-2}$ and $a^+(\chi) : \mathcal{H}_{N-2} \to \mathcal{H}_N$ are the annihilation respectively creation operator of the pair state χ. Physically, this means that the expected value of Γ_Ψ with respect to a pair state χ is the 'occupation number' of the pair state in the N-body system. As emphasized by Yang [7], the theorem then means that the maximum possible occupation number of a pair state is order N

not order N^2, i.e. only an asymptotically negligible fraction of the order N^2 pairs in the system can occupy the same state. This is of interest for understanding superconductivity because it calls into question a naive picture advocated in some textbooks in which a macroscopic fraction of electrons is said to form 'Cooper pairs' condensing into a single pair state.

3. Connection with quantum chemistry

Starting point is the fact that the electronic Schrödinger operator for an atom or molecule,

$$H = \sum_{i=1}^{N}\Big(-\frac{1}{2}\Delta_{x_i} + V(x_i)\Big) + \sum_{1\le i<j\le N}\frac{1}{|x_i - x_j|}$$

(where N is the number of electrons and $V(x) = -\sum_{\alpha=1}^{M} Z_\alpha/|x - R_\alpha|$ is the Coulomb potential exerted by an array of nuclei with positions R_α and charges Z_α), contains only two-body interactions. Said more abstractly, it is a 'two-body operator', i.e. of form $H = \sum_{i<j} h(i,j)$ for some $h : \mathcal{H}_2 \to \mathcal{H}_2$. (Simply take $h = \frac{1}{N-1}(-\frac{1}{2}\Delta_{x_1} - \frac{1}{2}\Delta_{x_2} + V(x_1) + V(x_2) + \frac{1}{|x_1-x_2|}$.) But from the definition of the two-body density matrix it is easy to see that expected values of two-body operators depend only on Γ_Ψ not Ψ,

$$\langle\Psi|H|\Psi\rangle = \operatorname{trace} h\,\Gamma_\Psi.$$

This yields the following elementary formula for the ground state energy.

Proposition: For any bounded selfadjoint two-body operator $H : \mathcal{H}_N \to \mathcal{H}_N$,

$$\inf\{\langle\Psi|H|\Psi\rangle \,|\, \Psi \in A_N\} = \inf\{\operatorname{trace} h\tilde{\Gamma} \,|\, \tilde{\Gamma} : \mathcal{H}_2 \to \mathcal{H}_2 \text{ N-representable}\}.$$

(The extension to unbounded operators is straightforward, by restricting the class of admissible Ψ in the above infimum and in the definition of representability.)

In other words, the ground state energy of an N-electron Schrödinger equation can be found by solving – instead of the usual quadratic minimization problem on $\mathcal{H}_N$ – a 'linear programming problem' on the set of N-representable two-body density matrices.

This is of computational interest because at fixed dimension of $\mathcal{H}_1$ (say K), the domain in the usual minimization problem has dimension $\dim\mathcal{H}_N = \binom{K}{N}$, i.e. scales exponentially in particle number, whereas in the new problem it has dimension dim(self-adjoint operators on $\mathcal{H}_2$) $= \frac{1}{2}\binom{K}{2}(\binom{K}{2}+1) \sim K^4$, independenly of particle number. Hence an, as yet unavailable, appropriate intrinsic characterization of the set of representable operators would give rise to a rigorous computational algorithm of sub-exponential complexity for atomic and molecular energy levels.

4. Recent progress for $N = 4$

The new result in case $N = 4$ [5] is an infinitude of new sharp necessary inequalities for representability. More precisely, for each pair state $\chi \in A_2$, we compute the optimal constant $c(\chi)$ in the inequality

$$\langle\chi|\tilde{\Gamma}|\chi\rangle \le c \text{ for all 4-representable } \tilde{\Gamma}.$$

Physically, the optimal constant is the maximum possible occupation number of χ in a 4-body system. It depends in an interesting way on how strongly χ is correlated (in particular it is lowest when χ is a Slater determinant); mathematically it has a somewhat surprising form, namely it turns out to be a nonlinear average of the eigenvalues of the one-body density matrix of χ.

The result can be re-interpreted as an explicit formula for the lowest eigenvalue of a class of interacting model Hamiltonians on $\mathcal{H}_4$, namely Hamiltonians $H = -\sum_{i<j} P_\chi(i,j) = -a^+(\chi)a(\chi)$, where P_χ is the rank-1 projector onto the span of $\chi \in A_2$. In fact we also obtain an explicit formula for the ground state, in particular it has the product form $\psi \wedge \varphi$ for some ψ, $\varphi \in A_2$ (both depending on χ). Note that this is reminiscent of, but different from, a BCS ansatz $\varphi \wedge \varphi$. The product structure of the ground state shows that the linear eigenvalue equation in 4-body space for the ground state is equivalent two a system of two nonlinear eigenvalue equations in 2-body space, obtained by making the above product ansatz in the Rayleigh-Ritz variational principle and computing the variational derivative with respect to each factor. The factorization result does not require the one-body state space $\mathcal{H}_1$ to be finite-dimensional, but when specializing to $dim\,\mathcal{H}_1 = K$ we see that it entails an intriguing computational complexity reduction, from $O(K^4)$ degrees of freedom (for the linear eigenvalue equation in 4-body space) to $O(K^2)$ degrees of freedom (for the nonlinear system in 2-body space).

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Quantum Hall Effect(s)

Jürg Fröhlich

(joint work with U. M. Studer, E. Thiran)

The Quantum Hall Effect is observed in $2D$ electron gases forming at the interface of a heterojuncture when a gate voltage is applied and an external magnetic field transversal to the plane of the gas is turned on. For certain values of the the

gate voltage and the magnetic field, a mobility gap opens in the specrum of the Hamiltonian of the electron gas above its ground state energy. One then speaks of an "incompressible Hall fluid". In these lectures, it is explained what the theory of the $(1+1)-$dimensional chiral anomaly and $3-$dimensional topological field theory have to teach us about incompressible Hall fluids – no kidding!

An outline of a general classification of such fluids is given. The possible plateau–values of Hall conductivity and the quantum numbers (fractional electric charge, spin, internal quantum numbers) of quasi–particles in incompressible Hall fluids are predicted. Examples of fluids with quasi–particles exhibiting non–electron braid statistics are presented. It is explained quite carefully how the theory of modular tensor categories and the theorie of certain odd–integral lattices are applied to the classification of incompressible Hall fluids.– No analysis–much abstract algebra! And long live M∪Φ.

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The spectrum of magnetic Schroedinger operators and on conformally cusp manifolds

Sylvain Golenia

(joint work with Sergiu Moroianu)

We consider open manifolds which are interiors of a compact manifold with boundary, and Riemannian metrics asymptotic to a conformally cylindrical metric near the boundary. We show that the essential spectrum of the Laplace operator on functions vanishes under the presence of a magnetic field which does not define an integral relative cohomology class. It follows that the essential spectrum is not stable by perturbation even by a compactly supported magnetic field.

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Limiting Absorption Principle and Local Decay for the Standard Model of Non-Relativistic QED

Marcel Griesemer

(joint work with Jürg M. Fröhlich, Israel M. Sigal)

Bohr's stationary states of an atom or molecule are, in general, unstable due to the interaction with photons. The exceptions are the ground state and some excited states that are preserved for reasons of symmetry (orto-helium). Within

non-relativistic QED the instability of excited states finds its mathematical expression in the migration of eigenvalues into the complex lower half-plane as the fine-structure constant α is increases to a non-zero value. As a result, the spectrum of the Hamiltonian becomes purely *absolutely continuous* in a neighborhood of the unperturbed excited eigenvalue [3, 2]. The ground state, however, is stable [1, 2, 4] and the methods that have been used to analyze the spectrum near unstable excited eigenvalues have either failed [3] or not been pushed far enough [2] to give any information on the spectrum in a neighborhood of the ground state energy. We have recently established results that close this gap [5]. We have established a family of Mourre estimates and corresponding limiting absorptions principles for intervals near the least point of the energy spectrum. It follows that the spectrum is purely absolutely continuous near the ground state energy. As a Corollary of the Mourre estimates we also we also get local decay of the photon dynamics.

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Initiation to Witten Laplacians Methods in Statistical Mechanics

Bernard Helffer

In these lectures we shall analyze with techniques coming mainly from partial differential equations (PDE) and of semi-classical analysis problems coming from statistical mechanics.

Our main object of analysis is a (family of) measure(s) representing the probability of presence of m particles in interaction and having the form

$$d\mu^{(m)} := Z(m,h)^{-1} \exp -\frac{\Phi^{(m)}(X)}{h}\, dX$$

($m \in \mathbb{N}$) where

- $Z(m,h)$ is a normalization constant,
- $\Phi^{(m)}$ is a C^∞ function defined on $\mathbb{R}^m$, tending to ∞ at ∞, with a specific structure coming from statistical mechanics (usually a perturbation of $\sum_{j=1}^m \varphi(x_j)$ taking account of the interaction between nearest neighbours),
- h is a strictly positive parameter playing the role of an effective planck constant,

- dX is the Lebesgue measure on $\mathbb{R}^m$
- the integer m represents the cardinality of a set Λ in the lattice $\mathbb{Z}^d$ which will tend infinity.

We have consequently two main parameters h and m. The limit $h \to 0$ corresponds to the so called semi-classical limit h (which can actually corresponds to the temperature) and $m \to +\infty$ corresponds to the so called thermodynamic limit, when a large number of particles is involved.

Our aim is to explain how the technique of the Witten Laplacian approach which gives a new light and suggests new proofs for the analysis of Poincaré estimates and log-Sobolev estimates in relation with the measure of the decay of correlations. The main difficulty will be to control constants or remainders independently of the dimension. We shall show in this context how techniques coming from the theory of partial differential equations and applied to the Witten Laplacian on one-forms can be efficient for showing the uniformity of some constants.

We will also show how the Witten Laplacians appear for other problems in the analysis of the metastability and will discuss some recent results on the splitting of the two lowest eigenvalues for Witten Laplacians occuring in the analysis of the Fokker-Planck operator.

My lectures will cover results obtained by various subgroups of the set :

Bach, Bodineau, Helffer, Hérau, Jecko, Matte, Moeller, Nier, Sjöstrand, Witten.

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Epstein Glaser approach to perturbative quantum field theory and the Quantum Noether method

Tobias Hurth

The Epstein-Glaser (Bogoliubov-Shirkov-Stückelberg) approach is directly based the axioms of relativistic quantum field theory. It clarifies how the fundamental axioms guide the perturbative construction of the scattering matrix. It is also an explicit construction method for the most general perturbation series compatible with causality and Poincaré invariance.

With the help of the causality condition the well-known problem of ultraviolet divergences is reduced to a well-defined problem, namely, the splitting of an operator-valued distribution with causal support into a distribution with advanced and a distribution with retarderd support; alternatively, the continuation of time-ordered products to coincidence points. In fact, every consistent renormalization scheme solves this problem. In this sense, the explicit EGBSS approach should not be regarded as a special renormalization scheme, but as a general framework in which the condition posed by the fundamental axioms of QFT on any renormalization scheme are built in, by construction.

The Quantum Noether conditions are model-independent symmetry conditions which implement a classical symmetry on the tree and quantum level of a quantum theory. They show that the only input information one needs in order to construct a perturbative quantum field theory with a global (local) symmetry is a set of free fields realizing the asymptotic part of this symmetry. It is shown with the help of the gauge-fixed cohomology that the restrictions on local contributions and local anomalies by the quantum Noether conditions in the EGBSS approach of gauge theories are equivalent to the restrictions imposed by the BRST invariance on local contributions and anomalies in the standard Lagrangian approach.

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The Fröhlich Polaron revisited

Jacob Schach Møller

We study a model of an electron in a ionic crystal derived by Herbert Fröhlich in 1954 [1]. It is called the Fröhlich (or large) polaron model. A *polaron* means here a charged particle travelling together with a polarization of the underlying crystal lattice. The polarization of the lattice is modelled by longitudinal optical phonons in a second quantized formalism. The talk is based on [3] and some work in progress.

The class of Hamiltonians we study acts in the Hilbert space $L^2(\mathbb{R}^\nu_{\mathrm{x}}) \otimes L^2(\mathbb{R}^\nu_k)$. The first

component is the electron space, and the latter is the symmetric Fock-space of the phonons. The free part of the operator is

$$H_0 = \Omega(p) \otimes \mathbf{1}_{\Gamma(L^2(\mathbb{R}^\nu_k))} + \mathbf{1}_{L^2(\mathbb{R}^\nu_{\mathrm{x}})} \otimes d\Gamma(\omega),$$

where $p = \frac{1}{\mathrm{i}} \nabla_{\mathrm{x}}$. The interacting Hamiltonian is

$$H = H_0 + \alpha \int_{\mathbb{R}^\nu_k} \{v(k) e^{-\mathrm{i}k\cdot\mathrm{x}} \mathbf{a}^*(k) + \bar{v}(k) e^{\mathrm{i}k\cdot\mathrm{x}} \mathbf{a}(k)\} dk,$$

where α is a coupling constant and $v \in L^2(\mathbb{R}^\nu)$ is a coupling function. In the polaron model of H. Fröhlich one takes $\Omega(\eta) = \eta^2/m_{\mathrm{eff}}$, where m_{eff} is the effective mass of an electron in the periodic background potential of a static crystal. For

phonon dispersion one takes $\omega(k) \equiv \omega_0 > 0$, a constant. The coupling function is proportional to $1/|k|$ (if $\nu = 3$), which is not in L^2, so we need an ultraviolet cutoff. The coupling constant here is typically not small.

The Hamiltonians above commute with the operator of total momentum $P = p \otimes \mathbf{1} + \mathbf{1} \otimes d\Gamma(k)$, and hence fibers $H \sim \oint_{\mathbb{R}^\nu} H(\xi) d\xi$. We have

$$H(\xi) = d\Gamma(\omega) + \Omega(\xi - d\Gamma(k)) + \Phi(v) \quad \text{on } \Gamma(L^2(\mathbb{R}^\nu_k)),$$

where $\Phi(v) = \int \{v(k)\mathbf{a}^*(k) + \bar{v}(k)\mathbf{a}(k)\} dk$ is a field operator. (We put $\alpha = 1$.)

The assumptions we impose are

(1) $\Omega \in C^2(\mathbb{R}^\nu)$, $\Omega \geq 0$, $|\nabla\Omega(\eta)| \leq C\Omega(\eta) + C$, and $\sup_\eta \|\nabla^2\Omega(\eta)\| < \infty$.
(2) $\omega \in C^0(\mathbb{R}^\nu)$ and $\omega(k) \geq \omega_0 > 0$.
(3) $v \in L^2(\mathbb{R}^\nu)$.

Let us furthermore for simplicity specialize to the case $\sup \omega(k) < \infty$ and assume $\Omega(\eta) \to \infty$, $|\eta| \to \infty$, which contains the polaron model. We have the following basic results on the bottom of the energy-momentum spectrum $\{(\xi, E) | \xi \in \mathbb{R}^\nu, E \in \sigma(H(\xi))\}$:

An HVZ-theorem: $\sigma_{\text{ess}}(H(\xi)) = [\Sigma_{\text{ess}}(\xi), \infty)$, where $\Sigma_{\text{ess}}(\xi)$ is the smallest energy you can get by adding free phonons to a polaron ground state while keeping the total momentum equal to ξ. That is,

$$\Sigma_{\text{ess}}(\xi) = \min_{n \in \mathbb{N}} \inf_{k_1, \ldots, k_n \in \mathbb{R}^\nu} \left\{ \Sigma_0(\xi - k_1 - \cdots - k_n) + \omega(k_1) + \cdots + \omega(k_n) \right\},$$

where $\Sigma_0(\xi) = \inf \sigma(H(\xi))$ is the bottom of the spectrum for the fiber Hamiltonian. For ω which are subadditive (and unbounded from above), the hard part of this goes back to J. Fröhlich [2].

Uniqueness of groundstates: If $\Sigma_0(\xi)$ is an isolated eigenvalue, then it is non-degenerate. If $v \neq 0$ a.e., and $\Sigma_0(\xi)$ is an embedded ground state, then it is also non-degenerate. The result for non-negative v goes back to [2].

For the last two results we furthermore assume ω to be strictly subadditive, $v \neq 0$ a.e., and $|v|^{-1} \in L^\infty_{\text{loc}}(\mathbb{R}^\nu)$.

Existence of isolated groundstates: If $\nu = 1, 2$, then for all $\xi \in \mathbb{R}^\nu$, $\Sigma_0(\xi)$ is a n isolated ground state for $H(\xi)$. The proof is an extension and polished version of one given by Spohn [4].

Non-existence of embedded ground states: If $\nu = 3, 4$, and $\Sigma_0(\xi) = \Sigma_{\text{ess}}(\xi)$, then $\Sigma_0(\xi)$ is not an eigenvalue.

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Introduction to perturbative QED

Ettore Remiddi

The 4 lectures of the minicourse begin with a short description of the historical development of QED, starting from the Planck formula for the black body radiation (1900), till the Lamb shift and electron magnetic moment experiments of 1947 and the formalism of perturbative QED which was quickly worked out for explaining them. As a consequence of radiative corrections, the socalled bare quantities (masses and other parameters) appearing in the free Lagrangian cannot be identified with the corresponding physical quantities, but can be related to them through suitable renormalization constants, which must be themselves evaluated with the radiative corrections. The radiative corrections, when naively considered, are given in the form of divergent integrals; a suitable regularization procedure is required to give them a non ambiguous mathematical meaning, after which radiative corrections and renormalization constants can be evaluated and combined into finite physical quantities.

The socalled continuous dimensional regularization scheme (or d-regularization) of 't Hooft and Veltman is reviewed in some detail. As all integrals are properly defined in the d-regularization, one can consider all the scalar integrals associated to a given Feynman graph and write for all the relations due to symmetry consideration and, in particular, the socalled integration by parts identities. One obtains in that way a set of infinite linear equations for the infinite numebr of scalar integrals associated to the graph; the equation can be solved [1], reducing as a result the totality of the integrals to a linear combination, with polynomial coefficients, of a few irreducible Master Integrals.

The Master Integrals are found to satisfy a linear system of non homogeneous differential equations [2] in any of the Mandelstam variable or square masses entering in the graph; the study of the equation provides an enormous amount of information on the Master Integrals, arriving in some cases to their evaluation in closed analytic form.

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The Bogoliubov-Dirac-Fock approximation in no-photon QED

Éric Séré
(joint work with Christian Hainzl, Mathieu Lewin)

The Bogoliubov-Dirac-Fock (BDF) model has been designed for the study of relativistic electrons near a heavy nucleus. It is derived from the no-photon QED Hamiltonian in presence of an external electrostatic field φ. This Hamiltonian is normal-ordered with respect to a fixed reference: a translation-invariant projector $\mathcal{P}^0$, which represents the Dirac sea when the external field is not present. Then the QED energy is restricted to Hartree-Fock states of the electron-positron field, the one-body density of which is a Hilbert-Schmidt operator Q such that $\mathcal{P}^0 + Q$ is a projector. The resulting BDF energy functional takes the form

$$\mathcal{E}(Q) = \mathrm{tr}(\mathcal{D}^0 Q) - \alpha \int \rho_Q \varphi + \frac{\alpha}{2} \iint \frac{\rho_Q(x)\rho_Q(y)}{|x-y|}\, dx\, dy - \frac{\alpha}{2} \iint \frac{|Q(x,y)|^2}{|x-y|}\, dx\, dy$$

with $\rho_Q(x) = \mathrm{tr}_{\mathbb{C}^4} Q(x,x)$. This functional has been introduced by Chaix and Iracane [1] as a possible alternative to the Dirac-Fock functional, which is widely used in numerical computations but is not bounded below. The BDF model has several advantages as compared to Dirac-Fock: it is more accurate (taking into account vacuum polarization effects), its physical derivation is more convincing, the energy is bounded below [2], and the ground state solutions have a simple definition as mimimizers [1]. However, note that Chaix and Iracane take the negative spectral projector P^0 of D^0 as reference for normal ordering. This choice is not correct, as shown recently by Hainzl, Lewin and Solovej [5], who obtain the right $\mathcal{P}_0$ by a thermodynamical limit of QED in a box of very large size. Note also that in order to define correctly the BDF energy [3, 4], one has to introduce a momentum cut-off, and a generalized trace "$\mathrm{tr}_{\mathcal{P}_0}$" has to be defined, since the operator Q is not trace-class.

For $0 \leq N \leq Z$ and α small enough, the BDF energy has a minimizer [3, 4, 6] in the charge sector N (i.e. under the constraint $\mathrm{tr}_{\mathcal{P}_0}(Q) = N$), which solves a self-consistent equation of the form $Q = \chi_{(-\infty;\mu)}(D_Q) - \mathcal{P}_0$, where $D_Q := \mathcal{D}^0 + \alpha(\rho_Q * \frac{1}{|\cdot|} - \varphi) - \alpha \frac{Q(x,y)}{|x-y|}$ is the mean-field Hamiltonian, $\mu \in (0;1)$ is a Lagrange multiplier associated with the charge constraint, and interpreted as a chemical potential. Moreover, one can rewrite the equation in the form $P = Q + \mathcal{P}_0 = \Pi + \sum_{k=1}^{N} |\psi_k\rangle\langle\psi_k|$ .Here, $\Pi = \chi_{(-\infty,0)}(D_Q)$ is the projector on the (polarized) Dirac sea, and the mono-electronic wave functions ψ_k are solutions of the Dirac-Fock equations, perturbed by vacuum polarization terms : $D_Q\psi_k = \varepsilon_k\psi_k$, $0 < \varepsilon_k < 1$, $1 \leq k \leq N$.

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Two-body scattering at low energies

Erik Skibsted

(joint work with Jan Derezinski)

We give an account of various recent results obtained with Søren Fournais[1] [2] and Jan Dereziński[2] [1] related to low-energy scattering for a class of long-range potentials containing the attractive Coulombic one. This includes the construction of wave operators of Isozaki–Kitada type ([3], [4]) diagonalizing the *whole* continuous part of the Hamiltonian. The corresponding S–matrix is strongly continuous (although not differentiable) at zero energy. We derive a relationship to the analogous Dollard type constructions, and show that the location of the singularities of the scattering kernel $S(\lambda)(\omega,\omega')$ experiences an abrupt change at $\lambda = 0$. Thus, for example, for the purely Coulombic case the set of singularities jumps (as the energy goes down) from the set of coinciding outgoing and incoming angles, $\omega = \omega'$, to the set of oppositely oriented angles, $\omega = -\omega'$, reflecting the fact that the classical orbits at zero energy in this case are parabolas.

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