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Mini-Workshop: Dynamics of Cocycles and One-Dimensional Spectral Theory

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Abstract. Many spectral questions about one-dimensional Schrödinger operators with quasi-periodic potentials can be reduced to dynamical questions about certain quasi-periodic $\mathrm{SL}(2,\mathbb{R})$-valued cocycles. This connection has recently been employed to prove a number of long-standing conjectures. The aim of this mini-workshop was to bring together people from both spectral theory and dynamical systems in order to further develop and exploit the dynamical approach to quasi-periodic Schrödinger operators.

Mathematics Subject Classification (2000): 34C20, 37E45, 47B80.

Introduction by the Organisers

The mini-workshop *Dynamics of Cocycles and One-Dimensional Spectral Theory*, organised by David Damanik (Pasadena), Russell Johnson (Firenze) and Daniel Lenz (Chemnitz), was held November 13–19, 2005.

There have been a number of recent breakthroughs in the spectral theory of one-dimensional Schrödinger operators with quasi-periodic potentials that were accomplished using sophisticated dynamical systems methods; especially by establishing reducibility properties of certain quasi-periodic $\mathrm{SL}(2,\mathbb{R})$-valued cocycles.

The most popular example of a one-dimensional Schrödinger operators with quasi-periodic potential is given by the almost Mathieu operator,

$$[Hu]_n = u_{n+1} + u_{n-1} + 2\lambda\cos(2\pi(n\alpha+\theta))u_n,$$

where $\lambda \neq 0$ and α is irrational. Using the connection with dynamics, it was recently shown for all parameter values that the spectrum of H is a Cantor set of Lebesgue measure $|4-4|\lambda||$.

It was the objective of the mini-workshop to bring together experts from both spectral theory and dynamical systems to learn from each other and to further explore potential applications of dynamical systems methods in the context of quasi-periodic Schrödinger operators. Special attention was paid to having many young participants. This was made easy by the fact that currently there are many excellent graduate students and postdocs working on problems located at the interface between the two areas. Consequently, about two thirds of the participants belonged to the age group 35 years or younger.

The talks presented by the participants reflected the current developments in this area. Among other things, there is now an improved understanding of analytic potentials with non-perturbatively small coupling, there are extensions of some results known for analytic potentials to certain classes of non-analytic potentials, while phenomena different from those in the analytic case may occur for potentials of low regularity, and there is improved understanding of the case of Liouville frequencies.

Among the highlights of the discussions outside the talks one could mention the "joining" of independent and closely related work of Bjerklöv and Jäger and the solution by Avila and Damanik of one of the few questions about the almost Mathieu operator that were still left open after the recent advances triggering the mini-workshop.

Mini-Workshop: Dynamics of Cocycles and 1D Spectral Theory

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Abstracts

Nonperturbative reducibility and irreducibility

Joaquim Puig

In this talk we consider discrete, one-dimensional Schrödinger operators with real analytic potentials and Diophantine frequencies

$$(H_{V,\omega,\theta}x)_n = x_{n+1} + x_{n-1} + V(2\pi\omega n + \theta)x_n, \qquad n \in \mathbb{Z}. \tag{1}$$

and the relation with the reducibility problem of the associated *quasi-periodic cocycle*, which is the following dynamical system on $SL(2,\mathbb{R}) \times \mathbb{T}^d$,

$$\begin{array}{rccc} (A_{a,V},\omega): & SL(2,\mathbb{R}) \times \mathbb{T}^d & \longrightarrow & SL(2,\mathbb{R}) \times \mathbb{T}^d \\ & (X,\theta) & \mapsto & (A(\theta)X, \theta + 2\pi\omega)\,. \end{array} \tag{2}$$

with

$$A_{a,V}(\theta) = \begin{pmatrix} a - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}.$$

As many recent works have outlined, there is a close relationship between spectral properties of (1) and dynamics of the cocycle (2) which arise from the fact that the latter is the matrix first-order system associated to the eigenvalue equation of (1),

$$x_{n+1} + x_{n-1} + V\left(2\pi\omega n + \theta\right) x_n = a x_n, \quad n \in \mathbb{Z}.$$

Among dynamical properties which are relevant for the spectral description of (1), conjugacy of cocycles is a key tool, since it allows to classify different dynamical types. Two cocycles (A,ω) and (B,ω) are *conjugated* if there is a *conjugation* $Z : \mathbb{T}^d \to SL(2,\mathbb{R})$ such that $(A,\omega) \circ (Z,0) = (Z,0) \circ (B,\omega)$. The notion of conjugacy, once a regularity class for the transformation has been imposed, allows to classify dynamically quasi-periodic cocycles. A particularly important class is that of *reducible* cocycles, which are those conjugated to a cocycle with constant coefficients, which is called a *Floquet matrix*.

In our case of interest, i.e. when the frequencies are Diophantine, $\omega \in DC(c,\tau)$ for some c,τ,

$$|\langle \mathbf{k}, \omega \rangle| \geq \frac{c}{|\mathbf{k}|^\tau}, \quad \mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\},$$

and the potential V is real analytic, $C^a_\rho(\mathbb{T}^d,\mathbb{R})$ for some $\rho > 0$ with the norm

$$|V|_\rho := \sup_{|\Im\theta|<\rho} |V(\theta)| < \infty,$$

Johnson [7] showed that a is not in the spectrum of (1) if, and only if, the cocycle (2) is reducible to constant coefficients with hyperbolic Floquet matrix (possibly halving the frequency). In the spectrum, however, the situation is much more involved.

Eliasson [6], among other results, showed that, once ρ, c and τ positive have been fixed, there is a $\varepsilon_E = \varepsilon_E(c,\tau,\rho) > 0$ such that if $|V|_\rho < \varepsilon_E$, then the cocycle $(A_{a,V},\omega)$ is analytically reducible to constant coefficients for Lebesgue almost all

$a \in \mathbb{R}$. Moreover, for a generic V, with $|V|_\rho < \varepsilon$ the spectrum of (1) is a Cantor set and it contains a subset of a's for which $(A_{a,V}, \omega)$ is not reducible to constant coefficients.

This result is optimal for $d > 1$ due to an example of Bourgain [2] and, in fact, it gives a characterization of the set of "reducible energies a" in terms of their rotation number or integrated density of states. When $d = 1$, we can use localization results by Bourgain & Jitomirskaya [3] and Aubry duality to show the following.

Theorem 1 ([10])**.** *Let $\rho > 0$. Then,there is a $\varepsilon = \varepsilon(\rho) > 0$ such that if*

$$|V|_\rho < \varepsilon$$

and $\omega \in DC(c, \tau)$ then the following holds

(i) *$(A_{a,V}, \omega)$ is* reducible *to constant coefficients for* almost all $a \in \mathbb{R}$.
(1) *For a* generic *$V \in C^a_\rho(\mathbb{T}, \mathbb{R})$, the spectrum $\sigma(V, \omega)$ is a Cantor set.*
(2) *If $\sigma(V, \omega)$ is a Cantor set, there is a residual set in $\sigma(V, \omega)$ for which $(A_{a,V}, \omega)$ is not reducible to constant coefficients.*

The theorem above is not a full extension of Eliasson's result, since the set of reducible energies has no characterization in terms of the IDS (see the talk by Avila and Jitomirskaya on these reports for an answer to this question), but it clarifies the relationship between Cantor spectrum and irreducibility (which was an ingredient in the proof of [6]).

As said before the proof of the first item is an application of [3] for which it suffices to realize that the (analytic) reducibility of a Schrödinger cocycle with Floquet matrix in $SO(2, \mathbb{R})$ is equivalent to the existence a pair of linearly independent *Bloch waves* for the Harper-like equation. Passing to the equation in Fourier space, one needs to find exponentially localized eigenvalues of a long-range operator (with exponentially decaying coefficients) and a small cosine potential. Such solutions are provided by Bourgain & Jitomirskaya. To prove full measure, one only needs to apply the bounds for the growth of the density of states when the Lyapunov exponent is zero [5].

Items (ii) and (iii) in Theorem 1 allow a d-dimensional version. Indeed, using the analyticity of m-functions, as in Avila & Jitomirskaya [1], one can show that, whenever the spectrum has an open interval with zero Lyapunov exponent, then the corresponding cocycle is reducible to constant coefficients and the Floquet matrix is a rotation. In such a case, an application of Moser & Pöschel [8] (cf. [10]) shows that an arbitrarily small and generic perturbation opens up all collapsed gaps in the interval and the spectrum in the interval becomes a Cantor set. When $d = 1$, Cantor spectrum is supposed to be generic without the smallness hypothesis or, even, to hold always when the Lyapunov exponent is positive (see Sinai [11] for results in this direction), but there is no proof in this generality.

The relationship between Cantor spectrum and irreducibility can also be generalized to more general potentials (such as ergodic potentials, see [9]). For definiteness, let us consider continuous quasi-periodic potentials on $\mathbb{T}^d$ with irrational frequency vector ω and Cantor spectrum where the Lyapunov exponent vanishes.

Then, there is a residual G_δ in the spectrum where the corresponding cocycle is not reducible to constant coefficients by a continuous transformation. In fact such a transformation cannot be square integrable either. The latter is related to the existence of *Kotani eigenstates* (pair of solutions with norms given by the absolute values of an L^2 function along the trajectory of the ergodic transformation) which Kotani considered in the set where the Lyapunov exponent vanishes in the ergodic setting. De Concini & Johnson [4] showed that when this set contains an interval then there are Kotani eigenstates for all points in the interval. Our result shows that whenever the spectrum is a Cantor set and the Lyapunov exponent vanishes there, then there is a topologically significant set for which these eigenstates do not exist.

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Schrödinger cocycles at non-perturbatively small coupling

Artur Avila, Svetlana Jitomirskaya

We are interested in one-dimensional quasiperiodic Schrödinger operators $H = H_{v,\alpha,\theta}$ defined on $l^2(\mathbb{Z})$

$$(Hu)_n = u_{n+1} + u_{n-1} + v(\theta + n\alpha)u_n \tag{1}$$

where $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is the potential, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is the frequency and $\theta \in \mathbb{R}$ is the phase. The most important example is given by the almost Mathieu operator, when $v(x) = 2\lambda \cos(2\pi x)$.

In [6], Eliasson obtained a very precise description of such operators for α Diophantine, in the case of small analytic potentials in the perturbative regime (this means that a smallness condition depends on α, and thus the analysis of a given potential, however small, can only be carried out for a positive Lebesgue measure set of α). His work actually dealt with more general, multi-frequency potentials (that is, v is an analytic function on $\mathbb{R}^d/\mathbb{Z}^d$, $d \geq 1$).

More recently less precise non-perturbative results have been obtained by Jitomirskaya [8] and Bourgain-Jitomirskaya [4]. It was also shown by Bourgain that such results can not be extended to more than one frequency.

Our goal is to extend Eliasson's results to a non-perturbative setting, with the optimal smallness requirement in the case of the almost Mathieu operator.

1. The almost Mathieu operator

Theorem 1. *Let α be Diophantine, $v = 2\lambda \cos 2\pi\theta$, $\lambda \neq -1, 0, 1$. Then the integrated density of states is $1/2$-Hölder continuous.*

Previously there had been results by Bourgain [2] (almost 1/2-Hölder continuity in the perturbative regime), Goldstein-Schlag [7] (almost 1/2-Hölder non-perturbatively) and Sana Ben Hadj Amor (1/2-Hölder in the perturbative regime). This result is quite sharp in several respects. Under the conditions of the theorem, there are square-root singularities for the integrated density of states, so 1/2-Hölder can not be improved, it is known that for fixed $|\lambda| > 0$ and generic α the integrated density of states is not Hölder, and Bourgain [3] showed that for $|\lambda| = 1$ even a fairly mild Diophantine condition is not enough to guarantee Hölder continuity.

Theorem 2. *The spectral measures are absolutely continuous for $v(\theta) = 2\lambda \cos 2\pi\theta$, α Diophantine, $0 < |\lambda| < 1$ and all phases.*

This is an important partial advance in the direction of Problem 6 of [10] (which conjectures that this should hold without any hypothesis on α).

Previously, this was known for almost every phases [8]. The case of all θ is considerably more subtle, and was only known in the perturbative regime, due to the work of Eliasson.

It was proved in [1], that the spectrum of the almost Mathieu operator is a Cantor set for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\lambda \neq 0$. This is the Ten Martini Problem of Kac-Simon. A strengthening of it, the so-called dry version of the Ten Martini Problem is to show that all gaps predicted by the Gap-Labelling Theorem are open. This was obtained for generic (λ, α) [5], as well as for a set of (λ, α) of positive Lebesgue measure [9]. Here we are able to deal with almost every (λ, α).

Theorem 3. *The dry version of the Ten Martini Problem holds for $\alpha \in$ DC, $\lambda \neq -1, 0, 1$.*

2. Analytic potentials

We consider now the case of small analytic potential λv where $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is analytic.

Theorem 4. *If α be Diophantine and $0 < \lambda < \lambda_0(v)$. Then the integrated density of states is $1/2$-Hölder.*

Theorem 5. *If α is Diophantine and $0 < \lambda < \lambda_0(v)$ then the spectral measures are absolutely continuous for all phases.*

3. Almost localization and almost reducibility

The result of Eliasson is based on dynamical systems considerations. A cocycle (α, A) is defined by $\alpha \in \mathbb{R}$ and an analytic map $A : \mathbb{R}/\mathbb{Z} \to SL(2,\mathbb{R})(2,\mathbb{R})$. It is viewed as a linear skew-product $(x, w) \mapsto (x + \alpha, A(x) \cdot w)$, $x \in \mathbb{R}/\mathbb{Z}$, $w \in \mathbb{R}^2$. Eliasson's theory describes the dynamics of (α, A) when α is Diophantine and A is close to a constant. The precise closeness quantifier defines the Eliasson's perturbative regime.

The connection to Schrödinger operators is clear when

$$A(x) = S_{v,E}(x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix}, \tag{2}$$

since a solution $Hu = Eu$ satisfies $A(\theta + n\alpha) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix}$.

We say that two analytic cocycles $(\alpha, A^{(i)})$, $i = 1, 2$, are analytically conjugate if there exists an analytic map $B : \mathbb{R}/2\mathbb{Z} \to SL(2,\mathbb{R})(2,\mathbb{R})$ such that

$$A^{(2)}(x) = B(x + \alpha)A^{(1)}(x)B(x)^{-1}. \tag{3}$$

The dynamical properties of cocycles are preserved by conjugacies.

Let us say that a cocycle which is analytically conjugate to one in Eliasson's perturbative regime is almost reducible. The idea is that all dynamical properties and spectral consequences that can be established in Eliasson's perturbative regime should still hold for almost reducible cocycles and operators whose associated cocycles are almost reducible.

One feature of the previous non-perturbative results we mentioned [4, 8] is that most of the analysis is concerned with the dual model $\hat{H} = \hat{H}_{v,\alpha,\theta}$ defined on $l^2(\mathbb{Z})$

$$(\hat{H}\hat{u})_n = \sum \hat{v}_k \hat{u}_{n-k} + 2\cos(2\pi\theta + n\alpha)\hat{u}_n, \tag{4}$$

where $\hat{v}_k$ are the Fourier coefficients of $v(x) = \sum \hat{v}_k e^{2\pi i k x}$. Localization (pure point spectrum with exponentially decaying eigenfunctions) for the dual model corresponds to reducibility for the original cocycle. Reducibility means that the cocycle is analytically conjugate to a constant and is strictly stronger than almost reducibility.

Unfortunately reducibility fails generically even in Eliasson's perturbative regime, and this corresponds to failure of localization (for some phases) of the dual model. We are still able to prove that the dual models exhibit what we call almost localization (which can not imply point spectrum, but is stronger than what was known before) in the non-perturbative regime. Our work shows that such estimates are sufficient to get almost reducibility.

Theorem 6. *Assume that $\{\hat{H}_{v,\alpha,\theta}\}_{\theta\in\mathbb{R}}$ exhibits almost localization. Then the cocycle associated with $\{H_{v,\alpha,\theta}\}_{\theta\in\mathbb{R}}$ is analytically conjugate to a cocycle in Eliasson's perturbative regime.*

We should point out that our dynamical analysis is sufficiently precise to obtain many features of the perturbative regime (such as 1/2-Hölder continuity of the integrated density of states) directly by non-perturbative analysis.

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Strange non-chaotic attractors and non-smooth saddle-node bifurcations

Tobias Jäger

A *quasiperiodically forced (qpf)* interval or circle map is a continuous map $T : \mathbb{T}^1 \times X \to \mathbb{T}^1 \times X$ (with X either a compact interval $[a,b]$ or the circle $\mathbb{T}^1$, respectively) which has the following skew-product structure:

$$T(\theta, x) = (\theta + \omega, T_\theta(x)) \tag{1}$$

where $\omega \in \mathbb{T}^1$ is irrational and $T_\theta(x) := \pi_2 \circ T(\theta, x)$ are called the *fiber maps*. An *invariant graph* is a function $\varphi : \mathbb{T}^1 \to X$ that satisfies $T_\theta(\varphi(x)) = \varphi(\theta + \omega)\ \forall \theta \in \mathbb{T}^1$.

One phenomenon which has evoked considerable interest in qpf systems in theoretical physics is the existence of non-continuous invariant graphs with negative Lyapunov exponents,[1] which were called *'strange non-chaotic attractors' (SNA)* due to their combination of non-chaotic dynamics with a fractal geometry [1, 2, 3, 4, 5]. However, despite the vast amount of numerical results, the only examples where the existence of these objects was shown rigorously so far are so-called pinched skew-products - a simple type of model systems introduced in [1]

[1] Given that the fiber maps are all differentiable, the Lyapunov exponent of an invariant graph is defined as $\lambda(\varphi) := \int_{\mathbb{T}}^1 \log T'_\theta(\varphi(\theta))\ d\theta$.

(see also [6]) - and quasiperiodic $\mathrm{SL}(2,\mathbb{R})$-cocycles, with the particular case of quasiperiodic Schrödinger cocycles. For the later, Herman described in [7], amongst other results, how at the top of the spectrum of the associated almost-Mathieu operator SNA's are created by the collision of a stable and an unstable invariant curve. Unfortunately, these arguments depend crucially on the linear structure of the cocycles and cannot be extended to other types of qpf systems, where numerical studies indicate that such *'non-smooth saddle-node bifurcations'* are quite common as well (e.g. [5]).

In [8] the parameter family

$$(\theta, x) \mapsto (\theta + \omega, \arctan(\alpha x) - \beta \cdot (1 - \sin(\pi\theta))) \tag{2}$$

is studied,[2] and a purely dynamical proof for the existence of SNA created in non-smooth saddle-node bifurcations is derived. The results can be summarized as follows:

Theorem 1. *Suppose ω is diophantine and α is sufficiently large (depending on ω). Then there exists a unique bifurcation parameter β_0, such that for $\beta < \beta_0$ the system (2) has exactly three invariant graphs, all of which are continuous, and for $\beta > \beta_0$ there exists only one (continuous) invariant graph. At $\beta = \beta_0$ there exist three invariant graphs $\varphi^- < \psi \leq \varphi^+$, of which only φ^- is continuous. $\lambda(\varphi^+)$ is negative, such that φ^+ is an SNA, whereas $\lambda(\psi) > 0$. Further, the topological closures of the point sets corresponding to ψ and φ^+ are equal.*

In the following, we shall briefly describe the dynamical mechanism responsible for such non-smooth bifurcations, and which determines the strategy for the proof of the above result in [8]. In order to do so, we will concentrate on the behaviour of the upper invariant graph φ^+ as the parameter β is varied and approaches β_0 from below (see Figure 1): The characteristic pattern which precedes the collision in Figure 1 seems to be quite general and can be observed similarly in many other parameter families, including quasiperiodic Schrödinger cocycles. (See [8] for more examples.) Figure 2 gives an heuristic explanation for this behaviour.

When this basic idea is translated into a rigorous proof of Theorem 1, a crucial role is played by so-called *'sink-source orbits'* — orbits that have a positive Lyapunov exponent both forwards and backwards in time. The existence of such a-typical orbits is already known from quasiperiodic Schrödinger cocycles, where it is equivalent to Anderson localization of the associated operators. Now the same phenomenon is found in a purely dynamical setting. Sink-source orbits imply the existence of SNA (see Thm. 3.5 in [8]), such that the non-smoothness of the bifurcation in Theorem 1 is obtained as consequence of the following

Lemma 2. *If ω is diophantine and α is sufficiently large (depending on ω), then there exists a parameter β_0 such that (2) with $\beta = \beta_0$ has a sink-source orbit. (Of course, this β_0 is the same as in Theorem 1.)*

[2] In fact, the same results hold under more general conditions. In particular the functions $F(x) = \arctan \alpha x$ and $g(\theta) = (1 - |\sin(\pi\theta)|)$ in (2) could be replaced by $\mathcal{C}^1$-distortions.

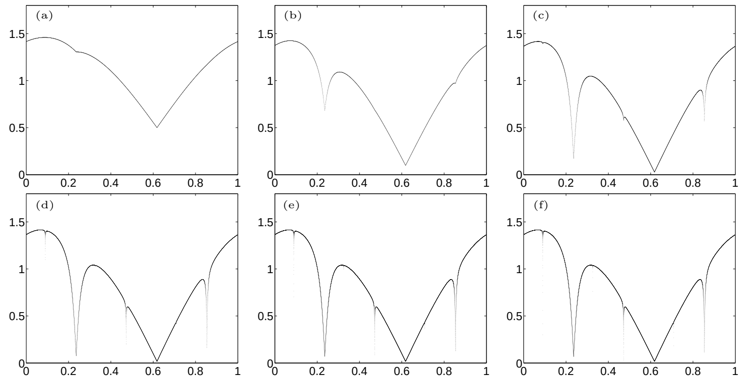

Figure 1. The upper invariant graphs in the parameter family given by (2) with ω the golden mean and $\alpha = 10$. In (a) ($\beta = 1$) the invariant graph has a global minimum, which we refer to as the first *'peak'*. As β is increased to 1.4, a second peak appears (b), which is steeper and grows faster than the first one. More peaks appear in (c) $\beta = 1.47$, (d) ($\beta = 1.48$), (e) $\beta = 1.4808$ and (f) ($\beta = 1.48095$). This pattern continues all up to the collision.

The described approach also allows to prove the existence of SNA in symmetric systems such as $(\theta, x) \mapsto (\theta + \omega, \arctan(\alpha x) + \beta(1 - 4d(\theta, 0)))$. Due to the inherent symmetry $T_\theta(-x) = -T_\theta(x)$, any collision of invariant curves must involve three curves at the same time, such that in this situation there exist two SNA which embrace a non-continuous invariant graph with positive Lyapunov exponent. Again, the topological closures of all three objects coincide.

Finally, it should be mentioned that the results presented here should only be understood as a first step towards more general criteria which ensure the existence of strange non-chaotic attractors in a great variety of parameter families. Concerning this goal, great impetus should also be expected from recent work by Bjerklöv ([9] and [10]) who uses similar ideas in order to study quasiperiodic Schrödinger cocycles. As his approach is purely dynamical and does not really make use of the specific linear structure of cocycles, it should allow to treat more general cases in almost the same way.

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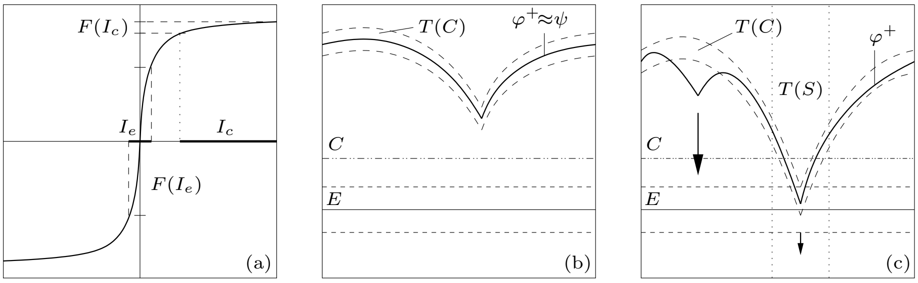

FIGURE 2. For large α, the map $F : x \mapsto \arctan(\alpha x)$ is strongly s-shaped. Thus, a small interval I_e around the repelling fixed point $x = 0$ is highly expanded, whereas an interval I_c above is strongly contracted (a). For the skew-product system, this gives rise to an expanding region E around the 0-line and a contracting region C above. Likewise, there is a contracting region below the 0-line, but as the forcing in (2) acts only downwards this is always a trapping region, and any interesting interaction takes place between E and C. As long as the forcing parameter β is not too large, the contracting region is still mapped inside itself, such that $\bigcap_{n\in\mathbb{Z}_+} T^n(C)$ is the point set of an attracting and continuous invariant graph inside $T(C)$ (b). Thus, the this upper invariant graph will be close to the line $x_c - \beta(1-|\sin(\pi\theta)|)$, where x_c is the upper fixed point of F, and consequently it has a first peak corresponding to the maximum of the forcing function. As the forcing parameter is increased, the tip of this first peak will eventually enter the expanding region. At this point it will induce a second peak, which is steeper and thinner than the first one and moves faster by a factor corresponding to the expansion constant in E. As soon as this second peak reaches the expanding region as well, it induces a third one and so on .... Hence, it is not hard to imagine that this process gives rise to a non-continuous limit object as the bifurcation parameter is approached.

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Dynamics of the quasi-periodic Schrödinger cocycle at the lowest energy in the spectrum

Kristian Bjerklöv

We consider the discrete quasi-periodic Schrödinger equation

$$-(u_{n+1}+u_{n-1})+\lambda V(\theta+n\omega)u_n=Eu_n,\quad n\in\mathbb{Z}, \tag{1}$$

for C^2-functions $V:\mathbb{T}^d\to\mathbb{R}$ ($d\geq 1$) with unique, non-degenerate, minimum, Diophantine frequency vectors $\omega\in\mathbb{T}^d$ and large coupling constants λ. The dynamics of this equation can be studied via the Schrödinger cocycle

$$F_E:(\theta,X)\in\mathbb{T}^d\times\mathbb{R}^2\mapsto\left(\theta+\omega,\begin{pmatrix}0 & 1\\ -1 & \lambda V(\theta)-E\end{pmatrix}X\right)\in\mathbb{T}^d\times\mathbb{R}^2$$

and its projectivization

$$G_E:(\theta,r)\in\mathbb{T}^d\times\mathbb{P}\mapsto\left(\theta+\omega,\lambda V(\theta)-E-\frac{1}{r}\right)\in\mathbb{T}^d\times\mathbb{P}.$$

Here we have defined $\mathbb{P}=\mathbb{R}\cup\{\infty\}$.

It is well-known that there exists a real number E_0 such that F_E is uniformly hyperbolic for all $E<E_0$ and not uniformly hyperbolic for $E=E_0$. This value of E coincides with the lowest energy in the spectrum of the Schrödinger operator associated with equation (1) and acting on $l^2(\mathbb{Z})$.

If the Lyapunov exponent $\gamma(E_0)$ is positive, then the projective mapping G_{E_0} has a unique minimal set M [8, 9]. Moreover, $M\subset\mathbb{T}^d\times[1/c,c]$, for some constant $c>1$ [9]. Thus we can define the two functions

$$l^+(\theta)=\max\{r:(\theta,r)\in M\},\quad l^-(\theta)=\min\{r:(\theta,r)\in M\}.$$

In [9], M. Herman asked the following question:

$$\text{Is } M=\{(\theta,r):l^-(\theta)\leq r\leq l^+(\theta)\}? \tag{Q}$$

Theorem 1. [3] *Assume that $V:\mathbb{T}^d\to\mathbb{R}$ is C^2 and has a unique non-degenerate minimum. Assume moreover that $\omega\in\mathbb{T}^d$ is Diophantine. Then there exists a $\lambda_0>0$ such that the following holds for all $\lambda>\lambda_0$:*

i) The Lyapunov exponent $\gamma(E_0)>\frac{\log\lambda}{4}$;
ii) There exists a $\theta\in\mathbb{T}^d$ and a vector $u\in l^2(\mathbb{Z})$, which is exponentially decaying at $\pm\infty$ and which satisfies equation (1);
iii) The answer to question (Q) is: YES.

Remark: Statement *ii*) says that for some phase θ, the lowest energy in the spectrum is an eigenvalue for the Schrödinger operator. This generalizes Soshnikov's result [12] to the case of $d>1$. To the authors knowledge, (*iii*) is the first answer to Herman's question. For other types of dynamics at the lowest energy in the spectrum we refer to [2]. Examples when G_E is minimal but not uniquely ergodic can be found in [1].

Open problems: The general answer to Herman's question (Q) is still open. Namely, assuming only that V is sufficiently regular, $\omega = (\omega_1, \dots, \omega_d)$ is rationally independent and that $\gamma(E_0) > 0$, what can then be said about the minimal set M? Can there be examples with Cantor sets in the fiber?

Other general open problems are whether the Lyapunov exponent $\gamma(E)$ is positive for all $E \in \mathbb{R}$, provided that λ is sufficiently large and V is finitely smooth. This is related to questions concerning the existence of Anderson localization. For an introduction to this vast field, we refer to [4, 5, 6, 7, 9, 10, 11] and the references therein.

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The relation between dynamics and the spectrum of one-dimensional Schrödinger operator

Roberta Fabbri

In this talk results for the one-dimensional Schrödinger operator with a quasi-periodic potential are presented. Consider the Schrödinger operator H defined by

$$H\varphi := \left(-\frac{d^2}{dt^2} + q(t)\right)\varphi, \quad \varphi \in L^2(\mathbb{R})$$

where q is bounded and continuous. Then we have that the spectral problem $H\varphi = E\varphi$ is equivalent to the differential system

$$(*) \qquad \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -E + q(t) & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}$$

The spectral theory of the operator H can be fruitfully studied considering the corresponding system $(*)$. In fact the presence of an exponential dichotomy for the system is equivalent to the statement that the energy E belongs to the resolvent of the operator H. Suppose that q is a quasi-periodic function, i.e.

$$q(t) = Q(\psi_1 + \gamma_1 t, \ldots, \psi_k + \gamma_k t) = Q(\psi + \gamma t)$$

where $Q : \mathbb{T}^k \to \mathbb{R}$ is continuous, $k \geq 2$ and the vector $\gamma = (\gamma_1, \ldots, \gamma_k) \in \mathbb{R}^k$. Let $\Gamma \subset \mathbb{R}^k$ be the set of frequency vectors with rationally independent components. Consider the quasi-periodic operator $H_\psi := -\frac{d^2}{dt^2} + Q(\psi)$ on $L^2(\mathbb{R})$ and the corresponding system

$$(1) \qquad \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -E + Q(\psi + \omega t) & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}$$

First we consider the question of the positivity of the Lyapunov exponent $\beta(E)$.

Take the set of frequencies Γ and let $C^r(\mathbb{T}^k) = C^r$ be the Banach space of Hölder continuous potentials where $0 \leq r < 1$. Indicate by W_r the following product: $W_r = \Gamma \times C^r$. We prove the following (see [4])

Theorem 1. *Suppose that $0 \leq r < 1$. There is a residual subset $W_* \subset W_r$ such that if $\omega = (\gamma, Q) \in W_*$, then either* (1) *admits an exponential dichotomy (E.D.) or $\beta(\omega) = 0$.*

In the proof we use the robustness properties of the exponential dichotomy concept. Indicate by $\mathcal{D} = \{(\gamma, Q) \in W_r \mid (1) \text{ admits an E.D.}, \forall \psi \in \mathbb{T}^k\}$. We prove that there exists a residual subset $W_r \subset \Gamma \times C^r$ such that if $\omega = (\gamma, Q) \in W_r \setminus \mathcal{D}$, then $\beta(\omega) = 0$ when $0 \leq r < 1$. This result is actually true for generic $SL(2, \mathbb{R})$-cocycles [2].

Regarding the study of the spectrum of the operator we prove that, if the frequencies are of Liouville-type then, for a residual set of $C^0(\mathbb{T}^k) = C(\mathbb{T}^k)$ potentials, the operator has Cantor spectrum. This holds for all $k \geq 3$. The result is the following (cf. [3])

Theorem 2. *There is a residual subset $\Gamma_1 \subset \Gamma$ with the following property. For each $\gamma \in \Gamma_1$, there is a residual subset $\mathcal{D}_\gamma \subset C(\mathbb{T}^k)$ such that: if $Q \in \mathcal{D}_\gamma$, then the spectrum $\sigma(H_\psi)$ of the operator $H_\psi : -\frac{d^2}{dt^2} + Q(\psi + \gamma t)$ is a Cantor set.*

In the proof a genericity result regarding the exponential dichotomy property in the C^0 topology for a $sl(2, \mathbb{R})$ system is used. [2]. The methods and techniques used are based on the concepts and properties of rotation number and exponential dichotomy. They are not of perturbative type and no assumptions on the location of the energy E are made. Our frequencies are of Liouville-type: they are "close" to rational numbers.

Remarks (a) It would be interesting to prove (or disprove) that, for each $\gamma \in \Gamma$, the set $\mathcal{D}_\gamma$ is residual in $C(\mathbb{T}^k)$.
(b) It would also be interesting to prove the Cantor spectrum result for the discrete Schrödinger operator with a quasi-periodic potential.

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Are Liouvilian cocycles almost reducible?

Raphaël Krikorian
(joint work with Bassam Fayad)

Let us consider $\alpha \in \mathbf{R}/\mathbf{Z} - \mathbf{Q}/\mathbf{Z}$, and $A : \mathbf{R}/\mathbf{Z} \to SL(2,\mathbf{R})$ a smooth or analytic map. We are interested in the dynamics of the cocycle (α, A) on $\mathbf{R}/\mathbf{Z} \times SL(2,\mathbf{R})$

$$\begin{aligned}(\alpha, A) : \mathbf{R}/\mathbf{Z} \times SL(2,\mathbf{R}) &\to \mathbf{R}/\mathbf{Z} \times SL(2,\mathbf{R}) \\ (x,y) &\mapsto (x+\alpha, A(x)y)\end{aligned}$$

This kind of problem occurs in symplectic dynamics when one wants to understand Hamiltonian dynamics in the vicinity of lower dimensional invariant tori and also in the study of the spectral properties of one dimensional quasi-periodic Schrödinger operators

$$(Hu)_n = u_{n+1} + u_{n-1} + V(x+n\alpha)u_n, \qquad (u_n) \in l^2(\mathbf{Z}).$$

In the latter case, it is the relevant to understand the dynamics of the one parameter family of cocycles (the so-called Schrödinger cocycles) $(\alpha, S_{E,V}(\cdot))$ where

$$S_{E,V}(x) = \begin{pmatrix} E - V(x) & -1 \\ 1 & 0 \end{pmatrix}$$

(E is the spectral parameter). The basic spectral properties of this operator are related in a deep and non-trivial way to the dynamical properties of the cocycles $(\alpha, S_{E,V})$ ($E \in \mathbf{R}$) namely Floquet reducibility and positivity of the Lyapunov exponent. We say that a cocycle (α, A) is *reducible* if there exist $A_0 \in SL(2,\mathbf{R})$ and a smooth or analytic $B : \mathbf{R}/\mathbf{Z} \to SL(2,\mathbf{R})$ such that $A(\cdot) = B(\cdot+\alpha)A_0B(\cdot)^{-1}$ that is to say $(\alpha, A) = (0,B) \circ (\alpha, A_0) \circ (0,B)^{-1}$. (In fact, one has to generalize this definition to include the case where B is only defined on $\mathbf{R}/2\mathbf{Z}$.) We say that a cocycle has positive Lyapunov exponent L if

$$L := \lim_{n\to\infty} \frac{1}{n} \int_{\mathbf{R}/\mathbf{Z}} \log \|A_n(x)\| dx > 0, \qquad (A_n(\cdot) = A(\cdot + (n-1)\alpha) \cdots A(\cdot)).$$

In that case there exist measurable stable and unstable bundles invariant by the dynamics (Oseledec theorem). If these bundles are not continuous, we say that the cocycle is *not uniformly hyperbolic* (N.U.H)

When the frequency α is diophantine

$$\forall k \in \mathbf{Z} - \{0\}, \qquad \|k\alpha\| := \inf_{l \in \mathbf{Z}} |k\alpha - l| \geq \frac{\gamma}{|k|^{\sigma}},$$

the situation is to some extent well understood at least in the *local* situation:

Theorem 1 (Eliasson, [4]). *Assume that A is real analytic on a strip of width h and α diophantine. There exists ϵ_0 such that if $\|A\|_h \leq \epsilon_0$*

i) the cocycle (α, A) is always almost reducible (see definition below).

ii) If $\tau > 0$ and the fibered rotation number of (α, A) is τ-diophantine w.r.t α (that means that there exists K such that for any $k \in \mathbf{Z} - \{0\}$, $\|2\{-\}k\alpha\| \geq (K/|k|^{\tau})$) or rational w.r.t α ($\exists k_0 \in \mathbf{Z}$, $2\{=\}k_0\alpha \mod 1$) then (α, A) is reducible.

iii) For the Schrödinger family, when V is analytic and small enough item ii) holds for a set of total Lebesgue measure of E.

Remarks: a) Though the theorem is stated in the analytic category it also holds in the smooth case.
b) The proof of this theorem which relies on a KAM scheme is local.

We should explain what means almost reducible. A smooth (or analytic) cocycle (α, A) is said to be *almost reducible* if there exist sequences $A_n \in SL(2, \mathbf{R})$ and $B_n : \mathbf{R}/2\mathbf{Z} \to SL(2, \mathbf{R})$ such that $B_n(\cdot + \alpha)^{-1} A(\cdot) B(\cdot) - A_n$ converges to zero in the smooth topology (in the analytic case the definition has to be more quantitative).

In the *global* situation, when α is still diophantine, the following result is true

Theorem 2 ([2]). *If α is recurrent diophantine and A is homotopic to the identity and smooth or analytic, then, for almost every $\theta \in \mathbf{S}^1$ (resp., a.e $E \in \mathbf{R}$), the cocycle $(\alpha, R_\theta A(\cdot))$, where R_θ is the rotation matrix by the angle θ (resp., the cocycle $(\alpha, S_{E,V})$) is either non uniformly hyperbolic or is reducible (the conjugacy is smooth or analytic).*

When (α, A) is not homotopic to the identity a similar result is true (cf. [2]). The recurrent diophantine condition is a condition of full Lebesgue measure (α is said to be *recurrent diophantine* if its images under iterates of the Gauss map $G(x) = \{x^{-1}\}$, are infinitely many times in a diophantine set of positive measure) but the result is also true if α is only diophantine. The proof of the preceding theorem goes as follows: if the Lyapunov exponent is zero on a set of positive measure then a.e on this set one can use renormalization to reduce the global situation to a local one and apply Eliasson's theorem.

It is natural to try to understand what happens when α does not satisfy any diophantine condition, that is when α is of *Liouville type.* This should be more difficult since in that case no local theory is available. Also, since even a smooth cocycle of rotations (which means with values in $SO(2, \mathbf{R})$) is not automatically reducible we have to change our definition of reducibility and almost reducibility. We shall say that a cocycle (α, A) is *rotations-reducible* if there exist a smooth

$B : \mathbf{R}/\mathbf{Z} \to SL(2,\mathbf{R})$ and a smooth $U : \mathbf{R}/\mathbf{Z} \to SO(2,\mathbf{R})$ such that $A(\cdot) = B(\cdot+\alpha)U(\cdot)B(\cdot)^{-1}$ and we shall say that (α, A) is *almost rotations-reducible* if there exist sequences of smooth maps $B_n : \mathbf{R}/\mathbf{Z} \to SL(2,\mathbf{R})$ and $U_n : \mathbf{R}/\mathbf{Z} \to SO(2,\mathbf{R})$ such that $B_n(\cdot+\alpha)^{-1}A(\cdot)B_n(\cdot) - U_n(\cdot)$ goes to 0 in the smooth topology.

Question 1 *Is the following dichotomy true: If α is Liouville and A smooth then for Lebesgue almost every θ the cocycle $(\alpha, R_\theta A)$ is either non uniformly hyperbolic or is rotations-reducible?*

We can prove with B. Fayad the following result:

Theorem 3. *i) Let α be irrational and A be C^0-conjugated to a cocycle of rotations (by a theorem of Yoccoz, this is equivalent to the fact that the fibered product of A along $x \mapsto x+\alpha$ are uniformly bounded). Then, there exists a sequence of integers n_k (depending on α) such that if the fibered rotation number ρ of (α, A) satisfies*

$$\limsup \|\frac{\rho}{\beta_{n_k}}\| > 0$$

(where $\beta_n = \alpha_0 \cdots \alpha_n$ and α_n is the n-th iterate of α under the Gauss map), then (α, A) is almost rotations-reducible.
ii) If α is super-Liouville (which means $\liminf(\log q_{n+1})/q_n = \infty$), then for almost every θ, $(\alpha, R_\theta A(\cdot))$ is either non uniformly hyperbolic or is almost rotations-reducible.

Remark a) The condition in item i) on the fibered rotation number is of full Lebesgue measure.
b) One ingredient of the proof of this result is the fact that renormalizations of cocycles that are L^2-conjugated to rotations converge to constants (cf. [3]).

The natural question which should be easier than Question 1 is then:

Question 2 *Is item ii) of the preceding result true for any α irrational?*

Let us mention one corollary of the above theorem:

Corollary 4. *If α is irrational, then the set of cocycles (α, A) with positive Lyapunov exponent is C^∞-dense.*

Remarks: a) In the case where α is in a set of full Lebesgue measure, this was proven in [5].
b) Artur Avila has a different proof of the corollary which allows for a generalization ([1]): for example, he can prove C^∞-density of cocycle with positive Lyapunov exponent for cocycles defined on higher dimensional tori (multi-frequency case).

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Remarks on isospectral deformation theory

Russell Johnson

In recent years, the isospectral deformation theory of certain one-dimensional operators has been studied using the theory of algebraic curves. This is true of the Schrödinger operator, the Jacobi operator, and the AKNS operator. More recently, the isospectral deformation problems corresponding to the difference operator defining orthogonal polynomials on the unit circle and to the Sturm-Liouville operator defining the Camassa-Holm equation have also been studied using Riemann surface theory.

It turns out that the Weyl m-functions and Kotani-type results can be used to formulate these isospectral deformation problems when the operators in question have bounded non-periodic coefficients.

Consider for example the case of orthogonal polynomials on the unit circle $K \subset \mathbb{C}$. Let σ be a probability measure on K whose support is infinite. The corresponding family of orthogonal polynomials
$\{\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots\}$ is obtained by performing the Gram-Schmidt process on the set $\{1, z, z^2, \ldots\} \subset L^2(K, \sigma)$.

It is well-known ([1], [3]) that σ defines a unique family of Verblunsky coefficients $\{\alpha_n \mid n \geq 1\}$. There are numbers in the open unit disc $D \subset \mathbb{C}$, with the property that

$$(*) \qquad \begin{pmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{pmatrix} = T_z(n) \ldots T_z(1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (n \geq 1)$$

where $T_z(k) = \left(1 - |\alpha_k|^2\right)^{-\frac{1}{2}} \begin{pmatrix} z & \alpha_k \\ z\bar{\alpha}_k & 1 \end{pmatrix}$ $(k \geq 1)$ and $\varphi_n^*(z) = z^n \bar{\varphi}_n(\frac{1}{z})$ is the dual polynomial for each $n = 0, 1, 2, \ldots$. One has $\alpha_n = \frac{\varphi_n(0)}{\varphi_n^*(0)}$ for $n \geq 1$.

Let us now assume that $|\alpha_n|$ is a stationary ergodic sequence with $\sup_n |\alpha_n|_\infty < 1$. There is no loss of generality in assuming that

$$\alpha_n(\omega) = g(\tau^{n-1}(\omega)) \quad (\omega \in \Omega)$$

where Ω is a compact metric space, $g : \Omega \to \Omega$ is continuous, $\tau : \Omega \to \Omega$ is a homeomorphism, and μ is a Radon probability measure on Ω which is τ-ergodic.

Let $0 \neq z \in \mathbb{C}$. The matrix product $T_z(\omega, n) \ldots T_z(\omega, 1)$ defines a cocycle $T_z : \Omega \times \mathbb{Z} \to GL(2, \mathbb{C})$. If $|z| = 1$ then T_z takes values in $U(1, 1)$. If $|z| \neq 1$, then T_z admits an exponential dichotomy. For each $\omega \in \Omega$, there are complex lines $l^{\pm}_{\omega,z}$ defined by the contracting direction as $n \to \infty$ resp. $n \to -\infty$. These lines are parametrized by the Weyl m- functions:

$$l^{\pm}_{\omega,z} = Span \begin{pmatrix} 1 \\ m_{\pm}(\omega, z) \end{pmatrix}.$$

We pose an inverse spectral problem. Let $\Sigma \subset K$ be the set of nonisolated points of $\sigma = \sigma_\omega$ for μ- a.a. ω. Assume that

$$\Sigma = [z_0, z_1] \cup \ldots [z_{2g-2}, z_{2g-1}]$$

is a union of finitely many closed intervals. Further assume that the Lyapunov exponent $\beta(z)$ of T_z equals zero on Σ. The problem is to reconstruct $\{\alpha_n\}$.

Somewhat surprisingly, this problem can be solved explicitly. First one proves a Kotani-type result, and extends $m_\pm(\omega, \cdot)$ holomorphically through Σ. It turns out that, as one crosses from $\mathbb{C} \setminus \bar{D}$ to D through an interval in Σ, m_+ extends to m_- and viceversa. This implies that, for each $\omega \in \Omega$, the functions $m_\pm(\omega, \cdot)$ are branches of a meromorphic function on the algebraic curve C defined by $w^2 = \Pi_{i=0}^{2g-1}(z - z_i)$.

This observation is the starting point for the reconstruction of α_n. It turns out that

$$\alpha_n = F(v_0 + vn)$$

where v_0 and v are vectors in $\mathbb{C}^g$, and F is a meromorphic function on a generalized Jacobian of C. In particular $n \to \alpha_n$ is quasi-periodic. See [2] for details.

In an analogous way, one can pose and solve an inverse spectral problem for the Sturm-Liouville equation

$$(**) \qquad -\varphi'' + \varphi = Ey(t)\varphi$$

where $y(t)$ is a positive function to be determined from the following conditions.

- The set $\Sigma = \{E \in \mathbb{R} \mid$ the system $\begin{pmatrix} \varphi \\ \varphi \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -Ey(t) & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}$ does not admit an exponential dichotomy $\}$ is a finite union of intervals: $\Sigma = [E_0, E_1] \cup \ldots \cup [E_{2g}, \infty)$.
- The Lyapunov exponent $\beta(E)$ of $(**)$ vanishes for $E \in \Sigma$.

One can give a concrete description of the possible y's [4]. In fact one can describe the triples (p, q, y) such that the more general equation

$$-(p\varphi')' + q\varphi = Ey\varphi$$

satisfies these conditions.

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Uniform existence of Lyapunov exponents for low-complexity subshifts

Daniel Lenz

We consider subshifts over a finite alphabet and the associated Schrödinger operators. This means we are given a finite set $A \subset \mathbb{R}$ and $\Omega \subset A^{\mathbb{Z}}$ closed and invariant under $T : A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$, $(Ts)(n) = s(n+1)$. Here, A carries the discrete topology and $A^{\mathbb{Z}}$ carries the product topology. We will assume that (Ω, T) is minimal and uniquely ergodic. For further details concerning the following discussion we refer to the cited literature.

To a subshift (Ω, T) we associate the family $(H_\omega)_{\omega\in\Omega}$ of operators $H_\omega : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ acting by

$$(H_\omega u)(n) = u(n+1) + u(n-1) + \omega(n)u(n).$$

These operators have attracted attention for several reasons. They arise in the quantum mechanical treatment of certain (dis)ordered solids called quasicrystals. They exhibit interesting mathematical features as they have a tendency to purely singularly continuous spectrum, Cantor spectrum of Lebesgue measure zero and anomalous transport. Recently, they have also been used in order to study certain almost periodic operators (see e.g. abstract of David Damanik in this report).

Here, we discuss a method to prove Cantor spectrum of Lebesgue measure zero for these operators. This method is based on ergodic theory and study of the cocycles of transfer matrices.

As is well known, much of the spectral theory of these operators is investigated by means of the solutions of the difference equation

$$u(n+1) + u(n-1) + (\omega(n) - E)u(n) = 0 \quad (*)$$

for $E \in \mathbb{R}$. These solutions can be expressed via the transfer matrices

$$M_E : \Omega \longrightarrow SL(2,\mathbb{R}), \;\; M^E(\omega) := \begin{pmatrix} E - \omega(1) & -1 \\ 1 & 0 \end{pmatrix}$$

and the corresponding cocycle $M_E : \mathbb{Z} \times \Omega \longrightarrow SL(2,\mathbb{R})$. More precisely, u is a solution of $(*)$ if and only if $\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = M^E(n,\omega) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}$ for all $n \in \mathbb{Z}$.

Kingman's subadditive ergodic theorem gives that the limit

$$\Lambda(M_E) = \lim_{n\to\infty} \frac{1}{n} \log \|M_E(n,\omega)\|$$

exists for almost every $\omega \in \Omega$ and is independent of it. This limit is called the Lyapunov exponent and denoted by $\gamma(E)$.

Definition 1. *A continuous map $A : \Omega \longrightarrow SL(2,\mathbb{R})$ is called uniform if the above limit exists for all $\omega \in \Omega$ and the convergence is uniform on Ω.*

The relevance of uniformity in our context comes from the fact [4] that the spectrum Σ of the H_ω (which does not depend on $\omega \in \Omega$ by minimality) is given by

$$\Sigma = \{E \in \mathbb{R} : \gamma(E) = 0\} \cup \{E \in \mathbb{R} : \; M_E \text{ is not uniform}\},$$

where the union is disjoint. Now, a result of Kotani from '89 says that the Lebesgue measure of

$$\{E : \gamma(E) = 0\}$$

is zero, if (Ω, T) is aperiodic. In this case we obtain that Σ is a Cantor set of Lebesgue measure zero if M_E is uniform for all $E \in \mathbb{R}$.

Uniformity of cocycles can be studied via combinatorial methods. More precisely, let $\mathcal{W}$ be the set of finite words associated to Ω. For $v, x \in \mathcal{W}$ denote the number of copies of v in x by $\sharp_v(x)$ and the length of v by $|v|$. A function $F : \mathcal{W} \longrightarrow \mathbb{R}$ is called subadditive if $F(xy) \leq F(x) + F(y)$ whenever $xy \in \mathcal{W}$.

Lemma 2. [3] *Let (Ω, T) be a minimal subshift. The following assertions are equivalent:*

(i) The limit $\lim_{|x|\to\infty} \frac{F(x)}{|x|}$ exists for every subadditive F.

(ii) There exists a $C > 0$ such that $\liminf_{|x|\to\infty} \frac{\sharp_v(x)}{|x|}|v| \geq C$ for every $v \in \mathcal{W}$.

The condition (ii) in the previous lemma will be denoted as (PW) for positivity of weights. The lemma and the discussion above allow one to prove the following.

Theorem 3. [4] *Let (Ω, T) be an aperiodic subshift satisfying (PW). Then, Σ is a Cantor set of Lebesgue measure zero.*

Condition (PW) is a requirement on words of all length scales. As we are dealing with subadditive functions coming from matrices, it is possible to weaken this requirement via the so-called avalanche principle. The avalanche principle was introduced by Goldstein/Schlag in '01 and later strengthened by Bourgain/Jitomirskaya in '02. Roughly speaking it says that a product of matrices has large norm if each matrix has large norm and the product of each pair of consecutive matrices has large norm.

Definition 4. *A subshift (Ω, T) is said to satisfy the Boshernitzan condition (B) if it is minimal and there exists a sequence (l_n) in $\mathbb{Z}_+$ with $l_n \longrightarrow \infty$ for $n \to \infty$ and a constant $C > 0$ such that $\liminf_{|x|\to\infty} \frac{\sharp_v(x)}{|x|}|v| \geq C$ for every $v \in \mathcal{W}$ with $|v| = l_n$ for some $n \in \mathbb{Z}_+$.*

This condition was introduced by Boshernitzan in his study of unique ergodicity. The main results of [1] now reads as follows.

Theorem 5. [1] *If (Ω, T) satisfies (B), then every locally constant $A : \Omega \longrightarrow SL(2, \mathbb{R})$ is uniform.*

This has the following spectral theoretic consequence.

Corollary 6. [1] *If (Ω, T) is aperiodic and satisfies (B), then Σ is a Cantor set of Lebesgue measure zero.*

A detailed study of examples of subshifts satisfying (B) is carried out in [2]. Examples include (almost all) circle maps and interval exchange transformations.

The above considerations raise various questions.

(1) Does uniformity of locally constant cocycles hold for all uniquely ergodic minimal subshifts? (See [Conjecture 12.8.2] in [5] as well.)
(2) Does every uniquely ergodic minimal dynamical system admit a non-uniform continuous $SL(2,\mathbb{R})$ cocycle? (See corresponding question in [6] as well.)
(2') Does every uniquely ergodic minimal dynamical system admit a non-uniform continuous Schrödinger cocycle?

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Quasi-periodic Schrödinger cocycles of low regularity

David Damanik

We consider Schrödinger cocycles of the form

$$A_{f,E}(\omega) = \begin{pmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{pmatrix}$$

over irrational rotations $\omega \mapsto \omega + \alpha$ on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let us denote the associated Lyapunov exponent by $\gamma_f(E)$, that is,

$$\gamma_f(E) = \inf_{n \ge 1} \frac{1}{n} \int_{\mathbb{T}} \log \|A^n_{f,E}(\omega)\| \, d\omega,$$

where $A^n_{f,E}(\omega)$ is given by $A^n_{f,E}(\omega) = A_{f,E}((n-1)\alpha + \omega) \times \cdots \times A_{f,E}(\omega)$.

The dynamics of these cocycles are crucial in the study of the spectral properties of the discrete quasi-periodic Schrödinger operator

$$[H_{\alpha,f}\psi](n) = \psi(n+1) + \psi(n-1) + f(n\alpha + \omega)\psi(n).$$

In the physics literature, the following two cases are of especial interest. The almost Mathieu operator is obtained when $f(\omega) = \cos(2\pi\omega)$. The Fibonacci operator corresponds to $f(\omega) = \chi_{[1-\alpha,1)}(\omega)$ (and $\alpha = (\sqrt{5}-1)/2$). Of course, it is natural to extend results for each of these special cases to as large a class of f's containing the special case as possible.

Here we will consider f's of low regularity. In view of the previous paragraph, the motivation for this is twofold. First, we want to study the Fibonacci operator and understand to what classes we can extend the results for this model. Second, this will also show natural borderlines for results that hold in the highly regular case.

The following dynamical description of the spectrum of $H_{\alpha,f}$ is useful [7, 9]:

$$\sigma(H_{\alpha,f}) = \{E : \gamma_f(E) = 0\} \cup \{E : A_{f,E} \text{ is non-uniformly hyperbolic}\} \equiv \Sigma_0 \cup \Sigma_{\mathrm{nh}}.$$

We want to consider the case where f has low regularity properties and show that many properties that hold for nice f (e.g., analytic) break down. An important step in understanding this is to realize that the spectrum is often quite small when f is not regular:

- When f is a step function, we will regard the sequences $V_\omega = f(\cdot\alpha + \omega)$ as elements of a subshift. Then, by verifying the Boshernitzan condition in many cases, we will show that $\Sigma_{\mathrm{nh}} = \emptyset$.
- Using Kotani theory, we will show that, again in many cases, Σ_0 has zero Lebesgue measure. For example, when f is step function, when f has a discontinuity, or when f is a generic continuous function.

For added effect, introduce a coupling constant λ and consider the one-parameter family λf. Note that the properties required from f above are independent of the coupling constant and hence they hold for the entire family once they hold for f.

This gives rise to a situation, where

$$\sigma(H_{\alpha,\lambda f}) = \Sigma_0 \quad \text{and} \quad \mathrm{Leb}(\Sigma_0) = 0.$$

Thus, as opposed to the analytic case, localization fails for large λ and the existence of absolutely continuous spectrum fails for small λ.

For both types of results, it is important to pass to the induced measure on the sequence space. If the range of f is contained in a compact subset S of $\mathbb{R}$, we define the mapping

$$\mathbb{T} \to S^{\mathbb{Z}}, \quad \omega \mapsto (n \mapsto f(n\alpha + \omega))$$

and denote the push-forward of the Lebesgue measure on $\mathbb{T}$ by ν.

If f takes finitely many values, we take S to be the finite set $\mathrm{Ran} f$ and hence obtain a subshift $\Omega \equiv \operatorname{supp} \nu \subseteq S^{\mathbb{Z}}$. If w is a word of length n occurring in some $\omega \in \Omega$, denote the associated cylinder set by $[w]$, that is,

$$[w] = \{\omega \in \Omega : \omega_1 \ldots \omega_n = w\}.$$

The subshift Ω obeys the Boshernitzan condition (cf. [3]), denoted by (B), if

$$\limsup_{n\to\infty} n \cdot \mu(n) > 0,$$

where

$$\mu(n) = \min\{\nu([w]) : w \text{ is a word of length } n \text{ occurring in some } \omega \in \Omega\}.$$

Theorem 1 (Damanik-Lenz [5]). *If Ω obeys* (B), *then* $\Sigma_{\mathrm{nh}} = \emptyset$.

The Boshernitzan condition. Let $f : \mathbb{T} \to \mathbb{R}$ be a step function. That is,

$$f(\omega) = \sum_{m=1}^{M} v_m \chi_{[\beta_{j-1},\beta_j)}(\omega),$$

where $0 = \beta_0 < \beta_1 < \cdots < \beta_{M-1} < \beta_M = 1$. Denote the resulting subshift by $\Omega_{\alpha,\beta_1,\ldots,\beta_{M-1}}$.

Theorem 2 (Boshernitzan [2], Damanik-Lenz [6]). *Let α be irrational.*
(a) *If $\beta = m\alpha + n \bmod 1$ with integers m, n, then $\Omega_{\alpha,\beta}$ satisfies* (B).
(b) *If α has bounded partial quotients, then $\Omega_{\alpha,\beta}$ satisfies* (B) *for every $\beta \in (0,1)$.*
(c) *If α has unbounded partial quotients, then $\Omega_{\alpha,\beta}$ satisfies* (B) *for Lebesgue almost every* (*but not every*) *$\beta \in (0,1)$.*
(d) *If $\beta_1, \ldots, \beta_{M-1} \in \mathbb{Q}$, then $\Omega_{\alpha,\beta_1,\ldots,\beta_{M-1}}$ satisfies* (B).

Nondeterminism. It was shown by Kotani that for non-deterministic models, Σ_0 has zero Lebesgue measure. Consider the induced measure ν. It is called deterministic if $s_1, s_2 \in \operatorname{supp}\nu$ and $s_1|_{\mathbb{Z}_-} = s_2|_{\mathbb{Z}_-}$ or $s_1|_{\mathbb{Z}_+} = s_2|_{\mathbb{Z}_+}$ implies $s_1 = s_2$. Thus, one can define a bijection $\Phi : \operatorname{supp}\nu|_{\mathbb{Z}_-} \to \operatorname{supp}\nu|_{\mathbb{Z}_+}$. The measure ν is called topologically deterministic if Φ is a homeomorphism. Kotani showed that $\mathrm{Leb}(\Sigma_0) > 0$ implies that ν is topologically deterministic. Thus, to show $\mathrm{Leb}(\Sigma_0) = 0$, it suffices to find two distinct elements of $\operatorname{supp}\nu$ that coincide on some half-line.

Theorem 3 (Kotani [8]). *If f is non-constant and takes finitely many values, then* $\mathrm{Leb}(\Sigma_0) = 0$.

Theorem 4 (Damanik-Killip [4]). *If f has a non-removable discontinuity at some $\omega_0 \in \mathbb{T}$ but is continuous at all points in the backward* (*or forward*) *orbit of ω_0 under rotation by α, then* $\mathrm{Leb}(\Sigma_0) = 0$.

Theorem 5 (Avila-Damanik [1]). *The set $\{f \in C(\mathbb{T}) : \mathrm{Leb}(\Sigma_0) = 0\}$ is residual.*

Note that Theorems 3 and 4 automatically extend to the whole one-parameter family λf. The residual set in Theorem 5 will depend a priori on λ. Thus, the following result is of interest.

Theorem 6 (Avila-Damanik [1]). *The set $\{f \in C(\mathbb{T}) : \mathrm{Leb}(\Sigma_0(\lambda f)) = 0$ for almost every $\lambda\}$ is residual.*

A particular consequence of the results above is the following.

Corollary 7. *Suppose f is of the form*

$$f(\omega) = \sum_{m=1}^{M} v_m \chi_{[\beta_{j-1},\beta_j)}(\omega),$$

with $\beta_1, \ldots, \beta_{M-1} \in \mathbb{Q}$ *and the* v_m*'s not all equal. Then, for all* $\lambda \neq 0$,

$$\sigma(H_{\alpha,\lambda f}) = \{E : \gamma_{\lambda f}(E) = 0\}$$

and this set has zero Lebesgue measure. In particular, the spectrum is a Cantor set.

As was alluded to above, in the situation of Corollary 7, the phenomena that hold for analytic f's break down. That is, there is never any absolutely continuous spectrum, and in particular not for small λ. Moreover, localization never holds, in particular not for large λ. This leaves the question if there is any point spectrum with sub-exponentially decaying eigenfunctions. This does not seem to be the case. Using Gordon's lemma, one can show that, for a step function f, the point spectrum is empty for almost every α, almost every ω and every λ. In some cases (like the Fibonacci case and extensions) one can prove absence of point spectrum for all irrational α and all ω.

Open Problems. (a) Study the validity of the Boshernitzan condition or its consequences when the circle is replaced by the d-dimensional torus and $f : \mathbb{T}^d \to \mathbb{R}$ takes finitely many values.
(b) In the same situation as in (a), find criteria for the resulting potentials to be of Gordon type.
(c) Is it possible to extend uniformity results to potentials that do not take finitely many values?

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The density of states for the quasi-periodic Schrödinger equation in the perturbative regime

Håkan Eliasson

This is a report on the work of S. Ben Hadj Amor [1] which contains three results on the discrete quasi-periodic Schrödinger equation with Diophantine frequencies and a small analytic potential:

(i) almost reducibility and reducibility of the corresponding co-cycle;
(ii) $\frac{1}{2}$-Hölder regularity of the integrated density of states;
(iii) a sub-exponential estimate of the complement of the spectrum.

The Schrödinger operator and co-cycles. We consider the Schrödinger operator

$$(H_\theta u)_n = -(u_{n+1} + u_{n-1}) + V(\theta + n\omega)u_n$$

acting on $l^2(\mathbb{Z})$. The potential $V : \mathbb{T}^d \to \mathbb{R}$ is supposed to be analytic and

$$|V|_r = \sup_{|\Im\theta|<r} |V(\theta)| < \infty.$$

The frequency vector ω is Diophantine and

$$\left\| \frac{<n,\omega>}{2} \right\| =: \sup_{k\in\mathbb{Z}} \left| \frac{<n,\omega>}{2} + k \right| \geq \frac{\kappa}{|n|^\tau} \quad \forall n \in \mathbb{Z}^d \setminus 0.$$

Let $E \to k(E)$ be the integrated density of states of H_θ. It is a continuous non-decreasing function that does not depend on θ.

We also consider the quasi-periodic co-cycle

$$X_{n+1} = (A + F(\theta + n\omega))X_n, \qquad n \in \mathbb{Z}, \qquad (*)$$

where $F : \mathbb{T}^d \to gl(2, \mathbb{R})$ is analytic and

$$|F|_r = \sup_{|\Im\theta|<r} |F(\theta)| < \infty,$$

and

$$A,\ A + F(\theta) \in SL(2, \mathbb{R}).$$

Let ρ_{A+F} be the rotation number of $(*)$. It is determined modulo π and depends continuously on $A + F$ (in C^0-topology). In particular $\rho_A = \alpha \bmod \pi$, where $\{e^{\pm i\alpha} : 0 \leq \Re\alpha < \pi\}$ are the eigenvalues of A. We say that ρ is rational with respect to ω if

$$\rho = \frac{<n,\omega>}{2}$$

for some $n \in \mathbb{Z}^d$, and that ρ is Diophantine with respect to ω if, for some $\kappa', \tau' > 0$,

$$\left\| \rho - \frac{<n,\omega>}{2} \right\| \geq \frac{\kappa'}{|n|^{\tau'}} \quad \forall n \in \mathbb{Z}^d \setminus 0.$$

The equation

$$H_\theta u = Eu$$

can be written as the co-cycle

$$X_{n+1} = (\begin{pmatrix} -E & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} V(\theta+n\omega) & 0 \\ 0 & 0 \end{pmatrix})X_n, \qquad n \in \mathbb{Z}, \qquad (**)_E$$

and the integrated density of states $k(E)$ equals the rotation number of $(**)_E$ mod π.

Reducibility and almost reducibility. We say that $(*)$ is almost reducible if for any $\varepsilon > 0$ there is a mapping

$$W = W_\varepsilon : (2\mathbb{T})^d \to SL(2,\mathbb{R})$$

such that

$$W(\theta+\omega)^{-1}(A+F(\theta))W(\theta) = A' + F'(\theta),$$

where $A' = A_\varepsilon \in SL(2,\mathbb{R})$ is independent of θ and $F' = F_\varepsilon$ verifies

$$|F|_0 < \varepsilon.$$

$(*)$ is reducible if there exists such a mapping W with $F' = 0$. ($2\mathbb{T}$ denotes a 2-fold covering of $\mathbb{T}$, i.e. W is 4π-periodic in each variable. This is the natural requirement in $SL(2,\mathbb{R})$, but in other groups one would need to consider other coverings.) [1]

There are several results on quasi-periodic co-cycles, both time-discrete and time-continuous and not only on $SL(2,\mathbb{R})$, and they confirm that: analytic quasi-periodic co-cycles with Diophantine frequencies close to constant coefficients are always almost reducible. Reducibility is a much more delicate property and non-reducible co-cycles are expected to be dense. To ask for an individual co-cycle to be reducible does therefore not make much sense. A more natural question is to ask for reducibility for a.e. co-cycle in one-parameter families. This turns out often to be the case. For references to these questions, see for example [3] and [2].

[1] contains the following theorem.

Theorem 1. *There exist a constant $\varepsilon_0 = \varepsilon_0(\kappa,\tau,r,|A|)$ such that if $|F|_r < \varepsilon_0$ then $(*)$ is almost reducible. Moreover, if ρ_{A+F} is rational or Diophantine with respect to ω then $(*)$ is reducible.*

Applied to the Schrödinger equation, this result implies that, in the perturbative regime, $(**)_E$ is reducible for almost every E.

Regularity of the rotation number. An almost reducing transformation is not close to the identity even in the perturbative regime. It is close to a mapping of the form

$$H(\theta) = C^{-1}\begin{pmatrix} e^{i\frac{\langle m,\theta\rangle}{2}} & 0 \\ 0 & e^{-i\frac{\langle m,\theta\rangle}{2}} \end{pmatrix}C, \qquad m \in \mathbb{Z}^d,$$

or, more generally, to a product of such mappings.

[1]This version of reducibility is stronger than the original one introduced by Lyapunov, where the reducing transformation W is not required to be quasi-periodic but only bounded and invertible with a bounded inverse.

The use of conjugations of this type for $(*)$ makes it possible to change the eigenvalues of the constant part. Indeed, if

$$A = C^{-1} \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} C,$$

then

$$H(\theta+\omega)^{-1} A H(\theta) = C^{-1} \begin{pmatrix} e^{i(\alpha - \frac{<m,\omega>}{2})} & 0 \\ 0 & e^{-i(\alpha - \frac{<m,\omega>}{2})} \end{pmatrix} C.$$

This is an essential tool in order to control the small divisors when the eigenvalues of A are "very close" to a number $\frac{<m,\omega>}{2}$, but it is a very discontinuous tool. If the co-cycle $(*)$ depends smoothly on a parameter E, then any almost reducing transformations

$$W_\varepsilon(\theta+\omega)^{-1}(A + F(\theta))W(\theta) = A_\varepsilon + F_\varepsilon(\theta),$$

will be only piece-wise smooth in E.

The control of the size of these pieces I_ε and an upper bound of the first derivative of

$$trA_\varepsilon(E) = 2\cos(\alpha_\varepsilon(E))$$

is one important point of [1]. A second important point is the proof of a lower bound of the second derivative of $trA_\varepsilon(E)$. This permits to obtain a lower bound of the first derivative in certain cases. This is used to prove the following result.

Theorem 2. *Assume that $A(E)$ is C^2 in $E \in I$, that*

$$\left| \frac{\partial^\nu}{\partial E^\nu} A(E) \right| \leq C \quad \forall E \in I,\ \nu = 0, 1, 2,$$

and that

$$\left| \frac{\partial}{\partial E} trA(E) \right| \geq C' > 0 \quad \forall E \in I.$$

Assume also that the rotation number $\rho(E) = \rho_{A(E)+F(E)}$ is monotone on I.

Then there is a constant $\varepsilon_0 = \varepsilon_0(\kappa, \tau, r, C, C')$ such that if

$$\left| \frac{\partial^\nu}{\partial E^\nu} F \right|_r < \varepsilon_0 \quad \forall E \in I,\ \nu = 0, 1, 2,$$

then

(i) *$E \mapsto \rho_{A(E)+F(E)}$ is $\frac{1}{2}$-Hölder continuous,*
(ii) *for any $m \in \mathbb{Z}^d$*

$$Leb(\rho^{-1}(\frac{<m,\omega>}{2})) \leq C_1 e^{-C_2|m|^\beta}.$$

Here C_1, C_2 are positive constants that only depend on κ, τ, r and β is a number in $]0,1[$.

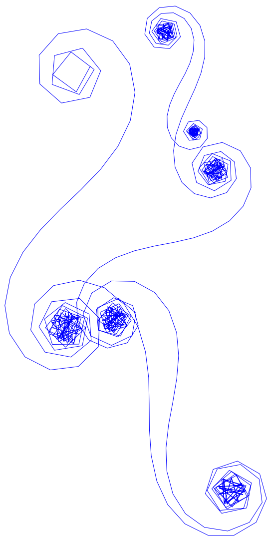

FIGURE 1. The graph of the exponential sum for $k = 2$ and $a = \pi^3$ (500 points).

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Renormalization of exponential sums and matrix cocycles

FRÉDÉRIC KLOPP

(joint work with Alexander Fedotov)

The work discussed in this talk was motivated by a talk by Michel Mendès-France given at the University Paris 12 in January, 2005. In his talk, M. Mendès-France discussed geometrical aspects of some exponential sums coming up in number theory (see e.g. [3, 6] and references therein).

1. RENORMALIZATION OF EXPONENTIAL SUMS

On the complex plane, one plots the sequence of points representing the successive values of the sums

$$S(N, a, k) = \sum_{0 \le n \le N-1} e\left(-\frac{an^k}{k}\right) \tag{1}$$

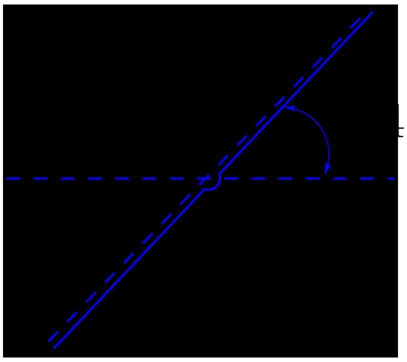

Figure 2. $l(\xi)$ and $\gamma(\xi)$.

where $N = 1, 2, 3 \ldots$, $e(z) = e^{2\pi i z}$, $0 < a < 1$ and $k > 1$ are fixed parameters. The points $S_1(a)$, $S_2(a)$, ... , being successively connected by line segments, form beautiful curves having a self similar structure, see Fig. 1.
We restrict ourselves to the case $k = 2$.

Consider the function $\mathcal{F} : \mathbb{C} \to \mathbb{C}$ defined by the formula:

$$\mathcal{F}(\xi, a) = \int_{\gamma(\xi)} \frac{e\left(\frac{p^2}{2a}\right)\, dp}{e(p - \xi) - 1}, \tag{2}$$

where the contour $\gamma(\xi)$ is going up from infinity along $l(\xi)$, the strait line $\xi + e^{i\pi/4}\mathbb{R}$, coming infinitesimally close to the point ξ, then, going around this point in the anti-clockwise direction along an infinitesimally small semi-circle, and, then, going up to infinity again along $l(\xi)$ (see Fig. 2). One proves

Lemma 1 ([5]). *For each $a > 0$, $\mathcal{F}$ is an entire function of ξ, and, for all $\xi \in \mathbb{C}$, one has*

$$\mathcal{F}(\xi, a) - \mathcal{F}(\xi - 1, a) = e\left(\frac{\xi^2}{2a}\right); \tag{3}$$

$$\mathcal{G}(\xi + a, a) - \mathcal{G}(\xi, a) = e\left(-\frac{\xi^2}{2a}\right), \tag{4}$$

$$\mathcal{G}(\xi, a) = c(a)\, e\left(-\frac{\xi^2}{2a}\right) \mathcal{F}(\xi, a), \quad c(a) = e(-1/8)\, a^{-1/2}. \tag{5}$$

As an easy consequence of Lemma 1, one proves

Theorem 2 ([5]). *Let $N \in \mathbb{Z}$ and $a \in \mathbb{R}$ be two positive numbers. Let*

$$\xi = \{aN\}, \quad N_1 = [aN], \quad a_1 = -\frac{1}{a}. \tag{6}$$

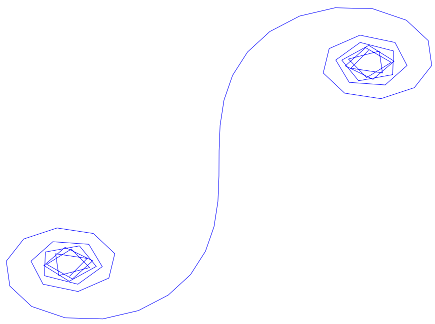

FIGURE 3. Sampling points.

Then,

$$S(N, a, 2) = c(a) \left[S(N_1, a_1, 2) + e\left(-\frac{aN^2}{2}\right) \mathcal{F}(\xi, a) - \mathcal{F}(0, a) \right]. \tag{7}$$

The asymptotics of $\mathcal{F}$ for a small are described by

Proposition 3 ([5]). *Let $-1/2 \leq \xi \leq 1/2$ and $0 < a \leq 1$. Then, $\mathcal{F}$ admits the representation:*

$$\mathcal{F}(\xi, a) = e(1/8)\, f(a^{-1/2}\xi) + O(a^{1/2}), \quad f(t) = e(t^2/2) \int_{-\infty}^{t} e(-\tau^2/2)\, d\tau, \tag{8}$$

where $O(a^{1/2})$ is bounded by $C\, a^{1/2}$, where C is an absolute constant.

We now turn to the discussion of the curlicue structures observed in Fig. 1. Therefore, let N_0 and N_1 be two positive integers such that $N_1 = [aN_0]$ and $a(N_0 - 1) < N_1$. The nex result, Lemma 4, shows that the curlicue structures of the graphs of the quadratic exponential sums on the complex plane are obtained by sampling points on the graph of the Fresnel integral $F(t) = \int_{-\infty}^{t} e(-\tau^2/2)\, d\tau$ (see Fig. 3).

Lemma 4 ([5]). *Let N_0 and N_1 be fixed as above. For a sufficiently small and $-1/2 \leq aN - N_1 \leq 1/2$, one has*

$$S(N, a, 2) = Const + a^{-1/2}\, e\left(\frac{N_1^2}{2a}\right) \int_{-\infty}^{(aN - N_1)/a^{1/2}} e(-\tau^2/2)\, d\tau + o(1) \tag{9}$$

where Const denotes an expression independent of N.

2. Renormalization of matrix cocycles

The renormalization done above for exponential sums can be extended to some matrix cocyles which we describe now.

2.1. **The matrix cocycle.** Consider the cylinder $\mathbb{R} \times \mathbb{T}$, where $\mathbb{T}$ is the one dimensional torus of length 1. Let M_2 be the matrix defined by

$$M_2(k) = \begin{pmatrix} \alpha & \beta\, e(-ak^2/2) \\ \bar{\beta}\, e(ak^2/2) & \bar{\alpha} \end{pmatrix}. \tag{10}$$

We interpret the matrix product

$$M_2(N-1) \cdots M_2(2)\, M_2(1)\, M_2(0) \tag{11}$$

as the matrix cocycle defined by the matrix valued function $\mathcal{M} : \mathbb{R}\times\mathbb{T} \to SL(2, \mathbb{C})$,

$$\mathcal{M}(x) = \begin{pmatrix} \alpha & \beta\, e(-x_2) \\ \bar{\beta} e(+x_2) & \bar{\alpha} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R} \times \mathbb{T}. \tag{12}$$

and $T(a)$, the skew shift on the cylinder i.e. the automorphism of the cylinder defined by

$$T(a)x = Jx + ae_1, \quad J = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

It is intimately related to the quadratic exponential sums, see [2]. We can write the matrix product (11) in the form

$$\mathcal{M}(T^{N-1}(a)\, x) \cdots \mathcal{M}(T(a)\, x)\, \mathcal{M}(x) \tag{13}$$

with $x_1 = a/2$, and $x_2 = 0$, i.e. consider it as a matrix cocycle defined on $\mathbb{R} \times \mathbb{T}$ by the pair $(\mathcal{M}, T)$. We denote this cocycle by $(\mathcal{M}, T)$.

2.2. **The monodromy matrix.** The analysis of the cocycle $(\mathcal{M}, T)$ is equivalent to the analysis of the equation

$$\Psi(T(a)x) = \mathcal{M}(x)\Psi(x), \quad x \in \mathbb{R} \times \mathbb{T}. \tag{14}$$

On the cylinder, the map $T(a)$ and the translation by e_1 commute; so, the space of solutions of (14) is invariant with respect to the translation $f(\cdot) \to f(\cdot{+}e_1)$. Hence, $\Psi(\cdot)$ and $\Psi(\cdot + e_1)$ both satisfy this equation. Moreover, the space of solutions of (14) is a two dimensional module over the ring of functions invariant with respect to the transformation $f(\cdot) \to f(T(a) \cdot)$. Assume that Ψ is a fundamental solution to (14), i.e., that $\det \Psi(x) = 1$ for all x. Then, this implies that, $\forall x \in \mathbb{R} \times \mathbb{T}$,

$$\Psi(x + e_1) = \Psi(x)\tilde{\mathcal{M}}^t(x), \tag{15}$$

where ^{t} denotes the transposition, and $\tilde{\mathcal{M}} : \mathbb{R} \times \mathbb{T} \to SL(2, \mathbb{C})$ is a matrix valued function satisfying the relations

$$\tilde{\mathcal{M}}(T(a)\, x) = \tilde{\mathcal{M}}(x), \quad \det \tilde{\mathcal{M}}(x) = 1, \quad x \in \mathbb{R} \times \mathbb{T}. \tag{16}$$

Define s_1, a shift on the cylinder, by $s_1(x) = x + e_1$. There is a simple relation between the matrix cocycles $(\mathcal{M}, T)$ and $(\tilde{\mathcal{M}}, s_1)$. To describe it, we pick $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{T}$ such that $0 \le x_1,\, x_2 < 1$. We fix N, a positive integer, and we represent $T^N(a)x$ in the form

$$T^N(a)x = me_1 + ne_2 + \xi,$$

where n, m are (non-negative) integers, $\xi \in \mathbb{T} \times \mathbb{R}$, and $0 \leq \xi_1, \xi_2 < 1$. It is easy to see that

$$m = [Na + x_1].$$

The definition of the monodromy matrix (15) immediately implies

Lemma 5. *Let Ψ be a fundamental solution to equation* (14)*, and let $\tilde{\mathcal{M}}$ be the corresponding monodromy matrix. Then,*

$$\begin{aligned} (17) \quad & \mathcal{M}(T^{N-1}(a)x) \cdots \mathcal{M}(T(a)x)\mathcal{M}(x) \\ & \qquad = \Psi(\xi) \left[\tilde{\mathcal{M}}(\xi + (m-1)e_1) \cdots \tilde{\mathcal{M}}(\xi + e_1)\tilde{\mathcal{M}}(\xi) \right]^t \Psi(x)^{-1}. \end{aligned}$$

2.3. **Self-similarity of the quadratic exponential matrix cocycle.** To formulate our results, we introduce more notations. Consider the matrix defined by (12). As this matrix is unimodular, one has $|\alpha|^2 = 1 + |\beta|^2$. Without loss of generality, we can and do assume that $\beta \geq 0$ as the matrix with parameters α and $|\beta|$ is similar to the matrix with parameters α and β. Then, we get

$$\beta = (|a|^2 - 1)^{1/2}.$$

With this in mind, we denote our matrix by $\mathcal{M}(x, a)$.
Our main technical result is

Theorem 6. *Let $0 < a < 1$. There exists Ψ, a fundamental (unimodular matrix) solution to* (14) *with $\mathcal{M} = \mathcal{M}(x, a)$, such that the corresponding monodromy matrix is*

$$\tilde{\mathcal{M}} = \begin{pmatrix} Au & Bv \\ \bar{B}v^{-1} & \bar{A}u^{-1} \end{pmatrix}, \qquad u = e\left(-\frac{x_1}{a}\right), \quad v = e\left(x_2 - \frac{x_1^2}{2a} - \frac{x_1}{a} + \frac{x_1}{2}\right),$$

where

$$A = -e\left(-1/(2a)\right) \, (\bar{\alpha})^{1/a}, \quad B = \left(|\alpha|^{2/a} - 1\right)^{1/2}.$$

The solution $x \mapsto \Psi(x)$ is an entire function of x.

Note that A and B are independent of x. The proof of Theorem 6 heavily relies on the theory of the minimal entire solutions of difference equations on the complex plane as developed in [1].

By means of Theorem 6 and relation (17), we get the main result of this note:

Theorem 7. *Let $0 < a < 1$. Define x, m and ξ as for Lemma 5. Then,*

$$\begin{aligned} & \mathcal{M}\left(T^{N-1}(a)\, x, \alpha\right) \cdots \mathcal{M}\left(T(a)\, x, \alpha\right)\, \mathcal{M}\left(x, \alpha\right) \\ (18) \quad & \quad = \Psi(\xi)\, D\left(\xi_1\right)^{-1} \\ & \qquad \cdot \left[\, \mathcal{M}\left(T^{m-1}(a_1)\, y,\, \alpha_1\right) \cdots \mathcal{M}\left(T(a_1)\, y,\, \alpha_1\right)\, \mathcal{M}(y, \alpha_1) \right]^t \\ & \qquad\qquad \cdot D\left(\xi_1 + m\right) \Psi(x)^{-1}, \end{aligned}$$

where $D(t) = Diag(e(-t^2/(2a)), e(t^2/(2a)))$ *and*

$$a_1 = -1/a \ (mod 2), \quad -1 < a_1 \leq 1,$$
$$m = [Na + x], \quad \alpha_1 = -(\bar{\alpha})^{1/a},$$
$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad y_1 = -(\xi_1/a + 1/2)\,(\,mod 1),$$
$$y_2 = -(\xi_1^2/(2a) + \xi_1/2 + \xi_2)\,(\,mod 1)$$

Relation (18) relates the matrix cocycles $(\,\mathcal{M}\,(\cdot,\alpha), T\,(a)\,)$ and $(\mathcal{M}\,(\cdot,\alpha_1), T\,(a_1)\,)$. It is a "two-dimensional" analog to (7), the exact renormalization formula relating the exponential sums.
Of course, to make (18) useful for applications, one has to obtain an effective description of the solution Ψ. Let us outline how this can be done. To study the input matrix product for large N, one makes many consecutive renormalizations; if a is irrational, at each step, one obtains a new constant a. One can carry out the renormalizations so that each of these constants satisfies $0 < a < 1$. But, then, as the absolute value of the input constant α is greater than 1, the absolute values of the new constants α will grow. Using standard results from the metric theory of numbers, one can see that, for almost all values of the input a, the sequence of the new α tends to infinity. In result, roughly, one can replace the solution Ψ by their asymptotics for $\alpha \to \infty$. We use this idea in [4].

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Discrete one-dimensional quasi-periodic Schrödinger operators with non-analytic potentials

Silvius Klein

Consider the discrete one-dimensional quasi-periodic Schrödinger operator defined on $l_2(\mathbb{Z}^d)$ by:

$$[H_\lambda(x)u]_n := -u_{n+1} - u_{n-1} + \lambda v(T^n x) u_n$$

where v is a real valued function on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, x is a parameter on $\mathbb{T}^d$, λ is a real number called the coupling constant and T is an ergodic map on $\mathbb{T}^d$. Because of ergodicity, the spectrum and the spectral types of the operator $H_\lambda(x)$ do not depend on the phase x almost surely.

Examples of such maps T are: $Tx = x + \omega$ (the one or multi frequency shift by an incommensurable frequency), $T(x, y) = (x + y, y + \omega)$ (the skew shift by an irrational frequency), $Tx = 2x$ (the doubling map), $Tx = 1 - ax^2$.

For now, let us consider only the case of the shift $Tx = x + \omega$ on the torus $\mathbb{T}$ of dimension one.

The eigenvalue equation:

$$[H_\lambda(x)u]_n = -u_{n+1} - u_{n-1} + \lambda v(T^n x) u_n = E u_n$$

is a second order finite difference equation. Solving it for $[u_n]$ yields

$$\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = M_n(x, E) \begin{bmatrix} u_1 \\ u_0 \end{bmatrix}$$

where

$$M_n(x, E) := \prod_{j=n}^{1} \begin{bmatrix} \lambda v(T^j x) - E & -1 \\ 1 & 0 \end{bmatrix}$$

is the Nth transfer matrix of the operator $H_\lambda(x)$.

Let

$$L_N(E) = \int \frac{1}{N} \log ||M_N(x, E)||\, dx$$

and

$$L(E) := \lim_{N\to\infty} L_N(E)$$

$L(E)$ is the Lyapunov exponent of the Schrödinger operator $H_\lambda(x)$. It measures the average exponential growth of the transfer matrices. Note that it is a nonnegative number.

Ergodicity implies

$$L(E) := \lim_{N\to\infty} \frac{1}{N} \log ||M_N(E, x)|| \quad \text{for a.e. } x$$

Let $E_N(x)$ be the set of eigenvalues of the eigenvalue equation restricted to the finite interval $[1, N]$ with Dirichlet boundary conditions (i.e. the spectrum of $H_\lambda(x)$ projected on the finite box $[1, N] \times [1, N]$).

The limit in the sense of distributions

$$N(E) = \lim_{N\to\infty} \frac{1}{N} \#[E_N(x) \cap (-\infty, E)]$$

exists, does not depend on x a.e. and is called the integrated density of states (IDS) of $H_\lambda(x)$.

The IDS is related to the Lyapunov exponent by the Thouless formula:

$$L(E) = \int \log |E - E'| \, dN(E')$$

We are interested in the following types of problems related to the operator $H_\lambda(x)$ and the quantities defined above.

- Determine the nature of the spectrum (e.g. point, absolutely continuous, singular continuous).
- Prove positivity of the Lyapunov exponent in certain regimes.
- Prove fine continuity properties of the IDS.

Let us remind the reader that an operator has pure point spectrum if there is a complete set of eigenfunctions, that is, the operator can be 'diagonalized'. Moreover, Anderson localization (AL) for an operator means pure point spectrum with exponentially decaying eigenfunctions.

There has been extensive work and important breakthroughs obtained in the last decade in the case of analytic potentials (see [3, 5, 6, 7, 4, 11, 12, 13, 15, 17]). There has also been work in the opposite regime, of low regularity of the potential function (see for instance [1, 9, 10]).

I am interested in the intermediate regime, of (ultimately) C^k potentials. Below are two results I have obtained in the Gevrey-class (see also [16]).

Let us remember that a function $v(x)$ belongs to the Gevrey class of order $s > 1$ if it is smooth, and its Fourier coefficients decay like:

$$|\hat{v}(k)| \leq M e^{-\rho |k|^{1/s}} \qquad \text{for all } k \in \mathbb{Z}, \text{ and for some } M, \rho > 0.$$

Theorem 1. *Consider the Schrödinger operator: $H_{\omega,\lambda}(x)$ defined above. Assume that $v(x)$ is in a Gevrey class of order $s > 1$ and that it satisfies a generic transversality condition (TC) of the form: for all $x \in \mathbb{T}$ there exists $m \geq 1$ so that $\partial^m v(x) \neq 0$ (that is, v is not flat at any point). Assume also a (rather strong) Diophantine condition (DC) on the frequency ω.*

Then there exists $\lambda_0 = \lambda_0(v, DC)$ so that if $|\lambda| > \lambda_0$, then the following hold:

- *For a.e. Diophantine frequency ω, $H_{\omega,\lambda}(x)$ satisfies AL for all $x \in \mathbb{T}$.*
- *The Lyapunov exponent is bounded away from zero for all energies: $L_{\omega,\lambda}(E) > \frac{\log |\lambda|}{2}$ for all $E \in \mathbb{R}$ and all ω Diophantine.*
- *The Lyapunov exponent $L(E)$ and the integrated density of states $N(E)$ are continuous functions in E with modulus of continuity at least $h(t) = C\, e^{-c |\log t|^\eta}$, for some positive constants C, c and η (where $\eta = 1-$).*

Compared to the analytic case, this result is perturbative. Here is a partial non-perturbative result:

Theorem 2. *Consider the operator $H_\omega(x) := -\Delta + v(x + n\omega)\delta_{n,n'}$, where the coupling constant λ is fixed.*

Assume ω Diophantine and $v \in G^s(\mathbb{T})$ with $s \in (1, 2)$. We do not assume a TC, but we assume positivity of the Lyapunov exponent for all Diophantine frequencies and for all energies: $L_\omega(E) \geq c_0 > 0$. Then we have:

- *$H_\omega(x_0)$ satisfies AL for all $x_0 \in \mathbb{T}$ and for a.e. Diophantine ω.*
- *The Lyapunov exponent: $L(E)$ and the integrated density of states $N(E)$ are continuous in E, with a modulus of continuity at least $h(t) = C\, e^{-c|\log t|^\eta}$ for some constants C, c and η (here $\eta \to 0$ as $s \to 2$).*

To prove these two theorems we follow some of the techniques developed for the case of analytic potentials. One important tool in the analytic case is the use of of subharmonic functions techniques, applied for instance to the function $x \overset{u}{\mapsto} \frac{1}{N} \log ||M_N(x)||$. If v is not analytic, then this function u does not have a subharmonic extension, and these techniques do not apply directly.

We use appropriate truncations of the function $v(x)$ and a more refined analysis of the norms of the transfer matrices to prove our results in the Gevrey class.

An interesting project is to prove nonperturbative positivity of the Lyapunov exponent in the Gevrey class (extending Sorets-Spencer result to this class). This could be achieved by using the above method together with a sharp estimate on small shifts of a subharmonic function as in [4].

As indicated in [16], our method seems to be optimal for the Gevrey class. Therefore, different ideas are needed to approach the more general case of C^k potentials.

Recently, Jackson Chan (see [8]; see also [2] for a related result) proved positivity of the Lyapunov exponent for "generic" C^3 potentials (in the 1-frequency case), where genericity is understood in a measure theoretical sense. His method uses separation of the eigenvalues at finite scale as in [13] and a method of avoiding multiple resonances by carefully eliminating small sets of frequencies and perturbing the potential.

My current project is to extend this method to the multi-frequency case, where the geometry of the sets of phases and frequencies leading to multiple resonances is more complicated. However, I believe the main ideas will be transfered over to this multi-frequency case.

I believe the following, more general conjecture to be true:

Conjecture 3. *Assume $v(x)$ is smooth and satisfies a TC. Assume ω satisfies a (weak) DC. Then there is $\lambda_0 = \lambda_0(v, DC)$ so that for every $|\lambda| > \lambda_0$, for all energies E and for all such frequencies ω, we have $L_{\omega,\lambda}(E) \geq \frac{\log|\lambda|}{4} > 0$.*

This conjecture would immediately imply, using Kotani's theory the absence of a.c. spectrum. We expect more to be true though, namely AL.

However, these results seem out of reach at this moment.

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Elimination of resonance and Anderson localization

Jackson Chan

Consider the one-dimensional quasi-periodic Schrodinger equation

$$-\varphi(n+1) - \varphi(n-1) + \lambda V(x+n\omega)\varphi = E\varphi(n)$$

The spectrum of this equation with zero boundary conditions $\varphi(a-1) = 0 = \varphi(b+1)$ consists of the eigenvalues of the matrix

$$H_{[a,b]}(x,\omega) = \begin{pmatrix} \lambda V(x+a\omega) & -1 & 0 & \dots & 0 \\ -1 & \lambda V(x+(a+1)\omega) & -1 & 0 & \dots \\ 0 & -1 & \ddots & \ddots & \\ \vdots & & \ddots & & -1 \\ 0 & \dots & 0 & -1 & \lambda V(x+b\omega) \end{pmatrix}$$

Multi-scale analysis, introduced by Goldstein and Schlag [3, 4] for analytic potential, is a KAM type approach that relates the eigenvalues and eigenfunctions of $H_{[-N,N]}(x,\omega)$ to those of $H_{[-\bar{N},\bar{N}]}(x,\omega)$, with $N \ll \bar{N}$. Using multi-scale analysis, we can prove the following theorem.

Theorem 1. *Let $V : \mathbb{T} \to \mathbb{R}$ be $\mathcal{C}^3$ such that $|V'(x)| + |V''(x)| \geq > 0$ for all $x \in \mathbb{T}$. Then there is $\lambda_0 = \lambda(V)$ such that for any $\lambda > \lambda_0$, the following holds: For "typcial" potential $\tilde{V} = V + \sum_{s=1}^{\infty} W^{(s)}$, $\|W^{(s)}\|_{\mathcal{C}^3} \leq exp(-N^{(s)})$, $N^{(s+1)} \asymp exp(N^{(s)})$, there exists a set $\Omega = \Omega(\lambda, \tilde{V}) \subset \mathbb{T}$, $\operatorname{mes}\Omega > 1 - \lambda^{-1/2}$, such that for any $\omega \in \Omega$, and for any $E \in \mathbb{R}$, the Lyapunov exponent $L(\omega, E) > c\log\lambda$.*

We will discuss the meaning of "typical" potential as defined in [2]. At each scale, we need to establish the followings.

(1) Localization in finite length.
(2) Esitmate the number of eigenvalues and separation of eigenvalues.
(3) $|\partial_x E| + |\partial_{xx} E| > exp(-[N^{(s)}]^{\sigma})$
(4) Elimination of multiple resonances.

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Classical motion in random potentials

Christoph Schumacher

In the 1950s Anderson suggested a model for a crystal with impurities, cf. [3]. Although since then *quantum Anderson Models* have been studied extensively in various versions there seems to be only very little literature concerning the classical analogue. We use the potential which arises in the Anderson Model for *Hamiltonian Mechanics*. Special attention is given to the ergodic theory of asymptotic velocity.

For the construction of the potential we need

- a lattice $\mathcal{L} := \mathrm{span}_{\mathbb{Z}}(\ell_1, \ldots, \ell_d) \subseteq \mathbb{R}^d$ and
- smooth short range *single site potentials* $W_j : \mathbb{R}^d \to \mathbb{R}$ $(j \in J, \#J < \infty)$

to define

- the Cantor set of configurations $\Omega := J^{\mathcal{L}}$ and
- the potential of the crystal
$$V : \Omega \times \mathbb{R}^d \to \mathbb{R},\ V(\omega, q) := \sum_{\ell \in \mathcal{L}} W_{\omega(l)}(q - \ell).$$

On Ω we fix a probability measure β such that $(\omega(\ell))_{\ell \in \mathcal{L}}$ are iid (or shift ergodic).

In order to introduce *Hamiltonian Dynamics* we fix $\omega \in \Omega$ and obtain a smooth potential $V_\omega : \mathbb{R}^d \to \mathbb{R}$, $V_\omega(q) := V(\omega, q)$. As usual in classical mechanics the dynamic is defined on

- the phase space $M := \mathbb{R}^d_p \times \mathbb{R}^d_q$ via
- the Hamiltonian $H_\omega : M \to \mathbb{R}$, $H_\omega(p, q) = \frac{1}{2}p^2 + V_\omega(q)$ and
- the Hamiltonian differential equations
$$\dot{p} = -\nabla_q H_\omega(p, q),\ \dot{q} = \nabla_p H_\omega(p, q),$$
which are solved by
- the Hamiltonian flow for initial conditions $(p(0), q(0)) = (p_0, q_0) \in M$
$$\Phi_\omega : \mathbb{R} \times M \to M,\ (t, p_0, q_0) \mapsto (p(t), q(t)).$$

Hamiltonian flows preserve Lebesgue measure λ on M, i.e.
$$\lambda \circ \Phi^t_\omega = \lambda \quad (t \in \mathbb{R}, \omega \in \Omega, \Phi^t_\omega := \Phi_\omega(t, \cdot)).$$

In order to attain a more global point of view, we unite over ω by defining

- the *augmented* phase space $P := \Omega \times M$,
- the augmented Hamiltonian $H : P \to \mathbb{R}$, $(\omega, p, q) \mapsto H_\omega(p, q)$,
- the augmented flow $\Phi : \mathbb{R} \times P \to P$, $(t, \omega, p_0, q_0) \mapsto \Phi_\omega(\omega, p_0, q_0)$ and
- the Liouville measure $\mu := \beta \otimes \lambda$ on P.

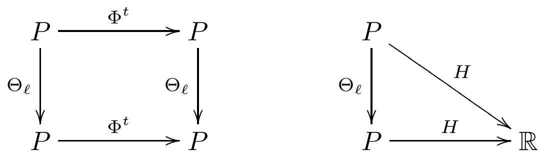
We are thus led to a *measure preserving dynamical system* $(\mathbb{R}, P, \mu, \Phi)$.

Consider the *asymptotic velocity*
$$\overline{v}^{\pm} : P \to \mathbb{R}^d,\ (\omega, p_0, q_0) \mapsto \lim_{t \to \pm\infty} \frac{q(t) - q_0}{t} = \lim_{t \to \pm\infty} \frac{1}{t} \int_0^t p(s) ds$$

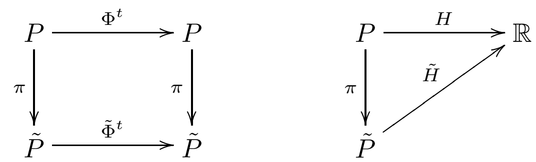
which happens to be a Birkhoff average. The Birkhoff ergodic theorem would give us almost sure existence of the limit, if the dynamical system was finite. In the following we modify the dynamical system in order to apply Birkhoff's theorem. We therefore introduce a group action Θ of $\mathcal{L}$ on P

$$\Theta\colon \mathcal{L} \times P \to P,\ (\ell, \omega, p, q) \mapsto (\vartheta_\ell(\omega), p, q - \ell)$$

with $\vartheta_\ell(\omega)(\ell') := \omega(\ell + \ell')$ for all $\omega \in \Omega$ and $\ell, \ell' \in \mathcal{L}$. Then the diagrams



commute and we can define the quotient $\pi\colon P \to \tilde{P} := P/\Theta$ as well as $\tilde{H}$ and $\tilde{\Phi}^t$ according to



Although Ω/θ is not even Hausdorff, due to the q–coordinate Θ acts freely and $\tilde{P}$ is metrizable. Now we introduce a measure $\tilde{\mu}$ on $\tilde{P}$ via

$$\tilde{\mu}(\tilde{A}) := \mu(\pi^{-1}(\tilde{A}) \cap \Omega \times \mathbb{R}^d \times \mathcal{D}) \qquad (\tilde{A} \subseteq \tilde{P} \text{ measurable})$$

with any fundamental domain $\mathcal{D}$ of $\mathcal{L}$. Since Θ_ℓ preserves μ, $\tilde{\mu}$ is independent of the particular choice of $\mathcal{D}$. By letting $\tilde{P}_E := H^{-1}(]-\infty, E])$ $(E \in \mathbb{R})$ we obtain a family of finite measure preserving dynamical systems $(\mathbb{R}, \tilde{P}_E, \tilde{\Phi}|_{\tilde{P}_E}, \tilde{\mu}|_{\tilde{P}_E})$.

Finally Birkhoff's ergodic theorem applies to

$$\overline{v}^{\pm}(\omega, p_0, q_0) = \lim_{t \to \pm\infty} \frac{1}{t} \int_0^t \tilde{\pi}_p \circ \tilde{\Phi}_E(s, \omega, p_0, q_0) ds,$$

(here: $\tilde{\pi}_p\colon \tilde{P} \to \mathbb{R}^d$, $\tilde{\pi}_p(\pi(\omega, p, q)) := p$) so the asymptotic velocity exists μ–almost everywhere and $\overline{v}^- = \overline{v}^+$ μ–almost everywhere.

In the following we present some properties of the asymptotic velocity. Another consequence of Birkhoff's ergodic theorem is

Theorem 1. *Let*

$$\Gamma_\omega\colon M \to \mathbb{R} \times \mathbb{R}^d, \quad (p, q) \mapsto (H_\omega(p, q), \overline{v}(\omega, p, q)),$$

$$Q_N := \left\{ \sum_{j=1}^d x_j \ell_j : x \in [-N, N[^d \right\},$$

$$\zeta_{N,\omega}(\cdot) := (2N)^{-d} \lambda(\Gamma_\omega^{-1}(\cdot) \cap (\mathbb{R}_p^d \times Q_N)).$$

Then the measure

$$\zeta := \lim_{N\to\infty} \zeta_{N,\omega}$$

on $\mathbb{R} \times \mathbb{R}^d$ *exists and is deterministic.*

In special cases more can be said. In dimension $d = 1$ the asymptotic velocity can be given explicitly.

Theorem 2. *Let* $d = 1$ *and* $\mathcal{L} = \mathrm{span}_{\mathbb{Z}}(\ell_1) \subseteq \mathbb{R}$. *For* $(\omega, p_0, q_0) \in P$ *with* $E := H(\omega, p_0, q_0) > \sup_{(\omega,q)\in\Omega\times\mathbb{R}} V(\omega, q)$ *there holds*

$$\bar{v}(\omega, p_0, q_0) = \frac{\ell_1}{\mathbb{E}^\beta(\tau)} \qquad \beta\text{–a. s.}$$

with

$$\tau(\omega, p_0, q_0) := \int_0^{\ell_1} \frac{\mathrm{sgn}(p_0)}{\sqrt{2(E - V(\omega, q))}} dq$$

and $\mathbb{E}^\beta$ *being the expectation with respect to* β.

τ is a random variable measuring the time needed to pass the fundamental domain $[0, \ell_1[$ of $\mathcal{L}$. The proof is again using Birkhoff's ergodic theorem.

We provide an example of a random potential for which the distribution of the asymptotic velocity is not concentrated at 0 for arbitrarily high energies.

Since the augmented phase space is not compact, there are different notions of ergodicity, see [1]. By the example mentioned above random potentials do not need to give rise to ergodic motion. However, in [5] ergodicity is shown for twodimensional periodic potentials with Coulomb singularities using hyperbolicity. See also [6].

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