Mathematisches Forschungsinstitut Oberwolfach

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Set Theory

Organised by
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Hugh Woodin (Berkeley)

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Abstract. This meeting covered all important aspects of modern Set Theory, including large cardinal theory, combinatorial set theory, descriptive set theory, connections with algebra and analysis, forcing axioms and inner model theory. The presence of an unusually large number (19) of young researchers made the meeting especially dynamic.

Mathematics Subject Classification (2000): 03Exx.

Introduction by the Organisers

This meeting was organised by Sy-David Friedman (University of Vienna), Menachem Magidor (Hebrew University, Jerusalem) and Hugh Woodin (University of California, Berkeley). Largely due to the generous EU support for the meeting, we were able to invite an unusually large number of young researchers (at most 10 years after the beginning of doctoral studies). We consequently chose an unusual format for the meeting, in which young people were given priority in the scheduling of lectures. This turned out to be a good choice, as the quality of these lectures was extremely high, giving us an excellent opportunity to learn about the new talent that has recently come into our field.

45-minute lectures were delivered by Krueger (Vienna), Dobrinen (Vienna), Camerlo (Torino), Zoble (Toronto), Geschke (Berlin), König (Paris), Viale (Paris), Asperó (Barcelona), Sharon (Irvine) and Lopez-Abad (Paris). These young people presented striking new results concerning combinatorial set theory, descriptive set theory, set-theoretic analysis and forcing axioms. One of the most impressive of these new results was that of Viale, who established a connection between two important set-theoretic principles, by showing that the singular cardinal hypothesis follows from the proper forcing axiom.

Shorter lectures (20 to 30 minutes) were given by the more senior researchers. In total, lectures were held by 29 of the 59 participants. By restricting the number and length of these lectures, we were able to leave substantial time for private discussion. The 15-minute breaks between lectures were especially appreciated, giving the meeting a relaxed feeling.

The traditional Wednesday afternoon walk was a success. The Schwarzwaldkirschtorte in Wolfach was superb, and despite the rain and getting (slightly) lost in the dark, everyone made it safely back to the institute in time for dinner.

Workshop: Set Theory

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Abstracts

Disjoint Club Sequences

John Krueger

A stationary set $S \subseteq \omega_2$ is *fat* if for every club set $C \subseteq \omega_2$, $S \cap C$ contains a closed subset with order type $\omega_1 + 1$. For example, if $A \subseteq \omega_2 \cap \mathrm{cof}(\omega_1)$ is stationary, then $A \cup \mathrm{cof}(\omega)$ is fat. Abraham and Shelah [1] proved that assuming CH, for any fat set $S \subseteq \omega_2$ there is a forcing poset which does not add any sets of ordinals with size less than ω_2 and adds a club subset to S.

Let T be a stationary subset of $P_{\omega_1}(\omega_2)$. We say that T is *thin* if for all $\beta < \omega_2$, $|\{a \cap \beta : a \in T\}| < \omega_2$. The existence of a thin stationary subset of $P_{\omega_1}(\omega_2)$ is a consequence of CH or the existence of a special Aronszajn tree on ω_2. Sy Friedman [2] proved that if there exists a thin stationary subset of $P_{\omega_1}(\omega_2)$, then for any fat set $S \subseteq \omega_2$ there is a forcing poset, consisting of finite conditions with models as side conditions, which adds a club subset to S without collapsing cardinals.

I introduced the following combinatorial property on ω_2 which implies not only that there does not exist a thin stationary subset of $P_{\omega_1}(\omega_2)$, but moreover that there is a fat subset of ω_2 which cannot acquire a club subset in any outer model with the same ω_2 (this answers a problem of Abraham and Shelah [1]). A *disjoint club sequence on* ω_2 is a sequence $\langle \mathcal{C}_\alpha : \alpha \in A \rangle$, where A is a stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$, each $\mathcal{C}_\alpha$ is a club subset of $P_{\omega_1}(\alpha)$, and $\mathcal{C}_\alpha \cap \mathcal{C}_\beta$ is empty for $\alpha < \beta$ in A. The existence of such a sequence follows from Martin's Maximum. If $\langle \mathcal{C}_\alpha : \alpha \in A \rangle$ is a disjoint club sequence, then A does not contain almost all ordinals of cofinality ω_1 (for example, A avoids ordinals which are the union of an internally approachable chain of countable sets). It follows that $A \cup \mathrm{cof}(\omega)$ is a fat set which cannot acquire a club subset without collapsing either ω_1 or ω_2.

The following theorem [3], joint with Sy Friedman, gives the exact consistency strength of these ideas.

Each of the following statements is equiconsistent with a Mahlo cardinal:

(1) There exists a disjoint club sequence on ω_2.
(2) There does not exist a thin stationary subset of $P_{\omega_1}(\omega_2)$.
(3) There is a fat subset of ω_2 which cannot acquire a club subset by any forcing poset which preserves ω_1 and ω_2.

The construction of a model with a disjoint club sequence using a Mahlo cardinal provides another proof of Mitchell's theorem [4] that the existence of a Mahlo cardinal implies a model with no special Aronszajn tree on ω_2. We iterate with countable support up to a Mahlo cardinal κ, forcing at each inaccessible $\alpha < \kappa$ first a Cohen real and then an ω_1-chain through the collection of countable subsets of ω_2 added by the real.

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Co-stationarity of the ground model

Natasha Dobrinen
(joint work with Sy-David Friedman)

This paper investigates when it is possible for a partial ordering $\mathbb{P}$ to force $(\mathscr{P}_\kappa(\lambda))^{V^{\mathbb{P}}} \setminus (\mathscr{P}_\kappa(\lambda))^V$ to be stationary in $(\mathscr{P}_\kappa(\lambda))^{V^{\mathbb{P}}}$. When this is the case, we say that $(\mathscr{P}_\kappa(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$. It follows from a result of Gitik that whenever $\mathbb{P}$ adds a new real, then $(\mathscr{P}_\kappa(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for each regular uncountable cardinal κ in $V^{\mathbb{P}}$ [2]. However, the following theorem of Magidor implies that when no new ω-sequences are added, large cardinals become necessary.

Theorem 1 (Magidor [3])**.** *Assume there are no ω_1-Erdös cardinals in K_{DJ}, the Dodd-Jensen core model. Then for every ordinal β one can define in K_{DJ} a countable collection of functions $\mathscr{C}$ on β such that every subset of β closed under $\mathscr{C}$ is a countable union of sets in K_{DJ}.*

Magidor's theorem and a generalization of it above any given cardinal yield (2) implies (1) in the next two equiconsistency results. We show that (1) implies (3). ((3) implies (2) is trivial.)

Theorem 2. *The following are equiconsistent:*

(1) *There is an ω_1-Erdös cardinal.*
(2) *If $\mathbb{P}$ is $\aleph_1$-Cohen forcing, then $(\mathscr{P}_{\aleph_2}(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for all $\lambda \geq \aleph_3$.*
(3) *If $\mathbb{P}$ adds a new subset of $\aleph_1$ and is $(\aleph_3, \aleph_3, \aleph_1)$-distributive, then $(\mathscr{P}_{\aleph_2}(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for all $\lambda \geq \aleph_3$.*

The proof of the consistency makes use of a construction of Baumgartner [1]. We generalize that construction to larger cardinals in order to prove (1) implies (3) of the next theorem.

Theorem 3. *The following are equiconsistent:*

(1) *There is a proper class of ω_1-Erdös cardinals.*
(2) *If $\mathbb{P}$ is $\aleph_1$-Cohen forcing, then $(\mathscr{P}_\kappa(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for all regular $\kappa \geq \aleph_2$ and all $\lambda > \kappa$.*

(3) *If $\mathbb{P}$ adds a new subset of $\aleph_1$ and satisfies the $(\kappa^+, \kappa^+, <\kappa)$-distributive law for all successor cardinals $\kappa \geq \aleph_2$ and is κ-c.c. for all strongly inaccessible cardinals κ, then $(\mathscr{P}_\kappa(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for all regular $\kappa \geq \aleph_2$ and all $\lambda > \kappa$.*

In the quest for the equiconsistency of the co-stationarity of the ground model for $\mathscr{P}_{\aleph_3}(\lambda)$, we need much more than an ω_1-Erdös cardinal. A result of Magidor shows that at least a measurable cardinal is necessary if we are to obtain the consistency of the co-stationarity of the ground model for $\mathscr{P}_{\aleph_3}(\lambda)$ for any $\aleph_3 < \lambda < \aleph_{\omega_2}$. In fact, we show that $\aleph_2$ measurable cardinals are necessary for any $\aleph_3 < \lambda \leq \aleph_{\omega_2}$, and generalize to higher cardinals.

Theorem 4. *Let $\kappa \geq \aleph_2$ be regular and assume that there is no inner model with κ measurable cardinals. Then there is a countable collection $\mathscr{C}$ of functions on $\aleph_\kappa$ such that every subset of $\aleph_\kappa$ closed under $\mathscr{C}$ is the union of $<\kappa$ sets in K_M, Mitchell's core model for sequences of measures.*

Using the previous theorem, a consistency result of Shelah for free subsets [4], and the fact that free subsets can be used for coding, we obtain the following equiconsistency.

Theorem 5. *The following are equiconsistent for regular κ:*

(1) *$\aleph_\kappa > \kappa$ and there are κ measurable cardinals.*
(2) *$\aleph_\kappa > \kappa$ and if $\mathbb{P}$ is κ-Cohen forcing, then $(\mathscr{P}_{\kappa^+}(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for all $\lambda \geq \aleph_\kappa$.*
(3) *$\aleph_\kappa > \kappa$ and if $\mathbb{P}$ is $(\aleph_\kappa, \aleph_\kappa, \kappa)$-distributive, then $(\mathscr{P}_{\kappa^+}(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for all $\lambda \geq \aleph_\kappa$.*

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Universal analytic preorders

Riccardo Camerlo

Given standard Borel spaces X, X' and n-ary relations R, R' on X, X' respectively, say that R is Borel reducible to R', in symbols $R \leq_B R'$ if and only if there exists a Borel function $g : X \to X'$ such that

$$\forall x_1, \ldots, x_n \in X \ (R(x_1, \ldots, x_n) \Leftrightarrow R'(g(x_1), \ldots, g(x_n))).$$

A *universal analytic preorder* on the standard Borel space X is an analytic preorder R on X such that, for all analytic preorders R' on some standard Borel space, the relation $R' \leq_B R$ holds. By [LR05], analytic universal preorders do exist.

Let LO be the Polish space of total orders on $\mathbb{N}$. So each element $LO \times \mathbb{N}^{\mathbb{N}}$ is a coloured total order. Given a preorder P on $\mathbb{N}$, for $(\sqsubseteq, \varphi), (\sqsubseteq', \varphi') \in LO \times \mathbb{N}^{\mathbb{N}}$ define $(\sqsubseteq, \varphi) \preceq_P (\sqsubseteq', \varphi')$ if and only if there exists an injection $g : \mathbb{N} \to \mathbb{N}$ such that:

- $\forall a, b \in \mathbb{N}\ (a \sqsubseteq b \Leftrightarrow g(a) \sqsubseteq' g(b))$;
- $\forall a \in \mathbb{N}\ \varphi(a) P \varphi' g(a)$.

So $\preceq_P$ is an analytic preorder. By [L71], if P is a bqo, then $\preceq_P$ is a bqo as well, so very far from being universal.

Theorem 1. *If S is an analytic preorder on $LO \times \mathbb{N}^{\mathbb{N}}$ such that $\preceq_= \subseteq S \subseteq \preceq_\geq$, then S is universal. Consequently, if P is not a wqo then $\preceq_P$ is an analytic universal preorder.*

Problem 2. Classify $\preceq_P$ when P is a wqo but not a bqo.

A function $g : \mathbb{Q} \to \mathbb{Q}$ is a *dense order preserving function* if it preserves the order and

$$\forall q_0, q_1, r_0, r_1 \in \mathbb{Q}\ (f(q_0) < r_0 < r_1 < f(q_1) \Rightarrow \exists q \in \mathbb{Q}\ r_0 < f(q) < r_1);$$

equivalently, there exists a continuous increasing function $h : \mathbb{R} \to \mathbb{R}$ such that $h(\mathbb{Q}) \subseteq \mathbb{Q}$ and g coincides with the restriction of h to the rationals.

Let P be a preorder on a set A. For $\varphi, \psi \in A^{\mathbb{Q}}$ let $\varphi \leq^P_{dop} \psi$ if and only if there is a dense order preserving function $g : \mathbb{Q} \to \mathbb{Q}$ such that $\forall q \in \mathbb{Q}\ \varphi(q) P \psi g(q)$.

Theorem 3. *If P is equality on $A = 2$, then $\leq^P_{dop}$ is a universal analytic preorder.*

Theorem 4. *If $=, \geq$ are the usual relations on $\mathbb{N}$ and S is an analytic preorder on $\mathbb{N}^{\mathbb{Q}}$ such that $\leq^=_{dop} \subseteq S \subseteq \leq^\geq_{dop}$, then S is a universal analytic preorder.*

Problem 5. If $\leq_\alpha$ is the usual order on the countable ordinal α, classify the relation $\leq^{\leq_\alpha}_{dop}$ on $\alpha^{\mathbb{Q}}$, for each $\alpha \in \omega_1$.

The above result can be applied to continuous embeddability of continua. A *continuum* is a compact connected metric space. A *dendrite* is a locally connected continuum that does not contain any simple closed curve.

Theorem 6. *The relation of continuous embeddability on dendrites whose branching points have order 3 is a universal analytic preorder.*

The relation of being continuous image trivialises on dendrites, since any two locally connected continua are continuous image of each other. However the following can be obtained.

Theorem 7. *The relation of being continuous image is a universal analytic preorder for continua.*

Theorem 8. *The relation of being continuous open image is a universal analytic preorder on dendrites.*

Theorems 1, 3 and 6 improve results of [MR04].

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Baire Reflection

Stuart Zoble

(joint work with Stevo Todorčević)

For a set A of ω-sequences of ordinals and a set of ordinals H the game $G(A, H)$ has two players who alternate playing ordinals from H. Player II wins if the cooperative play belongs to A and loses otherwise. A weak version of the Game Reflection Principle defined and studied in [5], which we denote $GRP_\omega(\theta)$ for an uncountable cardinal θ, asserts that for every $A \subset \theta^\omega$, player II has a winning strategy in $G(A, \theta)$ if and only if II has a winning strategy in $G(A, H)$ for an ω_1-club[1] of $H \in [\theta]^{\omega_1}$. The weakened principle obtained by requiring that each player to play finite sequences, rather than single ordinals (producing an element of θ^ω by concatenating the plays), is immediately equivalent to the Weak Baire Reflection Principle below.

Definition 1. $BRP^w(\theta)$ asserts that any $A \subset \theta^\omega$ is meager if and only if $A \cap H^\omega$ is meager in H^ω for an ω_1 club of $H \in [\theta]^{\omega_1}$.

A stronger version requires reflection of a failure of the Baire Property.

Definition 2. $BRP(\theta)$ asserts that any $A \subset \theta^\omega$ has the Baire Property in X if and only if $A \cap H^\omega$ has the Baire Property in H^ω for an ω_1 club of $H \in [\theta]^{\omega_1}$.

BRP^w and BRP will denote the global versions of these principles. We show that $BRP(\theta)$ is a consequence of $GRP_\omega(\theta)$ and and that both Baire Reflection and Weak Baire Reflection are equivalent to their versions for metric spaces, and spaces with point countable bases. A relationship with stationary reflection is established and used to show that BRP^w does not imply $BRP(\omega_2)$.

Definition 3. $WRP^c_{(2)}(\kappa)$ asserts that pairs of stationary sets $S, T \subset [\kappa]^\omega$ which are complimentary on a club reflect simultaneously to a cofinal set of $X \in [\kappa]^{\omega_1}$.

[1] Closed under ω_1-length increasing unions and cofinal in the $\subseteq$ ordering

We show that $BRP^w(\kappa)$ implies $WRP(\kappa)$, $BRP(\kappa)$ implies $WRP^c_{(2)}(\kappa)$, and $WRP^c_{(2)}(\kappa)$ plus $BRP^w(\kappa)$ implies $BRP(\kappa)$. We use this connection to produce a model in which BRP^w holds but $BRP(\omega_2)$ does not. We also show that BRP holds in the model obtained by the Mitchell collapse of a supercompact, which shows that BRP does not imply CH, in contrast to König's theorem that $GRP_\omega(\omega_2)$ does imply CH (see [5]). In the last section of this paper we show that Weak Baire Reflection fails under Martins Maximum in a variety of ways.

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Metric Baumgartner theorems and universality

Stefan Geschke

In a joint work with Menachem Kojman [2] we consider the possibility of getting metric analogues to a theorem of Baumgartner. Baumgartner proved that it is consistent with $2^{\aleph_0} > \aleph_1$ that any two $\aleph_1$-dense sets of reals are order isomorphic. Baumgartner's consistent statement implies the existence of a universal separable linear order of size $\aleph_1$.

Since there are too many distances between real numbers, we cannot expect to have a universal separable metric space of size $\aleph_1$ (with respect to isometric embeddings) unless CH holds. It follows that in order to get a metric version of Baumgartner's theorem, we have to consider a notion of structure preserving maps between metric spaces that is weaker than that of isometries.

Definition 1 (Kojman, Shelah [3])**.** Two metric spaces M and N are *almost isometric* if for all $K > 1$ there is a bijection $f : M \to N$ such that both f and f^{-1} are Lipschitz of constant K.

Kojman and Shelah showed that there are many pairwise not almost isometric $\aleph_1$-dense subsets of $\mathbb{R}$ of size $\aleph_1$ [3]. It follows that we also have to strengthen the largeness condition of $\aleph_1$-density in Baumgartner's theorem. We then obtain the following metric version:

Theorem 2. *If* ZFC *is consistent, then so is* ZFC + $2^{\aleph_0} = \aleph_2$ + *"for every metric space X, if X is isometric to Uryson's universal metric space or to some separable Banach space, then*

> $(\star)_X$ *X has a nowhere meager subset of size $\aleph_1$ and any two nowhere meager subsets of X of size $\aleph_1$ are almost isometric to each other."*

The proof of this theorem uses Oracle Forcing that has been introduced by Shelah for related purposes [4].

As a corollary, we obtain consistency results about the existence of almost isometry universal elements in different classes of metric spaces of size $\aleph_1$.

Theorem 3. *If* ZFC *is consistent, then so is* ZFC + $2^{\aleph_0} = \aleph_2$ + (a) +(b) + (c), *where:*

(a) *There is an almost isometry ultrahomogeneous and almost isometry universal element in the class of all separable metric spaces of size $\aleph_1$.*
(b) *For every separable Banach space B there is a conditionally almost ultrahomogeneous and almost isometry universal element in the class of subspaces of B of size $\aleph_1$.*
(c) *For every finite-dimensional Banach space B, the property of being almost isometry universal in the class of subspaces of B of cardinality $\aleph_1$ determines a space up to almost isometry.*

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Gödel logics and the continuous Fraissé conjecture

Martin Goldstern

We are interested in describing all *Gödel logics*. A Gödel logic is a fuzzy version of classical (first order predicate) logic, in which the truth values 1 and 0 (`true` and `false`) are replaced by "certainty values" that can lie strictly between 0 and 1.

For any closed set set $C \subseteq [0,1]$ containing 0 and 1 (such sets are called *Gödel sets*), the Gödel logic over C is defined as follows: For any language L of (first order) predicate logic and any Gödel set C, an (L,C)-model is a structure $\mathcal{M} = (M, f^{\mathcal{M}}, \ldots, R^{\mathcal{M}}, \ldots)$ where all k-ary relational symbols of L are interpreted by "fuzzy truth functions" $R^{\mathcal{M}} : M^k \to C$. (The function symbols are interpreted as functions, just like in classical logic.)

To each closed formula φ in L and each such structure $\mathcal{M}$ we can now naturally associate a "certainty value" $[\![\varphi]\!]^{\mathcal{M}}$; $\vee$ and $\wedge$ are interpreted as max and min, $\forall$ and $\exists$ as inf and sup, and $\to$ is interpreted by the "Gödel implication" (a natural adjoint to the function $\wedge = \min$):

$$\forall x, y \in [0,1]: \qquad (x \to y) := \sup\{z : z \wedge x \le y\}$$

which simplifies to

$$(x \to y) = \begin{cases} 1 & \text{if } x \le y \\ y & \text{otherwise} \end{cases}$$

We then define $\neg x = (x \to 0)$; so $\neg 0 = 1$, and $\neg x = 0$ for all $x > 0$.

Fix a language L; then the set $\Gamma_C = \Gamma_{C,L}$, the *Gödel logic of* C, is the set of all closed formulas φ in L which take the value 1 in every possible (L, C)-model $\mathcal{M}$.

Γ_C carries some information about C; for example, let P_n be unary relation symbols for $n = 1, 2, \ldots$. Then:

(1) the formula
$$\forall x \bigvee_{i=1}^{n} \big[P_i(x) \to P_{i+1}(x)\big]$$
is in Γ_C iff C has at most n elements.

(2) The formula $\big[\forall x\, \neg\neg P_1(x)\big] \to \big[\neg\neg \forall x\, P_1(x)\big]$ (which says: if all P_1-values are positive, then their infimum is positive) is in Γ_C iff 0 is an isolated point of C.

THEOREM: (joint work with Arnold Beckmann and Norbert Preining)

(1) There are only countably many Gödel logics.
(2) The set of Gödel logics, ordered by $\supseteq$, is better quasi-ordered ("BQO").
(3) For any recursive ordinal α there is a chain of length α of Gödel logics.

The proof of the theorem uses the following generalization of Laver's theorem, which we call the "continuous Fraissé conjecture":

- For two closed sets $A, B \subseteq [0,1]$, write $A \le B$ iff there is a **continuous monotone 1-1 embedding** $f : A \to B$. Let $\sim$ be the associated equivalence relation.
  Then $\le$ is a BQO, and there are exactly $\aleph_1$ equivalence classes of closed sets.
  (Note that there are only two classes of uncountable closed set: the class of the interval, and the class of the Cantor set.)
- If Q is a countable BQO, and ℓ_i are Q-labelings of closed *countable* sets A_i for $i = 1, 2$ (i.e., $\ell_i : A_i \to Q$), then we define $(A_1, \ell_1) \le (A_2, \ell_2)$ iff there is a continuous monotone 1-1 map $f : A \to B$ with $\ell_1(x) \le \ell_2(f(x))$ for all $x \in A$.
  Then again $\le$ is a BQO, and there are only $\aleph_1$ classes.

Alain Louveau has pointed out that such a theorem for continuous maps between *Borel* sets was earlier proved in an unpublished work of Khalid Kada.

A preprint of our paper is available at `http://arxiv.org/abs/math/0411117/`.

Absoluteness for universally Baire sets and the uncountable

Ilijas Farah and Paul Larson
(joint work with Richard Ketchersid, Menachem Magidor)

Cantor's Continuum Hypothesis was proved to be independent from the usual ZFC axioms of Set Theory by Gödel and Cohen. The method of forcing, developed by Cohen to this end, has lead to a profusion of independence results in the following decades. Moreover, Gödel's incompleteness theorems imply the existence of statements whose consistency with ZFC can be proved using only strong axioms of infinity, so-called *large cardinal axioms.* A classical example is Banach's 'Lebesgue measure has a σ-additive extension to all sets of reals.' While it is fairly easy to find a model in which this is false and there is no known ZFC-proof of its negation, proving the consistency of this statement requires assuming the existence of a *measurable cardinal.*

A remarkable result was proved by Shoenfield ([9]): every statement of the form $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})\phi(x, y)$, where all quantification in ϕ is over the natural numbers and all of its parameters are real numbers, is *absolute* between models of ZFC that are transitive and contain all countable ordinals. In this form Shoenfield's theorem is best possible, as it cannot even be improved by adding one more alteration of quantifiers ranging over $\mathbb{R}$. However, a corollary that the truth of any $\mathbf{\Sigma}^1_2$ statement (i.e., one of the above syntactical form) cannot be changed by forcing turned out to be susceptible to far-reaching generalizations. One of the more striking results in modern set theory is that the existence of suitable large cardinals implies that the theory of the inner model $L(\mathbb{R})$ (the smallest inner model of ZF, the usual axioms of Set Theory without the Axiom of Choice, containing all real numbers) cannot be changed by set forcing (see [4, 6]). In particular, a sentence with real parameters and any number of alterations of quantifiers ranging over $\mathbb{R}$, has a fixed truth value that cannot be changed by forcing. The impact of large cardinals to sets of reals goes well beyond $L(\mathbb{R})$ to imply absoluteness of the *universally Baire* sets of reals ([3]). A remarkable consequence is that the existence of large cardinals outright implies that all sets of reals in $L(\mathbb{R})$, and indeed all universally Baire sets, share all the classical regularity properties of Borel sets such as Lebesgue measurability.

This impact of large cardinals does not extend to $L(\mathcal{P}(\aleph_1))$ (the smallest inner model of ZF containing all subsets of the first uncountable cardinal, $\aleph_1$) or to the closely related set-model $H(\aleph_2)$ (the set of all sets whose transitive closure is of cardinality not greater than $\aleph_1$), since theories of these models are highly susceptible to forcing. Nevertheless, large cardinals do influence theory of 'the uncountable,' even in the context free of metamathematics. From Hausdorff's gap to Moore's L-space, many highly sophisticated objects on ω_1 have been constructed during the last century. Statements asserting the existence of these objects are absolute simply because they are true.

Lévy's absoluteness theorem ([7]) states that all Σ_1-statements, ones of the form $(\exists X)\phi(X)$, where all quantification in ϕ is over the natural numbers and all of its parameters are real numbers, are *absolute* between models of ZFC that are transitive. One consequence is that every Σ_1-statement true in the set-theoretic universe is already witnessed by a set in $H(\aleph_1)$ (the set of all sets whose transitive closure is at most countable), thus Σ_1-statements reflect to $L(\mathbb{R})$. An absoluteness result that is genuinely about ω_1 is a consequence of Keisler's completeness theorem for the logic $L_{\omega_1\omega}(Q)$ ([5]). Recall that $L_{\omega_1\omega}(Q)$ is the extension of the first-order logic by allowing countable disjunctions of formulas (with finitely many free variables) and quantifier Qx. A model $\mathfrak{A}$ of an $L_{\omega_1\omega}(Q)$ sentence is *standard* if $\mathfrak{A} \models Qx\phi(x)$ holds iff the set $\{x \mid \mathfrak{A} \models \phi(x)\}$ is uncountable. The consequence of Keisler's theorem relevant to our work is the fact that the Σ_1 statement 'ϕ has a standard model' of $H(\aleph_2)$ is forcing-absolute. The existence of a special Aronszajn tree, Hausdorff gap, or Countryman order are all equivalent to the assertions that a specific $L_{\omega_1\omega}(Q)$ sentence has a correct model. A (typically minor) technical obstacle to this method is the fact that, even if ϕ contains a large fragment of ZFC, the statement 'ϕ has a correct model whose ω_1 is well-founded' is not necessarily forcing absolute.

Absoluteness results. Our results can be expressed in the following form: if a sufficiently large rank initial segment of the universe of sets can be forced to satisfy a sentence ϕ, then there exists a model of ϕ already which has certain correctness properties. The properties we consider in [1] are the following.

(1) Containing any specified set of $\aleph_1$-many reals.
(2) Correctness about NS_{ω_1}.
(3) Correctness about any given universally Baire set of reals (with a predicate for this set added to the language).

As a consequence of (3), for every suitably definable partition of n-tuples of reals the statement 'there exists an uncountable homogeneous set' is forcing-absolute and therefore all suitably defined ccc forcings are productively ccc. The Borel case is a well-known result of Shelah; incidently, the only known proof of Shelah's theorem uses Keisler's completeness. Another simplifying consequence is that $\mathbf{\Pi}^1_1$-correctness implies well-foundedness of ω_1 in correct models.

Relativized absoluteness results. Assuming Jensen's $\diamondsuit$ principle in [2] we obtain the following property.

(4) Correctness about the existence of uncountable homogeneous sets for subsets of $[\omega_1]^{<\omega}$ and any $[\kappa]^{<\omega}$.

Note that (4) relates to (a slight extension of) results of Magidor and Malitz [8] in the same way that (3) relates to Keisler's theorem. It also implies correctness about the countable chain condition for partial orders and correctness about uncountable chains through (some) trees of height and cardinality ω_1.

These results are obtained using two main tools (both due to Woodin):

(a) iterable models (also called $\mathbb{P}_{\max}$-preconditions), introduced in [12],
(b) stationary-tower forcing ([6]), or more specifically, Woodin's proof of Σ^2_1-absoluteness ([11]) .

In both cases the construction roughly proceeds as follows. Assuming a given statement ϕ can be forced we obtain a model M satisfying ϕ with countable ω_1. A generic ultrapower $j\colon M \to M^*$ provides a model M^* with correct ω_1 satisfying ϕ. This ultrapower is constructed so as to assure the additional desired properties. In case of (4) the generic ultrapower is a direct limit of an iteration M_α ($\alpha < \omega_1$) obtained using $\diamondsuit$ as a modification of the construction from [8].

Method (b) requires more large cardinal strength than (a) (measurable Woodin cardinals instead of Woodin cardinals). On the other hand, it allows one to assure (1). Note that for every $\mathbb{P}_{\max}$ precondition (N, I) there exists a real that does not belong to any iterate of (N, I). Aside from (1) we can obtain all of these properties simultaneously using the method (a). Aside from (1) and (4) we can prove all of these properties simultaneously using the method (b). As a matter of fact, simultaneously obtaining (1) and a rather weak form of (4) would be of great interest as it would imply Σ^2_2-absoluteness conditioned on $\diamondsuit$, confirming a conjecture of John Steel (see [10]).

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Guessing Clubs in $\mathcal{P}_{\omega_1}(\omega_2)$

Bernhard König

Club-guessing principles have been studied intensively in the literature, a major source is [1]. But in all these references, the guessing sequences anticipate clubs of ordinals, e.g. clubs in ω_2. The purpose of this note is to introduce principles that guess clubs $\mathcal{D} \subseteq \mathcal{P}_{\omega_1}(\omega_2)$. The following assertion seems to be in that spirit:

if $E \subseteq \omega_2$ is stationary, let $\mathrm{TCG}(E)$ be the statement that there is a sequence $\langle \mathcal{F}^\delta : \delta \in E \rangle$ such that

(1) $\mathcal{F}^\delta = (F^\delta_\xi : \xi < \omega_1)$ is a continuous, $\subseteq$-increasing and unbounded chain in $\mathcal{P}_{\omega_1}(\delta)$ for all δ in E.

(2) for all clubs $\mathcal{D} \subseteq \mathcal{P}_{\omega_1}(\omega_2)$ there is a club $C \subseteq \omega_2$ such that for all $\delta \in C \cap E$ there is $\xi_0 < \omega_1$ with $F^\delta_\xi \in \mathcal{D}$ whenever $\xi > \xi_0$, i.e. a tail of $\mathcal{F}^\delta$ is contained in $\mathcal{D}$.

The following facts help determining the status of $\mathrm{TCG}(E)$:

Lemma 1.

(1) $\diamond^*(E)$ *implies* $\mathrm{TCG}(E)$.

(2) $\mathrm{TCG}(E)$ *is preserved by ccc-forcings.*

The next theorem is due to the author and Yasuo Yoshinobu.

Theorem 2. *The following are equivalent:*

(1) $\mathrm{TCG}(\omega_2 \cap \mathrm{cof}(\omega))$.

(2) *The density of the ω_1-club filter is $\aleph_2$.*

These new club-guessing principles are not without applications:

Corollary 3. *Assume that the density of the ω_1-club filter is $\aleph_2$. Then every stationary $\mathcal{E} \subseteq \mathcal{P}_{\omega_1}(\omega_2)$ can be thinned out to a stationary $\mathcal{E}_0 \subseteq \mathcal{E}$ such that $\mathcal{E}_0 \cap \mathcal{P}_{\omega_1}(\delta)$ is non-stationary for every $\delta \in \omega_2 \cap \mathrm{cof}(\omega)$.*

We also note the following results for $\mathrm{TCG}(\omega_2 \cap \mathrm{cof}(\omega_1))$. The first one is using iteration techniques from [2].

Theorem 4. GCH *does not imply* $\mathrm{TCG}(\omega_2 \cap \mathrm{cof}(\omega_1))$.

Theorem 5. $\mathrm{GCH} + \mathrm{TCG}(\omega_2 \cap \mathrm{cof}(\omega_1))$ *implies the existence of an $\aleph_2$-Suslin-tree.*

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A covering property, PFA **and** SCH

Matteo Viale

We show that the Proper Forcing Axiom implies the Singular Cardinal Hypothesis. The proof is by interpolation and introduces a new property of forcing axioms (and large cardinals).

Recall that SCH asserts that for all $\kappa > 2^{\operatorname{cof}\kappa}$, $\kappa^{\operatorname{cof}\kappa} = \kappa^+$. Our notation is standard and follows [1]. We just specify that for regular cardinals $\lambda < \kappa$, $S^{\leq\lambda}_{\kappa}$ denotes the subset of κ of points of cofinality $\leq \lambda$ and that a family $\mathcal{D}$ is covered by a family $\mathcal{E}$ if for every $X \in \mathcal{D}$ there is a $Y \in \mathcal{E}$ such that $X \subseteq Y$.

For a cardinal κ of countable cofinality, $(C_\eta : \eta < \kappa^+)$ is a ladder system, if for all η of uncountable cofinality, C_η is a club in η of order type less than κ. $\mathcal{D} = \{K(n,\beta) : n < \omega\,, \beta \in S^{\omega}_{\kappa^+}\} \subseteq [\kappa^+]^{<\kappa}$ is a good matrix associated to the sequence $(C_\delta : \delta < \kappa^+)$ if it satisfies the following requirements:

(i) There is an increasing sequence of regular cardinals $\kappa_n < \kappa$ such that $\kappa = \sup_n \kappa_n$ and $|K(n,\beta)| = \kappa_n$,
(ii) $K(n,\beta) \subseteq K(m,\beta)$ for $n < m$,
(iii) if $\eta < \beta$ and $|C_\eta| < \kappa$, there is n such that $C_\eta \subseteq K(n,\beta)$,
(iv) $K(n,\beta)$ is a closed subset of $\beta + 1$,
(v) $\beta + 1 = \bigcup_n K(n,\beta)$,

First remark that good matrixes exist on all successors of a cardinal of countable cofinality: given a ladder system on κ^+, one can define a good matrix $\mathcal{D}$, for example, as follows. For all $\eta < \kappa^+$ let $\phi_\eta : \kappa \to \eta$ be a surjection. Fix also $\{\kappa_n : n < \omega\}$ increasing sequence of regular cardinals cofinal in κ. Now let for all β of countable cofinality $K(n,\beta)$ be the following set:

$$\overline{\phi_\beta[\kappa_n]} \cup \overline{\bigcup\{C_\eta : \eta \in \phi_\beta[\kappa_n]\ \&\ |C_\eta| \leq \kappa_n\}}$$

Definition 1. ($\mathsf{CP}(\kappa)$) κ has the "Covering Property" if for every good matrix $\mathcal{D}$ on κ^+, there is C club, such that $[C \cap S^{\omega}_{\kappa^+}]^\omega$ is covered by $\mathcal{D}$.

CP is the statement: *"*$\mathsf{CP}(\kappa)$ *for all singular* κ *of countable cofinality".*

Theorem 2. *Assume* CP*. Then* $\lambda^{\aleph_0} = \lambda$*, for every* $\lambda \geq 2^{\aleph_0}$ *of uncountable cofinality.*

Proof: proceed by induction. The only non trivial case being the case $\lambda = \kappa^+$ for κ singular of countable cofinality. In this case use CP and the inductive hypothesis. $\square$

The above is enough to get SCH. So we will be done once we show that:

Theorem 3. PFA *implies* CP.

Sketch of proof: The strategy of the proof is the following: we let P be the poset of closed countable sets c contained in κ^+, such that $c \cap C_\alpha$ is finite for all α of uncountable cofinality, $c \leq d$ iff c end extends d. We will show that if $\mathsf{CP}(\kappa)$

fails, then P is proper in the following stronger sense: for every M containing all relevant information and for all $c \in P \cap M$ there is a $d \leq c$ (M,p)-generic such that $d \in D$ for all nonempty dense open subsets D of P which are in M. Once this is achieved, a simple argument shows that for all $\alpha < \omega_1$, $D_\alpha = \{c : \mathrm{otp}(c) \geq \alpha\}$ is dense. Now assume that PFA holds and let G be a P-generic filter such that for all α, $G \cap D_\alpha$ is non empty. Then $C = \cup G$ is a club in $\delta = \sup C$. So $C \cap C_\delta$ is a club in δ. If $c \in G$ is long enough, then $c \cap C_\delta$ is infinite so it is not a condition!!

We are left with the proof that P is proper in the above stronger sense, if $\mathsf{CP}(\kappa)$ fails. Let M be a countable elementary submodel containing everything relevant of some $H(\theta)$ with θ regular and large enough. Let $\delta_M = \sup M \cap \kappa^+$ and β_M be large enough in order that for all $\gamma < \kappa^+$ there is $\eta < \beta_M$ of such that $C_\gamma \cap M = C_\eta \cap M$. Let $c \in P \cap M$. We need to build an (M,P)-generic d extending c and belonging to all nonempty dense open sets of M. List all nonempty dense open subsets of P which are in M, as $(D_n)_n$. We build a decreasing chain of conditions $c = p_0 \geq p_1 \geq p_2 \geq p_3 \geq \cdots$ arranging that

(1) $p_{n+1} \in (D_n \cap M)$,
(2) $(p_{n+1} \setminus p_n) \cap K(n, \beta_M) = \emptyset$.

Once this is achieved let $d = \bigcup_n p_n \cup \{\delta_M\}$. d is the desired (M,P)-generic condition: A density argument yields that d is closed and countable. So we just need to check that d has finite intersection with any C_γ. This is so because for any γ of uncountable cofinality, $(d \cap C_\gamma \setminus \{\delta_M\}) \subseteq M \cap C_\gamma = M \cap C_\eta$ for some $C_\eta \subseteq K(n, \beta_M)$, by choice of β_M. So we get that $d \cap C_\gamma \setminus \{\delta_M\} = d \cap C_\eta \setminus \{\delta_M\}$. Now use the fact that for some n, $C_\eta \subseteq K(n, \beta_M)$ and condition (2) on the sequence of p_n-s to get that $(d \cap C_\gamma) \setminus \{\delta_M\} = p_n \cap C_\eta$ is finite.

So, it is enough to define the inductive step from p_n to p_{n+1}. Work in M and define a function $f : \kappa^+ \to \kappa^+$ in M as follows: Take $\xi \in (\max p_n, \kappa^+)$, Look at $p \cup \{\xi\}$. It is a condition, so there is some condition, say $r_\xi \in D_n$, extending it. Let $f(\xi) = \max(r_\xi)$. Let $C \in M$ be the set of closure points of f. M knows that $C \cap S^\omega_{\kappa^+}$ has a countable subset of points of countable cofinality, say $X \in M$, which is not covered by any member of the good matrix. So there is $\gamma \in C \cap M \cap X$, such that $\gamma \notin K(n, \beta_M)$. $K(n, \beta_M)$ is closed and $\gamma \in M$ is of countable cofinality and this is why we can find $\xi \in M$ such that $[\xi, \gamma] \cap K(n, \beta_M) = \emptyset$. Now, look at the condition $p \cup \{\xi\}$ and let $r_\xi \in M \cap D_n$ be an extension of it witnessing that $f(\xi) = \max r_\xi$. Since $\gamma \in C$, we have that $\max(r_\xi) < \gamma$. Check that $p_{n+1} = r_\xi$ is the desired extension of p_n. $\square$

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Simply definable (without parameters) well–orders of $H(\omega_2)$, together (or not) with forcing axioms

David Asperó

1. Main starting questions

The work presented here deals mostly with the problem of finding optimal definitions of well–orders of the reals and other objects. More precisely, it addresses the following two questions.

Question 1: Suppose A is a subset of ω_1. Suppose we are given the task of going over to a set-forcing extension preserving stationary subsets of ω_1 in which A admits a definition $\Phi(x)$, without parameters, over the structure $\langle H(\omega_2), \in\rangle$ (or over some natural extension of this structure, like $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$). What is the lowest degree of logical complexity that can be attributed to a definition $\Phi(x)$ for which we can perform the above task?

Question 2: What is the lowest possible degree of logical complexity for which there is a formula $\Phi(x, y)$ (again without parameters) with that complexity and with the property that we can go over to a set-forcing extension in which the set of real numbers admits a well–order defined by $\Phi(x, y)$ (again over the structure $\langle H(\omega_2), \in\rangle$ or over some natural extension of it)?

We will measure logical complexity by means of the familiar Levy hierarchy of formulas.[1]

2. Results that don't mention forcing axioms

Let $\mathcal{L}$ be the first order language of the structure $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$. Let us say that two $\mathcal{L}$–formulas $\Phi_0(x)$ and $\Phi_1(x)$ are *ZFC–provably incompatible over $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$* if ZFC proves that there is no $x \in H(\omega_2)$ such that $\langle H(\omega_2), \in, NS_{\omega_1}\rangle \models \Phi_0(x) \wedge \Phi_1(x)$. Also, for an $\mathcal{L}$–formula in two free variables $\Phi(x, y)$, let us say that $\Phi(x, y)$ is *ZFC–provably antisymmetric over $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$* if ZFC proves that there are no x, y in $H(\omega_2)$, $x \neq y$, such that $\langle H(\omega_2), \in, NS_{\omega_1}\rangle \models \Phi(x, y) \wedge \Phi(y, x)$.

The main results in this section are the following.

Theorem 1. *There are Σ_3 $\mathcal{L}$–formulas $\Phi_0(x)$ and $\Phi_1(x)$ and Π_3 formulas $\Psi_0(x)$ and $\Psi_1(x)$ with the following two properties.*

[1] For formulas of a language extending the language of set theory.

(1) $(\Phi_0(x), \Phi_1(x))$ *and* $(\Psi_0(x), \Psi_1(x))$ *are two pairs of* ZFC*–provably incompatible formulas over the structure* $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$.
(2) *For every* $A \subseteq \omega_1$ *there is a proper poset forcing that*
  (a) A *is defined, over* $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$*, by* $\Phi_0(x)$ *and by* $\Psi_0(x)$*, and*
  (b) $\omega_1 \backslash A$ *is defined, over* $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$*, by* $\Phi_1(x)$ *and by* $\Psi_1(x)$.

Theorem 2. *There is a* Σ_3 $\mathcal{L}$*–formula* $\Phi(x, y)$ *and a* Π_3 $\mathcal{L}$*–formula* $\Psi(x, y)$ *with the following two properties.*

(1) $\Phi(x, y)$ *and* $\Psi(x, y)$ *are* ZFC*–provably antisymmetric formulas over the structure* $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$.
(2) *If there is an inaccessible cardinal, then there is a proper poset* $\mathcal{P}$ *forcing the existence of a well–order* $\leq$ *of* $H(\omega_2)$ *of order type* ω_2 *such that* $\leq$ *is defined, over* $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$*, both by* $\Phi(x, y)$ *and by* $\Psi(x, y)$.

In fact, Theorem 1 follows[2] from the following result.

Theorem 3. *There is a* Σ_2 $\mathcal{L}$*–formula* $\Phi(x)$ *with the property that for every* $A \subseteq \omega_1$ *there is a proper poset forcing*

$$A = \{\xi < \omega_1 \,:\, \langle H(\omega_2), \in, NS_{\omega_1}\rangle \models \Phi(\xi)\}$$

The proofs of Theorems 3 and 2 involve the manipulation, by forcing, of certain weak club–guessing properties for club–sequences defined on subsets of ω_1, in such a way that the Σ_2 theory of $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$ with countable ordinals as parameters codes any prescribed subset of ω_1.

On the other hand, by appealing mainly to results of Woodin, one can prove that Theorems 1 and 2 are optimal from the point of view of the Levy hierarchy. More precisely, one can prove that, in the presence of sufficiently strong large cardinals,[3] 3 cannot be replaced by 2 in the statement of either Theorem 1 or Theorem 2. In fact, one cannot prove a version of either Theorem 1 or Theorem 2 in which Σ_3 (equivalently, Π_3) is replaced by Π_2.

One may ask wether the use of the predicate NS_{ω_1} in the statement of either Theorem 1 or 2 can be avoided. Concerning this question, there is a version of Theorems 1 and 2 with $\langle H(\omega_2), \in\rangle$ replacing the more expressive $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$. The coding techniques employed in the proof of these theorems are quite different from the ones used in the proofs of Theorems 1 and 2. Finally, these theorems use ZFC + "There is an inaccessible limit of measurable cardinals" as base theory, rather than just ZFC or ZFC + "There is an inaccessible cardinal".

3. One result mentioning PFA

Theorem 4. *Suppose* κ *is a supercompact cardinal. Then there is a semiproper partial order* $\mathcal{P} \subseteq V_\kappa$ *such that*

(1) $\mathcal{P}$ *forces* PFA^{++}*, and*

[2] By taking $\Phi_0(x)$ and $\Psi_0(x)$ to be $\Phi(x)$ and by taking $\Phi_1(x)$ and $\Psi_1(x)$ to be $\neg Phi(x)$.
[3] For example a proper class of Woodin cardinals.

(2) *$\mathcal{P}$ forces the existence of a well–order of $H(\omega_2)$ definable, over the structure $\langle H(\omega_2), \in\rangle$, by a formula without parameters.*

This time the proof involves the manipulation of certain guessing properties of functions $F : S \longrightarrow \mathcal{P}(\omega_1)$[4] with respect to canonical functions.[5]

As far as I know, the following important questions remain open.

Question 5. Assume there is a supercompact cardinal (or some other reasonable large cardinal assumption). Is it possible to force in such a way that Martin's Maximum holds in the extension, together with the existence of a well–order of $H(\omega_2)$ definable, over $\langle H(\omega_2), \in\rangle$, by a formula without parameters (or even by a formula with a real number as parameter)?

Does Martin's Maximum imply that there is a well–order of $H(\omega_2)$ definable, over $\langle H(\omega_2), \in\rangle$, by a formula with at most a real number as parameter?

Capacities and Ramsey theory

Jindřich Zapletal

For a subadditive capacity c on a Polish space X let $I(c)$ be the sigma-ideal of sets of zero c-mass. I investigate the problem of properness and other forcing properties of the forcing $P(c)$ of Borel $I(c)$-positive sets ordered by inclusion. It turns out that some of these partial orders are proper, others are not, some add splitting reals, some do not, some preserve the outer Lebesgue measure, others do not, depending mostly on the measure-theoretic properties of the capacity c. The forcings, if proper, are bounding.

I isolate a measure theoretic property of stability of capacities and state the related theorems:

Theorem 1. *If c is stable then the forcing $P(c)$ is proper.*

Theorem 2. *All capacities used in potential theory are stable.*

I further isolate a measure theoretic property of Ramseyness of capacities and state the following theorems:

Theorem 3. *If c is Ramsey and the forcing $P(c)$ is proper then $P(c)$ does not add splitting reals.*

Theorem 4. *The Hausdorff content capacity associated with the Davies-Rogers example of a Hausdorff measure with only zero and infinite values is a Ramsey capacity.*

[4]Where $S \subseteq \omega_1$ and where, for every $\nu \in S$, $ot(F(\nu))$ is in some prescribed interval of countable ordinals.

[5]Where the fact that a function F as above guesses canonical functions means that $\{\nu \in S : g(\nu) \in F(\nu)\}$ is stationary for every $\alpha < \omega_2$ and every canonical function g for α.

Splitting Squares

Otmar Spinas

Theorem 1. *The partition relation*

$$2^\omega \longrightarrow_{Borel} \left[\text{closed countably splitting}\right]^n_{\mathbb{R}}$$

holds for each $1 \leq n < \omega$; that is:
For every Borel $f : [2^\omega]^n \to \mathbb{R}$ there exists a closed countably splitting A such that $f''[A]^n$ is a proper subset of $\mathbb{R}$.

Here $A \subseteq 2^\omega$ is countably splitting iff

$$\forall \text{countable } F \subseteq [\omega]^\omega \ \exists x \in A \ \forall a \in F \ \exists^\infty i \in a \ \exists^\infty j \in a \ \ x(i) = 0 \ \& \ x(j) = 1.$$

The approachability property and the Failure of SCH

Assaf Sharon

The Singular Cardinals Hypothesis (SCH) asserts that if $2^{cf(\kappa)} < \kappa$ then $\kappa^{cf(\kappa)} = \kappa^+$. The SCH plays an important role in cardinal arithmetic in trying to determine the possible values of the exponential function κ^λ. In fact SCH together with the values of the Gimmel function on regular cardinals (which, by a well known result of Easton [1], can admit any reasonable value as long as monotonicity and König's lemma are satisfied) determine the exponential function.

It turns out that the failure of SCH implies the existence of some non-trivial combinatorial principles. Let us give a few examples illustrating this phenomenon.

Theorem 1 (Shelah [3])**.** *Assume that κ is a singular cardinal of cofinality ω and that SCH fails at κ. Then there exists a non reflecting stationary set in $[\kappa]^\omega$.*

Theorem 2 (Shelah [5]))**.** *Assume that SCH fails at κ. Then there is a better scale at κ.*

Theorem 3 (CFM [2])**.** *The failure of SCH at κ implies the combinatorial principle ADS_κ.*

It is natural to ask whether there are other combinatorial principles which can be derived from ¬SCH. It is particulary interesting to analyze those principles on κ^+ whose failure is derived from the existence of strongly compact cardinals below κ, since by a classical result of Solovay [6] the SCH holds above a strongly compact cardinal. In this talk we shall consider one principle of this type called the Approachability Property.

Definition 4 (Shelah [4])**.** Assume that κ is a cardinal. We say that the Approachability Property holds at κ if there exists a sequence $\langle C_\alpha \mid \alpha < \kappa^+ \rangle$ with the following properties:

(1) If $\alpha < \kappa^+$ is a limit ordinal then C_α is club in α and $o.t.(C_\alpha) = cf(\alpha)$.
(2) There exists a club D of κ^+ such that for every $\alpha \in D$ and every $\beta < \alpha$ there is $\gamma < \alpha$ such that $C_\alpha \cap \beta = C_\gamma$.

The following theorem implies that the Approachability Property is not a consequence of the failure of SCH.

Theorem 5 (Gitik-Sharon)**.** *Assume that κ is a supercompact cardinal. Then there is a forcing extension in which κ is a strong limit cardinal of cofinality ω, $2^\kappa = \kappa^{++}$ and the approachability property fails at κ.*

It is also possible to show that full reflection at κ^+ and very good scales cannot be derived from $\neg$SCH.

Definition 6. Let κ be a regular cardinal. We say that full reflection holds at κ if for every S stationary in κ there exists $\alpha < \kappa$ such that $cf(\alpha) > \omega$ and $S \cap \alpha$ is stationary in α.

Theorem 7 (Gitik-Sharon)**.** *Assume that κ is a limit of a sequence of supercompact cardinals. Then there is a forcing extension in which κ is a strong limit cardinal of cofinality ω, $2^\kappa = \kappa^{++}$ and full reflection holds at κ^+.*

Definition 8. Let κ be a singular cardinal of cofinality ω.

(1) Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of regular cardinals such that $\cup \kappa_n = \kappa$. A sequence $\langle f_\alpha \mid \alpha < \kappa^+ \rangle \subseteq \prod \kappa_n$ is called a very good scale in $\prod \kappa_n$ iff
   (a) $\langle f_\alpha \mid \alpha < \kappa^+ \rangle$ is a scale in $\prod \kappa_n$.
   (b) for every $\alpha < \kappa^+$ such that $cf(\alpha) > \omega$ there exists a club C in α and $n < \omega$ such that
$$\forall \beta < \gamma \in C \, \forall m \geq n \, f_\beta(m) < f_\gamma(m).$$
(2) There is a very good scale at κ iff there are $\langle \kappa_n \mid n < \omega \rangle$ and $\langle f_\alpha \mid \alpha < \kappa^+ \rangle$ such that $\langle f_\alpha \mid \alpha < \kappa^+ \rangle$ is a very good scale at $\prod \kappa_n$.

Theorem 9 (Sharon)**.** *Assume that κ is a limit of a sequence of κ^{++} strong cardinals and that $\kappa_0 < \kappa$ is supercompact. Then there is a forcing extension in which SCH fails at κ and there is no very good scale at κ.*

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Upper bounds on the groupwise density number

Heike Mildenberger

For $f, g \in {}^\omega\omega$ we write $f \leq g$ iff $(\exists k)(\forall n \geq k)(f(n) \leq g(n))$. A family $B \subseteq {}^\omega\omega$ is *unbounded* iff for every $g \in {}^\omega\omega$ there is some $f \in B$ such that $f \not\leq^* g$. The *bounding number* $\mathfrak{b}$ is the smallest cardinal of an unbounded family $B \subseteq {}^\omega\omega$.

For a filter $\mathscr{F}$ on ω we regard the reduced order $f \leq_{\mathscr{F}} g$ iff $\{n : f(n) \leq g(n)\} \in \mathscr{F}$.

For $A, B \in [\omega]^\omega$, we write $A \subseteq^* B$ iff $A \setminus B$ is finite. A subset $\mathscr{G}$ of $[\omega]^\omega$ is called *groupwise dense* if $(\forall X \in \mathscr{G})(\forall Y \subseteq^* X)(Y \in \mathscr{G})$ and for every partition of ω into finite intervals $\Pi = \langle \pi_i : i < \omega \rangle$ there is an infinite set A such that $\bigcup\{[\pi_i, \pi_{i+1}) : i \in A\} \in \mathscr{G}$. The *groupwise density number*, $\mathfrak{g}$, is the smallest number of groupwise dense families with empty intersection. The *groupwise density number for ideals*, $\mathfrak{g}_f$, is the smallest number of groupwise dense ideals with empty intersection. The mininum cofinality of a reduced ultrapower $\omega^\omega/\mathscr{U}$ is called $\mathfrak{mcf}$. The inequalities $\mathfrak{g} \leq \mathfrak{g}_f \leq \mathfrak{mcf}$ are easy to see. Moreover $\mathfrak{mcf} \geq \mathfrak{b}$, whereas $\mathfrak{g}$ and $\mathfrak{g}_f$ can be strictly below $\mathfrak{b}$.

A family D is *finitely dominating* iff for every $g \in {}^\omega\omega$ there is some n, $f_0, \dots, f_{n-1} \in D$ such that $g \leq^* \max(f_i : i < n)$, the maximum is meant pointwise. The minimum number of not finitely dominating families whose union is dominating, $\mathrm{cov}(\mathscr{D}_{\mathrm{fin}})$, is [4] the same as

$$\min\{\kappa : (\exists \langle \mathscr{U}_\alpha, f_\alpha : \alpha < \kappa \rangle)(\mathscr{U}_\alpha \text{ is an ultrafilter and } f_\alpha \in {}^\omega\omega \wedge (\forall g \in {}^\omega\omega)(\exists \alpha)(f_\alpha \geq_{\mathscr{U}_\alpha} g))\}.$$

Theorem 1. [3] *It is consistent relative to ZFC that*

$$\aleph_1 = \mathfrak{g} = \mathfrak{b} < \mathrm{cov}(\mathscr{D}_{\mathrm{fin}}) = \mathfrak{mcf} = \mathfrak{c} = \aleph_2.$$

The model is constructed by oracle chain condition forcing. Answering a question of Taras Banakh, we show

Theorem 2. [1] *Also* $\mathfrak{g}_f = \aleph_1$ *in the models from* [3].

In joint work with Shelah [2]. we show that $\aleph_2 \leq \mathfrak{b} < \mathfrak{g}$ is consistent. There is nothing specific about $\aleph_2$. However, our forcing techniques give only $\mathfrak{g} = \mathfrak{b}^+$ so far. There is a reason for this. We let $\mathfrak{d}_\mathfrak{b}$ denote the dominating number of the eventual domination order in ${}^\mathfrak{b}\mathfrak{b}$.

Theorem 3. *(M., Shelah)* $\mathfrak{g} \leq \mathfrak{d}_\mathfrak{b}$.

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Smoke and Mirrors: Nonstationary Ideals and Inner Models with Huge Cardinals

Matthew Foreman

This talk reports combinatorial properties of Ideals and Strong Chang's Conjectures on ω_3 and ω_4 whose existence is equiconsistent with very large cardinals such as supercompact and huge cardinals. A suggestion is made about how to show that MM implies the existence of an inner model with a cardinal κ that is κ^+-supercompact.

Barriers and near unconditionality

Jordi Lopez-Abad

The aim of this talk is to present some results of a joint work with S. Todorčević about weakly null sequences in Banach spaces using Ramsey theory of families of finite subsets of $\mathbb{N}$. Recall that the Ramsey theory on families of finite subsets of $\mathbb{N}$ was developed in a series of papers of Nash-Williams in the 60's, a theory that is today naturally embedded in the more familiar infinite-dimensional Ramsey theory. The affinities between the infinite-dimensional Ramsey theory and some problems of the Banach space theory and especially those dealing with Schauder basic sequences have been explored for quite some time, starting perhaps with Farahat's proof of Rosenthal's ℓ_1-theorem (see [10] and [15]). The Nash-Williams' theory though implicit in all this was not fully exploited in this context. We shall therefore try to demonstrate the usefulness of this theory by applying it to the classical problem of finding (weak or full) unconditional basic-subsequence of a given normalized weakly null sequence in some Banach space E.

Recall that Bessaga and Pelczynski [5] have shown that every normalized weakly null sequence in a Banach space contains a subsequence forming a Schauder basis for its closed linear span. However, as demonstrated by Maurey and Rosenthal [12] there exist weakly null sequences in Banach spaces without unconditional basic subsequences. So one is left with a task of finding additional conditions on a given weakly null sequence guaranteeing the existence of unconditional subsequences. One such condition, given by Rosenthal himself around the time of publication of [12] (see also [15]), when put in a proper context reveals the connection with the Nash-Williams theory. It says that if a weakly null sequence (x_n) in some space of the form $\ell_\infty(\Gamma)$ is such that each x_n takes only the values 0 or 1, then (x_n) has an unconditional subsequence. To see the connection, consider the family

$$\mathcal{F} = \{\{n \in \mathbb{N} : x_n(\gamma) = 1\} : \gamma \in \Gamma\}$$

and note that $\mathcal{F}$ is a pre-compact family of finite subsets of $\mathbb{N}$. As pointed out in [15], Rosenthal result is equivalent saying that there is an infinite subset M of $\mathbb{N}$ such that the trace $\mathcal{F}[M] = \{t \cap M : t \in \mathcal{F}\}$ is *hereditary*, i.e., it is downwards closed under inclusion. On the other hand, recall that the basic notion of the Nash-Williams' theory is the notion of a *barrier*, which is simply a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ no two members of which are related under the inclusion which has the property that an arbitrary infinite subset of $\mathbb{N}$ contains an initial segment in $\mathcal{F}$. Thus, in particular, $\mathcal{F}$ is a pre-compact family of finite subsets of $\mathbb{N}$. Though the trace of an arbitrary pre-compact family might be hard to visualize, a trace $\mathcal{B}[M]$ of a barrier $\mathcal{B}$ is easily to compute as it is simply equal to the downwards closure of its restriction $\mathcal{B} \upharpoonright M = \{t \in \mathcal{B} : t \subseteq M\}$. A further examination of Rosenthal's result shows that for every pre-compact family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ there is an infinite set M such that the trace $\mathcal{F}[M]$ is actually equal to the downwards closure of a uniform barrier $\mathcal{B}$ on M, or in other words that the $\subseteq$-maximal elements of $\mathcal{F}[M]$ form a uniform barrier on M. As it turns out, this fact holds considerably more information that the conclusion that $\mathcal{F}[M]$ is merely a hereditary family which is especially noticeable if one need to perform further refinements of M while keeping truck on the original family $\mathcal{F}$. This observation was the starting point of the research of this paper. Further extensions of Rosenthal's result required however analysis of not only pre-compact families of finite subsets of $\mathbb{N}$ but also maps from barriers into pre-compact families of finite subsets of $\mathbb{N}$, or into weakly pre-compact subsets of c_0. In fact, our more general results deal with partial maps from FIN $\times$ c_0 into the reals whose domains project onto weakly pre-compact subsets of c_0. Recall, that the equivalence relations associated to arbitrary maps defined on barriers have been characterized by Pudlak and Rödl [16]. Here we show that for certain maps one can say considerably more. For example, we show that for every mapping h from a barrier $\mathcal{B}$ into a weakly pre-compact subset of c_0 and every $\varepsilon > 0$ there is an infinite subset M of $\mathbb{N}$ such that $\sum_{i\in M\setminus s} |h(s)(i)| < \varepsilon$ for every $s \in \mathcal{B} \upharpoonright M$. This sort of a combinatorial result has shown to be quite useful in studying weakly-null sequences in Banach spaces. In fact, using a variation on this result, we show that if (x_n) is a normalized weakly-null sequence of $\ell_\infty(\Gamma)$ with the property that

$$\inf\{|x_n(\gamma)| : n \in \mathbb{N}, \gamma \in \Gamma\} = \delta > 0, \tag{1}$$

then (x_n) has an unconditional subsequence. More precisely, (x_n) has a subsequence which is $\delta/4$-equivalent to the basis (e_i) of a $\mathcal{F}$-Schreier space associated with the downwards closure $\mathcal{F}$ of a barrier on $\mathbb{N}$. (The original construction of Schreier uses the family $\{s : |s| \leq \min(s) + 1\}$ (see [7])). An exposition of this result appeared first in Part II of [3], a result that was proved independently from recent articles of Arvantakis [4] and Gasparis, Odell and Wahl [9] who use different approaches to prove a similar result.

Deeper applications of the combinatorics of finite sets of integers that we develop lead us to new forms of *near-unconditionality* and *convex-unconditionality*. Our near-unconditionality result says that for every normalized weakly-null sequence (x_n) and for every $\varepsilon > 0$ there is an infinite subset M of $\mathbb{N}$ such that for every $(a_i)_{i\in M}$ such that $\sup_{i\in M} |a_i| \leq 1$ and every finite subset $s \subseteq M$,

$$\|\sum_{i\in s} a_i x_i\| \leq \frac{2+\varepsilon}{\min_{i\in s} |a_i|} \|\sum_{i\in M} a_i x_i\|. \tag{2}$$

This should be compared with the well-known near-unconditionality result of Elton [8] (see also [15]). We also prove our convex-unconditionality result which says that given a normalized weakly-null sequence (x_n) in some Banach space X then for every $\varepsilon > 0$ there is an infinite subset M of $\mathbb{N}$ such that for every sequence of scalars $(a_i)_{i\in M}$ such that $\sup_{i\in M}|a_i| \leq 1$ and every subset $N \subseteq M$ such that $\sum_{i\in N}|a_i| \leq 1$,

$$\|\sum_{i\in N} a_i x_i\| \leq (4+\varepsilon)\sqrt{\|\sum_{i\in M} a_i x_i\|}. \tag{3}$$

This in turn should be compared with the corresponding well-known convex-unconditionality result of Argyros, Mercourakis and Tsarpalis [2] originally obtained by dualizing the argument of Elton [8].

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Near Coherence Classes of Ultrafilters: Recent Advances and a Retreat

Andreas Blass

1. Introduction

In this report, all filters are understood to be on ω and to contain all cofinite sets. So ultrafilters are nonprincipal ultrafilters on ω. The image of a filter $\mathcal{F}$ under a function $f : \omega \to \omega$ is

$$f(\mathcal{F}) = \{X \subseteq \omega : f^{-1}(X) \in \mathcal{F}\}.$$

Two filters $\mathcal{F}$ and $\mathcal{G}$ are *coherent* if $\mathcal{F} \cup \mathcal{G}$ generates a filter. They are *nearly coherent* if, for some finite-to-one f, $f(\mathcal{F})$ and $f(\mathcal{G})$ are coherent. For ultrafilters, coherence is equality and near-coherence is an equivalence relation.

The question addressed in this report is: **How many equivalence classes of ultrafilters are there, with respect to near coherence?**

By results of Mioduszewski [5], building on earlier work of Rudin [7] and Bellamy[2], this question admits a topological reformulation: How many composants are there in the Stone-Čech remainder of a closed half-line $[0, \infty) \subseteq \mathbb{R}$?

2. Previous work

The number of near-coherence classes of ultrafilters is clearly at least 1 and at most $2^{\mathfrak{c}}$ (the total number of ultrafilters). It is $2^{\mathfrak{c}}$ under CH or Martin's axiom or various weaker hypotheses that provide $2^{\mathfrak{c}}$ selective ultrafilters [7]. It can consistently be 1 [3].

If there are simple P_κ-points (i.e., ultrafilters generated by an almost-decreasing κ-sequence) for two different regular cardinals κ, then there are exactly two near-coherence classes of ultrafilters [4]. Until recently, it was "known" that there is a model with simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points, but the construction, in [3, Section 6], was recently found, by Alan Dow, to have a serious error. Shelah has proposed a new construction of such a model; it is still being digested and checked.

Until 2004, nothing was known about any other possibilities for the number of near-coherence classes; every cardinal in the range from 3 (inclusive) to $2^{\mathfrak{c}}$ (exclusive) was an open problem.

3. New results

Toward the end of 2004, I obtained the first negative result in this area, i.e., the first result saying that certain cardinals cannot be the number of near-coherence classes. I showed that if the number of near-coherence classes is infinite, then it is at least $\mathfrak{u}$ (the minimum number of generators for an ultrafilter) and at least $\mathfrak{d}$ (the dominating number, i.e., the cofinality of the eventual-majorization ordering on ${}^{\omega}\omega$). In particular, there cannot be exactly $\aleph_0$ near-coherence classes.

Very soon afterward, Jason Aubrey and Taras Banakh independently pointed out that $\mathfrak{u}$ in this result can be improved to $\mathfrak{u}^+$. Subsequently, Banakh and I jointly solved the problem completely as far as infinite cardinals are concerned [1]:

Theorem 1. *If the number of near-coherence classes is infinite, then it is* $2^{\mathfrak{c}}$.

The problem remains open for finite cardinals, but some constraints are known. It is shown in [1] that, if there are only finitely many near-coherence classes of ultrafilters then $\mathfrak{u} < \mathfrak{d}$.

4. Ingredients of the proof

The proof of the theorem quoted above from [1] involves numerous lemmas, including one that applies a result from [6] to give a general method for constructing $2^{\mathfrak{c}}$ non-nearly-coherent ultrafilters as limits of suitable countable sequences of ultrafilters. That lemma is too technical to warrant inclusion in this report, but here are a few more quotable lemmas.

Definition 2. Let $\mathcal{F}$ be a filter, use it to pre-order the functions $\omega \to \omega$ by

$$f \leq_{\mathcal{F}} g \iff \{n : f(n) \leq g(n)\} \in \mathcal{F},$$

and let $\mathfrak{d}(\mathcal{F})$ be the dominating number for this order.

Lemma 3. *For any filter $\mathcal{F}$, there is a family (a* test family *over $\mathcal{F}$) of at most $\mathfrak{d}(\mathcal{F})$ finite-to-one functions such that, whenever two ultrafilters extending $\mathcal{F}$ are nearly coherent, then one of these functions witnesses it.*

Lemma 4. *If two filters, $\mathcal{F}$ and $\mathcal{G}$, are not nearly coherent, then $\mathcal{G}$ cannot be generated by fewer than $\mathfrak{d}(\mathcal{F})$ sets.*

This was stated in [1] under the extra assumption that $\mathcal{G}$ is an ultrafilter; Heike Mildenberger pointed out that the proof didn't use this assumption. In fact, the proof also shows that $\mathcal{G}$ cannot even have a pseudo-base of size smaller than $\mathfrak{d}(\mathcal{F})$.

Lemma 5. *If countably many filters $\mathcal{U}_n$ are each not nearly coherent with a P-point $\mathcal{V}$, then the filter $\bigcap_n \mathcal{U}_n$ is also not nearly coherent with $\mathcal{V}$.*

Lemma 6. *If $\mathfrak{u} < \mathfrak{d}$ and $\mathcal{U}_n$ are countably many, pairwise not nearly coherent ultrafilters, then their closure in $\beta\omega$ contains an infinite closed set (hence of cardinality $2^{\mathfrak{c}}$) of pairwise not nearly coherent ultrafilters.*

This last result holds also if $\mathfrak{u} > \mathfrak{d}$, but we don't know whether it holds for $\mathfrak{u} = \mathfrak{d}$.

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Two Remarks on Determinacy and Inner Models

Stephen Jackson

We mention two results which are at the interface of determinacy and inner model theory. We work throughout in $L(\mathbb{R})$ and assume AD. The first result concerns an initial segment property for inner models of $L(\mathbb{R})$. Specifically, we make the following definition. Let W_1^1 denote the normal measure on ω_1 (in $L(\mathbb{R})$). Let W_1^n denote the n-fold product of the normal measure W_1^1.

Definition 1. Let $M \subseteq L(\mathbb{R})$ be a transitive model of ZFC. We say M has the ω_2 initial segment property if whenever $f : \omega_1 \to \omega_1$ is in M and $\alpha < [f]_{W_1^1}$, then there is a $g\colon \omega_1 \to \omega_1$ in M with $\alpha = [g]_{W_1^1}$.

In other words, M has the initial segment property if the set of ordinals below ω_2 (in $L(\mathbb{R})$) which are represented by functions in M form an initial segment of ω_2.

We note that the set of $\alpha < \omega_2$ which are represented by functions in M has size ω_1, and so is bounded below ω_2. It is not immediately clear, however, under what conditions this set is an initial segment of ω_2.

To put this question in context, and for the second question below, we recall a coding of the ordinals below ω_ω. Let WO_1 denote the standard set of $x \in \omega^\omega$ which code well-orderings of ω. So, WO_1 is $\mathbf{\Pi}^1_1$. For $n > 1$, let WO_n be the set of reals z coding tuples $z = \langle w, x_1, \dots, x_{n-1}\rangle$ where $w \in \mathrm{WO}_1$, and for $1 \leq i \leq n-1$, T_{x_i} is well-founded. Here T denote the Kunen tree, that is, T is a tree on $\omega \times \omega_1$ such that for every $h\colon \omega_1 \to \omega_1$ there is an $x \in \omega^\omega$ such that T_x is well-founded and for all infinite $\alpha < \omega_1$, $h(\alpha) < |T_x \restriction \alpha|$ (here T_x denotes the section of T at x). $z \in \mathrm{WO}_n$ codes the ordinal $|z| < \omega_n$ as follows: $|z| = [g]_{W_1^{n-1}}$, where $g(\alpha_1, \dots, \alpha_{n-1}) = |T_{x_{n-1}} \restriction (\alpha_{n-1})(\gamma_{n-2})| =$ the rank of γ_{n-2} in the well-ordering $T_{x_{n-1}} \restriction \alpha_{n-1}$ (we are identifying T_x with a well-ordering of ω_1 here), and in general, $\gamma_i = |T_{x_i} \restriction \alpha_i(\gamma_{i-1})|$, and finally, $\gamma_0 = |w| < \omega_1$. WO_n is $\mathbf{\Pi}^1_2$ for each n. Let $\mathrm{WO}_\omega = \bigcup_n \mathrm{WO}_n$.

Our first result is:

Theorem 2. *Let $M \subseteq L(\mathbb{R})$ be a transitive inner model M of ZFC which is built via a directed system from an iterable mouse using a definable iteration strategy and satisfying a minimality condition. Then M will satisfy the ω_2 initial segment condition.*

Examples include $M = L$, $M = L[U]$ for U a normal measure on ω_1, $M = M_n$ the canonical system built from a minimal iterable model for n-Woodin cardinals, and $M = \mathrm{HOD}$ (or actually a large initial segment of this model).

As a corollary, we get this result about $L(\mathbb{R})$.

Corollary 3. *If $f : \omega_1 \to \omega_1$ is in HOD and $\alpha < [f]_{W_1^1}$, then there is a $g \in$ HOD with $\alpha = [g]_{W_1^1}$.*

A second result concerns a question asked by Steel concerning the supercompactness measure, and which arose from inner-model theory arguments. Recall that assuming AD, ω_1 is $< \Theta$ supercompact, and in fact by a result of Woodin for all $\lambda < \Theta$ there is a unique supercompactness measure on $\mathcal{P}_{\omega_1}(\lambda)$.

Recall that the Kechris-Martin theorem asserts that Π^1_3 is closed under ordinal quantification of length ω_ω. More precisely, If $P \subseteq \omega^\omega \times \mathrm{WO}_\omega$ is Π^1_3 and invariant in the codes (that is, $P(x, z)$ and $|z'| = |z|$ implies $P(x, z')$), then $Q(x) \Leftrightarrow \exists z \in \mathrm{WO}_\omega\ P(x, z)$ is also Π^1_3.

In a similar manner we can let C be the set of reals coding countable subsets of ω_ω and let $P \subseteq \omega^\omega \times C$ be invariant, and ask if $Q(x) \Leftrightarrow \forall^*_{\nu_\omega} S\ P(x, S)$ is necessarily Π^1_3 ($P(x, S)$ means $P(x, z)$ for any $z \in C$ coding S, and ν_ω denotes the supercompactness measure on $\mathcal{P}_{\omega_1}(\omega_\omega)$. We can also ask the version of this question using the measure ν_n on $\mathcal{P}_{\omega_1}(\omega_n)$.

Our second result is a negative one, namely:

Theorem 4. *$\boldsymbol{\Pi}^1_3$ is not closed under quantification by ν_ω.*

We do not know if $\boldsymbol{\Pi}^1_3$ is closed under quantification by ν_n.

Note that in the case of the Kechris-Martin theorem, there is no essential difference in ordinal quantification over ω_n for each n, and quantification over ω_ω. We suspect the situation is different for the supercompactness measure. Also, a positive result for quantification by ν_n would be a strengthening of the Kechris-Martin theorem.

Actions of Borel groups and topologies on such groups

Sławomir Solecki

1. By a *Borel group* we understand a group with a metric second countable group topology that is a Borel subset of any (equivalently, all) of its metric completions. For a Borel group H, a *Polish H-space* is a Polish space X on which H acts continuously. (A *Polish topology* is a second countable metric complete topology.) In this situation, by E^X_H we denote the equivalence relation on X induced by the partition of X into orbits of the action of H. We call this equivalence relation the *obit equivalence relation.*

General Question. *Let H be a Borel group. Consider all orbit equivalence relations induced by continuous actions of H on Polish spaces. Does complexity* *of these equivalence relations determine whether or not H admits a compact second countable group topology containing the original topology, or a locally compact second countable such topology, or a Polish one?*

Complexity of equivalence relations is to be calibrated here by comparing them, in terms of Borel reducibility, with some canonical equivalence relations. We say that an equivalence relation E on a Polish space X is *Borel reducible* to an equivalence relation F on a Polish space Y, in symbols $E \leq F$, if there exists a Borel function $f : X \to Y$ such that $x_1 E x_2$ if and only if $f(x_1) F f(x_2)$ for any $x_1, x_2 \in X$. One can paraphrase it by saying that there is an injection from X/E to Y/F that lifts to a Borel function from X to Y. The canonical equivalence relations that will be relevant to the problem at hand are E_0 on $2^{\mathbb{N}}$, $E_0^{\mathbb{N}}$ on $(2^{\mathbb{N}})^{\mathbb{N}}$, and E_1 on $(2^{\mathbb{N}})^{\mathbb{N}}$ which are defined as follows:

$$
\begin{aligned}
xE_0y &\iff \exists m \forall n > m\ x(n) = y(n)\\
(x_n)E_0^{\mathbb{N}}(y_n) &\iff \forall n\ x_n E_0 y_n\\
(x_n)E_1(y_n) &\iff \exists m \forall n > m\ x_n = y_n.
\end{aligned}
$$

For more on these equivalence relations see [4].

In what follows, H stands for a Borel group.

2. The compact case of General Question is the simplest one. In fact, if there is a compact second countable group topology on a Borel group H which contains the original topology, then the two topologies are equal. The compact case was solved in [9], where the following theorem was proved.

Theorem 1 ([9])**.** *H is not compact if and only if $E_0 \leq E_H^X$ for some Polish H-space X.*

Strictly speaking the implication from left to right in Theorem 1 was proved in [9] only for Polish groups H. But if H is Borel and not Polish, then an application of [1, Theorem 3.4.5] to the action described in $(*)$ below yields immediately the conclusion of the theorem.

3. The following question, which is essentially due to Kechris, covers the locally compact case.

Question 2. Is it true that H does not have a locally compact second countable group topology including the original topology if and only if $E_0^{\mathbb{N}} \leq E_H^X$ for some Polish H-space X?

Kechris proved in [5] that the implication from right to left holds. Progress on proving the opposite implication was recently made by Thompson in [10]. However, Question 2 remains open.

4. We turn now to the case of Polish topology in General Question. After [6], we call a Borel group H *polishable* if there is a Polish group topology on H that contains the original one. This is equivalent to saying that the Polish group topology has the same Borel structure as the original topology. An important fact is the canonicity of such a Polish group topology: if it exists, then it is unique. For more on polishable groups see [8] and [3]. The following question, which covers the Polish topology case of General Question, is related to some problems raised by Kechris and Louveau in [6].

Question 3. Is it true that H is not polishable if and only if $E_1 \leq E_H^X$ for some Polish H-space X?

The implication from right to left was proved by Kechris and Louveau in [6]. Therefore, to answer Question 3 in the affirmative, one needs to construct for any non-polishable Borel group H a Polish H-space X with $E_1 \leq E_H^X$. There is a natural candidate for such an H-space which is produced as follows. Each metric second countable group has a metric completion that is a Polish group, see e.g. [1, Theorem 1.1.2]. Thus, each Borel group H is a Borel dense subgroup of a Polish group G. We can consider then the natural action of H on G defined by

$$(*) \qquad H \times G \ni (h, g) \to g \cdot h^{-1} \in G$$

whose orbits are left cosets of H in G. We denote the orbit equivalence relation of this action by $E_{G/H}$. One may suspect that if H is Borel and not polishable, then E_1 Borel reduces to the orbit equivalence relation of the form $E_{G/H}$ for some Polish group G. The theorem below, on which I gave a talk during the 2005 Oberwolfach meeting, proves this to be true in two cases, thereby giving affirmative answers to Question 3 in these cases.

Theorem 4. *Let H be a Borel subgroup of a Polish group G.*

(i) *If H is abelian, then either H polishable or $E_1 \leq E_{G/H}$.*
(ii) *If H is the union of an increasing sequence of polishable groups, then either H is polishable or $E_1 \leq E_{G/H}$.*

The above theorem sharpens results from [6], [7], and [2].

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Monadic Definability of Ordinals

Itay Neeman

A formula φ is *monadic second order* (*monadic* for short) if each of its variables is assigned a type, either the type "first order" or the type "second order". In defining the truth value of a formula in a structure $\langle A; \ldots \rangle$ we take the first order variables to range over elements of A, and take the second order variables to range over subsets of A.

Note that monadic formulae do not allow, at least not directly, talking about sets of *pairs* of elements of A. In particular they need not introduce Gödel sentences, and they need not allow defining cardinality.

Let ON be the class of all ordinals. The following are examples of statements about sets of ordinals that can be expressed in the monadic language over the structure $\langle \mathrm{ON}; < \rangle$:

- "α is a limit ordinal".
- "C is unbounded in α".
- "C is closed and unbounded in α".
- "$\mathrm{cof}(\alpha) \geq \omega$", expressed simply by "$\alpha$ is a limit ordinal".
- "$\mathrm{cof}(\alpha) \geq \omega_{n+1}$", expressed by the formula formalizing the statement $(\forall C)[(C$ closed unbounded in $\alpha) \rightarrow (\exists \beta)(\beta \in C \wedge \mathrm{cof}(\beta) \geq \omega_n)]$.
- "$\alpha = \omega_n$" expressed by $(\mathrm{cof}(\alpha) \geq \omega_n) \wedge (\forall \beta < \alpha)(\mathrm{cof}(\beta) \not\geq \omega_n)$.

In particular, for each $n < \omega$, ω_n is definable over $\langle \mathrm{ON}, < \rangle$ through a monadic formula.

By a result of Magidor it is consistent, assuming large cardinals, that $\omega_{\omega+1}$ is definable.

My talk addressed the question of the definability of ω_ω:

Theorem. ω_ω is *not* definable. In fact *no* singular cardinal is definable.

The proof uses finite state automata, acting on infinite strings, to convert monadic formulae over the ordinals to formulae of a specific kind that allows talking about clubs, stationary sets, and reflection, but does not allow quantifying over individual ordinals.

Rainbow Ramsey theory

James Cummings
(joint work with Uri Abraham)

This report presents a part of some joint work with Uri Abraham (Ben-Gurion University, Beersheva) on the subject variously known as "polychromatic Ramsey theory" or more colourfully as "rainbow Ramsey theory". In standard Ramsey theory a common slogan is "Complete disorder is impossible", but in rainbow Ramsey theory we may say "Complete disorder is inevitable".

Historical note: Theorems 1 and 2 below are old remarks by Galvin made in letters to Stevo Todorčević.

We begin with some definitions:

- $P \subseteq X$ is *polychromatic* for $f : [X]^n \to C$ iff $f \restriction [P]^n$ is 1-1.
- $f : [X]^n \to C$ is *κ-bounded* iff $|f^{-1}[\{c\}]| \leq \kappa$ for all $c \in C$, similarly $< \kappa$-bounded has the obvious meaning.
- Notation: $\lambda \to^* (\alpha)^n_{\kappa-\mathrm{bdd}}$ iff for every κ-bounded colouring of $[\lambda]^n$ there is a polychromatic set of order type α.

There is a painless way to obtain positive results in rainbow Ramsey theory.

Theorem 1 (Galvin). *Let $k < \omega$. If $\lambda \to (\alpha)^n_k$ then $\lambda \to^* (\alpha)^n_{k-\mathrm{bdd}}$.*

Proof. Let $f : [X]^n \to C$ be k-bounded. Fix a linear ordering of $[X]^n$, define $f^* : [X]^n \to k$ by setting $f^*(a) = i$ for the unique $i < k$ such that a is the i^{th} n-tuple of colour $f(a)$. It is easy to see that any monochromatic set for f^* is polychromatic for f. □

Dualising some popular Ramsey theorems we get

(1) (Finite Ramsey) For all $k, l, m \in \omega$ there is $n \in \omega$ such that $n \to^* (l)^k_{m-\mathrm{bdd}}$
(2) (Infinite Ramsey) $\omega \to^* (\omega)^k_{m-\mathrm{bdd}}$ for all $k, m \in \omega$.
(3) (Baumgartner-Hajnal) $\omega_1 \to^* (\alpha)^2_{2-\mathrm{bdd}}$ for all countable α.

What about $\omega_1 \to (\omega_1)^2_{2-\mathrm{bdd}}$? Dualising standard Ramsey theorems no longer helps because of Todorčević's result that $\omega_1 \not\to [\omega_1]^2_2$. On the other hand there is some evidence that it is easier to build polychromatic sets than it is to build monochromatic ones: for example in the finite theory it is a result of Rödl that the least n such that $n \to^* (l)^k_{m-\mathrm{bdd}}$ grows only polynomially as a function of l.

Theorem 2 (Galvin). *CH implies $\omega_1 \not\to^* (\omega_1)^2_{2-\mathrm{bdd}}$.*

Proof. Enumerate subsets of ω_1 with order type ω as X_α for $\alpha < \omega_1$. Define $f \restriction [\beta]^2$ by induction on β. If $\beta = \alpha + 1$ then enumerate the sets X_η such that $\eta < \alpha$ and $X_\eta \subseteq \alpha$ as Y_n for $n < \omega$, then run through $n \in \omega$ arranging that there are distinct $\beta, \beta' \in Y_n$ with $f(\beta, \alpha) = f(\beta', \alpha)$.

Suppose for contradiction that $Z \in [\omega_1]^{\omega_1}$ is polychromatic. Let $X = X_\eta$ be the first ω elements and find $\alpha \in Z$ so large that $X_\eta \subseteq \alpha$ and $\eta < \alpha$. Contradiction! □

Mirna Dzamonja pointed out that the weak guessing principle "stick", which is consistent with large continuum, would suffice for Theorem 2. Answering a question by Galvin, Todorčević showed

Theorem 3 (Todorčević). *It is consistent that $\omega_1 \to^* (\omega_1)^2_{2-\mathrm{bdd}}$.*

The proof actually shows a stronger fact to be consistent: for any $< \omega$-bounded coloring of $[\omega_1]^2$, ω_1 is the union of countably many polychromatic sets.

Sketch. We sketch of a proof (of the stronger statement) from PFA. Let f be a $< \omega$-bounded colouring of $[\omega_1]^2$. Breaking ω_1 into ω fast growing pieces we may assume that two pairs of the same colour have the same maximum point. Now force with conditions $(p, \vec{M})$ where

(1) p is a finite partial function from ω_1 to ω with $p^{-1}[\{i\}]$ polychromatic for each i.
(2) $\vec{M}$ is a finite $\in$-chain of countable submodels of H_{ω_2}.
(3) p is 1-1 on $(M_{i+1} \setminus M_i) \cap \omega_1$ for each i.

We argue it's proper and apply PFA. □

It is natural to ask whether a weaker forcing axiom would suffice.

Theorem 4 (Abraham and C.)**.** *MA* $+\ \omega_1 \not\to^* (\omega_1)^2_{2-\mathrm{bdd}}$ *is consistent.*

Sketch. The argument is in two phases. In phase one we add a 2-bounded colouring which has no uncountable polychromatic set in any ccc extension, and in phase two we force MA in the usual way. More details of Phase One: add by finite support iteration a 2-bounded colouring $c : [\omega_1]^2 \to \omega_1$ such that

(1) Two pairs with same colour have same maximum.
(2) For every uncountable $A \subseteq \omega_1$, every $f : A \times \omega_1 \to A$ such that $\forall \alpha \in A\ \forall \beta \in \omega_1\ f(\alpha, \beta) \geq \beta$, and every club $C \subseteq \omega_1$ there exists $\langle \alpha_i, \beta_i, \gamma_i \rangle$ such that
  (a) For all i, $\alpha_i \in A < \beta_i \in C \leq \gamma_i = f(\alpha_i, \beta_i)$.
  (b) For all $i < j$, $\gamma_i < \alpha_j$.
  (c) Either $c(\alpha_i, \alpha_j) = c(\gamma_i, \alpha_j)$ or $c(\alpha_i, \gamma_j) = c(\gamma_i, \gamma_j)$.

To show that this is preserved by ccc forcing, use the easy fact that if $\mathbb{P}$ is ccc and $\langle p_i : i \in \omega_1 \rangle$ are conditions in $\mathbb{P}$ then some $p \in \mathbb{P}$ forces that $p_i \in G$ for unboundedly many i.

To show it implies that c has no uncountable polychromatic set, let A be uncountable,let $f(\alpha, \beta) = \min(A \setminus \beta)$ and use clause 2c.

Remark: The proof is a descendant of an argument by Abraham and Todorčević that MA is consistent with the existence of an S-space. □

Now we go up one cardinal and look at 2-bounded colourings of $[\aleph_2]^2$. From Theorem 3 it is clear that under PFA there will be many polychromatic subsets of type ω_1 which are stationary in their supremum. We consider a stronger assertion with the same flavour.
$(*)$: "For every 2-bounded colouring of $[\omega_2]^2$ there is a **closed** polychromatic set of order type ω_1".

Theorem 5 (Abraham and C.)**.** $(*)$ *follows from MM*

Sketch. Let f be a 2-bounded colouring of $[\omega]^2$. Consider a two step iteration where the first step adds a generic continuous cofinal map $G : \omega_1 \to \omega_2$, and the second step adds (with finite conditions of some kind) a club in ω_1 which is polychromatic for $f \circ G$. Second forcing resembles Baumgartner's finite club forcing and also the poset for Theorem 3. Argue the iteration is stationary preserving and apply MM. □

Theorem 6 (Abraham and C.)**.** $(*)$ *is independent of PFA.*

Sketch. We introduce a combinatorial principle with the same flavour as square, which has been crafted to be strong enough to kill $(*)$ yet weak enough to be consistent with PFA.

Principle P: there is a function g with domain $\omega_2 \cap cof(\omega)$ such that

- For all γ there exist α, β such that $\alpha < \beta < \gamma$ and $g(\gamma) = \{\alpha, \beta\}$.
- For every $\delta \in \omega_2 \cap cof(\omega_1)$ and every $\alpha < \beta < \delta$, $g(\gamma) = \{\alpha, \beta\}$ for a stationary set of $\gamma \in \delta \cap cof(\omega)$.

It is not hard to see that P can be used to build a colouring such that if $g(\gamma) = \{\alpha, \gamma\}$ then the pairs $\{\alpha, \beta\}$ and $\{\beta, \gamma\}$ get the same colour. This is easily seen to contradict $(*)$.

Remarks: The principle P implies that for any $\alpha < \beta < \omega_2$, $g(\gamma) = \{\alpha, \beta\}$ for stationarily many $\gamma \in \omega_2 \cap cof(\omega)$. Recalling that MM implies every stationary set contains a closed copy of ω_1, MM is incompatible with principle P.

Now let $\mathbb{P}$ be the natural poset to add a witness to Principle P, we will show that forcing over a model of PFA with $\mathbb{P}$ preserves PFA.

Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a proper poset and $\dot{D}_i$ for $i < \omega_1$ be names for dense sets. Applying PFA produces a filter on $\mathbb{P} * \dot{\mathbb{Q}}$ containing for each i a pair $(p_i, \dot{q}_i)$ such that $p_i \Vdash \dot{q}_i \in \dot{D}_i$ BUT the p_i may have no lower bound (which is what we need to get a condition forcing the existence of a suitably generic filter).

Solution: design $\dot{\mathbb{R}} \in V^{\mathbb{P} * \dot{\mathbb{Q}}}$ so that $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is proper and $\mathbb{R}$ adds a suitable lower bound for the $\mathbb{P}$-generic. The proof has a family resemblance to Beaudoin's argument for the consistency of PFA and a non-reflecting stationary set. $\square$

We finish with some directions for future research which we proposed during our talk, and some comments made by conference participants.

- Can we find a rainbow version of the Galvin-Prikry theorem?
  Steve Jackson: The dual of a 2-bounded Borel colouring is Borel so the dual colouring trick applies.
  Otmar Spinas, Stevo Todorčević: The canonisation theorem of Promel and Voigt can be used to derive this result also.
- Exponent 3 at $\aleph_1$?
  Stevo Todočević: Galvin discussed questions of this kind.
- ZFC results at $\aleph_2$?
- Graph Ramsey theory?
- Large cardinals?

Jindřich Zapletal: Look for canonical Ramsey theorems that will have the rainbow theorems as special cases.

Lajos Soukup: Theorem 4 can be derived from some results of his on MA and topology, also descending from the work of Abraham and Todorčević.

Borel ideals and isomorphism of quotient Boolean algebras

Su Gao
(joint work with Michael R. Oliver)

Farah [1] asked the question: How many Boolean algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$, where $\mathcal{I}$ is a Borel ideal, are there up to isomorphism? In [2] Oliver proved that there are $2^{\aleph_0}$ many Borel ideals with pairwise non-isomorphic quotient Boolean algebras. Two noticeable features of this result are (1) it is provable within ZFC and (2) in fact all these ideals can be analytic P-ideals, and thus $\mathbf{\Pi}^0_3$, i.e., the ideals considered are of simple descriptive complexity.

We consider the new question: Which equivalence relations can be reducible to the isomorphism relation of quotient Boolean algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$, $\mathcal{I}$ a Borel ideal? The rigorous meaning of the reducibility is as follows: for an equivalence relation E on a standard Borel space X, we need a Borel function

$$\theta : X \to \omega^\omega$$

such that each $\theta(x)$ is a Borel code for a Borel ideal on ω, and such that

$$xEy \iff \mathcal{P}(\omega)/B_{\theta(x)} \cong \mathcal{P}(\omega)/B_{\theta(y)}.$$

In fact Oliver in [2] did show that the answer is yes for E_0 as well as for $\mathrm{id}(\omega_1)$. However, noticing that the descriptive complexity of the isomorphism relation on the right is enormous compared to either E_0 or $\mathrm{id}(\omega_1)$, we speculate that the results should be extendable to many more equivalence relations.

The main theorem of this report is the following:

Main Theorem *There is an assignment, which is Borel in the codes, of a Borel ideal $\mathcal{I}_A$ for each Borel set A of real numbers such that*

$$A = B \iff \mathcal{P}(\omega)/\mathcal{I}_A \cong \mathcal{P}(\omega)/\mathcal{I}_B.$$

Corollary *Any $\mathbf{\Sigma}^1_1$ equivalence relation is Borel reducible to the isomorphism relation of quotient Boolean algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$, where $\mathcal{I}$ is a Borel ideal.*

These results are again provable in ZFC. However, the Borel ideals occurring in the proof are of unbounded Borel complexity, in contrast to the ideals used in the proof of [2]. An open question remains whether it is possible to give the same result with analytic P-ideals.

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Correctness, definability, and self-iterabilty for ω-small mice

John Steel
(joint work with Mitch Rudominer and Ralf Schindler)

We shall show that if $\mathcal{M}$ is a fully iterable ω-small mouse such that $\mathbb{R} \cap L(\mathbb{R} \cap \mathcal{M}) \subseteq \mathcal{M}$, then $L(\mathbb{R} \cap \mathcal{M}) \models$ "there is a wellorder of $\mathbb{R}$". In fact:

Theorem 1. *Assume that $M_\omega^\sharp$ exists and is fully iterable. Let $\mathcal{M}$ be a fully iterable, ω-small mouse such that $\mathbb{R} \cap L(\mathbb{R} \cap \mathcal{M}) \subseteq \mathcal{M}$; then there is a β such that*

(a) *letting θ be the supremum of the lengths of prewellorders of $\mathbb{R}^M$ in $J_\beta(\mathbb{R}^M)$, we have that for some γ, $J_\theta(\mathbb{R})^{\mathcal{M}} \prec_1 J_\gamma(\mathbb{R})$;*
(b) *$J_\beta(\mathbb{R})^{\mathcal{M}} \models \mathsf{AD}$, and*
(c) *$J_{\beta+1}(\mathbb{R})^{\mathcal{M}} \models$ "there is a wellorder of $\mathbb{R}$".*

Part (b) follows easily from (a).

As a corollary to the proof, one gets

Theorem 2. *Assume that $M_\omega^\sharp$ exists and is fully iterable. Let $\mathcal{M}$ be a fully iterable, ω-small, proper class mouse. Then for some α, $\mathcal{M}$ knows how to iterate itself for set-sized trees with all critical points $> \alpha$.*

This shows that any such $\mathcal{M}$ satisfies "I am K over $\mathcal{M}|\alpha$", for some α. We believe that in fact, one can show any such $\mathcal{M}$ breaks into finitely many intervals $[\alpha, \beta]$ such that $\mathcal{M}$ knows how to iterate itself for iteration trees in $\mathcal{M}|\beta$ with all critical points $> \alpha$, and thus $\mathcal{M}|\beta$ satisfies "I am K over $\mathcal{M}|\alpha$".

The fact that $V = K$ holds in $\mathcal{M}$ (in the sense hinted at above) would let one extend proofs involving covering arguments to $\mathcal{M}$. For example, let $\diamond^*_{\kappa,\lambda}$ be the assertion: There is a function F with domain $P_\kappa(H_\lambda)$ such that for all $X \in P_\kappa(H_\lambda)$, $|F(X)| \le |X|$, and for all $A \subseteq H_\lambda$, there are club many $X \in P_\kappa(\lambda)$ such that $A \cap X \in F(X)$. Kanamori showed L satisfies $\forall \kappa, \lambda \diamond^*_{\kappa,\lambda}$. Granted that $\mathcal{M}$ satisfies $V = K$ in the sense hinted at above, one can apply the proof of covering for larger core models inside an $\mathcal{M}$ as above, so as to show $\mathcal{M}$ satisfies $\forall \kappa, \lambda \diamond^*_{\kappa,\lambda}$.

Finally, we believe these results generalize to tame mice, replacing $L(\mathbb{R})$ by $K(\mathbb{R})$.

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The Ground Axiom

Joel David Hamkins

Many interesting models of set theory are not obtainable by nontrivial forcing over an inner model. This includes, for example, the constructible universe L, the canonical model $L[\mu]$ of a measurable cardinal and many instances of the core model K (although Schindler has observed that the least inner model M_1 of a Woodin cardinal actually is a nontrivial forcing extension of an inner model). To hightlight this phenomenon, my student Jonas Reitz and I introduced the Ground Axiom, which asserts that the universe is not a set forcing extension of any proper inner model.

Ground Axiom (H, Reitz)**.** *The universe is not a forcing extension of any inner model by nontrivial set forcing. Specifically, if $W \subsetneq V$ is a transitive inner model of* ZFC *and $G \subseteq \mathbb{P} \in W$ is W-generic, then $V \neq W[G]$.*

Despite the *prima facie* second order nature of this assertion, the Ground Axiom is actually first order expressible in the language of set theory.

Theorem 1 (Reitz, Woodin)**.** *The Ground Axiom is first order expressible in the language of* ZFC*.*

This theorem is the starting point of Reitz's dissertation [Rei], but an essentially equivalent assertion was observed independently by Woodin [Woo]. Reitz's proof makes use of ideas arising in Laver's [Lav] recent result that a ground model is always definable in its forcing extensions.

Theorem 2 (Laver)**.** *If $V \subseteq V[G]$ is a set forcing extension, then V is a definable class in $V[G]$, using parameters in V.*

This result was also observed independently by Woodin [Woo]. Laver's proof is connected with my recent theorem showing the extent to which embeddings in a forcing extension must be lifts of ground model embeddings.

Key Definition 3.

(1) $V \subseteq V[G]$ exhibits *δ-covering* if every set of ordinals in $V[G]$ of size less than δ is covered by a set of size less than δ in V.
(2) $V \subseteq V[G]$ exhibits *δ-approximation* if whenever $A \in V[G]$, $A \subseteq V$ and $A \cap a \in V$ for all $a \in V$ with $|a|^V < \delta$, then $A \in V$.

Such forcing extensions are abundant in the large cardinal literature. Any forcing notion of size less than δ has δ-approximation and δ-covering. More generally, any forcing of the form $\mathbb{P} * \dot{\mathbb{Q}}$, where $\mathbb{P}$ is nontrivial, $|\mathbb{P}| < \delta$ and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is

$<\delta$-strategically closed, exhibits δ-approximation and δ-covering. Therefore, such forcing as the Laver preparation or the canonical forcing of the GCH exhibit approximation and covering for many values of δ.

A special case of the main theorem of [Ham03] is:

Theorem 4. *If $V \subseteq V[G]$ exhibits δ-approximation and δ-covering, then every ultrapower embedding $j : V[G] \to M[j(G)]$ above δ in $V[G]$ is the lift of an embedding $j \restriction V : V \to M$ definable in V.*

In particular, $M \subseteq V$ and $j \restriction A \in V$ for all $A \in V$. The full theorem applies to all sufficiently closed embeddings, including many types of extender embeddings. The general conclusion is that extensions with δ-approximation and δ-covering have no new large cardinals above δ. The proofs of Theorems 2 and 4 make similar and extensive iterated use of the approximation and cover properties in their arguments that the respective classes are definable.

Returning to the Ground Axiom, one observes that the natural models of GA, such as L and $L[\mu]$, exhibit the GCH and many other regularity features. Are these a consequence of the Ground Axiom? The answer is no.

Theorem 5 (Reitz)**.** *If ZFC is consistent, then $\mathsf{ZFC} + \mathsf{GA} + \neg\mathsf{CH}$ is consistent.*

The method is flexible and shows that if σ is any Σ_2 assertion consistent with ZFC, then $\mathsf{ZFC} + \mathsf{GA} + \sigma$ is consistent. These theorems are proved by forcing, which is a bit paradoxical as GA asserts that the universe is not a forcing extension. Specifically, resolving the paradox, they are proved by *class* forcing. Using McAloon's [McA71] methods to force strong versions of $V = \mathsf{HOD}$, one codes the universe into the continuum function, and then GA holds with any desired V_α left intact. The hypothesis $V = \mathsf{HOD}$, however, by itself does not imply GA. Conversely, at the Set Theory Workshop at the Mathematische Forshungsinstitut Oberwolfach (0549, December 4-10, 2005), Woodin suggested a very promising line of argument to show that the Ground Axiom is consistent with $V \neq \mathsf{HOD}$, which is now being investigated. Reitz has proved, using large cardinal indestructibility results, that the Ground Axiom is consistent with nearly any kind of large cardinal, from measurable to strong to supercompact and beyond.

Theorem 6. *If the existence of a supercompact cardinal is consistent with ZFC, then it is consistent with $\mathsf{ZFC} + \mathsf{GCH} + \mathsf{GA}$.*

These theorems fit very well into the long-standing set theoretic program, advanced by Woodin and others, to obtain the features of the canonical inner models of large cardinals, but to obtain them by forcing over arbitrary models of those large cardinals. The Ground Axiom is such a feature.

Ordinarily, one imagines forcing as a way to reach out into larger mathematical universes. Here, however, we are reaching from a given universe down into the possible ground models of which it is a forcing extension. Given a model of set theory, perhaps we can strip away a top layer of forcing and be left with a ground model, a bedrock model if you will, that is not itself obtainable by forcing from any smaller inner model. In this case, the original universe satisfies:

Bedrock Axiom. *The universe V is a set forcing extension $V = W[G]$ of an inner model W of* $\mathsf{ZFC} + \mathsf{GA}$.

The model W is a bedrock model for V in the sense that it is a minimal ground model for V, having no ground model below it. This axiom is first order expressible for the same reasons that the Ground Axiom was. Since $V = W$ is allowed, we have $\mathsf{GA} \implies \mathsf{BA}$. A common feature of the models of GA and their forcing extensions, of course, is that they are all forcing extensions of a model of GA, and hence themselves models of BA. Are there any other models? Yes.

Theorem 7 (Reitz)**.** *If* ZFC *is consistent, then* $\mathsf{ZFC} + \neg\mathsf{BA}$ *is consistent. Indeed, if σ is any Σ_2 assertion consistent with* ZFC*, then* $\mathsf{ZFC} + \mathsf{BA} + \sigma$ *is consistent.*

Perhaps the main open question here is:

Question 8. Is the bedrock model unique when it exists?

Several attacks on this question were suggested by various participants at the Oberwolfach workshop, and a promising investigation has now ensued.

The theme of current work is to investigate the spectrum of possible ground models of the universe, the spectrum of inner models W of which the universe V is a forcing extension $V = W[G]$. The results above provide a uniform definition for these ground models W in V. By varying the parameters in this definition, one obtains in effect a class enumeration of the possible ground models W for V. That is, the class I of parameters p giving rise to a ground model W_p such that V is a forcing extension $V = W_p[G_p]$ is definable, and the corresponding meta-class $\{ W_p \mid p \in I \}$ of possible ground models is in effect definable as $\{ \langle p, x\rangle \mid x \in W_p \ \& \ p \in I \}$. Thus, the treatment of the spectrum of possible ground models is entirely a first order affair of ZFC. Another theme is to restrict attention to a particular class of forcing notions, with such axioms as $\mathsf{GA}_{\mathrm{CCC}}$, which asserts that the universe is not a nontrivial forcing extension of an inner model by c.c.c. forcing. We can produce models, for example, of $\neg\mathsf{GA} + \mathsf{GA}_{\mathrm{CCC}} + \sigma$, for any consistent Σ_2 assertion σ; these models are forcing extensions of an inner model, but are not obtainable by c.c.c. forcing. Similar questions and results abound here.

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Coherent Sequences and Threads

Ernest Schimmerling

It was known that the failure of $\square(\aleph_2)$ is equiconsistent with the existence of a weakly compact cardinal. I proved that if both $\square(\aleph_2)$ and $\square_{\aleph_2}$ fail, then there is an inner model with a proper class of strong cardinals. And I proved that if $2^{\aleph_1} = \aleph_2$ and both $\square(\aleph_2)$ and $\square_{\aleph_2}$ fail, then for all $n < \omega$, there exists an inner model with n Woodin cardinals. These results generalize to cardinals $\geq \aleph_2$. A corollary to this work and earlier theorems of others is that the Proper Forcing Axiom for posets of cardinality $\mathfrak{c}^+$ implies Projective Determinacy.

Distributivity numbers of $\mathcal{P}(\omega)/\mathrm{fin}$ and its friends

Jörg Brendle

Let $\mathbb{P}$ be a separative partial order. The *distributivity number* (or *height*) of $\mathbb{P}$, $\mathfrak{h}(\mathbb{P})$, is the least size of a family $\mathcal{D}$ of open dense subsets of $\mathbb{P}$ such that $\bigcap \mathcal{D}$ is not dense. Equivalently, $\mathfrak{h}(\mathbb{P})$ is the least size of a family $\mathcal{A}$ of maximal antichains of $\mathbb{P}$ which has no common refinement. From the forcing-theoretic point of view, $\mathfrak{h}(\mathbb{P})$ is the minimal cardinal κ such that there are $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{f}$ for a function from κ to the ground model V such that $p \Vdash_{\mathbb{P}} \dot{f} \notin V$. Clearly, $\mathfrak{h}(\mathbb{P})$ is an invariant of $\mathbb{P}$ as a forcing notion, that is, it does not depend on the particular realization of $\mathbb{P}$.

If $\mathbb{P}$ is *homogeneous*, that is, if $\mathbb{P}_p := \{q \in \mathbb{P} : q \leq p\}$ is forcing equivalent with $\mathbb{P}$ for all $p \in \mathbb{P}$, then $\mathfrak{h}(\mathbb{P})$ is the least size of a family $\mathcal{D}$ of open dense subsets of $\mathbb{P}$ with $\bigcap \mathcal{D} = \emptyset$. Equivalently, $\mathfrak{h}(\mathbb{P})$ is the least κ such that $\Vdash_{\mathbb{P}} \dot{f} \notin V$ for some $\mathbb{P}$-name $\dot{f} : \kappa \to V$.

$\mathfrak{h}(\mathbb{P})$ is easily seen to be a regular cardinal. Also, $\mathbb{P} \lessdot \mathbb{Q}$ implies $\mathfrak{h}(\mathbb{P}) \geq \mathfrak{h}(\mathbb{Q})$ where we write $\mathbb{P} \lessdot \mathbb{Q}$ if there is a *complete embedding* from $\mathbb{P}$ into $\mathbb{Q}$.

For a Boolean algebra $\mathbb{A}$, let $\mathbb{A}^\omega/\mathrm{fin} := \{[f] : f \in \mathbb{A}^\omega\}$ where $[f] = \{g \in \mathbb{A}^\omega : \forall^\infty n\ (f(n) = g(n))\}$, ordered by $[f] \leq [g]$ if $f(n) \leq g(n)$ holds for almost all n. The *reduced power* $\mathbb{A}^\omega/\mathrm{fin}$ is again a Boolean algebra. If $\mathbb{A} \lessdot \mathbb{B}$ then $\mathbb{A}^\omega/\mathrm{fin} \lessdot \mathbb{B}^\omega/\mathrm{fin}$ and thus $\mathfrak{h}(\mathbb{A}^\omega/\mathrm{fin}) \geq \mathfrak{h}(\mathbb{B}^\omega/\mathrm{fin})$. If $\mathbb{A}$ is the trivial algebra $\{\mathbf{0}, \mathbf{1}\}$, we see $\mathbb{A}^\omega/\mathrm{fin} \cong \mathcal{P}(\omega)/\mathrm{fin}$. In particular $\mathfrak{h}(\mathbb{B}^\omega/\mathrm{fin}) \leq \mathfrak{h}$ for any Boolean algebra $\mathbb{B}$ where $\mathfrak{h} := \mathfrak{h}(\mathcal{P}(\omega)/\mathrm{fin})$. Let $\mathfrak{h}_2 := \mathfrak{h}(\mathcal{P}(\omega)/\mathrm{fin} \times \mathcal{P}(\omega)/\mathrm{fin})$. Clearly $\mathfrak{h}_2 \leq \mathfrak{h}$.

Shelah and Spinas [SS1] proved the consistency of $\mathfrak{h}_2 < \mathfrak{h}$. In fact, they showed $\mathfrak{h}_2 < \mathfrak{h}$ holds in the iterated Mathias model (the ω_2-stage countable support iteration of Mathias forcing over a model of CH). Similarly, Dow [Do] obtained the consistency of $\mathfrak{h}(\mathbb{C}^\omega/\mathrm{fin}) < \mathfrak{h}$ in the iterated Mathias model. Here $\mathbb{C}$ denotes the Cohen algebra, that is, the forcing for adding one Cohen real. He asked whether $\mathfrak{h}(\mathbb{C}^\omega/\mathrm{fin} \times \mathbb{C}^\omega/\mathrm{fin}) = \mathfrak{h}(\mathbb{C}^\omega/\mathrm{fin})$. Balcar and Hrušák [BH] proved $\mathfrak{h}(\mathbb{C}^\omega/\mathrm{fin}) \leq \mathsf{add}(\mathcal{M})$ and thus obtained Dow's Theorem as a corollary. Namely, it is well-known (and much easier to prove than Dow's argument for

$\mathfrak{h}(\mathbb{C}^\omega/\mathrm{fin}) = \aleph_1$) that $\mathsf{add}(\mathcal{M}) = \aleph_1$ in the iterated Mathias model. They asked whether $\mathfrak{h}(\mathbb{C}^\omega/\mathrm{fin}) < \min\{\mathfrak{h}, \mathsf{add}(\mathcal{M})\}$ is consistent.

Both questions can be answered with basically the same method.

Theorem 1. [Br3] *CON*$(\mathfrak{h}(\mathbb{C}^\omega/\mathrm{fin}) < \min\{\mathfrak{h}, \mathsf{add}(\mathcal{M})\})$.

Theorem 2. [Br3] *CON*$(\mathfrak{h}_2 < \mathfrak{h}(\mathbb{C}^\omega/\mathrm{fin}))$.
Since $\mathfrak{h}(\mathbb{C}^\omega/\mathrm{fin} \times \mathbb{C}^\omega/\mathrm{fin}) \leq \mathfrak{h}_2$ *in ZFC, the consistency of* $\mathfrak{h}(\mathbb{C}^\omega/\mathrm{fin} \times \mathbb{C}^\omega/\mathrm{fin}) < \mathfrak{h}(\mathbb{C}^\omega/\mathrm{fin})$ *follows.*

Notice that the converse, namely, the consistency of $\mathfrak{h}_2 > \mathfrak{h}(\mathbb{C}^\omega/\mathrm{fin})$, follows from the consistency of $\mathfrak{h}_2 > \mathsf{add}(\mathcal{M})$ established by Shelah and Spinas [SS2] and from the Balcar-Hrušák Theorem.

Unlike earlier results on the independence of distributivity numbers, our results are obtained by finite support iteration of ccc forcing.

We briefly discuss the distributivity number of other structures related to $\mathcal{P}(\omega)/\mathrm{fin}$.

Let Dense($\mathbb{Q}$) denote the family of dense subsets of the rationals $\mathbb{Q}$, let nwd stand for the nowhere dense sets of rationals, and consider the quotient structure Dense($\mathbb{Q}$)/nwd. Let $\mathfrak{h}_\mathbb{Q} = \mathfrak{h}(\mathrm{Dense}(\mathbb{Q})/\mathrm{nwd})$. Balcar, Hernández and Hrušák [BHH] proved $\mathfrak{h}_\mathbb{Q} \leq \mathsf{add}(\mathcal{M})$ and thus obtained the consistency of $\mathfrak{h}_\mathbb{Q} < \mathfrak{h}$ in the iterated Mathias model. On the other hand, we have

Theorem 3. [Br2] *CON*$(\mathfrak{h} < \mathfrak{h}_\mathbb{Q})$.

Let $(\omega)^\omega$ denote the collection of *infinite partitions* of ω (i.e. the partitions into infinitely many blocks). For $A, B \in (\omega)^\omega$, write $A \leq B$ if A is *coarser* than B iff all blocks of A are unions of blocks of B. Say X is a *finite coarsening* of A if X is gotten from A by merging finitely many blocks of A. Write $A \leq^* B$ if there is a finite coarsening X of A such that $X \leq B$. Consider the *dual structure* $((\omega)^\omega, \leq^*)$ and let $\mathfrak{h}_d = \mathfrak{h}((\omega)^\omega, \leq^*)$. It is easy to see that $\mathcal{P}(\omega)/\mathrm{fin} {<\!\circ}\, ((\omega)^\omega, \leq^*)$, and thus $\mathfrak{h}_d \leq \mathfrak{h}$. Halbeisen [Ha] observed that $\mathfrak{h}_d > \aleph_1$ is consistent (namely, $\mathfrak{h}_d = \aleph_2$ holds in the iterated dual Mathias model), and Spinas [Sp] proved the consistency of $\mathfrak{h}_d < \mathfrak{h}$ in the iterated Mathias model. Furthermore

Theorem 4. [Br1] *CON*$(\mathfrak{h}_d = \aleph_1 + MA + \neg CH)$.

Note that this gives an alternative proof of the consistency of $\mathfrak{h}_d < \mathfrak{h}$.

The General Philosophy behind the results obtained so far is that distributivity numbers are independent unless there is an order relationship for trivial reasons, namely, unless there is a complete embedding between the partial orderings. Indeed, in all cases investigated so far, either $\mathbb{P} {<\!\circ}\, \mathbb{Q}$ or *CON*$(\mathfrak{h}(\mathbb{P}) < \mathfrak{h}(\mathbb{Q}))$ has been established.

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Countable quotients of combinatorial structures

Alain Louveau

In this talk, I considered the following problem:

Let $\mathcal{L}$ be a finite relational language, and $\mathcal{C}$ a class of $\mathcal{L}$-structures. Given a structure $\mathcal{X}$ in $\mathcal{C}$, when is there a countable quotient of $\mathcal{X}$ which is still in $\mathcal{C}$?

Here by a quotient of $\mathcal{X}$, I mean the following: If E is an equivalence on the domain X of $\mathcal{X}$, the quotient structure $\mathcal{X}/E$ has domain the quotient X/E, and relations the images of the relations of $\mathcal{X}$ by the quotient map. So I assume in this definition no particular properties of E wrt $\mathcal{X}$.

I got interested in the above problem via considerations in descriptive set theory. During the last two decades, a lot of information has been obtained on various classes of analytic structures, i.e. structures as above but with Polish domain and analytic relations on it. Among the first steps in studying these classes, dichotomy results usually play an important role. And this is particularly true with the "Cantor-type" dichotomies of the form "countable versus perfect", like

- the classical Suslin-Luzin perfect set theorem (for sets),
- the Silver dichotomy for coanalytic equivalence relations ([S]),
- the Harrington-Marker-Shelah dichotomy for Borel quasi-orderings ([HMS]),
- the Kechris-Solecki-Todorčević dichotomy for analytic graphs ([KST]).

Trying to find a common frame for all these results, I noticed some time ago that the "countable" side of the dichotomies corresponds in all cases to the existence of a Borel countable quotient (i.e. a countable quotient by a Borel equivalence relation) for an associated analytic structure (which may not be the natural one: e.g. for Silver's result, one has to consider the complement of the equivalence relation). Using this remark, I was able to prove similar dichotomies for more and more classes of analytic structures. But to understand how far one can go in this direction, the first natural step is to forget about Borelness, and study the purely set-theoretic problem above.

In the talk, I considered the case of classes defined by a finite set of forbidden finite configurations. To simplify the exposition and the notations, let us assume that the language has only one binary predicate. If $\mathcal{F}$ is a finite set of finite structures of form $\mathcal{D} = (D, H)$ with $H \subseteq D^2$, say that $\mathcal{X} = (X, R)$ is $\mathcal{F}$-free if for every $\mathcal{D}$ in $\mathcal{F}$ there is no morphism from $\mathcal{D}$ to $\mathcal{X}$, i.e. no map $h : D \to X$ sending H into R (note that we do not impose injectivity on the map h, so that the notion of freeness is stronger than just asking that there is no copy of $\mathcal{D}$ as a subgraph of $\mathcal{X}$).

The main result of the talk concerned the classes of $\mathcal{F}$-free structures, when $\mathcal{F}$ consists of connected finite structures. I showed that it is possible in that case to associate to such a family $\mathcal{F}$ another finite set $\mathcal{F}'$ of connected finite structures in an expanded language (by one binary predicate interpreted as an equivalence relation), in such a way that for an $\mathcal{F}$-free structure $\mathcal{X} = (X, R)$, the following are equivalent:

(1) $\mathcal{X}$ admits a countable $\mathcal{F}$-free quotient
(2) For any (D, H, E) in $\mathcal{F}'$, there is a countable coloring c of X such that for any morphism $h : (D, H) \to (X, R)$, the composition $c(h)$ is not constant on at least one E-class.

The exact definition of $\mathcal{F}'$ is a bit too technical to be given here. The proof of the above equivalence also gives a finite version (replacing countable by finite in both (1) and (2)), and also an effective version, assuming X is Polish, replacing countable quotient by countable Borel quotient in (1), and countable coloring by countable Borel coloring in (2). The strength of the result lies in the direction (2) implies (1), and comes from the fact that it is enough to check (2) for a finite set of connected structures.

In the second part of the talk, I gave a few applications of the main result (and its variants).

First I showed how it allows to give a quite different proof of the following result of Cherlin, Shelah and Shi (([CSS]): If $\mathcal{F}$ is a finite set of finite connected graphs, there is a universal countable $\mathcal{F}$-free graph, i.e. such that any other countable $\mathcal{F}$-free graph is isomorphic to an induced subgraph of it. The original proof in [CSS] is model-theoretic, and there has also been a combinatorial proof "from the inside" by Nesetril, using amalgamation of finite structures. The previous result allows a construction of the universal object "from the outside", as a quotient structure.

As a second application, a very similar technique allows to reprove (and generalize) results of Komarek and Nesetril-Tardif (see [NT]) on the existence of a finite $\mathcal{F}$-free graph which is maximum in the sense that any $\mathcal{F}$-free graph admits a morphism into it, in case $\mathcal{F}$ consists of oriented trees.

Finally, going back to the original motivation, I showed that in the case when $\mathcal{F}$ consists of connected graphs with at most one cycle, one can get from the main theorem a dichotomy result:

There is a finite list $G_0, G_1, ..., G_k$ of Borel $\mathcal{F}$-free graphs which do not admit countable Borel $\mathcal{F}$-free quotients, and such that for any analytic $\mathcal{F}$-free graph $\mathcal{X}$, either $\mathcal{X}$ admits a countable Borel $\mathcal{F}$-free quotient, or else one of the G_i's admits a Borel morphism into $\mathcal{X}$.

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Poset algebras over well quasi-ordered posets

Uri Abraham

The talk presented a background (including work of Robert Bonnet, Maurice Pouzet, Wiesław Kubiś, and Matatyahu Rubin) to a forthcoming paper by Abraham, Bonnet, and Kubis.

A new class of partial order-types, class $\mathcal{G}^+_{bqo}$ is defined. A poset P is in that iff the poset algebra $F(P)$ is generated by a better quasi-order G that is included in $L(P)$.

The class $\mathcal{G}_{wf}$ contains all the posets P such that $F(P)$ is well-generated. The class $\mathcal{G}_{wqo}$ contains all the posets P such that $F(P)$ is well quasi-ordered generated. The class $\mathcal{G}_{bqo}$ contains all the posets P such that $F(P)$ is better generated.

The free Boolean algebra $F(P)$ contains the partial order P and is generated by it: $F(P)$ has the following universal property. If B is any Boolean algebra and f is any order-preserving map from P into a Boolean algebra B, then f can be extended to an homomorphism $\hat{f}$ of $F(P)$ into B. We also define $L(P)$ as the sublattice of $F(P)$ generated by P.

We prove that if P is any well quasi-ordering, then $L(P)$ is well founded, and is a countable union of well quasi-orderings. We prove that the class $\mathcal{G}^+_{bqo}$ is contained in the class of well quasi-ordered sets. We prove that $\mathcal{G}^+_{bqo}$ is preserved under homomorphic image, finite products, and lexicographic sum over better quasi-ordered index sets. We prove also that every countable well quasi-ordered set is in $\mathcal{G}^+_{bqo}$. We do not know, however if $\mathcal{G}^+_{bqo}$ is the class of well quasi-ordered sets.

The class $\mathcal{G}_{wf}$ was completely characterized by the following theorem.

Let P be a poset. The following conditions are equivalent:
(i) *$P \in \mathcal{G}_{wf}$, that is $F(P)$ is well generated.*
(ii) *P is scattered and narrow.*
(iii) *$F(P)$ is superatomic.*

We tend to believe that well quasi ordering have greater affinity to better quasi ordering than the definitions would allow us to think. We ask the following two questions:

(1) Is any well quasi ordering is a countable union of better quasi orderings?
(2) If P is a well quasi ordering, is then $P \in \mathcal{G}_{bqo}$ (or at least $P \in \mathcal{G}_{wqo}$)?

Concerning the second question above, we prove a positive answer for every countable well quasi ordering. In fact, we prove a stronger result for these orderings: not only that the poset algebra of a countable well quasi ordering is better generated, but there is a sublattice of $L(P)$ that generates $F(P)$ and is a better quasi ordering.

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