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Deformations and Contractions in Mathematics and Physics

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Abstract. Deformations of mathematical structures are not only important in most parts of mathematics but also to a large extend in physics. Contractions are in some respect dual to deformations. The workshop brought together world experts, but also young post-docs, in these complementary topics with the goal to forster further interactions between the different scientific communities.

Mathematics Subject Classification (2000): 16xx, 17xx, 22xx, 53xx, 70xx, 81xx, 83xx.

Introduction by the Organisers

Deformations of mathematical structures are not only important in most parts of mathematics but also to a large extend in physics. Contractions are in some respect dual to deformations. The aim of the proposed workshop was to bring together world experts in these complementary topics of deformations and contractions of various algebraic structures. Deformations and contractions have been investigated by researchers who had different approaches and goals. Tools such as cohomology, gradings, etc. which are utilized in the study of one concept, are likely to be useful for the other concept as well. At this meeting there were mathematicians, mathematical physicists and physicists as well. The organizers hope that the meeting was of benefit to all groups.

Because various fields in mathematics and physics exist in which deformations are used, it was necessary to focus the topic of the workshop. The meeting mainly considered deformations of algebras (in particular, of Lie algebras), groups, and related algebraic structures, the corresponding contractions, and their applications to problems in physics. Nevertheless, other fields with strong relations to the central topic were present too. One such field, discussed in detail at the workshop, with tight interaction was deformation quantization. But also other topics like quantum groups, deformation of Hopf algebras, q-deformed physics, fuzzy spaces, quantum systems as deformations of classical systems, etc showed up.

As the workshop had an interdisciplinary character it was considered to be useful to start with some introductory talks on

(1) Deformations in mathematics and physics,
(2) Contractions of Lie algebras in physics,
(3) Cohomology and deformations,
(4) Deformation quantization,

with the aim to introduce the necessary concepts which were not always well-known to all the different communities present. For more details on the concepts, see the corresponding extended abstracts in this Oberwolfach report.

The following is a (non-exhaustive) list of topics which were discussed at the workshop.

(1) The concept of rigidity and deformations in its different versions; relations to cohomology, moduli spaces of algebras and existence of versal families; for formally rigid infinite dimensional algebras there exist nevertheless global deformations which are locally non-trivial; the deformations of enveloping algebras.
(2) Contractions and its relations to deformations considered from a mathematical point of view; the different concepts of contractions, generalized Inönü - Wigner contractions, graded contractions, degenerations, orbit closure, jump deformations, expansions; invariants of Lie algebra.
(3) Contractions and their physical implications; macroscopic quantum systems, local current algebras, supergravity, regularisations, symmetry of the hydrogen atom.
(4) Deformation quantization; its application to field theory, algebraic varieties, superformality, unimodular vector fields.
(5) Deformations of vector field algebras; its relation to geometric moduli spaces, algebras of Krichever-Novikov type, supergeometry
(6) Non-commutative spaces and related algebraic objects; differential geometry of noncommutative spaces, Hopf algebras, operads, quantum groups, curvature, elliptic gamma functions and triptic curves.

The workshop was attended by 44 participants from all over the world. The official program consisted of 24 lectures. Two evening sessions of informal presentations were organised. Beside the official program, there was ample time for the participants for further activities, such as self-organised sessions and discussion groups.

Workshop: Deformations and Contractions in Mathematics and Physics

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Abstracts

Deformations in mathematics and physics

Alice Fialowski

1. The notion of deformation. The theory of deformations originated with the problem of classifying all possible pairwise non-isomorphic complex structures on a given differentiable real manifold. The fundamental idea, which should be credited to Riemann, was to introduce an analytic structure therein. The notion of local and infinitesimal deformations of a complex analytic manifold first appeared in the work of Kodaira and Spencer (1958). In particular, they proved that infinitesimal deformations can be parametrized by the corresponding cohomology group. The deformation theory of compact complex manifolds was devised by Kuranishi (1965) and Palamodov (1976). Shortly after the work of Kodaira and Spencer, algebro-geometric foundations were systematically developed by Artin (1960) and Schlessinger (1968). Formal deformations of arbitrary rings and associative algebras, and the related cohomology questions, were first investigated by Gerstenhaber, in a series of articles (1964–1968). The notion of deformation was applied to Lie algebras by Nijenhuis and Richardson (1966–68).

Because various fields in mathematics and physics exist in which deformations are used, we focused on the topic of the conference. We mainly consider here deformations of algebras (in particular, of Lie algebras), groups, and related algebraic structures and their applications to problems in physics. But beside the central topic, to open up fertile interaction, we invited also researchers from neighbouring disciplines. One such topic with tight interaction is deformation quantization. But there will also be others, like quantum groups, deformation of Hopf algebras, Leibniz and dialgebras, infinity algebras, q-deformed physics, fuzzy spaces, quantum systems as deformations of classical systems etc.

Deformation is one of the tools used to study a specific object, by deforming it into some families of "similar" structure objects. This way we get a richer picture about the original object itself. But there is also another question approached via deformation. Roughly speaking, it is the question, can we equip the set of mathematical structures under consideration (may be up to certain equivalence) with the structure of a topological or geometric space. In other words, does there exist a moduli space for these structures. If so, then for a fixed object the deformations of this object should reflect the local structure of the moduli space at the point corresponding to this object.

2. Definitions. For simplicity, consider the Lie algebra case.

Let $\mathcal{L}$ be a Lie algebra with Lie bracket μ_0 over a field $\mathbb{K}$.

a) *Intuitive* definition: a deformation of $\mathcal{L}$ is a one-parameter family $\mathcal{L}_t$ of Lie algebras with the bracket

$$\mu_t = \mu_0 + t\varphi_1 + t^2\varphi_2 + \dots$$

where φ_i are $\mathcal{L}$-valued 2-cochains, i.e. elements of $\mathrm{Hom}_{\mathbb{K}}(\Lambda^2\mathcal{L},\mathcal{L}) = C^2(\mathcal{L};\mathcal{L})$, and $\mathcal{L}_t$ is a Lie algebra for each $t \in \mathbb{K}$. Two deformations, $\mathcal{L}_t$ and $\mathcal{L}'_t$ are equivalent if there exists a linear automorphism $\widehat{\psi}_t = \mathrm{id} + \psi_1 t + \psi_2 t^2 + \dots$ of $\mathcal{L}$ where ψ_i are linear maps over $\mathbb{K}$, i.e. elements of $C^1(\mathcal{L},\mathcal{L})$ such that

$$\mu'_t(x,y) = \widehat{\psi}_t^{-1}\big(\mu_t\big(\widehat{\psi}_t(x),\widehat{\psi}_t(y)\big)\big) \quad \text{for } x,y \in \mathcal{L}.$$

The Jacobi identity for the algebras $\mathcal{L}_t$ implies that the 2-cochain φ_1 is indeed a cocycle, i.e. $d_2\varphi_1 = 0$. If φ_1 vanishes identically, the first nonvanishing φ_i will be a cocycle. If μ'_t is an equivalent deformation with cochains φ'_i, then

$$\varphi'_1 - \varphi_1 = d_1\psi_1,$$

hence every equivalence class of deformations defines uniquely an element of $H^2(\mathcal{L},\mathcal{L})$.

b) Consider now a deformation $\mathcal{L}_t$ not as a family of Lie algebras, but as a Lie algebra over the algebra $\mathbb{K}[[t]]$. The natural generalization is to allow more parameters, or to take in general a commutative algebra over $\mathbb{K}$ with identity as base of a deformation. Let us fix an augmentation $\varepsilon : A \to \mathbb{K}$, $\varepsilon(1) = 1$, and set $\mathrm{Ker}\,\varepsilon = m$, which is a maximal ideal.

Definition. *A deformation λ of $\mathcal{L}$ with base (A,m) is a Lie A-algebra structure on the tensor product $A \otimes_{\mathbb{K}} \mathcal{L}$ with bracket $[\ ,\ ]_\lambda$ s.t.*

$$\varepsilon \otimes \mathrm{id} : A \otimes \mathcal{L} \to \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$$

is a Lie algebra homomorphism.

A deformation with base A is called *local* if the algebra A is local, and it is called infinitesimal if, in addition to this, $m^2 = 0$. For general commutative algebra base, we call the deformation *global*.

c) *Formal deformations.* Let A be a complete local algebra (completeness means that $A = \varprojlim_{n\to\infty}(A/m^n)$, where m is the maximal ideal in A). A formal deformation of $\mathcal{L}$ with base A is a Lie A-algebra structure on the completed tensor product $A\widehat{\otimes}\mathcal{L} = \varprojlim_{n\to\infty}\big((A/m^n)\otimes\mathcal{L}\big)$ s.t.

$$\varepsilon\widehat{\otimes}\mathrm{id} : A\widehat{\otimes}\mathcal{L} \to \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$$

is a Lie algebra homomorphism.

d) *Formal versal deformations.* It is known that in the category of algebraic varieties the quotient by a group action does not always exist (Hartshorne). Specifically, there is no universal deformation in general of a Lie algebra $\mathcal{L}$ with a commutative algebra base A with the property that for any other deformation of $\mathcal{L}$ with base B there exists a unique homomorphism $f : B \to A$ that induces an equivalent deformation. If such a homomorphism exists (but not unique), we call the deformation of $\mathcal{L}$ with base A *versal*.

The classical one-parameter deformation theory did not study the versal property of deformations. A more general deformation theory of Lie algebras follows from Schlessinger's work (1968). Namely, for complete local algebra base deformations, under some minor restriction, there exists a so-called miniversal deformation:

A formal deformation η of a Lie algebra $\mathcal{L}$ with a complete local algebra base B is called *miniversal*, if

i) for any formal deformation λ of $\mathcal{L}$ with any complete local base A there exists a homomorphism $f : B \to A$ s.t. the deformation λ is equivalent to the push-out of η by f;

ii) if A satisfies $m^2 = 0$, then f is unique.

The situation is much worse for global deformations, where we lose the cohomology tool for obtaining deformations, and there is no way so far to get a versal object.

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Contractions of Lie algebras in mathematics and physics

Marc de Montigny

Hereafter I briefly review the concept of contraction of Lie algebras and some of its applications in physics. The relation with deformations is that non-trivial contractions induce inverse "jump deformations" (defined in other contributions to this workshop). First, I recall the general definition of a contraction from Ref. [1], which contains more details about contractions and deformations of Lie algebras. Consider a Lie algebra $\mathfrak{g}$ of dimension N over an arbitrary field k (physicists work mostly with the real and the complex fields). Let us denote the basis elements of $\mathfrak{g}$ by $\{x_1, \cdots, x_N\}$ and write the Lie brackets as

$$[x_a, x_b] = C^c_{ab} x_c, \tag{1}$$

with structure constants C^c_{ab}. A contraction is defined via a non-singular linear transformation of $\mathfrak{g}$ denoted $U_\varepsilon \in \mathrm{GL}(N, k)$, which depends on a parameter ε. The commutation relations of a *contracted Lie algebra*, or *contraction*, $\mathfrak{g}'$ of $\mathfrak{g}$ are given by the following expression:

$$[x,y]' \equiv \lim_{\varepsilon \to \varepsilon_0} U_\varepsilon^{-1}([U_\varepsilon(x), U_\varepsilon(y)]), \tag{2}$$

where ε_0 is a singularity point (often equal to zero) of the inverse U_ε^{-1}.

In mathematical terms, the orbits under the action of $\mathrm{GL}(N,k)$ are the Lie algebra isomorphism classes, and a Lie bracket $[\cdot,\cdot]'$ is a contraction of $[\cdot,\cdot]$ if it is in the Zariski closure of the orbit of $[\cdot,\cdot]$ [2]. In addition to "orbit closures", contractions are also referred to as "degenerations" and "perturbations" in the mathematics literature. The notion of orbit closure arises in many areas of mathematics, where algebraic or topological transformation groups are considered, such as invariant theory (Hilbert (1893)), representation theory (Kostant (1963)), etc. For algebraic structures on a fixed finite dimensional vector space, the orbits under the action of U_ε are the isomorphism classes and, therefore, the orbit closure coincides with the closure of isomorphism classes. Passing to the orbit closure is a degeneration of the algebraic structure.

In physics, contractions were introduced fifty years ago by Inönü and Wigner [3]. Their motivation (a forerunner for many articles to follow) was to relate in terms of a limit process two well-known kinematical algebras: the Poincaré algebra (underlying Einstein's special relativity) and the Galilei algebra (which governs Newtonian physics). Further contractions of kinematical algebras have been investigated in Ref. [4]. These results are fundamental since the modern concept of a quantum particle rests on representations of such kinematic algebras. In Ref. [3], Inönü and Wigner have used a transformation of basis linear in ε:

$$U_\varepsilon \equiv \mathrm{id}_{\mathfrak{g}_0} + \varepsilon\, \mathrm{id}_{\mathfrak{g}_1},$$

where $\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{g}$. For historical reasons, this particular choice is often referred to as a "simple contraction" or "Inönü-Wigner contraction", whereas a plethora of so-called "generalized contractions" (for instance, the Saletan contraction [5]) were studied and simply amount to choosing specific transformations U_ε in Eq. (2). Henceforth, by "contraction", I simply mean *any* transformation matrix (diagonal or not, linear or non-linear) U_ε such that the Lie brackets (2) are well defined. I will not dwell on such generalizations and rather focus on various physical applications.

Loosely speaking, contracted Lie algebras are generally associated to an approximate or phenomenological theory. Also, the structure of a contracted Lie algebra $\mathfrak{g}'$ and its representations is typically more complicated than the parent algebra $\mathfrak{g}$. Well-known examples include the contractions from the three-dimensional rotation group into the Euclidean group in the plane, from the de Sitter and anti-de Sitter algebras to the Poincaré algebra, as well as from various Cartan classical Lie algebras into direct sums of special unitary and Heisenberg-Weyl Lie algebras of interest in the description of collective phenomena, etc. Many examples are discussed in Gilmore's book [6].

For the remainder of this contribution, I will enumerate various applications without any attempt to be comprehensive. My purpose is to convey an idea of the breadth of physical applications, and hopefully, motivate further developments. Various scattering processes of particles and symmetry breaking have been interpreted in terms of contractions in Ref. [7]. An interesting connection between contractions and classical lens optics was noted in Ref. [8]. At the quantum level, asymptotic limits of Wigner functions correspond to Lie algebra contractions, with possible applications in quantum interferometry and quantum information theory [9]. In condensed matter physics, contractions have allowed to relate some Hamiltonian models of superfluidity [10]. Of physical interest also are the results for special function theory and the problem of separation of variables obtained in Ref. [11]. Also it was noted that contractions of Lie algebras can be seen as the mathematical structure underlying limiting distributions in probability theory [12].

Physicists also needed contractions involving other algebraic structures than finite-dimensional Lie algebras. Infinite dimensional Lie algebras have been first contracted in Ref. [13], where contractions affine Kac-Moody algebras were motivated by conformal field theory. A nice result regarding Lie superalgebras as been the analysis of new supersymmetric extentions of the Poincaré algebra obtained by Inönü-Wigner contractions of the anti-de Sitter and de Sitter superalgebras [14]. In Ref. [15], a contraction procedure is applied to the super anti-de Sitter group, in connection with superstring actions. Contractions have also been used to construct a new non-reductive superconformal algebra in Ref. [16]. Another application related to sypersymmetric systems are the contractions of non-linear sigma models in connection with effective supergravity theories [17]. A generalization of Lie superalgebras called "Lie algebra of order F" also benefited from contraction methods in order to obtain new F-algebras [18]. Contractions of quantum groups and quantum algebras are discussed, for instance, in Refs. [19]. Generalizations of the contraction procedure to other algebraic structures such as Lie algebroids, n-ary products, coproducts, etc. are discussed in Ref. [20].

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Cohomology and Deformations

Martin Schlichenmaier
(joint work with Alice Fialowski)

1. Introduction. It is the general impression that deformation problems can always be described in cohomological terms. It was the goal of this talk to show that the connection between deformations of algebraic structures and the corresponding cohomology is not that close as one might naively assume. We consider deformations of Lie algebras. There this close connection is true for finite-dimensional algebras, but fails for infinite dimensional ones. We construct geometric families of infinite dimensional Lie algebras over the moduli space of complex one-dimensional tori with marked points. These algebras are algebras of Krichever-Novikov type which consist of meromorphic vector fields of certain type over the tori. The families are non-trivial deformations of the (infinite dimensional) Witt algebra, and the Virasoro algebra respectively, despite the fact that the cohomology space associated to the deformation problem of the Witt algebra vanishes, and hence the algebra is formally rigid. A similar construction works for current algebras and affined algebras respectively. The presented results are jointly obtained with Alice Fialowski [4], [5].

2. Definition of Deformations. The following is an intuitive definition of a deformation, which is sufficient for our purpose. Let $\mathcal{L}$ be a Lie algebra over $\mathbb{C}$ with Lie bracket $\mu_0 : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$; $\mu_0(x, y) = [x, y]$. Consider on the same vector space $\mathcal{L}$ is modeled on, a family of Lie structures

$$\mu_t = \mu_0 + t \cdot \phi_1 + t^2 \cdot \phi_2 + \cdots \ , \tag{1}$$

with bilinear maps $\phi_i : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$, such that $\mathcal{L}_t := (\mathcal{L}, \mu_t)$ is a Lie algebra and $\mathcal{L}_0$ is the Lie algebra we started with. The family $\{\mathcal{L}_t\}$ is a *deformation* of $\mathcal{L}_0$ with parameter t. Depending on the nature of the parameter one has different types of deformations:

(1) If t is a variable which allows to plug in numbers $\alpha \in \mathbb{C}$ then, for those values for which (1) is defined, one obtains a Lie algebra $\mathcal{L}_\alpha$. This gives a deformation over the affine line $\mathbb{C}[t]$ (or over the convergent power series $\mathbb{C}\{\{t\}\}$). The deformation is called a *geometric* (or an *analytic deformation* respectively).

(2) If t is a formal variable, and if one allows infinitely many terms in (1) one obtains a *formal deformation* over $\mathbb{C}[[t]]$.

(3) If t is an infinitesimal variable ($t^2 = 0$), one obtains *infinitesimal deformations* defined over $\mathbb{C}[X]/(X^2) = \mathbb{C}[[X]]/(X^2)$.

Clearly we have always a trivial deformation and the notion of equivalence between two deformations. A deformation is called *rigid* if every deformation μ_t of μ_0 is locally equivalent to the trivial family. Again, one has to specify whether one talks about rigidity in the geometric, analytic, formal or infinitesimal sense.

3. Cohomology. Given a Lie algebra $\mathcal{L}$, for Lie algebra deformations the relevant cohomology space is $\mathrm{H}^2(\mathcal{L}, \mathcal{L})$, the space of Lie algebra two-cohomology classes with values in the adjoint module $\mathcal{L}$. Given (1) the first non-vanishing element ϕ_i will be a two-cocycle. Equivalent families have cohomologous ϕ_i.

The space $\mathrm{H}^2(\mathcal{L}, \mathcal{L})$ classifies infinitesimal deformations. If this space is finite-dimensional, then the classes of formal deformations can be realized in $\mathrm{H}^2(\mathcal{L}, \mathcal{L})$, [1], [3]. In particular, if $\mathrm{H}^2(\mathcal{L}, \mathcal{L}) = 0$, then $\mathcal{L}$ is formally rigid. If $\mathcal{L}$ is finite-dimensional, then $\mathrm{H}^2(\mathcal{L}, \mathcal{L}) = 0$ implies that $\mathcal{L}$ is also rigid in the geometric and analytic sense [1].

As our examples show, this is not true anymore in infinite dimension. For the Witt algebra $\mathcal{W}$ one has $\mathrm{H}^2(\mathcal{W}, \mathcal{W}) = 0$ ([2], see also [4]). Hence, it is formally rigid. For the classical current algebras $\overline{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z^{-1}, z]$ with $\mathfrak{g}$ a finite-dimensional simple Lie algebra, Lecomte and Roger [7] showed that $\overline{\mathfrak{g}}$ is formally rigid. Nevertheless, for both types of algebras including their central extensions we have deformations which are both locally geometrically and analytically non-trivial [4], [5].

4. The geometric families. The *Witt algebra* is the Lie algebra consisting of those meromorphic vector fields on the Riemann sphere $\mathbb{P}^1(\mathbb{C}) = S^2$, which are holomorphic outside $\{0, \infty\}$. It has the basis $\{l_n = z^{n+1}\frac{d}{dz} \mid n \in \mathbb{Z}\}$ and the structure equations $[l_n, l_m] = (m-n)\, l_{n+m}$. By defining $\deg(l_n) := n$, it becomes a graded Lie algebra. The *Virasoro algebra* is its universal central extension.

The *Krichever-Novikov vector field algebras* [6], [8] are generalization of the Witt algebra to arbitrary higher genus compact Riemann surfaces X. They consist of meromorphic vector fields on X, which are holomorphic outside a certain finite set of points. In general the algebras are not graded anymore, but only almost-graded (see [8], [9] for the details and further references).

The examples considered are one-dimensional complex tori $T = \mathbb{C}/L$ where L is the lattice $L = \langle 1, \tau \rangle_{\mathbb{Z}}$, (resp. compact Riemann surfaces of genus one, resp. elliptic curves over $\mathbb{C}$). We consider the subalgebra of those vector fields, which are holomorphic outside of $\bar{z} = \bar{0}$ and $\bar{z} = \overline{1/2}$. Here τ is the complex moduli parameter and we obtain families of curves and associated families of vector field algebra by varying τ. In fact it is more appropriate to use Weierstrass $\wp$ functions and its derivative to embed the torus into the projective plane. Now everything can be expressed in terms of the two-division points $e_1, e_2, e_3 = -(e_1 + e_2)$. The two points where poles are allowed are ∞ and $(e_1, 0)$. A basis of such vector fields is given (with $k \in \mathbb{Z}$) by $V_{2k+1} := (X - e_1)^k Y \frac{d}{dX}$ and $V_{2k} := \frac{1}{2} f(X)(X-e_1)^{k-2}\frac{d}{dX}$.

For the algebra we obtain

$$
(2) \quad [V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd}, \\ (m-n)\big(V_{n+m} + 3e_1 V_{n+m-2} \\ \qquad + (e_1 - e_2)(e_1 - e_3)V_{n+m-4}\big), & n, m \text{ even}, \\ (m-n)V_{n+m} + (m-n-1)3e_1 V_{n+m-2} \\ \qquad + (m-n-2)(e_1-e_2)(e_1-e_3)V_{n+m-4}, & n \text{ odd}, \ m \text{ even}. \end{cases}
$$

In fact these relations define Lie algebras for every pair (e_1, e_2). We denote by $\mathcal{L}^{(e_1,e_2)}$ the Lie algebra corresponding to (e_1, e_2).

Proposition 1. ([4, Prop. 5.1]) *For $(e_1, e_2) \neq (0,0)$ the algebras $\mathcal{L}^{(e_1,e_2)}$ are not isomorphic to the Witt algebra $\mathcal{W}$, but $\mathcal{L}^{(0,0)} \cong \mathcal{W}$.*

Theorem 1. *Despite its infinitesimal and formal rigidity, the Witt algebra $\mathcal{W}$ admits deformations $\mathcal{L}_t$ over the affine line with $\mathcal{L}_0 \cong \mathcal{W}$, which restricted to every (Zariski or analytic) neighbourhood of $t = 0$ are non-trivial.*

5. The geometric reason behind. If we take $e_1 = e_2 = e_3$ we obtain the (singular) cuspidal cubic E_C, with affine part given by the polynomial $Y^2 = 4X^3$. It has a singularity at $(0,0)$ and the desingularization is given by the projective line $\mathbb{P}^1(\mathbb{C})$. The vector fields can be degenerated to E_C and pull-backed to vector fields on $\mathbb{P}^1(\mathbb{C})$. The point $(e_1, 0)$, where a pole is allowed, moves to the cusp. The other point stays at infinity. In particular, by pulling back the algebra we obtain the algebra of vector fields with two possible poles, which is the Witt algebra.

There are other interesting geometric interpretations for subfamilies of (2) corresponding to nodal cubics, see [4].

6. The current/affine algebra case. A similar construction works in the current algebra case. In particular we obtain (locally) nontrivial deformations of the current algebras $\overline{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z^{-1}, z]$, despite the fact that $\overline{\mathfrak{g}}$ is formally rigid. See [5] for further details and references. In a suitable Krichever-Novikov basis $\{A_n\}_{n\in\mathbb{Z}}$ of the function algebra and with $x, y \in \mathfrak{g}$ we obtain

$$(3)\quad [x \otimes A_n, y \otimes A_m] = \begin{cases} [x,y] \otimes A_{n+m}, & n \text{ or } m \text{ even},\\ [x,y] \otimes A_{n+m} + 3e_1[x,y] \otimes A_{n+m-2} & \\ \quad +(e_1 - e_2)(2e_1 + e_2)[x,y] \otimes A_{n+m-4}, & n \text{ and } m \text{ odd}. \end{cases}$$

If we let e_1 and e_2 (and hence also e_3) go to zero, we obtain the classical current algebra as degeneration.

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Group contraction in quantum field theory

Giuseppe Vitiello

In the Lhemann Symanzik Zimmermann (LSZ) formalism of Quantum Field Theory (QFT) the dynamics (the Lagrangian) is described in terms of the Heisenberg fields. The observables are given in terms of the asymptotic or physical fields (the quasiparticle fields in many body physics). The dynamical map expressing the Heisenberg fields in terms of the physical fields is known as the Haag expansion and is a weak relation in the sense that it only holds between expectation values over the physical state space. The set of the physical fields is supposed to be an irreducible set of fields and it may include bound states fields. Therefore, in general there is no one-to-one correspondence between Heisenberg fields and physical fields.

When the Lagrangian and the Heisenberg field equations are invariant under some continuous symmetry group G and such a symmetry is spontaneously broken (i.e. the vacuum is not invariant under G, but at most under one of its subgroups) the group under which the physical field equations are invariant is found to be the Inönü-Wigner group contraction of G. This result has been obtained in several papers (see the refs. below) by (model independent) functional integration techniques, by projective geometry methods, by group theoretical considerations and it has been also confirmed by explicit computations in specific models. Applications have been made in a wide range of physical problems, in high energy physics, condensed matter physics, quantum optics, coherent state, living matter physics. In all the cases where a subgroup (with the associated closed subalgebra) is preserved in the process of symmetry breakdown (the so-called stability group or little group of residual symmetry of the ground state), the group which is relevant to the observations (the phenomenological symmetry group) has been found to be the group contraction of the original invariance group of the Lagrangian of the theory. We are referring to continuous compact invariance groups and in the case the invariance group is a compact simple Lie group, the stability group is assumed to be a maximal subgroup. Contraction of group representations have been also studied.

The meaning of the group contraction mechanism in QFT is that the non-compact (abelian) subgroup arising due to the contraction implies that a gapless field must be present in the set of the physical field and the non-compact subgroup describes translation transformations of such a field: a field *translation* is an invariant transformation for the field equation if and only if that field is massless, indeed. It is clear that the gapless field is the Nambu-Goldstone (NG) boson field and that the statement that spontaneous breakdown of symmetry implies that the invariance group for the physical field equations is the group contraction of G is equivalent to the Goldstone theorem. The contraction mechanism offers a powerful tool to compute the number of NG modes. Actually, it implies that the NG fields must form an irreducible representation of the invariance group of the theory.

The group contraction mechanism completely determines the algebraic structure of the vacuum and, most important, it provides the group which is relevant to the observables. Many results of the current algebra formalism are direct consequences of the contraction, e.g. low energy theorems emerge from the functional dependence of the S-matrix directly implied by the contraction.

A key point in the proof of the contraction is realizing that at the level of the physical fields one always deals with localized observations and thus infinite volume (infrared) contributions are missing. When these contribution are formally considered the original group G is recovered. The group contraction is therefore *intrinsic* to "making physics" since at phenomenological level observations are always local as compared to the system volume.

When the field is translated by a topologically non-trivial space-dependent function one has non-homogeneous boson condensation and then it has been possible to describe extended objects such as vortices, monopoles, sphalerons as macroscopic (i.e. solutions of classical equations) envelopes of localized boson condensate. By tuning the condensate one may study symmetry restoration due to thermalization and how extended objects behave and possibly disappear due to changes in the temperature. The mechanism of group contraction thus plays an important rôle in the passage to the macroscopic phenomena: Abelian (boson) transformations introduced through group contraction regulate classical macroscopic phenomena through boson condensation. When a large number of bosons is condensed, observable symmetry patterns appear in ordered states, the quantum fluctuations become very small and the system behaves as a classical one. In this sense, we obtain *macroscopic quantum systems*. These are quantum systems not in the trivial sense that they are made, as any other physical system, by quantum components, but in the sense that their macroscopic features, such as ordering and stability, cannot be explained without recurse to the undergoing quantum dynamics. The results here presented thus seem to support the conjecture that the passage from quantum to classical physics involves some group contraction phenomena.

Moreover, the possibility of boson condensation, which determines the vacuum structure, can only occur in QFT since a pre-requisite for such an occurrence is the existence of infinitely many unitarily inequivalent representations of the canonical commutation relations. Provided mathematical care is adopted, field translations may then depict transitions among different (physically inequivalent) representations (a possible road to the study of phase transitions).

Applications have been also worked out in biological systems and in the mathematical modelling of brain functions. For a general overview see the book: Giuseppe Vitiello, My Double unveiled, John Benjamins, Amsterdam 2001.

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Quantization is deformation

Daniel Sternheimer

1. Epistemological introduction

Mathematics and Physics are two communities separated by a common language: mathematics. In research, a scientist should try to answer three questions: *why, what, how*. Work is 1% inspiration and 99% perspiration. But inspiration (which relates to the first two questions) is essential. Mathematicians, physicists, and mathematical physicists have different approaches to the three questions, even when dealing with physics. As said by Göthe, *mathematicians are like Frenchmen: they translate everything into their own language, and henceforth it is something entirely different*. Nevertheless, mathematics and physics have always been in strong interaction with one another. Physical ideas and concepts are often seminal in mathematics, and mathematics is the natural language of physics, even if it is spoken with a different accent and physicists often make very liberal use of it. Mathematical physicists aim at precise formulations and solutions of physical problems. Quantization is a perfect subject to examplify these different approaches.

2. Quantization and the three questions

Why do we introduce quantization? In physics, because a century ago there appeared experimental need for it. In mathematics, because physicists need it (and it gives nice mathematics). In mathematical physics, in my opinion, because of the *deformation philosophy* developed since around 1970 by Moshé Flato.

Physical theories have their domain of applicability defined by the relevant distances, velocities, energies, etc. involved. But the passage from one domain (of distances, etc.) to another does not happen in an uncontrolled way: experimental phenomena appear that cause a paradox and contradict accepted theories. Eventually a new fundamental constant enters and the formalism is modified: the attached structures (symmetries, observables, states, etc.) *deform* the initial structure to a new structure which in the limit, when the new parameter goes to zero, "contracts" to the previous formalism. The question is therefore, in which category do we seek for deformations? Usually physics is rather conservative and if we start e.g. with the category of associative or Lie algebras, we tend to deform in the same category. But there are important examples of generalization of this principle: e.g. quantum groups are deformations of Hopf algebras.

The discovery of the non-flat nature of Earth may be the first example of this phenomenon. Closer to us, the paradox coming from the Michelson and Morley experiment (1887) was resolved in 1905 by Einstein with the special theory of relativity: in our context, one can express that by saying that the Galilean geometrical symmetry group of Newtonian mechanics is deformed to the Poincaré group, the new fundamental constant being c^{-1} where c is the velocity of light in vacuum.

What is quantization? In a nutshell, one can say that in (theoretical) physics, it expresses "quantum" phenomena appearing (usually) in the microworld. In mathematics, it appears generally as a passage from commutative to noncommutative structures. In (our) mathematical physics, it is deformation quantization.

How do we quantize? In physics, via the so called correspondence principle introduced around 1927 when Heisenberg, Schrödinger, Weyl and Bohr (among others) "translated into German" the physical ideas of de Broglie's *mécanique ondulatoire*, that waves and particles are two aspects of the same physical reality. For many mathematicians (Weyl, Berezin, Kostant, ...), we can say that it is a functor (between categories of algebras of "functions" on phase spaces and of operators in Hilbert spaces): they take physicists' formulation for God's axiom. However physicists are neither God nor Jesus, but when the best of them "walk over mathematical waters," they sense very well where are the stones hidden under the water. Mathematicians will tackle the "mission impossible" to build a bridge across the lake, but good mathematical physicists will identify the aim and find their way to it around the lake. In mathematical physics, we consider quantization as a deformation (of composition laws of physical observables).

NOTE. The present Report[1] (a long Abstract) is based on two recent papers [3, 4] and an extensive review of deformation quantization [2]. It is meant as a gateway to these (to which we refer for further references and some more details), as self-contained as possible, with an opening towards future developments.

3. Deformation quantization: a tachyonic overview

Deformation quantization is (see e.g. [1, 2]) a viable alternative, autonomous and conceptually more satisfactory, to conventional quantum mechanics and ultimately to the theory of quantized fields. On the mathematical side it builds on the 1964 Gerstenhaber theory of deformation of algebras, in this case of "functions" on phase-space, and was initiated in the 1970's [1]. On the physical side it expresses the feeling, that was "in the back of the mind" of many, that the "classical limit" is a kind of contraction, in the limit when the Planck constant $\hbar$ vanishes.

However the Copenhagen probabilistic interpretation of quantum mechanics, successful and effective especially when the forces can be directly observed (as in atomic and molecular physics, where it originated), has a hard time answering Einstein's heterodox idea (shared by de Broglie and more recently approached in a very original way by 't Hooft) that *God does not play dice with the Universe.*

Classical mechanics, whether one starts from general equations (e.g. Newton's $F = ma$) or Lagrangian formulation, is often written in Hamiltonian form, on a (finite-dimensional) symplectic (or Poisson) manifold W with a distinguished function, the Hamiltonian H expressing the dynamics. Classical observables are usually functions $f \in C^\infty(W)$ and their time evolution is $\frac{df}{dt} = P(H, f)$ where P is the Poisson bracket. Roughly speaking the considerably harder case of an infinity of degrees of freedom (field theory) proceeds in a similar manner.

In the conventional formulation, quantization introduces the noncommutativity via a complete change in the nature of observables, which in quantum mechanics become operators on a Hilbert space of "states." In deformation quantization we keep the same observables as in classical theory but *deform* their composition law, commutative product and/or Poisson bracket: *Quantization is deformation*.

In this way one can treat in an *autonomous* way not only many paradigms of conventional quantum mechanics (e.g. harmonic oscillator, angular momentum and hydrogen atom) but also *general* phase spaces, i.e. Poisson manifolds, finite (e.g. complete existence and classification results were obtained) or infinite dimensional. Covariance (symmetries and their representations) is also treated and we can incorporate the "Hopf avatar" of deformation quantization a.k.a. quantum groups. The approach is being extended to algebraic varieties, "manifolds" with singularities, and field theory. It proves seminal in a variety of areas of mathematics, going from e.g. algebraic geometry to index theorems to number theory.

4. A speculative epilogue: deforming and quantizing space-time

Deformation quantization may be viewed as laying the ground for a (so far, not enough developed) noncommutative version of the Gelfand isomorphism (between commutative algebras and algebras of functions). Its related and very elaborate "avatar," noncommutative geometry, proceeds from a similar strategy: formulate conventional (commutative) theory in a nonconventional way so as to be able to "plug in" noncommutativity in a natural way. Quantized manifolds are presently the subject of frontier research (e.g. noncommutative spheres by Connes).

In view of the interesting phenomena that occur when deforming Minkowski space-time with a tiny negative curvature to anti de Sitter AdS_4, e.g. the fact that photons can be considered as composite (in a way compatible with quantum electrodynamics) of two Dirac "singletons," naturally confined, one is tempted to deform and quantize simultaneously space-time into $q\mathrm{AdS}_4$, an ultrahyperbolic version of quantized spheres. These could be mostly at the edge of the universe and behave like black holes (à la 't Hooft) wherefrom 2-singleton states in 3 flavors and colors would emerge [4] and be massified by interaction with e.g. dark energy, a picture of a universe in constant creation. So far that is Science Fiction but in any case the formidable mathematical problems it raises are worth studying.

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Quantization as unfinished regularization

David Finkelstein

Deformation by Quantization of Universal Envelopping Algebras

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(joint work with Abdenacer Makhlouf, Toukaiddine Petit)

1. Introduction

Let (A, μ_0) an associative (resp. Lie) algebra over a field $\mathbb{K}$. Recall that a *formal associative (resp. Lie) deformation* of (A, μ_0) is a formal power series $\mu = \mu_0 + \sum_{r=1}^{\infty} t^r \mu_r$ of bilinear maps of $A \times A$ into A such that the $\mathbb{K}[[t]]$-module $A[[t]]$ equipped with the multiplication μ is an associative (resp. Lie) algebra over the ring $\mathbb{K}[[t]]$. In case $\mu = \mu_0$ the deformation is called *trivial*. Two formal associative (resp. Lie) deformations μ' and μ of (A, μ) are said to be *equivalent* iff there is a formal series $\phi := \mathrm{Id_A} + \sum_{r=1}^{\infty} t^r \phi_r$ of linear maps $A \to A$ with $\mu' = \phi^{-1} \circ \mu \circ (\phi \otimes \phi)$. An associative (resp. Lie) algebra is called *rigid* iff any formal associative (resp. Lie) deformation μ is equivalent to a trivial deformation. For associative deformations it is known that μ_1 is always a Hochschild 2-cocycle, and there are the following sufficient criteria related to Hochschild cohomology

$$\mathbf{H}^2_H(A, A) = \{0\} \quad \Longrightarrow \quad A \text{ is rigid,} \tag{1}$$

$$[\mu_1] \neq 0 \quad \Longrightarrow \quad A \text{ is not rigid.} \tag{2}$$

For details, see for instance [7].

Let $(\mathfrak{g}, [\ ,\ ])$ be a Lie algebra of finite dimension n over a field $\mathbb{K}$ of characteristic 0 which in most of the cases will be the field of complex numbers $\mathbb{C}$. The *universal envelopping algebra of* $\mathfrak{g}$, $\mathcal{U}\mathfrak{g}$, is defined to be the quotient of the free (tensor) algebra of the vector space $\mathfrak{g}$ modulo the two-sided ideal generated by the words $x \otimes y - y \otimes x - [x, y]$ for all $x, y \in \mathfrak{g}$ (and therefore is a an associative algebra with unit). The *Poincaré-Birkhoff-Witt theorem* (see e.g. [6]) states that the standard symmetrization map ω of the symmetric algebra $\mathcal{S}\mathfrak{g}$ into $\mathcal{U}\mathfrak{g}$ is a linear isomorphism. Now $\mathcal{U}\mathfrak{g}$ is a $\mathfrak{g}$-module by means of the induced adjoint representation, as is the symmetric algebra $\mathcal{S}\mathfrak{g}$, and it is known that ω is a $\mathfrak{g}$-module isomorphism, see e.g. [6]. Regarding $\mathcal{U}\mathfrak{g}$ as a bimodule over $\mathcal{U}\mathfrak{g}$ in the usual way, we can look at the *Hochschild cohomology* of $\mathcal{U}\mathfrak{g}$ with values in $\mathcal{U}\mathfrak{g}$: it is well-known that

$$\mathbf{H}^k_H(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g}) := \mathbf{Ext}^k_{\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}^{\mathrm{opp}}}(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g}) \cong \mathbf{Ext}^k_{\mathcal{U}\mathfrak{g}}(\mathbb{K}, \mathcal{U}\mathfrak{g}) \cong \mathbf{H}^k_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g}), \tag{3}$$

where the latter is the Lie algebra (*Chevalley-Eilenberg*) *cohomology* of $\mathfrak{g}$ with values in $\mathcal{S}\mathfrak{g}$, see e.g. [3] for a proof.

We shall study formal associative deformations of universal envelopping algebras $\mathcal{U}\mathfrak{g}$ and give criteria and examples for rigid algebras. If $\mathcal{U}\mathfrak{g}$ is rigid we shall call the Lie algebra $\mathfrak{g}$ *strongly rigid*. We classify all those Lie algebras up to dimension 6. This report is based on our paper [2].

2. Results

It is well-known that for the following Lie algebras $\mathbf{H}^2_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g})$ vanishes whence they are strongly rigid according to eqn (1): $\mathfrak{g} = \{0\}$, $\mathfrak{g} = \mathbb{K}$ and every *semisimple Lie algebra* thanks to the Whitehead Lemma. Moreover for the *affine Lie algebras* $\mathfrak{aff}(m, \mathbb{K}) := \mathfrak{gl}(m, \mathbb{K}) \oplus \mathbb{K}^m$ (semidirect sum) for $m \in \mathbb{N} \setminus \{0\}$ there is the

Proposition 1. *We have*

$$\forall\ k \in \mathbb{N} : \ \mathbf{H}^k_{CE}\big(\mathfrak{aff}(m, \mathbb{K}), \mathcal{S}\mathfrak{aff}(m, \mathbb{K})\big) \cong \mathbf{H}^k_{CE}\big(\mathfrak{gl}(m, \mathbb{K}), \mathbb{K}\big)$$

whence in particular $\mathbf{H}^2_{CE}\big(\mathfrak{aff}(m, \mathbb{K}), \mathcal{S}\mathfrak{aff}(m, \mathbb{K})\big)$ vanishes and the affine Lie algebras are strongly rigid.

In order to *rule out certain Lie algebras as strongly rigid*, we want to use the criterion eqn (2), but for this we have to show the *existence of a nontrivial deformation* starting with a nontrivial Hochschild 2-cocycle μ_1.
We have the following two criteria:

Proposition 2.

$$\text{(4)} \qquad \textit{If } \mathfrak{g} \textit{ is not rigid} \quad \Longrightarrow \quad \mathcal{U}\mathfrak{g} \textit{ is not rigid.}$$

$$\text{(5)} \qquad \textit{If } \mathbf{H}^2_{CE}(\mathfrak{g}, \mathbb{K}) \neq \{0\} \quad \Longrightarrow \quad \mathcal{U}\mathfrak{g} \textit{ is not rigid.}$$

For example, statement (5) rules out all *nilpotent Lie algebras* of dimension ≥ 2 as strongly rigid by Dixmier's theorem on the cohomology of nilpotent Lie algebras [5].
In both cases, the nontrivial deformation of $\mathcal{U}\mathfrak{g}$ is obtained by constructing the universal envelopping algebra over the deformed Lie algebra (resp. over a central extension of $\mathfrak{g}$) and then passing to the t-adic completion (resp. after having divided by the ideal generated by 1 minus the new central element of $\mathfrak{g}$).

In order to generalize the two preceding criteria (4) and (5), recall that a *polynomial Poisson structure* P on an n-dimensional vector space V is an element $P = \sum_{j,k=1}^n (1/2) P_{jk} e^j \wedge e^k \in \mathcal{S}V^* \otimes \Lambda^2 V$ (where V^* denotes the dual space of V and $e^1, \ldots, e^n$ is a base of V) such that the *Poisson bracket* of two polynomials $f, g \in \mathcal{S}V^*$ given by ($x \in V$):

$$\text{(6)} \qquad \{f, g\}(x) := \{f, g\}_P(x) := \sum_{j,k=1}^n P_{jk}(x) \frac{\partial f}{\partial x_j}(x) \frac{\partial g}{\partial x_k}(x)$$

is a Lie bracket on $\mathcal{S}V^*$, i.e. satisfies the Jacobi identity. It is clear that any such $\{f, g\}_P$ satisfies the Leibniz rule. It is well-known that in the case where V is the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ there is the *linear Poisson structure* defined by $P^{(0)} := [\ ,\ ] \in \mathcal{S}^1\mathfrak{g} \otimes \Lambda^2 \mathfrak{g}^*$ or $P^{(0)}_{jk}(x) = \sum_{i=1}^n x_i c^i_{jk}$ using the structure constants of $\mathfrak{g}$ in some base.
In the theory of *deformation quantization*, see [1], one tries to construct formal

associative deformations $*$ (the so-called star-products) of the commutative associative algebra $\mathcal{S}V^*$ (or more generally on the space of smooth real or complex-valued functions on a differentiable manifold which we shall not need), such that the antisymmetric part of the first order term μ_1 is equal to the given Poisson bracket $\{\ ,\ \}_P$. Here $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. In his celebrated *formality theorem*, M. Kontsevich could prove the existence of star-products for any Poisson structure P by giving an explicit formula, see [10].

It can be shown that $\mathcal{U}\mathfrak{g}$ can be seen as a converging (in the formal parameter λ) formal deformation of $\mathcal{S}\mathfrak{g}$ equiped with the linear Poisson structure, see [9] and also [4] for a proof that Kontsevich's formula gives an equivalent converging deformation.

In order to apply this theory, we consider *formal polynomial Poisson deformations of the linear Poisson structure* $P^{(0)}$ *on* $\mathfrak{g}^*$, i.e. formal series $P = P^{(0)} + \sum_{r=1}^{\infty} t^r P^{(r)}$ of elements of $\mathcal{S}\mathfrak{g} \otimes \Lambda^2\mathfrak{g}^*$ such that the bracket (6) is Poisson, i.e. satisfies the Jacobi identity on $\mathcal{S}\mathfrak{g}[[t]]$. This is equivalent to a sequence of at most quadratic conditions (so-called *Schouten brackets*) on the $P^{(r)}$, the first being the fact that $P^{(1)}$ has to be a 2-*cocycle in the Chevalley-Eilenberg cohomology of* $\mathfrak{g}$. In the preceding two cases (4) and (5) all the $P^{(r)}$ had been linear (resp. constant) polynomials. By using Kontsevich's formula as a two-parameter deformation (in t and in λ) of $\mathcal{S}\mathfrak{g}$ and then proving convergence for $\lambda = 1$ we get the following

Theorem 1. *Let* $P \in \mathcal{S}\mathfrak{g} \otimes \Lambda^2\mathfrak{g}^*$ *be a formal polynomial Poisson deformation of the linear Poisson structure on* $\mathfrak{g}^*$ *such that the first order term* $P^{(1)}$ *is a non-trivial Chevalley-Eilenberg* 2*-cocycle of* $\mathfrak{g}$*. Then there exists a nontrivial formal associative deformation of* $\mathcal{U}\mathfrak{g}$*, hence* $\mathfrak{g}$ *is not strongly rigid.*

We have obtained the following list of strongly rigid complex Lie algebras up to dimension 6 as follows: it suffices to look at rigid Lie algebras (see e.g. [8] for some list), and we ruled out two 6-dimensional Lie algebras by constructing quadratic nontrivial deformations of their linear Poisson structures and applying Theorem 1. We get:

$$\{0\}; \mathbb{C}; \mathfrak{aff}(1,\mathbb{C}); \mathfrak{sl}(2,\mathbb{C}); \mathfrak{gl}(2,\mathbb{C}); \mathfrak{aff}(1,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C});\ \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C});\ \mathfrak{aff}(2,\mathbb{C})$$

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Deformation quantization of classical algebraic field theory

Klaus Fredenhagen

Deformations of Lie algebras of vector fields arising from families of schemes

Friedrich Wagemann

The goal of the present work in progress is to construct examples of global deformations of vector field Lie algebras in a conceptual way. Let me express thanks to the algebraic geometry group in Nantes, and to Alice Fialowski for inviting me to Strasbourg where the base to the present constructions has been laid in our discussions.

Fialowski and Schlichenmaier [2] construct global deformations of the infinitesimally and formally rigid Lie algebra of meromorphic vector fields on the two-pointed Riemann sphere and of its central extension, the Virasoro algebra. For this, they used an explicitly given family of pointed elliptic curves which degenerates to the cuspidal cubic as a special member. Passing to the normalization above the singular fiber, one constructs a family of Lie algebras (parametrized by the affine line $\mathbb{C}[t]$) of meromorphic vector fields with prescribed poles (one of them passes above the singularity). As the genus strictly drops in normalization, the underlying fiber above the singularity becomes a two-pointed $\mathbb{C}P^1$, while the other members of the family lead to a non-trivial deformation of the Lie algebra of meromorphic vector fields on the two-pointed Riemann sphere, and this although the latter is infinitesimally and formally rigid.

In the attempt of producing deformations of vector field Lie algebras from deformations of the underlying pointed algebraic variety, we are led to a notion of global deformation which is different from Fialowski-Schlichenmaier's [2]. Let A be an augmented algebra over $\mathbb{C}$. Their notion of deformation considers A-Lie algebra structures on the tensor product $A \otimes \mathfrak{g}_0$ in order to deform $\mathfrak{g}_0$, while we prefer to consider A-Lie algebra structures on a general A-module, not necessarily on a free one. We argue that this latter notion takes better care of global phenomena.

The conceptual approach to global deformations consists now in the following: at first, we endow the category of affine $\mathbb{C}$-schemes $\texttt{Aff}/\mathbb{C}$ (which, admittedly, is nothing but the opposite category to the category of commutative unital associative $\mathbb{C}$-algebras) with the Grothendieck topology given by faithfully flat quasi-compact morphisms (also called the ffqc-topology). We define the $\mathbb{C}$-stack Def of global deformations to be the lax functor associating to an affine scheme $\mathrm{Spec}(A)$ the groupoid of A-Lie algebras, morphisms being only the isomorphisms between A-Lie algebras. It is an easy consequence of Grothendieck's theorem of faithfully flat descent that Def is a $\mathbb{C}$-stack. As the ffqc-topology is the finest of the Grothendieck topologies usually considered on the big affine site, Def is also a stack in the fffp- (faithfully flat morphisms of finite presentation) or the étale topology.

The point is that we are now on the same footing as the moduli stack $\mathcal{M}_{g,n}$ of flat families of smooth genus g projective curves with n marked points (usually taken in the étale topology). The marked points define a divisor on the total space of the family. In order to define a morphism of stacks from $\mathcal{M}_{g,n}$ to Def, we show that the divisor is ample, and we get therefore an affine family of curves by extracting the marked points. The morphism $I : \mathcal{M}_{g,n} \to Def$ is now defined by taking the Lie algebra of A-linear derivations on the affine family over $S = \mathrm{Spec}(A)$. The 2-categorical compatibility condition on the thus defined functors $I(S)$ in order to be a morphism of stacks is fulfilled in the fffp-topology, because we need a finite presentation of the module of Kähler differentials here.

In the last part of the talk, we observed that the theory of Pursell-Shanks, Amemiya [1], Omori, Grabowski [3] and Siebert [4] according to which the Lie algebra determines the variety, leads in our setting to some kind of injectivity of the morphism I. Indeed, using methods of Amemiya, Grabowski and Siebert, we show that for smooth affine schemes $U = \mathrm{Spec}(B)$, $U' = \mathrm{Spec}(B')$ over $S = \mathrm{Spec}(A)$, the derivation algebras $\mathrm{Der}_A(B)$ and $\mathrm{Der}_A(B')$ are isomorphic as A-Lie algebras if and only if B and B' are A-isomorphic as algebras. This means that the existence of a morphism between two objects in the groupoid $Def(S)$ which are in the image of $I(S)$ implies the existence of a morphism between the corresponding objects in $\mathcal{M}_{g,n}(S)$.

Further questions and research directions explore, for example, whether Def is an algebraic stack, whether it is Deligne-Mumford, or Artin, whether one can say more about the morphism I. It seems clear to us, that the theory can be extended to families of curves with stable singularities, but this needs to be written up, and extends thus to the boundary of $\mathcal{M}_{g,n}$. Even then, the theory does not include Fialowski-Schlichenmaier's example of the cuspidal cubic (which is not stable); does one have to operate then with a more general object than $\mathcal{M}_{g,n}$?

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Expansions of (super)algebras and D=11 CJS supergravity

José A. de Azcárraga

1. Lie algebras and superalgebras from given ones

There are three well known methods of obtaining new Lie algebras and superalgebras from given ones (we only consider here finite (super)algebras): (a) *Contractions* [1] (see also [2] and refs. therein); (b) *Deformations* [3] (see also [4]) and (c) *Extensions* (see *e.g.* [5] for details and references). All of these processes are dimension preserving *i.e.*, the new Lie algebras have the same dimension as the original ones for (a) and (b) or, for (c), it is the sum of the dimensions of the two algebras involved. To these procedures, we would like to add a new one,

(d) *Expansions* (of Lie algebras and superalgebras).

Under a different name, Lie algebra expansions were first used in [6], and then the method was studied in general in [7]. The idea is to consider the Maurer-Cartan (MC) equations of the initial Lie algebra $\mathcal{G}$ satisfied by the invariant MC forms on the group G manifold, and then to rescale some of the group coordinates g^i $(i = 1, \dots, \dim \mathcal{G})$ by a parameter λ; the resulting MC one-forms $\omega^i(g, \lambda)$ are then expanded as series in λ. Inserting these polynomials in λ into the original MC equations for $\mathcal{G}$, one obtains a set of equations from each power of λ. The problem now is how to cut the series expansions of the different ω^i's in such a way that the resulting MC-like equations be closed under d, so that they define the MC equations of a new, *expanded* Lie algebra. We shall not enumerate all the possibilities here and refer to [7] instead for details. Let us divide the $\{\omega^i\}$ MC forms into $n+1$ sets $\{\omega^{i_s}\}$ subordinated to a splitting of $\mathcal{G}$ into subspaces V_s, $\mathcal{G} = \oplus_{s=0}^n V_s$ $(s = 0, \dots, n;\ i_s = 1, \dots, \dim V_s)$. When the conditions $c_{i_p j_q}^{k_s} = 0$ if $s > p+q$ (*i.e.*, $[V_p, V_q] \subset \oplus_s V_s,\ s \le p+q$) for the generalized IW contractions [2] are satisfied, and after the rescaling $g^{i_p} \mapsto \lambda^p g^{i_p}$ of the group coordinates, the forms $\omega^{i_p}(g, \lambda)$ in each $V_p \subset \mathcal{G}$ have expansions of the form [7]

$$(1) \quad \omega^{i_p}(\lambda) = \sum_{s=p}^{\infty} \omega^{i_p,s} \lambda^s \ , \qquad \text{i.e.} \qquad \omega^{i_p}(\lambda) = \lambda^p \omega^{i_p,p} + \lambda^{p+1} \omega^{i_p,p+1} + \dots \quad .$$

If one demands that the maximum power in the expansion of the forms $\{\omega^{i_p}\}$ in the p–th subspace is $N_p \ge p$, the d-closure MC condition requires $N_{q+1} = N_q$ *or* $N_{q+1} = N_q + 1$ $(q = 0, 1, \dots, n-1)$. The new Lie algebras, generated by

$$(2) \quad \{\omega^{i_0,0}, \omega^{i_0,1}, \overset{N_0+1}{\dots}, \omega^{i_0,N_0};\ \omega^{i_1,1}, \overset{N_1}{\dots}, \omega^{i_1,N_1};\ \dots;\ \omega^{i_n,n}, \overset{N_n-n+1}{\dots}, \omega^{i_n,N_n}\} \ ,$$

are labelled $\mathcal{G}(N_0, N_1, \dots, N_n)$ and define *expansions* of the original Lie algebra $\mathcal{G}$. The case $N_p = p$, $\mathcal{G}(0, 1, \dots, n)$, coincides with the generalized contraction [2] for which $\dim \mathcal{G}(0, 1, \dots, n) = \dim \mathcal{G}$; thus, the generalized contraction is a *particular* expansion. In all other cases the expanded algebra $\mathcal{G}(N_0, N_1, \dots, N_n)$ is larger than $\mathcal{G}$ [specifically, $\dim \mathcal{G}(N_0, \ldots, N_n) = \sum_{p=0}^{n}(N_p - p + 1) \dim V_p$ when all forms in (2) are present], so that in general the expansion process is *not* dimension preserving (hence its name).

Of particular interest is the case of Lie superalgebras with splittings $\mathcal{G} = V_0 \oplus V_1$ or $\mathcal{G} = V_0 \oplus V_1 \oplus V_2$ and such that V_0 or $V_0 \oplus V_2$ contain the even generators and V_1 contains the odd ones. Then, the expansions (1) of the one-forms of V_1 (V_0 and V_2) only contain odd (even) powers of λ [7]. The consistency conditions for the existence of $\mathcal{G}(N_0, N_1)$-type expanded superalgebras require that $N_0 = N_1 - 1$ *or* $N_0 = N_1 + 1$ and, for the $\mathcal{G}(N_0, N_1, N_2)$ case, that one of the three following possibilities holds: $N_0 = N_1 + 1 = N_2$; $N_0 = N_1 - 1 = N_2$; $N_0 = N_1 - 1 = N_2 - 2$.

2. On the gauge structure of Cremmer-Julia-Scherk D=11 supergravity

We are now interested in the underlying gauge symmetry of $D = 11$ CJS supergravity [8], where the possible relevance of $OSp(1|32)$ was already raised. It was specially considered by D'Auria and Fré [9], who looked at it as a search for a composite structure of the CJS three-form field $A_3(x)$. Indeed, while two of the supergravity fields (the graviton $e^a = dx^\mu e_\mu^a(x)$ and the gravitino $\psi^\alpha = dx^\mu \psi_\mu^\alpha(x)$) are given by *one*-form spacetime fields and thus can be considered, together with the spin connection ($\omega^{ab} = dx^\mu \omega_\mu^{ab}(x)$), as gauge fields for the standard super-Poincaré group, the additional $A_{\mu_1\mu_2\mu_3}(x)$ abelian gauge field in $D = 11$ CJS supergravity is not associated with any superPoincaré algebra generator or MC *one*-form since it rather corresponds to a *three*-form A_3. However, one may ask whether it is possible to introduce a set of additional one-form spacetime fields associated to MC forms such that they, together with e^a ($a = 1, \ldots, 11$) and ψ^α ($\alpha = 1, \ldots, 32$), can be used to express A_3 in terms of one-forms. If so, the 'old' e^a, ψ^α and the 'new' fields may be considered as *gauge* fields of a larger supersymmetry group, with A_3 expressed in terms of them; its compositeness makes the gauge structure of CJS supergravity manifest. It turns out that the solution of this problem is equivalent to trivializing a standard $D = 11$ supersymmetry algebra $\mathfrak{E}^{(11|32)}$ cohomology four-cocycle ω_4 (structurally equivalent to dA_3) on a *larger* supersymmetry algebra $\tilde{\mathfrak{E}}$, that of a larger supersymmetry group $\tilde{\Sigma}$.

It is shown in [10] that there is a whole one-parameter family of enlarged supersymmetry algebras $\tilde{\mathfrak{E}}(s), s \neq 0$ (those of the enlarged rigid superspace groups $\tilde{\Sigma}(s)$), that trivialize the $\mathfrak{E}^{(11|32)}$ four-cocycle ω_4 (dA_3) (two specific cases of the $\tilde{\mathfrak{E}}(s)$ family, $\tilde{\mathfrak{E}}(3/2)$ and $\tilde{\mathfrak{E}}(-1)$, were already found in [9]). Hence, and adding the $D = 11$ Lorentz $SO(1, 10)$ group, the underlying gauge supergroup of CJS supergravity can be described by any representative of a *one-parametric family of supergroups,* $\tilde{\Sigma}(s) \rtimes SO(1, 10)$ *for* $s \neq 0$. These may be seen as deformations of $\tilde{\Sigma}(0) \rtimes SO(1, 10) \subset \tilde{\Sigma}(0) \rtimes Sp(32)$. The singularity of $\tilde{\mathfrak{E}}(0)$ is natural since, for $\tilde{\Sigma}(0)$, the $SO(1, 10)$ automorphism group of $\tilde{\Sigma}(s)$ ($s \neq 0$) is enhanced to $Sp(32)$. This clarifies the connection with the orthosymplectic $OSp(1|32)$: it is seen that [10] that $\tilde{\Sigma}(0) \rtimes SO(1, 10)$ is an expansion of $OSp(1|32)$, $\tilde{\Sigma}(0) \rtimes SO(1, 10)$ $\approx OSp(1|32)(2, 3, 2)$. $\tilde{\Sigma}(0) \rtimes Sp(32)$ is also an expansion, $OSp(1|32)(2, 3)$.

The enlarged supersymmetry algebras $\tilde{\mathfrak{E}}(s)$ are central extensions of the M-algebra by an additional fermionic generator Q'_α [*ignoring* its $SO(1,10)$ automorphisms part, the M-superalgebra [11] (of generators P_a, Q_α, Z_{ab}, $Z_{a_1...a_5}$), is a maximal central extension of the abelian $D = 11$ supertranslations algebra (see [12, 13, 11, 14]); including $SO(1,10)$, the *full* M-algebra is the expansion $osp(1|32)(2,1,2)$ [7]). Trivializing the $\mathfrak{E}^{(11|32)}$ cohomology four-cocycle ω_4 on the larger $\tilde{\mathfrak{E}}(s)$, so that $\omega_4 = d\tilde{\omega}_3$ where *now* $\tilde{\omega}_3$ is $\tilde{\Sigma}(s)$-invariant (this is why ω_4 becomes a *trivial* cocycle for $\tilde{\mathfrak{E}}(s)$, $s \neq 0$; see *e.g.* [5]), is tantamount to finding a composite structure for the three-form field A_3 of CJS supergravity in terms of one-form gauge fields, $A_3 = A_3(e^a\,,\,\psi^\alpha\,;\,B^{a_1a_2}, B^{a_1\cdots a_5}, \eta^\alpha\,)$, associated to MC forms of $\tilde{\mathfrak{E}}^{(11|32)}$. The compositeness of A_3 is given by the same expression that provides the $\tilde{\omega}_3$ trivialization: the $\tilde{\mathfrak{E}}(s)$ MC forms are simply replaced by 'soft' one-forms -spacetime one-form fields- obeying a free differential algebra with curvatures.

The presence of the additional one-form gauge fields associated with the new generators in $\tilde{\mathfrak{E}}(s)$ might be expected. The field $B^{a_1\cdots a_5}(x)$, associated to the $Z_{a_1...a_5}$ M-algebra generator, is needed [15] for a coupling to BPS preons, the hypothetical basic constituents of M-theory [16]. In a more conventional perspective, one can notice that the generators $Z_{a_1a_2}$ and $Z_{a_1...a_5}$ can be treated as topological charges [13] of the M2 and M5 superbranes. The additional fermionic 'central' charge Q'_α is also present in the Green algebra [17] (see also [18, 19, 14] and references therein). We would like to conclude with a few comments:

• The supergroup manifolds $\tilde{\Sigma}(s)$ define rigid *extended* superspaces. The fact that all the *spacetime* fields appearing in the above description may be associated to the various coordinates of $\tilde{\Sigma}(s)$ is suggestive of an *enlarged superspace variables/spacetime fields correspondence* principle for $D = 11$ CJS supergravity.

• It may be seen [14] that one may introduce an *enlarged superspace variables/worldvolume fields correspondence* principle for superbranes, by which one associates all *worldvolume* fields, including the Born-Infeld (BI) ones [14, 20] in the various D-brane actions, to fields corresponding to forms defined on suitably enlarged superspaces $\tilde{\Sigma}$ (the actual worldvolume fields are the pull-backs of these forms to the worldvolume of the extended supersymmetric object). The worldvolume BI fields, as the spacetime A_3 field of CJS supergravity above, become composite fields. Moreover, the Chevalley-Eilenberg Lie algebra cohomology analysis [21, 14, 22] of the Wess-Zumino terms of many different superbrane actions determines the possible ones and how the ordinary supersymmetry algebra has to be extended (see also [23, 20]).

• Thus, could there be an *enlarged superspace variables/fields correspondence principle in M-theory*?

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Miniversal Deformations and Moduli Spaces of Lie Algebras

Michael Penkava
(joint work with Alice Fialowski)

A complex Lie algebra of dimension n is the same thing as a quadratic codifferential on the symmetric coalgebra of the parity reversion of $\mathbb{C}^n$. This abstract definition has many advantages; in particular, the cohomology of a Lie algebra with coefficients in the adjoint representation, which determines the deformations of the algebra, is simply given by the induced coboundary operator on the space of coderivations of this symmetric coalgebra. The moduli space of n-dimensional Lie algebras of dimension n is the set of equivalence classes of codifferentials under the action of the general linear group of $\mathbb{C}^n$, and therefore, it should not be expected to have a nice topological characterization. The cohomology group H^2 can be thought of as the tangent space to a point on the moduli space, but this analogy is somewhat rough, since the moduli space is not a manifold.

A curve in this moduli space is given by a 1-parameter family of Lie algebras. If the algebras are all nonequivalent, this is a smooth family of deformations, while if all of the Lie algebras are isomorphic, except for the starting point, then the deformation family is called a jump deformation. These deformations determine how the moduli space is glued together. There is a type of deformation of the Lie algebra, called a miniversal deformation, which encodes all the information about the smooth and jump deformations, providing a more complete picture of the moduli space locally than is given by the cohomology group H^2. For example, it can happen that the cohomology is 2-dimensional, but the deformations lie only along two curves within this space.

In [2], the authors gave a construction of the miniversal deformation for any L_∞ algebra. Since then, in a series of papers [3, 7, 6, 5, 4, 1], the authors have been studying how moduli spaces of low dimensional Lie algebras are glued together using the miniversal deformations. It turns out that these spaces have a unique decomposition into strata consisting of orbifolds, with jump deformations as the glue connecting these strata.

Our main strategy in studying these moduli spaces is to construct versal deformations for each element in the moduli space. These deformations are given by multiple parameters, and there are relations on the parameters, the relations on the base of the miniversal deformation, so that when a coderivation is given by the miniversal deformation formula for parameters satisfying these relations, it determines a Lie algebra structure. Thus, we can determine the nearby elements of the moduli space by studying the miniversal deformations. We have used this idea to study moduli spaces of Lie algebras of dimension up to four, and the tools we have developed for calculation of cohomology and deformations are effective.

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Remarks on algebraic aspects of deformation quantization

V. P. Palamodov

Let $\mathbf{k}$ be a fields of zero characteristic, A be an associative commutative $\mathbf{k}$-algebra. Any closed bilinear skew-symmetric mapping $p : A \times A \to A$ that belongs to the bar-complex of the algebra A defines an element of the Hochschild cohomology $\mathrm{Hoch}^2(A, A)$. The space $\mathrm{Q}(A)$ of such classes is a A-submodule and is complemented by the module $\mathrm{Harr}^2(A, A)$ called Harrison cohomology. It is the submodule of $\mathrm{Hoch}^2(A, A)$ generated by symmetric cocycles $s : A \otimes A \to A$ so we have

$$\mathrm{Hoch}^2(A, A) = \mathrm{Harr}^2(A, A) \oplus \mathrm{Q}(A),$$

The first term is the group of commutative deformations up to term (λ^2) and the second one is that of deformation quantizations.

A quantization of A corresponding to a cocycle $p \in \mathrm{Q}(A)$ is star-product of the form

$$a * b = ab + \lambda p(a, b) + \lambda^2 p_2(a, b) + \lambda^3 p_3(a, b) + ...$$

where $\lambda \in \mathbf{k}$ is a parameter and $p_2, p_3, ...$ are bilinear mappings to be found. The star-product must be a an associative operation. Suppose that $\mathrm{Harr}^3(A, A)$ vanishes. If the cocycle p fulfils the Jacobi equation there exists of a mapping $p_2 : A \otimes A \to A$ such that the star product is associative up to a term (λ^3) and vice versa. In other words, the Jacobi sum is the first obstruction to extension of the cocycle to a star-product.

A skew-symmetric 2-cocycle p that satisfies the Jacobi equation is called Poisson bracket.

Analytic algebras. The quotient algebra $A = R/I$ is called analytic algebra, wher R is the $\mathbb{C}$-algebra of convergent power series of n variables for some n, and

I is an ideal in R. This is a the structure algebra of the germ at the origin of the analytic set

$$X = \{z \in \mathbb{C}^n, f_1(z) = ... = f_m(z) = 0\}.$$

where $f_1, ..., f_m$ be a set of generators of I. The algebra A is endowed by the canonical locally convex topology.

Proposition 1. *For any analytic algebra as above the $\mathbb{C}$-space of continuous skew-symmetric cocycles is isomorphic to*

$$\mathrm{Q}(A) \cong \mathrm{Ker}\{J : A^n \wedge A^n \to A^{nm}\},$$

where J is the mapping generated by the Jacobian matrix $J = \{\partial_i f_j\}_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$.

Take arbitrary $f_1, ..., f_m \in I$ and consider the Jacobian matrix

$$J(f_1, ..., f_m, a, b) \doteq \begin{pmatrix} \partial_1 f_1 & \partial_2 f_1 & ... & \partial_n f_1 \\ ... & ... & ... & ... \\ \partial_1 f_m & \partial_2 f_m & ... & \partial_n f_m \\ \partial_1 a & \partial_2 a & ... & \partial_n a \\ \partial_1 b & \partial_2 b & ... & \partial_n b \end{pmatrix}$$

Proposition 2. *Let K be a subset of $[1, ..., n]$ of $m + 2$ elements and $J_K(f_1, ..., f_m, a, b)$ be the corresponding minor of the Jacobian matrix. The mapping*

$$P_K(a, b) = J_K(f_1, ..., f_m, a, b)$$

is a Poisson bracket. Moreover, any **k***-linear combination of brackets P_K is a Poisson bracket.*
If the germ $X \backslash 0$ is non-singular, the A-module $\mathrm{Q}(A)$ *is generated by such brackets.*

Obstructions. Let $(X, \mathcal{O}_X)$ be a complex analytic space, any section of the coherent sheaf $\mathcal{Q}(X)$ defines a global star product which is associative up to term (λ^2). Let $\mathrm{P}(X) \subset \Gamma(X, \mathcal{Q}(X))$ be the set (a cone) of sections of that fulfils the Jacobi identity. There is a homogenous mapping $\mathrm{Ob}^0 : \mathrm{P}(X) \to H^1(X, \mathcal{Q}(X))$ such that vanishing of $\mathrm{Ob}^0 p$ is necessary for existence of extension of the Poisson bracket to a star-product. On the kernel $\mathrm{P}_0(X)$ of this mapping there is defined amapping $\mathrm{Ob}^1 : \mathrm{P}_0(X) \to H^1(X, \mathcal{Q}(X))$ such that the equations $\mathrm{Ob}^0 p = \mathrm{Ob}^1 p = 0$ are necessary and sufficient the star product as above admits an associative extension of the star up to a term (λ^3).

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Degenerations and Contractions of Lie algebras and Algebraic groups

Dietrich Burde

1. Degenerations, contractions and deformations of Lie algebras

Degenerations, contractions and deformations play an important role in mathematics and physics. Unfortunately there are many different definitions and special cases of these notions. We try to give a general definition which unifies these notions and shows the connections among them.

1.1. **Degenerations via orbit closures.** Let $\mathfrak{g}$ be an n-dimensional vector space over a field k. Denote by $\mathcal{L}_n(k)$ the *variety of Lie algebra laws*. The general linear group $GL_n(k)$ acts by base changes on $\mathfrak{g}$, and hence on $\mathcal{L}_n(k)$. One denotes by $O(\mu)$ the orbit of μ under the action of $GL_n(k)$, and by $\overline{O(\mu)}$ the closure of the orbit with respect to the Zariski topology.

Definition 1. *Let $\lambda, \mu \in \mathcal{L}_n(k)$ be two Lie algebra laws. We say that λ degenerates to μ, if $\mu \in \overline{O(\lambda)}$. This is denoted by $\lambda \to_{\text{deg}} \mu$.*

Degeneration defines an order relation on the orbit space of n-dimensional Lie algebra laws by $O(\mu) \leq O(\lambda) \iff \mu \in \overline{O(\lambda)}$. For example, every law $\lambda \in \mathcal{L}_n(k)$ degenerates to the abelian law $\lambda_0 \in \mathcal{L}_n(k)$. In general, it is quite difficult to see whether there exists a degeneration $\lambda \to_{\text{deg}} \mu$ between two Lie algebra laws $\lambda, \mu \in \mathcal{L}_n(k)$. It is also interesting to investigate the varieties $\mathcal{L}_n(k)$ and the orbit closures over $\mathbb{R}$ or $\mathbb{C}$ in low dimensions, see [1], [2], [3], [7],[8].

1.2. **Degenerations, contractions and deformations.** Let k be a field and A be a discrete valuation ring (DVR). Grunewald and O'Halloran [5] proved a result which shows that the definition of degeneration generalizes as follows:

Definition 2. *Let $\mathfrak{g}$ be a Lie algebra over k and A a discrete valuation k-algebra with residue field k. Then a Lie algebra $\mathfrak{a}$ over A is a degeneration of $\mathfrak{g}$ over A, if there exists a finite extension L/K of the quotient field K of A, such that $\mathfrak{a} \otimes_A L \cong \mathfrak{g} \otimes_k L$. The Lie algebra $\mathfrak{g}_0 := \mathfrak{a} \otimes_A k$ is called the limit algebra of the degeneration.*

Remark 3. The limit algebra here is also a degeneration in the sense of orbit closure. In this formulation we see that there is a close relationship between deformations and degenerations: If $\mathfrak{a}$ is a degeneration of $\mathfrak{g}$, and $\mathfrak{g}_0$ is isomorphic to the limit algebra of $\mathfrak{a}$ via $\varphi\colon \mathfrak{g}_0 \to \mathfrak{a} \otimes_A k$, then $(\mathfrak{a}, \varphi)$ is a deformation of $\mathfrak{g}_0$.

Definition 4. *Let $\mu_1 \in \mathcal{L}_n(k)$ and $\mathfrak{g} = (V, \mu_1)$. Let A be a discrete valuation k-algebra with residue field k and quotient field K. Let $\varphi \in \mathrm{End}(V_A) \cap GL(V_K)$. If $\mu = \varphi \cdot \mu_1$ is in $\mathcal{L}_n(A)$, and hence μ defines a Lie algebra $\mathfrak{a} = (V_A, \mu)$ over A, then $\mathfrak{a}$ is called a contraction of $\mathfrak{g}$ via φ. The Lie algebra $\mathfrak{g}_0 := \mathfrak{a} \otimes_A k$ is called the limit algebra of the contraction.*

Lemma 5. *Every contraction of $\mathfrak{g}$ is also a degeneration of $\mathfrak{g}$.*

Remark 6. The converse of this lemma in general is not clear. For $\mathbb{R}$ and $\mathbb{C}$ however every degeneration is isomorphic to a contraction (see [9]).

Definition 7. *Let* $\mathfrak{a}$ *be a degeneration of* $\mathfrak{g}$ *and* $\varphi\colon \mathfrak{a}\otimes_A K \to \mathfrak{g}\otimes_k K$ *be an isomorphism. Then the pair* $(\mathfrak{a},\varphi)$ *is called a generalized contraction of* $\mathfrak{g}$ *with* φ.

2. Degenerations and Contractions of algebraic groups

We want to transfer the notions to algebraic groups. Note that in the case of Lie algebras the underlying space does not change (under deformation, degeneration or contraction). This will be different for algebraic groups, where the underlying variety will also change. The main results here are due to C. Daboul, see [4].

2.1. **Affine group schemes.** Let A be a ring. The *spectrum* of A is the pair $(\mathrm{Spec}(A),\mathcal{O})$ consisting of the topological space $\mathrm{Spec}(A)$ together with its structure sheaf $\mathcal{O}$. If $\mathfrak{p}$ is a point in $\mathrm{Spec}(A)$, then the stalk $\mathcal{O}_{\mathfrak{p}}$ at $\mathfrak{p}$ of the sheaf $\mathcal{O}$ is isomorphic to the local ring $A_{\mathfrak{p}}$. Consequently, $\mathrm{Spec}(A)$ is a locally ringed space. Every sheaf of rings of this form is called an affine scheme:

Definition 8. *A locally ringed space* $(X,\mathcal{O}_X)$ *is called an affine scheme, if it is isomorphic to* $\mathrm{Spec}(R)$ *of some ring* R*, i.e., if* $(X,\mathcal{O}_X)\cong(\mathrm{Spec}(R),\mathcal{O}_{\mathrm{Spec}(R)})$.

An affine scheme X is called an A-*scheme*, if its coordinate ring is an A-algebra. For $\mathfrak{p}=(0)$ we call the fibre $X_{\mathfrak{p}}$ the *generic fibre of* X and denote it by X_K. The fibre $X_{\mathfrak{m}}$ is called the *special fibre* and is denoted by X_k. There is the notion of a *smooth affine* A-*scheme*, see [6]. Note that algebraic groups over a field k of characteristic zero are smooth affine k-schemes. If we have a smooth affine group scheme $\mathcal{G}$ over A then we can define its Lie algebra $\mathrm{Lie}(\mathcal{G})$ via $\mathcal{G}$-invariant derivations.

2.2. **Degenerations, contractions and liftings.** An affine group scheme over A can be considered as a family of affine group schemes over the residue fields k_t, where $t\in\mathrm{Spec}(A)$. Its fibres $\mathcal{G}_t$ are in fact affine group schemes with coordinate rings $k_t[\mathcal{G}_t]$. Hence we have $\mathcal{G}_t=\mathcal{G}_{k_t}$, and we use both notations. In particular we write $\mathcal{G}_K$ for the generic fibre of $\mathcal{G}$, where K is the quotient field of A.

Definition 9. *Let* A *be a discrete valuation* k*-algebra with residue field* k *and quotient field* K*. A degeneration of an affine algebraic group* G *over* k *is a smooth affine group scheme* $\mathcal{G}$ *over* A*, such that there is a field extension* L/K *of finite degree, such that* G_L *is isomorphic to* $\mathcal{G}_L$.

The special fiber $\mathcal{G}_k$ then is called the *limit group* of the degeneration.

Definition 10. *Let* A *be a discrete valuation* k*-algebra with residue field* k *and quotient field* K*. A generalized contraction of an affine algebraic group* G *over* k *is a pair* $(\mathcal{G},\Phi)$ *consisting of a degeneration* $\mathcal{G}$ *of* G *and an isomorphism of* K*-group schemes* $\Phi\colon\mathcal{G}_K\to G_K$*. The pair* $(\mathcal{G},\Phi)$ *is called a contraction, if in addition* $\Phi^{\#}(A[G])\subseteq A[\mathcal{G}]$*, where* $\Phi^{\#}$ *denotes the dual map.*

Let G be an affine algebraic group. If $(\mathcal{G}, \Phi)$ is a contraction of G then $(\mathfrak{a}, \varphi) = (\mathrm{Lie}(\mathcal{G}), d\Phi)$ is a contraction of $\mathfrak{g} = \mathrm{Lie}(G)$. The same is true for a generalized contraction. One can show that every generalized contraction of an affine algebraic group is isomorphic to a contraction. Hence each degeneration of a Lie algebra which corresponds to a degeneration of an affine algebraic group is isomorphic to a contraction.

Definition 11. *Let $\mathfrak{a}$ be a deformation or a degeneration of $\mathfrak{g}$ over A. Then a smooth A-group scheme $\mathcal{G}$ with $\mathrm{Lie}(\mathcal{G}) \cong \mathfrak{a}$ is called a lifting of $\mathfrak{a}$.*

If G is an affine algebraic group over k with Lie algebra $\mathfrak{g}$, and if $\mathfrak{a}$ is a degeneration of $\mathfrak{g}$ over a discrete valuation k-algebra A, then we would like to find a lifting of $\mathfrak{a}$ with generic fibre G.

Proposition 12. *Let $\mathfrak{a}$ be a degeneration of $\mathfrak{g}$. Suppose that there exists a conserved representation of $\mathfrak{a}$, which is the derivative of a faithful representation of G. Then we can construct a lifting of the degeneration.*

A main ingredient in the proof is the closure of representations in the sense of schemes. This result applies to many degenerations: if, for example, the center of the limit algebra is trivial, then the adjoint representation is conserved and the condition is satisfied. On the other hand one can use the Neron-Blowup for schemes to obtain the following result:

Proposition 13. *All Inönü-Wigner contractions of Lie algebras can be lifted to the group level.*

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Contractions, gen. IW-contractions and Deformations of finite-dimensional complex (resp. real) Lie Algebras

Evelyn Weimar-Woods

When the velocity of light goes to infinity, the Poincaré group goes to the Galilean group. Over 50 years ago, this idea led to the concept of contractions of Lie algebras [1], [2], [3].

Definition 1 Let $L = (V, \mu)$ be a finite-dimensional complex (resp. real) Lie algebra with vector space V and Lie product μ. For $T(\varepsilon) \in Aut(V)$; $0 < \varepsilon \leq 1$; we denote the transformed Lie product by

$$\mu_{T(\varepsilon)}\ (x,y) = T^{-1}(\varepsilon)\ \mu\ (T(\varepsilon)\ x, T(\varepsilon)\ y)\ ;\ x,y \in V\ . \tag{1}$$

Then for $\varepsilon > 0$ we have

$$L_{T(\varepsilon)} = (V, \mu_{T(\varepsilon)}) \simeq L \quad .$$

If the limit

$$\mu_T(x,y) = \lim_{\varepsilon \to 0} \mu_{T(\varepsilon)}(x,y) \tag{2}$$

exists for all $x, y \in V$, then μ_T is a Lie product and we call $L_T = (V, \mu_T)$ the **contraction** of L by $T(\varepsilon)$. Similarly, if $T_n \in Aut(V)$; $n \in \mathbb{N}$; we can define a **sequential contraction**.

Geometrically, a contraction is a path (resp. sequence) in the space of structure constants which runs through the orbit $O(L) \simeq Aut(V) \big/ Aut(L)$ of L and which has a limit point in its closure. Every point of the closure can be reached by a sequential contraction.

Definition 2 When $T(\varepsilon)$ has, with respect to some basis $e_1, e_2, \cdots e_N$ of V, the diagonal form

$$T(\varepsilon)_{ij} = \delta_{ij}\varepsilon^{n_j} \quad ; \quad \varepsilon > 0\ ;\ n_j \in \mathbb{R}; \tag{3}$$

we call $T(\varepsilon)$ a **generalized Inönü-Wigner-contraction** [4].

In terms of the structure constants given by

$$\mu(e_i, e_j) = C^k_{ij}\ e_k$$

we have

$$\mu_{T(\varepsilon)}(e_i, e_j) = \sum_{k=1}^{N} \varepsilon^{n_i + n_j - n_k}\ C^k_{ij}\ e_k\ .$$

Hence $T(\varepsilon)$ defines a contraction of L if and only if

$$C_{ij}^k = 0 \quad \text{if} \quad n_k > n_i + n_j \tag{4}$$

and the contracted L_T is then given by

$$\mu_T(e_i, e_j) = \sum_{n_k = n_i + n_j} C_{ij}^k \; e_k \quad . \tag{5}$$

For a **simple** IW**-contraction** [2] (i.e. $n_j = 0$ or 1 in Eq. (3)), Eq. (4) means that $(T(0)V, \mu)$ is a subalgebra.

Eqs. (4) and (5) show how easy it is to work with gen. IW-contractions. In contrast to this, even in the general linear case which Saletan [3] studied, the conditions are enormously complicated.

Furthermore, gen. IW-contractions are well suited to contract the representations, invariants, BCH formulas, special functions of L to those of L_T (see also [6] part II).

A few years ago I was finally able to prove [6]

Theorem Every contraction (resp. sequential contraction) is equivalent to a gen. IW-contraction with **integer** exponents.

This theorem has some immediate consequences. It resolves a long-standing problem from the physics literature in the 1960's by showing that any contraction is inverse to an analytic (in fact polynomial) deformation. And this reciprocity shows that $L_T \not\cong L$ implies that L_T is not rigid, thus generalizing earlier results by Segal and myself.

We have untertaken a detailed study of the general properties of **graded** contractions [5] (which are **not** contractions) in order to allow a comparison with the contraction method. In [6] part I we have obtained a complete classification of all complex (resp. real) G-graded contractions, where G is an arbitrary finite group, which shows that they exhibit an interesting mathematical structure. In [6] part II we obtain further structural results e.g. we characterize continuous graded contractions which are equivalent to a **proper** subset of contractions. A careful comparison of the two methods clearly shows the serious and insurmountable defects of graded contractions with respect to their applicability in physics. E. g. they can never relate two faithful self-adjoint representations.

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A deformation problem related to complex supermanifolds

Arkady L. Onishchik

The talk concerns classification problems for complex analytic supermanifolds. The methods used in this theory are closely related to those developped in the classical deformation theory of compact complex manifolds (Kodaira, Spencer, Kuranishi) and in the deformation theory of Lie algebras (Nijenhuis, Richardson).

1. *Complex supermanifolds and the retraction.*

A complex supermanifold of dimension $m|n$ is, by definition, a ringed space $(M, \mathcal{O})$, where M is a topological space and $\mathcal{O}$ is a sheaf of $\mathbb{Z}_2$-graded complex algebras on M, which is locally isomorphic to the model $(D, \mathcal{F}_D \otimes \bigwedge(\xi_1, \dots, \xi_m))$, where $\mathcal{F}_D$ is the sheaf of holomorphic functions on an open set $D \subset \mathbb{C}^n$ and the Grassmann algebra is $\mathbb{Z}_2$-graded in the usual way. Thus, in a neighborhood U of each point of M the even sections $x_i,\ i = 1, \dots, n$, and the odd sections $\xi_j,\ j = 1, \dots, m$, of $\mathcal{O}$ are given, called, respectively, the *even* and the *odd local coordinates.* If $V \subset M$ is another open set with local coordinates (y_i) and (η_j), then on $U \cap V$ the transition functions of the following form are given:

$$\begin{aligned} y_i &= \phi^{i0}(x) + \sum_{r<s} \phi^{i2}_{rs}(x)\xi_r\xi_s + \dots \\ \eta_j &= \sum_r \psi^{j1}_r(x)\xi_r + \sum_{r<s<t} \psi^{j3}_{rst}(x)\xi_r\xi_s\xi_t + \dots, \end{aligned} \tag{1}$$

where all ϕ^{i0}, $\phi^{i2}_{rs}, \dots,$ ψ^{j1}_r, $\psi^{j3}_{rst}, \dots$ are holomorphic functions in $x_1, \dots, x_n$.

As for complex manifolds, we may consider deformations and contractions of complex supermanifolds; they can be expressed by deforming the transition functions (1). The following contraction is very important for the theory. Let a covering of M by coordinate neighborhoods be chosen and let us set $\xi_j = t\tilde{\xi}_j$, where $t \in \mathbb{C}$, in each neighborhood. For any $t \neq 0$ we get the new coordinates $\tilde{\xi}_j$ on the same supermanifold, but the transition functions (1) will be deformed. One sees easily that these deformed functions have the following limit for $t \to 0$:

$$\begin{aligned} y_i &= \phi^{i0}(x), \\ \tilde{\eta}_j &= \sum_r \psi^{j1}_r(x)\tilde{\xi}_r. \end{aligned} \tag{2}$$

The first line of (2) gives the transition functions of a complex manifold $(M, \mathcal{F})$ of dimension n, called the *reduction* of $(M, \mathcal{O})$, while the second line determines a holomorphic vector bundle $\mathbf{E}$ of rank m over $(M, \mathcal{F})$, the odd coordinates ξ_j being interpreted as basic holomorphic sections of $\mathbf{E}$ over U and V. The limit supermanifold $(M, \tilde{\mathcal{O}})$ with the transition functions (2) is called the *retract of*

$(M, \mathcal{O})$; its structure sheaf $\mathcal{O}$ is naturally isomorphic to $\bigwedge_{\mathcal{F}} \mathcal{E}$, where $\mathcal{E}$ is the sheaf of holomorphic sections of $\mathbf{E}$; the sheaf $\mathcal{O}$ posseses a natural $\mathbb{Z}$-grading. Actually, this latter construction can be applied to any holomorphic vector bundle $\mathbf{E} \to M$, giving a supermanifold $(M, \bigwedge_{\mathcal{F}} \mathcal{E}$ with a $\mathbb{Z}$-graded stucture sheaf; the supermanifolds of this form are called *split* ones.

2. *The classification problem.*
The first natural complexification problem related to complex supermanifolds is the following one: to classify, up to isomorphy, all supermanifolds with a given retract. Clearly, all the supermanifolds of dimensions $n|1$ are split, and the simplest non-split example can be constructed in the dimension $1|2$. The following classification theorem was proved in [2]: let $(M, \mathcal{O}_{\mathrm{gr}})$ be the split supermanifold associated with a vector bundle $\mathbf{E}$; then the supermanifolds with retract $(M, \mathcal{O}_{\mathrm{gr}})$ are in a bijective correspondence with $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})/\mathrm{Aut}\,\mathbf{E}$, where $\mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}}$ is the sheaf of those automorphisms of $\mathcal{O}_{\mathrm{gr}}$ which are identical modulo $\bigoplus_{p\geq 2} \mathcal{O}^p_{\mathrm{gr}}$. In certain cases this cohomology set can be calculated.

As an interesting example, consider the cotangent bundle $\mathbf{E} = \mathbf{T}(M)^*$ of M. The corresponding split supermanifold is (M, Ω), where Ω is the sheaf of holomorphic forms. In the case, when M is an irreducible compact Hermitian symmetric space, all the supermanifolds with this retract are described in [4]; for the Grassmanians $M = \mathbf{Gr}_{n,k},\ 1 < k < n-1$, they form a family parametrized by $\mathbb{C}P^1$, and otherwise there exists precisely one non-split supermanifold with retract (M, Ω).

Generally, we can describe the cohomology set $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$ in terms of a certain complex of differential forms on M [5]. Namely, let $\mathcal{T}$ denote the tangent sheaf of $(M, \mathcal{O}_{\mathrm{gr}})$, i.e., the sheaf of derivations of $\mathcal{O}_{\mathrm{gr}}$; it is a graded sheaf of Lie superalgebras which is a locally free $\mathcal{F}$-module. Consider the Dolbeault complex $(\Phi^{0,*}, \bar{\partial})$ corresponding to its subsheaf $\bigoplus_{p\geq 1} \mathcal{T}^{2p}$. Then $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$ can be interpreted as the quotient of the set of "1-cocycles" $Z = \{\omega \in \Phi^{0,1} \mid \bar{\partial}\omega - \frac{1}{2}[\omega, \omega] = 0\}$ by a certain group action (here $[\,,\,]$ is a natural bracket). If M is compact, then the harmonic theory allows to construct the "Kuranishi family" $\mathbf{K} \subset Z$ which is a (finitely dimensional) moduli variety for $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})$. (We remind that the original Kuranishi family of complex structures on a compact smooth manifold describes only the complex structures which are sufficiently close to the given one.)

3. *Homogeneous compact complex supermanifolds.*
We use the following "infinitesimal" definition of homogeneous compact complex supermanifolds. The *tangent space* at a point $x \in M$ of a supermanifold $(M, \mathcal{O})$ is, by definition, the vector superspace $T_x(M, \mathcal{O}) = (m_x/m_x^2)^*$, where m_x is the maximal ideal of the local superalgebra $\mathcal{O}_x$. We have $T_x(M, \mathcal{O}) = T_x(M, \mathcal{O})_{\bar{0}} \oplus T_x(M, \mathcal{O})_{\bar{1}}$, where $T_x(M, \mathcal{O})_{\bar{0}}$ is naturally identified with the holomorphic tangent space $T_x(M, \mathcal{F})$ and $T_x(M, \mathcal{O})_{\bar{1}}$ with the fibre E^*_x of the bundle $\mathbf{E}^*$, dual to the vector bundle $\mathbf{E}$ which determines the retract. Let $\mathcal{T} = \mathcal{D}er\,\mathcal{O}$ denote the tangent sheaf of $(M, \mathcal{O})$; the set $\mathfrak{v}(M, \mathcal{O})$ of its global sections is a Lie superalgebra which is finite-dimensional whenever M is compact. For any $x \in M$, there is a natural

(even) linear evaluation map $\mathrm{ev}_x : \mathfrak{v}(M, \mathcal{O}) \to T_x(M, \mathcal{O})$. We say that the compact supermanifold $(M, \mathcal{O})$ is *homogeneous* if ev_x is onto for each $x \in M$.

If a compact supermanifold $(M, \mathcal{O})$ is homogeneous, then its reduction $M = (M, \mathcal{F})$ is homogeneous, too, and the corresponding vector bundle $\mathbf{E} \to M$ is a homogeneous vector bundle. Moreover, the dual bundle $\mathbf{E}^*$ is spanned by its global holomorphic sections. If, conversely, $\mathbf{E} \to M$ is a homogeneous vector bundle over a compact complex homogeneous manifold and if $\mathbf{E}^*$ is spanned by its global holomorphic sections, then the corresponding split supermanifold $(M, \bigwedge_{\mathcal{F}} \mathcal{E})$ is homogeneous, but non-split supermanifolds with this retract (if they exist) need not be homogeneous ones. E.g., the split supermanifolds (M, Ω) is homogeneous for any compact complex homogeneous manifold M. But (see [4]) in the case, when M is an irreducible compact Hermitian symmetric space, the only non-split homogeneous supermanifolds with retract (M, Ω) are the so-called Π-*symmetric super-Grassmannians* $\Pi\mathbf{Gr}_{n|n,k|k}$ with reductions $M = \mathbf{Gr}_{n,k}$, $1 \le k \le n-1$,. They bild a part of one of four series of flag superspaces which were defined in[3]; these supermanifolds are homogeneous, mostly non-split and have as reductions the classical flag manifolds. In [3], the following problem (in the special case $M = \mathbf{Gr}_{4,2}$) has been raised: to describe all the homogeneous supermanifolds $(M, \mathcal{O})$, whose reduction M is a given complex flag manifold M.

Let P be a parabolic subgroup of a complex semisimple Lie group G, and let $\mathfrak{p} \subset \mathfrak{g}$ denote the corresponding pair of Lie algebras. It is well known that homogeneous vector bundles over $M = G/P$ are the bundles $\mathbf{E}_\varphi$ associated with holomorphic linear representations $\varphi : P \to GL(E)$. A representation φ being fixed, let us consider the corresponding split supermanifold $(M, \bigwedge_{\mathcal{F}} \mathcal{E}_\varphi)$. Its tangent sheaf $\mathcal{T}_\varphi$ is associated with a graded homogeneous vector bundle $\mathbf{T}_\varphi \to M$ which is determined with another representation ψ of P. To apply the classification method exposed in Section 2, we consider the Dolbeault complex $(\Phi^{0,*}, \bar{\partial})$ corresponding to the subsheaf $\bigoplus_{p \ge 1} \mathcal{T}_\varphi^{2p}$. If $(M, \mathcal{O})$ is a homogeneous supermanifold with retract $(M, \bigwedge_{\mathcal{F}} \mathcal{E}_\varphi)$, then we may assume that $\mathfrak{v}(M, \mathcal{O})_{\bar{0}} = \mathfrak{g}$. By a result of [5], $(M, \mathcal{O})$ is represented by a K-invariant element of Z, where K is the compact real form of G. Thus, we should consider the subcomplex $(\Phi^{0,*}, \bar{\partial})^K$ of our Dolbeault complex which can be described in terms of Lie algebras. Namely, we have the semi-direct decomposition $\mathfrak{p} = \mathfrak{r} + \mathfrak{n}$, where $\mathfrak{r}$ is the reductive Levi subalgebra and $\mathfrak{n}$ is the nilradical of $\mathfrak{p}$. Then $(\Phi^{0,*}, \bar{\partial})^K$ is isomorphic to the complex $(C(\mathfrak{n}, E)^{\mathfrak{r}}, \delta)$ of the $\mathfrak{r}$-invariant T-valued cochains of the Lie algebra $\mathfrak{n}$, where T is the fibre of $\mathbf{T}_\varphi$ at the point $P \in M$. Such a complex was introduced in [1] in order to calculate the cohomology of G/P with values in the sheaf of holomorphic sections of a homogeneous vector bundle.

As an application of these methods and results, we formulate the following theorem (in the case $M = \mathbf{Gr}_{4,2}$ it was proved in [6]).

Theorem 1. *Suppose that φ is irreducible. If $M = \mathbf{Gr}_{n,k}$, where $3 \le k \le n-k$ or $k = 2$, $n = 4$, and if there exists a non-split homogeneous complex supermanifold $(M, \mathcal{O})$ with the retract $(M, \bigwedge \mathcal{E}_\varphi)$, then $\mathcal{E}_\varphi \simeq \Omega$. The same assertion is true for the manifold $M = \mathrm{Sp}_{2n}(\mathbb{C})/P$ of maximal isotropic subspaces in the symplectic* *vector space* $\mathbb{C}^{2n}$, $n \geq 2$. *Hence, in the first case* $(M, \mathcal{O}) \simeq \Pi\mathbf{Gr}_{n|n,k|k}$, *while in the second one non-split homogeneous complex supermanifolds with the retract* $(M, \bigwedge \mathcal{E}_\varphi)$ *do not exist.*

For other series of irreducible Hermitian symmetric spaces M, we have not yet found any example of non-split homogeneous complex supermanifolds with the retract $(M, \bigwedge \mathcal{E}_\varphi)$, where φ is irreducible .

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Elliptic gamma functions, triptic curves and $SL_3(\mathbb{Z})$

Giovanni Felder

(joint work with André Henriques, Carlo A. Rossi, Chenchang Zhu)

The elliptic gamma function [5] is a function of three complex variables obeying

$$\Gamma(z+\sigma,\tau,\sigma) = \theta_0(z,\tau)\Gamma(z,\tau,\sigma),\ \theta_0(z,\tau) = \prod_{j=0}^{\infty}(1-e^{2\pi i((j+1)\tau - z)})(1-e^{2\pi i(j\tau+z)}).$$

In [3] three-term relations for Γ involving $ISL_3(\mathbb{Z}) = SL_3(\mathbb{Z}) \ltimes \mathbb{Z}^3$ were discovered, generalizing the modular properties of theta functions under $ISL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.

Here we summarize the results of [2] where we show that these identities are special cases of a set of three-term relations for a family of gamma functions $\Gamma_{a,b}$, which are interpreted geometrically as giving a meromorphic section of a hermitian gerbe on the *universal triptic curve*. This result generalizes the fact that the theta function θ_0 is a section of a hermitian line bundle on the universal elliptic curve.

First, we describe the gerbe by the enlarged gamma function family. For a, b linearly independent in the set Λ_{prim} of primitive vectors (namely not multiples of other vectors) in the lattice $\Lambda = \mathbb{Z}^3$, there is a unique primitive $\gamma \in \Lambda^\vee_{prim}$ in the dual lattice such that $\det(a, b, \cdot) = s\gamma$ for some $s > 0$. For $w \in \mathbb{C}, x \in \Lambda \otimes \mathbb{C} = \mathbb{C}^3$ for which the products converge we define

$$\Gamma_{a,b}(w,x) := \prod_{\delta \in C_{+-}/\mathbb{Z}\gamma} (1 - e^{-2\pi i(\delta(x)-w)/\gamma(x)}) \prod_{\delta \in C_{-+}/\mathbb{Z}\gamma} (1 - e^{2\pi i(\delta(x)-w)/\gamma(x)})^{-1},$$

where $C_{+-} = C_{+-}(a,b) = \{\delta \in \Lambda^\vee | \delta(a) > 0, \delta(b) \leq 0\}$ and $\mathbb{Z}\gamma$ acts on it by translation. We set similarly $C_{-+}(a,b) = C_{+-}(b,a)$. We define $\Gamma_{a,\pm a} = 1$. The function

$\Gamma_{a,b}$ is meromorphic on $\mathbb{C}\times(U_a^+\cap U_b^+)$, where $U_a^+=\{x\in\mathbb{C}^3|\mathrm{Im}(\alpha(x)\overline{\beta(x)})>0\}$ for any oriented basis α, β of the plane $\delta(a)=0$. For linearly independent $a,b\in\Lambda_{prim}$, $\Gamma_{a,b}$ is a finite product of ordinary elliptic gamma functions:

$$\Gamma_{a,b}(w,x)=\prod_{\delta\in F/\mathbb{Z}\gamma}\Gamma\left(\frac{w+\delta(x)}{\gamma(x)},\frac{\alpha(x)}{\gamma(x)},\frac{\beta(x)}{\gamma(x)}\right), \tag{1}$$

for any $\alpha,\beta\in\Lambda^\vee$ satisfying $\alpha(b)=\beta(a)=0$ and $\alpha(a)>0$ $\beta(b)>0$, $F=\{\delta\in\Lambda^\vee|0\le\delta(a)<\alpha(a),0\le\delta(b)<\beta(b)\}$. In particular, we recover $\Gamma(z,\tau,\sigma)=\Gamma_{a,b}(z,(\tau,\sigma,1))$ for the standard basis vectors $a=e_1$, $b=e_2$.

The functions $\Gamma_{a,b}$ satisfy cocycle conditions generalizing the three-term relations of [3]:

$$\Gamma_{a,b}(w,x)\Gamma_{b,a}(w,x)=1,\quad x\in U_a^+\cap U_b^+, \tag{2}$$

$$\Gamma_{a,b}(w,x)\Gamma_{b,c}(w,x)\Gamma_{c,a}(w,x)=\exp\left(-\frac{\pi i}{3}P_{a,b,c}(w,x)\right),\quad x\in U_a^+\cap U_b^+\cap U_c^+,$$

where $P_{a,b,c}(w,x)\in\mathbb{Q}(x)[w]$ can be explicitly described in terms of the Bernoulli polynomial $B_{3,3}$, see [2]. Moreover the gamma functions obey cocycle identities related to the action of the group $ISL_3(\mathbb{Z})=SL_3(\mathbb{Z})\ltimes\mathbb{Z}^3$. Fix a *framing* of Λ_{prim}, namely for each $a\in\Lambda_{prim}$ a choice of oriented basis $(\alpha_1,\alpha_2,\alpha_3)$ of $\Lambda^\vee\otimes\mathbb{R}$ such that $\alpha_1(a)=1$, $\alpha_2(a)=\alpha_3(a)=0$. Let

$$\Delta_a((g,\mu);w,x)=\prod_{j=0}^{\mu(g^{-1}a)-1}\theta_0\left(\frac{w+j\alpha_1(x)}{\alpha_3(x)},\frac{\alpha_2(x)}{\alpha_3(x)}\right),$$

where $(g,\mu)\in ISL_3(\mathbb{Z})=SL_3(\mathbb{Z})\ltimes\mathbb{Z}^3$. Then we have

$$\frac{\Gamma_{g^{-1}a,g^{-1}b}(w+\mu(g^{-1}x),g^{-1}x)}{\Gamma_{a,b}(w,x)}=e^{\pi iP_{a,b}((g,\mu);w,x)}\frac{\Delta_a((g,\mu);w,x)}{\Delta_b((g,\mu);w,x)}, \tag{3}$$

$$\Delta_a(\hat g\hat h;w,x)=e^{2\pi iP_a(\hat g,\hat h;w,x)}\Delta_a(\hat g;w,x)\Delta_{g^{-1}a}(\hat h;w+\mu(g^{-1}x),g^{-1}x), \tag{4}$$

where $\hat g=(g,\mu)$, $\hat h=(h,\nu)$ and $P_{a,b}(\hat g;\cdot)$, $P_a(\hat g,\hat h;\cdot)$ are again in $\mathbb{Q}(x)[w]$.

Let X be the dual of the tautological line bundle of $\mathbb{C}P^2$ restricted to $\mathbb{C}P^2-\mathbb{R}P^2$. Then $ISL_3(\mathbb{Z})$ acts on $X=(\mathbb{C}\times(\mathbb{C}^3-\mathbb{R}^3))/\mathbb{C}^\times$ by $(g,\mu)\cdot[(w,x)]=[(w-\mu(x),g\cdot x)]$. Both $\Gamma_{a,b}$ and Δ_a descend to the quotient X. By equations (2) (3) (4) we have:

Theorem 1. *There is an $ISL_3(\mathbb{Z})$-equivariant Čech 2-cocycle*

$$(\phi_{a,b,c},\phi_{a,b},\phi_a)=(e^{-\frac{2\pi i}{3!}P_{a,b,c}(w,x)},e^{-\frac{2\pi i}{2!}P_{a,b}((g,\mu);w,x)},e^{-2\pi iP_a((g,\mu),(h,\nu);w,x)}),$$

in $C^2_{ISL_3(\mathbb{Z})}(\mathcal{V},\mathcal{O}^\times)$ where $\mathcal{V}$ is the equivariant covering of X made up by $V_a=\{(w,x)|x\in U_a^+\}/\mathbb{C}^\times$, $a\in\Lambda_{prim}$. The image of ϕ in the equivariant Čech complex with values in the sheaf $\mathcal{M}^\times$ of invertible meromorphic sections is the coboundary of the equivariant cochain $(\Gamma_{a,b},\Delta_a)\in C^1_{ISL_3(\mathbb{Z})}(\mathcal{V},\mathcal{M}^\times)$.

The **gamma gerbe** $\mathcal{G}$ is the holomorphic equivariant gerbe on X corresponding to ϕ. Equivalently, it is a holomorphic gerbe on the stack $\mathcal{X} = [X/ISL_3(\mathbb{Z})]$.

More geometrically, if we view gerbes over stacks as central extensions of groupoids, then $\mathcal{G}$ is presented by a groupoid $R \rightrightarrows U_0$ fitting in the central extension of groupoids over U_0:

$$1 \to \mathbb{C}^\times \times U_0 \to R \to U_1 \to 1,$$

where $U_0 = \sqcup V_a$, $U_1 = U_0 \times_X (ISL_3(\mathbb{Z}) \times X) \times_X U_0$, $R = \sqcup L_{a,b} \otimes L_b(g)^{-1}$ with $L_{a,b}$, $L_b(g)$, $(a, b \in \Lambda_{prim}, g \in ISL_3(\mathbb{Z})$ the holomorphic $\mathbb{C}^\times$-bundles with transition functions $\phi_{a,b,c}\phi^{-1}_{a,b,d}$ (on $(V_a \cap V_b) \cap V_c \cap V_d$) and $\phi_{b,b'}(g,\cdot)\phi^{-1}_{b,b''}(g,\cdot)$ (on $V_b \cap V_{b'} \cap V_{b''}$) respectively. Notice that $U_1 = \cup W_{g,a,g^{-1}b}$ where $W_{g,a,g^{-1}b} = \{(g,y) | y \in V_a, g^{-1}y \in V_{g^{-1}b}\}$. Then $\Gamma_{a,b}\Delta_b^{-1}$ provides a meromorphic groupoid homomorphism $U_1 \to R$ hence Γ's and Δ's can be viewed as a meromorphic section of $\mathcal{G}$. A *hermitian structure* on a gerbe in this language is simply a hermitian structure on the complex line bundle associated to the central extension. A holomorphic gerbe with hermitian structure has a canonical connective structure whose curvature represents its Dixmier–Douady class.

Theorem 2. *Using the notation in* (1)*, there is a hermitian structure* $h_{a,b}h_b^{-1}$ *on* $\mathcal{G}$ *with* $h_{a,b}(w,x) = \prod_{\delta \in F/\mathbb{Z}\gamma} h_3\left(\frac{w+\delta(x)}{\gamma(x)}, \frac{\alpha(x)}{\gamma(x)}, \frac{\beta(x)}{\gamma(x)}\right)$ *and* $h_a((g,\mu); w, x) = \prod_{j=0}^{\mu(g^{-1}a)-1} h_2\left(\frac{w+j\alpha_1(x)}{\alpha_3(x)}, \frac{\alpha_2(x)}{\alpha_3(x)}\right)$*, where* h_n *are defined by Bernoulli polynomials:* $h_n(z, \tau_1, \ldots, \tau_{n-1}) = \exp\left(-(4\pi/n!)B_{n-1,n}(\zeta, t_1, \ldots, t_{n-1})\right)$*,* $\zeta = \operatorname{Im} z$*,* $t_j = \operatorname{Im} \tau_j$*.*

Moreover, as with line bundles, we can construct the gamma gerbe $\mathcal{G}$ via (pseudo)-divisors. A *triptic curve* $\mathcal{E}$ is a holomorphic stack of the form $[\mathbb{C}/\iota(\mathbb{Z}^3)]$ with $\iota : \mathbb{Z}^3 \to \mathbb{C}$ a map of rank 2 over $\mathbb{R}$. An *orientation* of a triptic curve $\mathcal{E}$ is given by a choice of a generator of $H^3(\mathcal{E}, \mathbb{Z}) \cong \mathbb{Z}$. Then the stack $\mathcal{T}r := [(\mathbb{C}P^2 - \mathbb{R}P^2)/SL_3(\mathbb{Z})]$ is the moduli space of oriented triptic curves. The stack $\mathcal{X} = [X/ISL_3(\mathbb{Z})]$ is the total space of the universal family of triptic curves over $\mathcal{T}r$. Given an étale map $U \to \mathcal{E}$, let $Z_U = 0 \times_{\mathcal{E}} U$. Z_U is naturally a discrete subset of a principal oriented $\mathbb{R}$-bundle on U. A *pseudodivisor* on U is a function $D : Z_U \to \mathbb{Z}$ such that if $\lim y_n = +\infty$ (resp. $-\infty$) for a sequence y_n in Z_U with relatively compact image in U then $\lim D(y_n) = 1$ (resp. 0). The notion of positive/negative infinity is derived from the orientation of the fibres of the $\mathbb{R}$-bundle. We can globalize this to $\mathcal{X}$, namely for an étale map $U \to \mathcal{X}$, a *pseudodivisor* on U is a function $D : \mathcal{T}r \times_{\mathcal{X}} U \to \mathbb{Z}$ such that for every point $q \to \mathcal{T}r$ with corresponding fibre $\mathcal{E} = q \times_{\mathcal{T}r} \mathcal{X}$, the restriction to $q \times_{\mathcal{T}r} Z_U$ is a pseudodivisor on $U \times_{\mathcal{X}} \mathcal{E}$. Then for two such D_i's, the pushforward $p_*(D_1 - D_2)$ is a divisor on U, hence can be used to twist a line bundle L to $L(p_*(D_1 - D_2))$, where $p : \mathcal{T}r \times_{\mathcal{X}} U \to U$. Using the categorical description of gerbes in [1], we then have

Theorem 3. *The gamma gerbe $\mathcal{G}$ is a gerbe over $\mathcal{X}$ made up by the following data: for U with an étale open map $U \to \mathcal{X}$,*

$$Obj(\mathcal{G}_U) = \{(L, D)|\ L \text{ is a line bundle on } U \text{ and } D \text{ a pseudodivisor on } U\},$$
$$Mor(\mathcal{G}_U)\big((L_1, D_1) \to (L_2, D_2)\big) = \Gamma^\times \left(U,\ \big(L_1^* \otimes L_2\big)\big(p_*(D_2 - D_1)\big)\right),$$

the invertible holomorphic sections.

We also have the following theorems calculating various cohomology groups and Dixmier–Douady classes of the gamma gerbe and of its restriction to a fibre.

Theorem 4. *Let $\mathcal{E} = \mathbb{C}/\iota(\mathbb{Z}^r)$, where $x_j = \iota(e_j)$, the images of the standard basis vectors, are assumed to be $\mathbb{Q}$-linearly independent and to span $\mathbb{C}$ over $\mathbb{R}$. Then*

$$H^{i \leq r-2}(\mathcal{E}, \mathcal{O}^\times) = \wedge^i(\mathbb{C}^r/(x_1, ..., x_r)\mathbb{C})/\wedge^i(\mathbb{Z}^r),\ H^{r-1}(\mathcal{E}, \mathcal{O}^\times) = \mathcal{E} \times \mathbb{Z},$$

and $H^{\geq r}(\mathcal{E}, \mathcal{O}^\times) = 0$. In particular, for $r = 3$, the groups classifying holomorphic and topological gerbes on $\mathcal{E}$ are $H^2(\mathcal{E}, \mathcal{O}^\times) = \mathcal{E} \times \mathbb{Z}$ and $H^3(\mathcal{E}, \mathbb{Z}) = \mathbb{Z}$, respectively.

Theorem 5. *The Dixmier–Douady class $c(\mathcal{G}|_\mathcal{E})$ of the restriction of the gamma gerbe to $\mathcal{E}$ is a generator of $H^3(\mathcal{E}, \mathbb{Z}) = \mathbb{Z}$.*

Theorem 6. *$H^3(\mathcal{X}, \mathbb{Z})$ fits into the short exact sequence*

$$0 \to \mathbb{Z} \to H^3(\mathcal{X}, \mathbb{Z})/torsion \to H^3(\mathbb{Z}^3, \mathbb{Z}) \cong \mathbb{Z} \to 0.$$

The image of the Dixmier–Douady class $c(\mathcal{G}) \in H^3(\mathcal{X}, \mathbb{Z})$ of the gamma gerbe is a generator of $H^3(\mathbb{Z}^3, \mathbb{Z})$.

There should exist non-abelian versions of this story in the context of q-deformed conformal field theory [4].

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Twisted Gauge Theories

Julius Wess
(joint work with Paolo Aschieri, Marija Dimitrijević, Frank Meyer and Stefan Schraml)

The idea to introduce noncommutative coordinates (ncc) is almost as old as quantum field theory. It was W. Heisenberg who proposed ncc to solve the problem of divergent integrals in quantum field theory in a letter to Peierls [2]. This idea was propagated via W. Pauli to Oppenheimer's student H.S. Snyder. He then published the first systematic analysis of a quantum theory built on ncc [3]. Pauli called this work mathematically ingenious but rejected it for reasons of physics [4]. It was the mathematical success of quantum groups and quantum spaces pioneered by V.G. Drinfeld, L. Faddeev, M. Jimbo and Y.I. Manin [5, 6, 7, 8] that revived the interest of ncc in physics.

After this talk, A. de Azcarraga pointed out to me [9] that the idea to give up continuous coordinates for very short distances was already expressed by Riemann in his famous 1854 inaugural lecture. Let me quote the relevant passage, because even today there is no better motivation for investigating this ideas:

"Now it seems that the empirical notion on which the metric determination of Space are based, the concept of a solid body and a light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of Space in the infinitely small do not conform to the hypotheses of geometry; and in fact, one ought to assume this as soon as it permits a simpler way of explaining phenomena...

... An answer to these questions can be found only by starting from that conception of phenomena which has hitherto been approved by experience, for which Newton laid the foundation, and gradually modifying it under the compulsion of facts which cannot be explained by it. Investigations like the one just made, which begin from general concepts, can serve only to ensure that this work is not hindered by too restricted concepts, and that the progress in comprehending the connection of things is not obstructed by traditional prejudices".

In this spirit we shall show that gauge theories can be formulated on ncc. This will be done in the framework of deformation quantization as it was developed by Flato and Sternheimer [10]. The deformed space time structure will be defined by an associative but noncommutative product of C^∞ functions. Such products are known as star products; the best known is the Moyal-Weyl product [11, 12]. In this letter we shall deal with this product exclusively.

From previous work [13, 14, 15] we know that the usual algebra of functions and the algebra of vector fields can be represented by differential operators on the deformed manifold. The deformed diffeomorphisms have been used to construct a deformed theory of gravity. Here we shall show that along the same lines a deformed gauge theory can be constructed as well. The algebra, based on a Lie algebra, will not change but the comultiplication rule will. This leads to a deformed Hopf algebra. In turn this gives rise to deformed gauge theories because the construction of a gauge theory involves the Leibniz rule that is based on the comultiplication.

Covariant derivatives can be constructed with the help of a connection. Different to a usual gauge theory the connection cannot be Lie algebra valued. The construction of covariant tensor fields (curvature or field strength) and of an invariant Lagrangian is completely analogue to the undeformed case. Field equations can be derived. The consistency relation for these field equations require conserved currents. Such currents only exist if we allow enveloping algebra valued gauge fields. It is for the first time that it can be shown that conserved currents exist for a deformed symmetry. There is no Noether theorem in this situation.

The deformed gauge theory has interesting new features. We start with a Lie(G)-valued connection and show that twisted gauge transformations close in Lie(G), however consistency of the equation of motion requires the introduction of additional, new vector potentials. The number of these extra vector potentials is representation dependent but remains finite for finite dimensional representations. Concerning the interaction, the Lie algebra valued fields and the new vector fields behave quite differently. The interaction of the Lie algebra valued fields can be seen as a deformation of the usual gauge interactions; for vanishing deformation parameters the interaction will be the interaction of a usual gauge theory. The interactions of the new fields are deformations of a free field theory for vector potentials; for vanishing deformation parameters the fields become free. As the deformation parameters are supposed to be very small we conclude that the new fields are practically dark with respect to the usual gauge interactions.

Finally we discuss the example of a $SU(2)$ gauge group in the two dimensional representation.

The treatment introduced here can be compared with previous ones. In [16] the noncommutative gauge transformations for $U(N)$ have an undeformed comultiplication. The action is the same if we restrict our discussion, valid for any compact Lie group, to $U(N)$ in the n-dimensional matrix representation. In other terms we show that noncommutative $U(N)$ gauge theories have usual noncommutative gauge transformations and also twisted gauge transformations. In [17, 18, 19, 20, 21] the situation is different because we consider field dependent transformation parameters.

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Inönü - Wigner Contractions and Separation of Variables

George Pogosyan
(joint work with A.A.Izmest'ev, A.N.Sissakian, P.Winternitz)

It is well known that practically all properties of large classes of special functions can be obtained from the representation theory of Lie groups, making use of the fact that the special functions occur as basis functions of irreducible representations, as matrix elements of transformation matrices, as Clebsch-Gordon coefficients, or in some other guise. In this context, one very fruitful application of Lie theory is the algebraic approach to the separation of variables in partial differential equations. In this approach separable coordinate systems (for Laplace-Beltrami, Schrödinger and other invariant partial differential equations) are characterized by complete sets of commuting second order operators. These lie in the enveloping algebra of the Lie algebra of the isometry group, or in some cases of the conformal group, of the corresponding homogeneous space.

In the series of papers [1]-[4] have been presented a **new aspects** of the theory of Lie group and Lie algebra contractions: *the relation between separable coordinates systems in curved and flat spaces, related by the contraction of their isometry group.* These are the specefic realizations of the original Inönü - Wigner contractions. In particular in articles [1]-[3] we have considered the simplest meaningful exaples of two homogeneous spaces: two-dimensional sphere $S_2 \sim O(3)/O(2)$ and two-dimensional hyperboloid $L_2 \sim O(2,1)/O(2)$ was introduced a contraction procedure, called **analytic contractions**, namely, when the contraction parameter - radius of sphere or pseudosphere R, appears in the operators of the algebra, in the eigenvalues and eigenfunctions, and not only in the structure constants. Following this method it is possible to observe the contraction limit at $R \to \infty$ at all levels: the Lie algebra as realized by vector fields, the Lalace-Beltrami operators in the four homogeneous (sphere and hyperboloid from one hand side and Euclidean and pseudo-euclidean from other) spaces, the second order operators in the enveloping algebras, characterizeng separable systems, the separable coordinate systems themselves, the separated (ordinary) differential equations, the separated eigenfunctions of the invariant operators and interbases expansions.

In paper [4] the dimension of the space was arbitrary but only the simplest types were considered, namely subgroup coordinates. Furthemore, we introduce a *graphical method* of connecting subgroup-type coordinates on the sphere $S_n \sim O(n+1)/O(n)$ (characterized by tree diagrams) and on the Euclidean space E_n (characterized by claster diagrams) and give the rule relating the contraction limit $R \to \infty$ of the coordinates, eigenvalues and basis functions. Later on the analytic contractions from the rotation group $O(n+1)$ to the Euclidean group $E(n)$ are used to obtain the asymptotic relations for matrix elements between the eigenfunctions of the Laplace-Beltrami operator corresponding to separation of variables in the subgroup–type coordinates on S_n [5]-[6]. The contraction for non subgroup coordinates have been described in [7].

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Operads and props

Jean-Louis Loday

The notion of algebraic *operad* is an efficient tool to study various "types of algebras". It permits us to construct the (co)homology theories and to give a meaning to "algebras up to homotopy" for instance. In order to study different "types of bialgebras", the relevant tool is the notion of *prop*.

1. Types of algebras

Deformation theory of associative algebras makes use of the Hochschild cohomology. For commutative algebras we use Harrison cohomology and for Lie algebras it is Chevalley-Eilenberg cohomology. But there are some other interesting types of algebras : Poisson algebras, Leibniz algebras (a non-anti-symmetric version of Lie algebras [L1]), pre-Lie algebras (also called Vinberg algebras), dendriform algebras (modelling the formal properties of the shuffles [L2]), $2as$-algebras (algebras with two independent associative operations), for instance. The common features of these types is that they are generated by binary operations. But there are more complicated types which admit generating operations of any arity, for instance the brace algebras (a structure of the Hochschild cochains), the associative (resp. commutative, resp Lie) algebras up to homotopy, the $\mathbf{B}_\infty$-algebras (a structure of cofree Hopf algebras).

The following is a natural question: what is the cohomology theory which permits us to perform a deformation theory ? A second natural question is the following. In several instances we are dealing with a chain complex whose homology is equipped with some algebra type. Though the operations are well-defined on the chains, the relations are satisfied only "up to homotopy". In order to make precise this notion we need to say what is an algebra up to homotopy for the given type. This has been achieved by Jim Stasheff in the sixties for associative algebras : he gave the axioms for *associative algebra up to homotopy* (A_∞-algebra). Lie algebras up to homotopy play a key role in the proof of Deligne's conjecture by Kontsevich.

2. Algebraic operads

It turns out that the theory of operads gives rise to a solution of both problems in many instances. By definition an *algebraic operad* is a functor $\mathcal{P} : \mathrm{Vect} \to \mathrm{Vect}$ equipped with a monoid structure. In other words we are given two transformations of functors: $\iota : \mathrm{Id} \to \mathcal{P}$ and $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ such that γ is associative and ι is a unit for γ. By definition an *algebra over the operad* $\mathcal{P}$ is a vector space A

equipped with a map $\gamma_A : \mathcal{P}(A) \to A$ compatible with ι and γ in an obvious sense. So any operad determines a type of algebras. In the other direction, given a type of algebras, the associated operad is obtained by considering the free algebra over the vector space V as a functor in V.

In most useful examples the free algebra is of the form

$$\mathcal{P}(V) = \bigoplus_n \mathcal{P}(n) \otimes_{S_n} V^{\otimes n}$$

for some family of modules $\mathcal{P}(n)$ over the symmetric group S_n. Another way to describe the operad is to make explicit the composition map γ on the $\mathcal{P}(n)$'s.

$$\text{type of algebras} \quad \leftrightarrow \quad \begin{matrix} \text{algebraic operad} \\ = \\ \text{monad in Vect} \end{matrix} \quad \leftrightarrow \quad \begin{matrix} \text{S}-\text{module} \\ + \\ \text{composition} \end{matrix}$$

The "renaissance" of operad theory happened when Ginzburg and Kapranov [GK] managed to extend Koszul duality for associative algebras to algebraic operads. This theory associates to any quadratic operad $\mathcal{P}$ another operad $\mathcal{P}^!$ called its dual. Example :

$$As^! = As,\ Com^! = Lie,\ Lie^! = Com,$$
$$Leib^! = Zinb,\ Dend^! = Dias,\ Pois^! = Pois.$$

The interesting features of this construction are the following. First, for any $\mathcal{P}$-algebra A there is a natural differential on $\mathcal{P}^{!*}(A)$ and, if $\mathcal{P}^{!*}(\mathcal{P}(V))$ is trivial, then the complex $(\mathcal{P}^{!*}(A), d)$ gives rise to the homology of the $\mathcal{P}$-algebra A. In the classical cases As, Com, Lie we recover Hochschild, Harrison, Chevalley-Eilenberg homology respectively. In the Poisson case we get a new theory [F]. Second, mimicking the classical cobar-construction on coassociative coalgebras, one can define a cobar-construction on algebraic cooperads : $\mathcal{C} \mapsto B\mathcal{C}$ and it turns out that $B\mathcal{P}^{!*}$-algebra is precisely the notion of $\mathcal{P}$-algebra up to homotopy.

3. Types of bialgebras

The classical bialgebras (Hopf algebras) involve the associative operad and also the Lie operad since the primitive elements form a Lie algebra. There is a well-known structure theorem which can be phrased as follows.

Theorem. (PBW+CMM) *Let $\mathcal{H}$ be a cocommutative bialgebra over a characteristic 0 field. TFAE:*
a) $\mathcal{H}$ is connected,
b) $\mathcal{H}$ is isomorphic to $U(\text{Prim}\,\mathcal{H})$,
c) $\mathcal{H}$ is cofree as a connected cocommutative coalgebra.

a)$\Rightarrow$ b) is the Cartier-Milnor-Moore theorem and b) $\Rightarrow$ c) is the Poincaré-Birkhoff-Witt theorem.

Similar results hold for other operads. For instance there is a notion of As^c-*Dend*-bialgebra (dendriform as an algebra, coassociative as a coalgebra). The primitive part can be shown to be a brace algebra and there is an enveloping functor U : brace $\to$ *Dend*. The following structure theorem holds:

Theorem. [R] *Let $\mathcal{H}$ be an As^c-Dend-bialgebra. TFAE:*
a) $\mathcal{H}$ is connected,
b) $\mathcal{H}$ is isomorphic to $U(\mathrm{Prim}\,\mathcal{H})$,
c) $\mathcal{H}$ is cofree as a connected coassociative coalgebra.

To summarize these results we say that the triples of operads (Com, As, Lie) and $(As, Dend, brace)$ are "good" triples of operads. The triple $(As, 2as, \mathbf{B}_\infty)$ has been shown to be good in [LR]. This result is interesting because any (classical) bialgebra, which is cofree, is in fact an As^c-$2as$-bialgebra. A general theory of good triples of operads is under investigation, cf. [L3].

4. Props

In order to study and compare various types of bialgebras we need a tool analogous to algebraic operads. Unfortunately there is no such notion as a free bialgebra because the forgetful functor from bialgebras to vector spaces does not admit a left adjoint. However there is a way to bypass this flaw by constructing directly the analogue of the **S**-modules. They are **S**-bimodules $\mathcal{P}(m, n)$ equipped with a right S_n-action and a left S_m-action. Here the operations with n inputs and one output of the operad setting are replaced by operations with n inputs and m outputs. The composition of these operations do satisfy some obvious associative axiom and the whole structure is called a *prop* (also written PROP in the literature because it was coined first as an acronym). Taking into account what we know about operads, it is natural to ask if a Koszul duality theory holds for props. Indeed such a theory has been set forth for some "connected" props called *properads* by Bruno Vallette [V].

Here is a consequence of this point of view on generalized bialgebras. The Lie algebra of polynomials $\mathbb{K}[p_1, \dots, p_n, q_1, \dots, q_n]$ equipped with the Poisson bracket (Moyal product), can be seen as the symplectic Lie algebra of the operad Com: $sp_n(Com)$. Then, it is not too difficult to define similar Lie algebras for any cyclic operad and even for props. Applying the method of Loday and Quillen [LQ], which computes $H_*(gl_\infty(A))$ in terms of cyclic homology, to compute the homology of $sp_\infty(Com)$ gives Kontsevich graph-complex. Replacing the operad Com by an operad or a prop gives new types of graph-complexes.

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Intuitive introduction to twisted and untwisted affine Kac-Moody algebras via the symmetry of the hydrogen atom and Contractions

Jamil Daboul

In my talk I reviewed several interrelated topics based on some of my papers with different coauthors.

1 . I started by recalling the conserved generators of the hydrogen atom: these are the angular momentum vector $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ and the Laplace-Runge-Lenz vector, $\boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{L} - m\alpha\hat{\boldsymbol{r}}$. They commute as follows (here I am using Poisson brackets)

$$[L_i, L_j] = \epsilon_{ijk} L_k, \ [L_i, A_j] = \epsilon_{ijk} A_k, \ [A_i, A_j] = h\epsilon_{ijk} L_k, \quad \text{where} \quad h := -2mH.$$

In the last seventy years physicists identified the above symmetry "algebra" as $\mathfrak{so}(4), \mathfrak{so}(3,1)$ and $\mathfrak{e}(3)$ for energy values $E < 0, E > 0$ and $E = 0$, respectively. This identification follows if one replaces the Hamiltonian H in the third commutation relation by its energy eigenvalues E. In this way one obtains numerical factors $\varepsilon = -2mE$ instead of the operators h. But since h is an *operator* and NOT a number, the above commutation relations do NOT yield a closed Lie algebra. To obtain a closed algebra we must include all the following operators

$$\mathbb{H}_3 = \{h^n L_i, \ h^n A_i \mid n \geq 0, \ i = 1, 2, 3\} \tag{1}$$

By using the following identifications

$$L_i^{(2n)} = h^n L_i, \quad \text{and} \quad A_i^{(2n+1)} = h^n A_i, \tag{2}$$

the set (1) becomes an infinite Lie algebra which was first identified in [1] as the positive part of 'twisted affine Kac-Moody (KM) algebra of $D_2 = \mathfrak{so}(4)$'. However, there is no KM algebra which is denoted as $D_2^{(2)}$. A closer look shows that $\mathbb{H}_3$ is isomorphic to $A_1^{(2)}$ [2].

Since the Kac-Moody formalism is not familiar to most physicists, I reviewed in my talk the formalism of the affine twisted and untwisted Kac-Moody algebras [3]. I believe that this application of the KM formalism to a finite system, instead to applications to field or string theories, provides a simple and intuitive introduction to the Kac-Moody algebra. In fact, Fuchs and Schweigert made two exercises in their book [4] based on the above model.

2 . I then reviewed the generalization of the above formalism to the N-dimensional Kepler Hamiltonian [2]. The corresponding algebras were called *hydrogen algebras* and were denoted by $\mathbb{H}_N$. The standard finite-dimensional algebras $\mathfrak{so}(N +$

1), $\mathfrak{so}(N,1)$ and $\mathfrak{e}(N)$ can then be reproduced as factor algebras $\mathbb{H}_N/I_N(E)$, relative to an energy-dependent ideal $I_N(E)$. These factor algebras define the contraction from $\mathfrak{so}(N+1), \mathfrak{so}(N,1)$ to $\mathfrak{e}(N)$, if E is used as a contraction parameter, $E \Rightarrow 0$.

3 . Next I reviewed the symmetry algebra of the following two-dimensional Kepler system plus a perturbation potential

$$H(\beta) = \frac{p_1^2 + p_2^2}{2m} - \frac{\alpha}{r} - \beta\, r^{-1/2} \cos\left(\frac{\varphi - \gamma}{2}\right) . \tag{3}$$

Leach and coworkers [5] showed that the symmetry algebra for $E = 0$ for the above Hamiltonian is the Heisenberg-Weyl algebra $\mathfrak{w}_1$ rather than the Euclidean algebra $\mathfrak{e}(2)$.

I studied the symmetry algebras of (3) and their contraction via the Kac-Moody formalism in [6] and [7]. For this I defined two symmetry loop algebras $\mathfrak{L}_i(\beta),\ i = 1, 2$, by choosing the 'basic generators' differently. These $\mathfrak{L}_i(\beta)$ can be mapped isomorphically onto subalgebras of $\mathbb{H}_2$, of codimension 2 or 3, revealing the reduction of symmetry. Both factor algebras $\mathfrak{L}_i(\beta)/I_i(E,\beta)$, relative to the corresponding energy-dependent ideals $I_i(E,\beta)$, are isomorphic to $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1)$ for $E < 0$ and $E > 0$, respectively, just as for the pure Kepler case. However, they yield two different non-standard contractions as $E \to 0$, namely to $\mathfrak{w}_1$ [5, 7] or to an abelian Lie algebra [7], instead of $\mathfrak{e}(2)$ for the pure Kepler case. The above example suggests a general procedure for defining generalized contractions, and also illustrates the *'deformation contraction hysteresis'*, since the contractions of the factor algebras $\mathfrak{L}_i(\beta)/I_i(E,\beta)$ of the 'deformed Hamiltonian' $H(\beta)$ remain unchanged even if we let β go back to zero. (for a detailed mathematical study of deformation and contraction, see the doctoral thesis of my daughter [8])

4 . I also reviewed contractions of $\mathfrak{u}(N)$ and $\mathfrak{gl}(N,\mathbb{R})$, by using the $N-$ dimensional attractive and repulsive isotropic oscillators, with an additional constant force $\mathbf{f}$

$$H = \frac{\mathbf{p}^2}{2m} + \frac{k}{2}\mathbf{x}^2 - \mathbf{f} \cdot \mathbf{x}. \tag{4}$$

I showed that the *"quadrupole moments"* of the oscillator yield *saved realizations* of the symmetric generators of the above algebras. Two different contractions can be obtained, depending on the order of letting the two contraction parameters $(k, f = |\mathbf{f}|)$ go to zero, first for $f \to 0$ followed by $k \to 0$ [9], and then for $k \to 0$ followed by $f \to 0$ [10].

Also the non-commuting limits of the wave functions of the "deformed oscillator (4) were studied [11].

5 . Finally I presented several examples of contractions of affine Kac-Moody algebras relative to their twisted subalgebras $\mathfrak{g}^{(2)}$ where $\mathfrak{g} = A_\ell, D_\ell, E_6$ and $D_4^{(3)}$ and also more unusual contractions. I am still investigating these contractions in collaboration with Marc de Montigny [12].

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Superformality and deformation quantization

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(joint work with Giovanni Felder)

Deformation quantization [1] of a smooth manifold M is the study (of the existence and of the classification) of the associative deformations of the commutative unital algebra of smooth functions (see [12] for an overview of the history and developments of deformation quantization). More precisely, one looks for associative products (usually called *star products*) on $C^\infty(M)[[\hbar]]$ that still have $f \equiv 1$ as a unit and that yield the original product modulo $\hbar$; two star products are considered equivalent if they are related by a module automorphisms of $C^\infty(M)[[\hbar]]$ that starts with the identity. Locality properties are usually imposed: e.g., by requiring the star products to be expressed in terms of bidifferential operators. Since an infinitesimal deformation modulo equivalence is the same as a Poisson brackets on $C^\infty(M)$, one usually starts directly with a Poisson algebra of functions and considers only compatible deformations: viz., one also requires the commutator of the star product divided by $\hbar$ to be equal to the given Poisson bracket modulo $\hbar$. A smooth manifold whose algebra of functions is endowed with a Poisson bracket is called a Poisson manifold. Notice that Poisson brackets are in one-to-one correspondence with Poisson bivector fields: viz., bivector fields whose Schouten–Nijenhuis commutators with themselves vanish. A simple example is that of $\mathbb{R}^n$ endowed with a constant bivector field; in this case, the unique (up to equivalence) product is called the Moyal product and was actually discovered [9] much before deformation quantization was invented.

Deformation quantization is an algebraic version of actual quantization, but it is much weaker and hence exists in more general cases. First, because one is not concerned with convergence problems of the formal series in $\hbar$ (a deformation quantization with convergent series is call strict). Second, because one considers only the abstract algebra of quantized functions but does not require it to be represented on a Hilbert space. A first instance of the gained generality is that deformation quantization is possible also on Poisson manifolds, while actual quantization exists only on (certain) symplectic manifolds, i.e., when the Poisson bivector field is nondegenerate (but see later for further comments). Moreover, in the symplectic case, actual quantization may be defined (e.g., via geometric quantization) only if certain conditions are met, while deformation quantizations always exists, as proved independently by De Wilde and Lecomte [5] and by Fedosov [6]. In this case, there is also a classification of inequivalent star products in terms of de Rham cohomology. The reason why the symplectic case is particularly tractable is that locally one can always choose Darboux coordinates where the symplectic form is constant; so, locally, the Moyal product defines a deformation quantization. The nontrivial problem consists then in showing that one can actually glue the local star products to obtain a global one.

The Poisson case is much more complicated and had to wait many more years to be finally solved by Kontsevich [8]. Again, one has local star products to be glued together, but the local product is a highly nontrivial generalization of Moyal's. Kontsevich's formula is expressed in terms of configuration space integrals for the upper half plane and is very reminiscent of the perturbative expansion of some topological quantum field theory. As we proved in [2], this is actually the case and the TQFT is the so-called Poisson sigma model [7, 11].

A slight generalization of Kontsevich's formula leads [8] to a proof of the formality theorem which ultimately asserts that the space of multivector fields of a manifold is isomorphic as a Lie algebra (with Schouten–Nijenhuis bracket) to the Hochschild cohomology of multidifferential operators (with Hochschild bracket), generalizing the classical isomorphy of vector spaces by Kostant, Hochschild and Rosenberg. More precisely, there is an explicit L_∞-quasiisomorphism between multivector fields and multidifferential operators. The structure relations can also be recovered [2] as Ward identities for the Poisson sigma model.

The Poisson sigma model can also be given Dirichlet boundary conditions corresponding to a submanifold of the target space. We showed in [3] that, in order for this boundary conditions not to break the symmetry of the Poisson sigma model around the zero Poisson structure, the submanifolds have to be coisotropic. The analogue of Kontsevich's formula in this case are formulae, which exist if a certain class (the anomaly) in some appropriate cohomology vanishes, defining star product on the reduced spaces as well as bimodule structures related to the intersections of the submanifolds. In [4] we showed how to obtain these formulae from a generalization to supermanifolds of Kontsevich's L_∞-quasiisomorphism.

Our method, as outlined in [4], allows us in principle, in the absence of anomalies, to quantize not only the algebra of functions of the reduced space but, more generally, the algebra of sections of the normal bundle of it viewed as a Poisson algebra up to homotopy (as observed also in [10] and actually relying on a general construction by Voronov [13]). This might have interesting applications when the reduced space is singular and the algebra of invariant functions is too small.

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Behaviour and dependence problems in contracting invariants of Lie algebras

Rutwig Campoamor-Stursberg

The presented talk focused on the theory of generalized Inönü-Wigner contractions of Lie algebras [10, 12] in combination with their generalized Casimir invariants, in order to obtain generally closed formulae for the invariants of large classes of non-semisimple Lie algebras [3, 5]. Since for discrete contractions no continuous limiting procedure for (generalized) Casimir operators exists, the analysis has been focused mainly on continuous contractions. Physically interesting cases like inhomogeneous and semidirect products with Heisenberg-Weyl algebras have been analyzed in this direction, and various possible generalizations of the classical formulae of Perelomov, Popov, Gruber and O'Raifeartaigh, among others, have been obtained.

A first step is to find sufficient and applicable conditions to ensure that a generalized Inönü-Wigner contraction $\mathfrak{g} \longrightarrow \mathfrak{g}'$ preserves the number $\mathcal{N}(\mathfrak{g})$ of invariants [2, 4]. This problem is of interest when trying to derive closed expressions for affine and inhomogeneous algebras obtained by contraction of simple Lie algebras. A possible approach is considering the Maurer-Cartan equations of both the contraction and contracted algebra, and analyzing the behaviour of differential forms of maximal rank. There are mainly two possibilities, either that a generic element of maximal rank is preserved by the contraction (i.e., that it remains unchanged), or that the rank is preserved, while the differential form changes during the process. One advantage of this procedure is the geometrical interpretation of the involved quantities. For applications, such as the Harrison-Estabrook approach to symmetries of differential equations, the theory of exterior forms is also of great interest, for example for the analytical implementation of contractions, i.e., by the introduction of contraction parameters into realizations of Lie algebras [7, 8].

Another crucial point about contractions, specially for physical applications, is the contraction of the invariants, and more concretely of Casimir operators [1, 3, 11]. Many examples are known which preserve the independence of a fundamental system of invariants when contracting, as well as examples where dependence problems appear. In some cases it is not difficult to predict the appearance of dependence, for example when altering only a part of the Cartan subalgebra of a simple Lie algebra, but general criteria are still to be found. In this context, the structure theorem of contractions (showing its equivalence to generalized Inönü-Wigner contractions with integer exponents [12]) combined with the generalization of the Gel'fand matrix method for the Casimir invariants of Lie algebras constitutes an interesting tool, since it allows to obtain sufficiency criteria based only on matrix theory. These criteria are founded on various reductions of the matrix providing the Casimir operators of a Lie algebra $\mathfrak{g}$ before and after a contraction, and the annihilation of the contraction parameters as a result of these reductions. In this context, applications to the missing label problem (MLP) are quite natural, for reduction chains involving a common subalgebra of a Lie algebra and some contraction [9]. This analysis often provides additional information that gives alternative methods to derive or deduce the independence of Casimir invariants by contractions. An important fact is that, at least for inhomogeneous algebras, dependence problems usually appear not as a consequence of the contraction or the choice of basis for the algebra, but deeply related to the procedure employed to determine the invariants. This fact is illustrated for the special pseudo-unitary and pseudo-orthogonal Lie algebras and their contractions, using various procedures to determine the Casimir operators [6]. Some matrix criteria to ensure the independence of the contracted invariants were discussed.

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Algebraic Quantization Methods For Unimodular Vector Fields

Claude Roger

The vector fields on an oriented manifold X, with vanishing divergence are called "unimodular", they form a Lie subalgebra of the Lie algebra of vector fields on the manifold denoted by $SVect(X)$. The systematic study of the Lie algebra of vector fields, and of its main subalgebras (often called Cartan Lie algebras), from the point of view of their derivations, extensions and deformations has been undertaken in the early seventies by Lichnerowicz and several collaborators. The tools are mainly cohomological, the continuous cohomology of Lie algebras of vector fields, known as Gelfand-Fuks cohomology and extensively developed in the book by D.B. Fuchs.

It lead to famous results for the symplectic and Poisson case, and brought new developments in the theory of deformation quantization (theory of $*$ products, cf Lecomte De Wilde, Kontsevich). The last remaining case was precisely the unimodular one, it was solved by P. Lecomte and the author(see [1])below):

- If the dimension of X is strictly bigger than 3, then the second adjoint cohomology group of $SVect(X)$ vanishes and so the Lie algebra $SVect(X)$ is rigid and infinitesimally rigid.

-If $dim(X) = 3$, one has a generator in the adjoint cohomology in degree 2,so

one has an infinitesimal deformation,which doesn't admit any deformation; (For dimension 2, simply recall that being symplectic and being unimodular is the same thing).
So from the point of view of deformation theory in a formal sense, things are clear : non trivial deformations can exist only for hamiltonian vector fields ,(and so for Poisson brackets, hence the theory of $*$-products). All other Lie algebras of Cartan type are rigid.

The importance of $SVect(X)$, and of its associated "Lie group" $SDiff(X)$, in physics is well known since the pionneering work of V.I. Arnold in the late sixties; there were also further remarkable developments in recent years(cf the book of Arnold and Khesin). Another recent interest for algebraic and geometric properties of this algebra came from a totally different domain: several works by physicists (among others Matsuo, Awata, Hoppe, Schomerus, Pioline) contain different tentative versions of "deformations" of $SVect(X)$. In that case, the interest comes from the theory of branes. Briefly speaking, branes are embedded submanifolds of spacetime, and appear as boundary conditions for dynamical open strings, and moreover carry a volume form and have a dynamic of their own. A survey of general properties of $SVect(X)$ in the light of those quantization problems can be found in [2].

A natural idea is to try to generalize the construction of Poisson formalism to the unimodular case. The volume form allows lifting-lowering of the indices and to construct a dual version of De Rham complex in terms of contravariant tensors (Ω_*, δ), where δ is the codifferential. One then obtains a kind of a resolution $\Omega_* \longrightarrow SVect(X)$, as well as a bracket $\{,\}$ which satisfies: $\delta\{A, B\} = [\delta A, \delta B]$. But unfortunately, it doesn't satisfies Jacobi nor Leibniz identities. One has in fact that $(\Omega_*, \delta), * > 1$, is a L_∞-algebra, that is to say a Lie algebra up to homotopy (cf [3]).One could think of deforming this L_∞-algebra in the category of L_∞-algebras, but apparently all the obvious candidates for deformation cocycles, coming from algebraic homological computations for $SVect(X)$, do not admit prolongations. As far as I know, the following problem is open:

Find a Lie algebra L, which is rigid as a Lie algebra, but which admits deformations as a L_∞-algebra.

Another approach would be to use the ideas sketched in the work of Matsuo and Shibusa (hepth / 0010040); they use the same idea of resolutions as above, but with the Hochschild complex, which is no longer a resolution (precisely because of the presence of Hochschild cohomology !) but which allows more explicit bracket computations. In that approach, one is lead to generalize Poisson bracket to n-uples of functions, so introducing "local Lie algebras" in the sense of A.A. Kirillov. This approach has not yet be given a precise mathematical status.

One can try also a more global approach, inspired by Kontsevich's proof of quantization of Poisson structures. The truncated De Rham dual complex $(\Omega_*, \delta), * > 1$, considered above admits a structure of a Gerstenhaber algebra (G-Algebra), with its exterior product and Schouten bracket intrinsiquely defined. The choice of a volume form allows to define the codifferential δ, which endows $(\Omega_*), * > 1$

with a structure of Batalin-Vilkovisky algebra (BV- algebra).
So one can consider the set of such BV-structures inducing the fixed G-structure on the truncated complex. In other words,one has to consider the following set: $\{\delta : \Omega_* \longrightarrow \Omega_* |\ \delta \circ \delta = 0, dH(\delta) = \Sigma\}$, where Σ indicates the Schouten bracket, viewed as a 2-cochain on the space of contravariant tensors, and dH is the Hochschild differential for the associative graded algebra Ω_*. So the right notion of deformation quantization in the unimodular case should be the quantization of this set, by finding a homotopy equivalence with some other space of cochains, just as Kontsevich did for Poisson structures and $*$-products.
General methods of homotopical algebra, with operad techniques for considering up to homotopy structures, is certainly relevant here. One has then to study the relative operad $(BV) \longrightarrow (G)$, just as Loday did in his theory of triples of operads [4].

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Local Current Algebra and Deformation of the Heisenberg-Poincaré Lie Algebra of Quantum Mechanics

Gerald A. Goldin

(joint work with Sarben Sarkar)

A few years ago, Vilela Mendes [1] argued again for consideration of the combined Heisenberg and Poincaré Lie algebras as a kinematical algebra for relativistic quantum mechanics. This structure is "unstable", but allows a parameterized family of nontrivial deformations that are "stable" or "rigid"—in the sense that all the Lie algebras in an open neighborhood in the space of structure constants are mutually isomorphic. The nontrivial second cohomology of the original Lie algebra is a necessary condition for it to be deformable [2]. Vilela Mendes considers a deformation having two fundamental length scales, and then takes the larger of these to infinity. The positional spectrum in an irreducible representation is discrete (though unbounded), and thus we move in the direction of the talk earlier this week by David Finkelstein [3]. Recent work by Chryssomalakos and Okon discusses the full set of possible stable deformations of the Heisenberg-Poincaré algebra, with explanation of the relevant cohomology theory and detailed references [4].

This talk addresses the problem of defining an infinite-dimensional Lie algebra of local currents compatible with the nonrelativistic quantum kinematics associated with Vilela Mendes' proposal [5]. We first clarify the relation of the irreducible representations of a deformed subalgebra to those of the limiting Heisenberg algebra, concentrating on the case of one space dimension (the generalization to higher dimensions is straightforward). The construction of generalized kinetic energy and harmonic oscillator Hamiltonians in this framework leads to an answer different from that suggested by Vilela Mendes. Two approaches to local current algebra are then considered. One is to localize currents with respect to the discrete spectrum of the deformed position operator. Here, however, the resulting Lie algebra necessarily includes elements having arbitrarily wide support. The second approach is to extend the usual nonrelativistic local current algebra of scalar functions and vector fields (and, correspondingly, infinite-dimensional groups of scalar functions and diffeomorphisms), whose irreducible representations describe a wide variety of quantum systems [6].

The result is to localize with respect to an abstract single-particle configuration space having one dimension more than the original physical space; so that the deformed $(1+1)$-dimensional theory entails self-adjoint representations of an infinite-dimensional Lie algebra of nonrelativistic, local currents on (2+1)-dimensional space-time. However the local operators no longer act in a single irreducible representation of the (global, finite-dimensional) deformed Lie algebra, but connect the reducing subspaces in a direct integral of irreducible representations. Such an approach seems to open up some interesting new possibilities. For example, representations previously interpreted as describing N indistinguishable particles in two-space obeying the intermediate statistics of "anyons" [7, 8, 9] might also provide local currents for a deformed algebra describing N-particle quantum mechanics in one spatial dimension.

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Curvature, contractions and quantum groups

Francisco José Herranz
(joint work with Ángel Ballesteros, Orlando Ragnisco)

Contractions of Lie algebras in Mathematics and Physics is nowdays a well established and developed theory which began to be systematically formulated from the early works of Inönü–Wigner, Saletan and Segal (see [1] and references therein). Roughly speaking, the way to obtain a contracted Lie algebra g' from an inital one g is to define the generators of g' in terms of those of g in an "adequate" way by introducing a contraction parameter ε in such a manner that under the limit $\varepsilon \to 0$ the commutation relations of g reduce to those of g'.

For arbitrary dimension N, two well known examples are: (i) the *flat* contraction that goes from $so(N+1)$ to the Euclidean algebra $iso(N)$ and (ii) the *non-relativistic* contraction that carries the Poincaré algebra $iso(N-1,1)$ into the Galiean one $iiso(N-1)$. When looking at the underlying symmetrical homogeneous spaces associated to the above Lie algebras, one finds that these contractions can geometrically be interpreted in terms of the vanishment of some *constant curvature* of such spaces [2]. The former example relates the ND spherical space of points $SO(N+1)/SO(N)$ of constant curvature $+1/R^2$ (R is the radius of the sphere), with the flat Euclidean one $ISO(N)/SO(N)$, that is, the limit $\varepsilon \to 0$ corresponds to $R \to \infty$; this process keeps in both casis a space of lines of positive constant curvature, $SO(N+1)/\left(SO(N-1)\otimes SO(2)\right)$ and $ISO(N)/\left(SO(N-1)\otimes \mathbb{R}\right)$, respectively. The latter contraction relates two flat spaces of points but can also be regarded as the contraction from the $2(N-1)$D space of (time-like) lines $ISO(N-1,1)/\left(SO(N-1)\otimes\mathbb{R}\right)$, with curvature $-1/c^2$ (c is the speed of light) in the flat Minkowskian spacetime $ISO(N-1,1)/SO(N-1,1)$, to the flat space of worldlines $IISO(N-1)/\left(SO(N-1)\otimes\mathbb{R}\right)$ in the flat Galilean space of points $IISO(N-1)/ISO(N-1)$ under the limit $c\to\infty$. This interpretation of Lie algebra contractions in terms of zero-curvature limits for homogeneous spaces can widely be applied for many other casis fully covering contractions within the four Cartan families of real semisimple Lie algebras.

On the other hand, let us consider a quantum deformation of the Lie algebra g with a Hopf structure, that is, a quantum algebra $U_z(g)$ which is a completion of its universal enveloping algebra $U(g)$ built as formal power series in a deformation parameter z ($q = \mathrm{e}^z$) with coefficients in $U(g)$ [3]. In this case we know that if a Lie algebra contraction $g \to g'$ exists under the limit $\varepsilon \to 0$, then this can be implemented at the deformed level $U_z(g) \to U_{z'}(g')$ through a Lie bialgebra contraction [4] which keeps the same contraction map for the generators while adds some transformation for the contracted deformation parameter $z' = z/\varepsilon^n$, where n is a real number to be fixed for each specific contraction. In this way, for the two aforementioned contractions we find that $U_z(so(N+1)) \to U_{z'}(iso(N))$ and $U_z(iso(N-1,1)) \to U_{z'}(iiso(N-1))$ under the limit $\varepsilon \to 0$.

When a quantum deformation is introduced for a Lie algebra, one gains an extra "quantity" determined by the deformation parameter z which can further be interpreted in different ways depending on the specific model under consideration (as a fundamental scale, a lattice step, a coupling constant, etc.). However, at the same time one pays the price of loosing the Lie structure and therefore the corresponding Lie group together with the associated homogeneous spaces. Hence, in principle, the geometrical interpretation of contractions in terms of curvatures is lost.

Nevertheless if the Lie algebra contraction procedure is read in the reverse direction as a Lie algebra deformation, a new interpretation for quantum deformations arises in a "natural" way. The deformation $g' \to g$ corresponds to introducing a constant curvature in a formerly flat homogeneous space associated to g'. The very same process can again be applied from g to another "less" contracted Lie algebra up to arriving to a semisimple Lie algebra, for which all the associated homogeneous spaces (of points, lines, planes, etc.) involved in the deformation sequence would be endowed with a non-zero constant curvature. Consequently, since quantum algebras go beyond semisimple Lie algebras (generalizing them) the above ideas suggest that a quantum deformation might also be understood as the introduction of a type of curvature in some way. The aim of this report is to show that a quantum deformation also introduces a curvature on some space, generically *non-constant*, which is governed by z [5, 6]; hence the non-deformed limit $z \to 0$, under the which $U_z(g) \to U(g) \sim g$, can also be understood as a zero-curvature limit or contraction.

In order to clarify these points let us consider the non-standard quantum deformation of $sl(2,\mathbb{R})$ written as a Poisson coalgebra with (deformed) Poisson brackets, coproduct Δ_z and Casimir C_z given by [5]:

$$\{J_3, J_+\} = 2J_+ \cosh zJ_-, \quad \{J_3, J_-\} = -2\,\frac{\sinh zJ_-}{z}, \quad \{J_-, J_+\} = 4J_3, \tag{1}$$

$$\begin{aligned} \Delta_z(J_-) &= J_- \otimes 1 + 1 \otimes J_-, \\ \Delta_z(J_l) &= J_l \otimes \mathrm{e}^{zJ_-} + \mathrm{e}^{-zJ_-} \otimes J_l, \quad l = +, 3, \end{aligned} \tag{2}$$

$$C_z = \frac{\sinh zJ_-}{z}\, J_+ - J_3^2. \tag{3}$$

Starting from the following one-particle symplectic realization of (1):

$$J_-^{(1)} = q_1^2, \qquad J_+^{(1)} = \frac{\sinh zq_1^2}{zq_1^2}\, p_1^2, \qquad J_3^{(1)} = \frac{\sinh zq_1^2}{zq_1^2}\, q_1 p_1,$$

the coproduct provides a two-particle symplectic realization:

$$\begin{aligned} J_-^{(2)} &= q_1^2 + q_2^2, \qquad J_+^{(2)} = \frac{\sinh zq_1^2}{zq_1^2}\, p_1^2\, \mathrm{e}^{zq_2^2} + \frac{\sinh zq_2^2}{zq_2^2}\, p_2^2\, \mathrm{e}^{-zq_1^2}, \\ J_3^{(2)} &= \frac{\sinh zq_1^2}{zq_1^2}\, q_1 p_1\, \mathrm{e}^{zq_2^2} + \frac{\sinh zq_2^2}{zq_2^2}\, q_2 p_2\, \mathrm{e}^{-zq_1^2}. \end{aligned}$$

By substituting this in (3) we obtain the two-particle Casimir

$$C_z^{(2)} = \frac{\sinh zq_1^2}{zq_1^2}\,\frac{\sinh zq_2^2}{zq_2^2}\,(q_1p_2 - q_2p_1)^2\,\mathrm{e}^{-zq_1^2}\mathrm{e}^{zq_2^2},$$

which Poisson-commutes with the generators (1). *Any* smooth Hamiltonian function $H_z = H_z(J_-^{(2)}, J_+^{(2)}, J_3^{(2)})$ gives rise to an integrable system [7], for which $C_z^{(2)}$ is the constant of the motion. Thus we find a large family of integrable deformations of the free motion of a particle on the 2D Euclidean space defined by

$$H_z = \tfrac{1}{2} J_+^{(2)}\, f(zJ_-^{(2)}), \tag{4}$$

where f is an arbitrary smooth function such that $\lim_{z\to 0} f(zJ_-^{(2)}) = 1$, that is, $\lim_{z\to 0} H_z = \frac{1}{2}(p_1^2 + p_2^2)$. By writing the Hamiltonian (4) as a free Lagrangian, the metric on the underlying 2D space can be deduced and its Gaussian curvature is, in general, non-constant [5]; the non-deformed limit $z \to 0$ can then be identified with the flat contraction providing the proper Euclidean space. For instance:

• The simplest choice $H_z = \frac{1}{2} J_+^{(2)}$ defines a 2D Riemannian space with metric

$$\mathrm{d}s^2 = \frac{2zq_1^2}{\sinh zq_1^2}\,\mathrm{e}^{-zq_2^2}\,\mathrm{d}q_1^2 + \frac{2zq_2^2}{\sinh zq_2^2}\,\mathrm{e}^{zq_1^2}\,\mathrm{d}q_2^2,$$

whose non-constant Gaussian curvature reads

$$K = -z \sinh\left(z(q_1^2 + q_2^2)\right).$$

• The very "special" case $H_z = \frac{1}{2} J_+^{(2)}\,\mathrm{e}^{zJ_-^{(2)}}$ leads to a Riemannian metric of constant curvature which coincides with the deformation parameter, $K = z$, namely

$$\mathrm{d}s^2 = \frac{2zq_1^2}{\sinh zq_1^2}\,\mathrm{e}^{-zq_1^2}\mathrm{e}^{-2zq_2^2}\,\mathrm{d}q_1^2 + \frac{2zq_2^2}{\sinh zq_2^2}\,\mathrm{e}^{-zq_2^2}\,\mathrm{d}q_2^2.$$

Notice that different potentials on these curved spaces can be obtained by adding a term $U(zJ_-^{(2)})$ to the free Hamiltonian [8]. We also stress that the coproduct ensures the generalization of this construction to arbitrary dimension. This element enables to obtain an N-particle realization of the quantum algebra which, in this framework, corresponds to consider a particle in an ND curved space.

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Sessions of short talks

In the frame of two evening sessions the following short talks were presented:

(1) **Oleg Sheinman:** Krichever-Novikov type algebras in the context of deformations
(2) **Harald Grosse:** Quantum Field Theory on deformed Space-Time
(3) **Jiri Tolar:** Fine gradings of $sl(3, \mathbb{C})$ and the associated graded contractions
(4) **Nikolay Gromov:** Quantum deformations of constant curvature spaces and quantum kinematics
(5) **Abdenacer Makhlouf:** Degeneration, rigidity and irreducible components of Hopf algebras

Reporter: Yaël Frégier

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