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Mini-Workshop: L^2-Spectral Invariants and the Integrated Density of States

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Abstract. L^2-spectral invariants play an increasingly important role in the analysis of infinite geometric objects allowing for the action of a group. Typical such objects are covering spaces like Riemannian manifolds and graphs. The aim is to understand the group and the geometry of the object. The associated L^2-invariants can all be derived from the *integrated density of states* —also known as *spectral distribution function*— of a suitable geometrically induced equivariant Laplacian. On the other hand, the integrated density of states is also a most prominent quantity in the study of Laplacians with additional (dis)order included. Such operators arise in Mathematical Physics and are known as random Schrödinger operators or more general equivariant Hamiltonians. Here, the aim is to understand spectral consequences of the underlying (dis)order.

While overall aims and specific perspectives on the integrated density of states may be somewhat different, it turns out that typical questions in both contexts concern its computation by averaging procedures, its continuity features at certain points of the spectrum and some logarithmic integrals.

Mathematics Subject Classification (2000): IMU-classifications : 13, 4, 9.

Introduction by the Organisers

Both the study of L^2-spectral invariants in geometry and the investigation of the integrated density of states in mathematical physics have attracted much attention in recent years. While the two topics are strongly related, the corresponding communities are rather unaware of each others work and methods. The main aim of this mini-workshop was to bring together people from both fields and provide a basis for interaction.

Accordingly, the first two days of the conference were spent with survey talks solicited by the organizers to highlight concepts and methods. There were 9 such talks with durations between 60 and 90 minutes. The second half of the conference was devoted to more detailed investigations. Most participants used the opportunity to present their current research in the area of the meeting. There were 13 such talks.

The results presented in those talks contained significant contributions e.g. to the Atiyah conjecture about integrality of L^2-Betti numbers for a completely new class of groups by Peter Linnell, a mathematically rigorous derivation using von Neumann traces of the asymptotics of the specific heat near absolute zero by Mikhael Shubin, and approximation results for the integrated density of states in various new contexts.

Altogether the conference was attended by 17 participants.

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Abstracts

The Wegner estimate: an overview and recent results

Peter D. Hislop
(joint work with Jean-Michel Combes, Frederic Klopp)

The Wegner estimate plays a key role in the proof of the continuity properties of the integrated density of states (IDS) [5, 14], and in the proof of Anderson localization using multiscale analysis [1, 3, 16, 18, 17, 19]. It is a simple statement on the probability that the eigenvalues of a local random Hamiltonian are close to a given energy. In order to describe this, let H_ω be a random Schrödinger operator on $L^2(R^d)$. We let H_Λ denote the local random Hamiltonian obtained by restricting H_ω to Λ with self-adjoint boundary conditions. For a fixed energy E_0 and any $\epsilon > 0$ small, the Wegner estimate [24] is the bound

$$P\{\text{dist}(\sigma(H_\Lambda), E_0) < \epsilon\} \leq C_W \epsilon^\alpha |\Lambda|^\beta, \tag{1}$$

for exponents $0 < \alpha \leq 1$, and $0 < \beta$. For purposes of studying the IDS, the exponent $\beta = 1$ is necessary in order to control the thermodynamic limit. Under that condition, the exponent α determines the Hölder continuity of the IDS.

In Wegner's original work, estimate (1) with $\alpha = \beta = 1$ was proved for lattice models on $\ell^2(Z^d)$ with Hamiltonians of the form $H_\omega = -\Delta + V_\omega$. The nonrandom operator $H_0 = -\Delta$ is the finite-difference Laplacian, and V_ω is the random potential given by $(V_\omega f)(n) = \omega_n f(n)$. The family $\{\omega_n\}_{n\in Z^d}$ is a family of independent, identically distributed random variables. Wegner assumed that the random variable ω_0 is distributed with a density h_0 of bounded support so that the probability measure is $d\mu_0(s) = h_0(s)ds$. Much work in recent years has been devoted to proving Wegner's estimate for more general random Hamiltonians (see [20] for a review of some of the progress). These include Hamiltonians on the continuum with more general random potentials. For random families of Hamiltonians on $L^2(R^d)$ of the form $H_\omega = H_0 + V_\omega$, there are two basic classes of random potentials V_ω for which an estimate of the form (1) has been proven: Potentials that are Z^d-ergodic, and those that are R^d-ergodic. The first class includes Anderson-type random potentials, such as on the lattice, that are determined by a family of random variables $\{\omega_j\}$ and, for models on R^d, a single-site potential u. They have the form

$$V_\omega(x) = \sum_{j\in Z^d} \omega_j u(x-j). \tag{2}$$

The strongest results require that u be sign-definite, although there are results for nonsign definite u such as [13] and [21]. Much less is known about R^d-ergodic random potentials. A Wegner estimate of the form (1) was proven for Gaussian random potential in [11] and [15]. Recently, Bourgain and Kenig [2] proved Anderson localization for the Bernoulli-Anderson model for which the random variable ω_0 takes the values 0 or 1 with equal probability 1/2. This singular distribution forced these authors to develop a new Wegner-type estimate based on multiscale analysis. Another important R^d-ergodic model is the Poisson model for which the impurities are distributed according to a Poisson process. A classic Wegner estimate of the form (1) does not hold for this model either [12].

Returning the Z^d-ergodic Anderson-type potentials for random Schrödinger operators on R^d, a special case of our most recent result [6] is an optimal Wegner estimate for which $\beta = 1$ and α is the Hölder exponent of the single-site probability measure μ_0. This implies that the integrated density of states (IDS) of random Schrödinger operators with Anderson-type potentials on $L^2(R^d)$, for $d \geq 1$, is locally Hölder continuous at all energies with the same Hölder exponent $0 < \alpha \leq 1$ as the probability measure for the single-site random variable. In particular, if the probability distribution is absolutely continuous with respect to Lebesgue measure with a bounded density, then the IDS is Lipschitz continuous at all energies, that is $\alpha = \beta = 1$ as in Wegner's original lattice result. The single-site potential $u \in L_0^\infty(R^d)$ must be nonnegative and compactly-supported. The unperturbed Hamiltonian H_0 must be periodic and satisfy a unique continuation principle. This work also includes proofs of analogous continuity results for the IDS of random Anderson-type perturbations of the Landau Hamiltonian in two-dimensions, cf. [4, 8, 22, 23].

The new Wegner estimate for local random Hamiltonians with Z^d-ergodic Anderson-type potentials (2) and general single-site probability measures μ_0 has the following form. For any $\epsilon > 0$, we define, as in [19], $s(\mu_0, \epsilon)$ as

$$s(\mu_0, \epsilon) \equiv \sup_{E \in R} \mu_0([E, E + \epsilon]). \tag{3}$$

For Hölder continuous probability measures, that is, those for which $\mu_0([E_0, E_0 + \epsilon]) \leq C_{\mu_0} \epsilon^\alpha$, we have $s(\mu_0, \epsilon) \leq C_{\mu_0} \epsilon^\alpha$. In applications to continuity of the IDS or Anderson localization, the rate of vanishing of $s(\mu_0, \epsilon)$, as $\epsilon \to 0$, is essential. If, for example, μ_0 is concentrated on a discrete set, our results do not provide this control. Indeed, we note that for a probability measure μ_0, we have $s(\mu_0, \epsilon) \to 0$ when $\epsilon \to 0$ if and only if μ_0 is continuous with respect to the Lebesgue measure. We prove the following optimal Wegner estimate, generalizing (1),

$$\begin{aligned} P\{\operatorname{dist}(\sigma(H_\Lambda), E_0) < \epsilon\} &\leq E\{Tr E_\Lambda([E_0 - \epsilon, E_0 + \epsilon])\} \\ &\leq C_W s(\mu_0, 2\epsilon)|\Lambda|. \end{aligned} \tag{4}$$

This implies, under the assumption that the probability measure is Hölder continuous, the following continuity result on the IDS,

$$0 \leq N(E + \epsilon) - N(E) \leq C_I \, s(\mu_0, \epsilon). \tag{5}$$

The proof of this result (4) employs a new spectral averaging result (cf. [9]) of the following form. Let $u_j(x) = u(x - j)$, and let $H(\omega_j) = H_{j^\perp} + \omega_j u_j^2$, be a one-parameter family of operators, where $H_{j^\perp}$ is H_Λ with $\omega_j = 0$. Let $E_0 \in R$ be fixed and arbitrary, and consider an interval $\Delta_\epsilon = [E_0, E_0 + \epsilon]$, for some fixed $0 < \epsilon < \infty$. We then prove

$$\int_{\Delta_\epsilon} dE \int_0^1 d\mu_0(\omega) \; \Im \langle \phi, u_j^2 \left(\frac{1}{H_j^\perp + \omega_j u_j^2 - E - i\epsilon} \right) u_j^2 \phi \rangle \leq 2\pi s(\mu_0, \epsilon) \|\phi\|^2. \tag{6}$$

The proof of (6) is based on an abstract estimate of the form. Let A and B be two self-adjoint operators on a separable Hilbert space $\mathcal{H}$, and suppose that $B \geq 0$ is bounded. Then, for any $\phi \in \mathcal{H}$, we have the bound

$$(7) \qquad \sum_{n \in Z} \sup_{y \in [0,1]} \langle B\phi, \frac{1}{(A + (n+y)B)^2 + 1} B\phi \rangle \leq \pi(\|B\| + \|B\|^2)\|\phi\|^2.$$

An application of these results to pointwise bounds on the expectation of the spectral shift function (cf. [10]) is presented in [7].

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Survey of analytic L^2-invariants

Józef Dodziuk

L^2-Betti numbers were defined by Atiyah [1] in the following context. Let $\tilde{M}$ be a complete Riemannian manifold with an isometric, free action by a discrete group Γ so that the quotient $M = \Gamma \setminus \tilde{M}$ is compact. The space $\mathcal{H}^p_{(2)}(\tilde{M})$ of L^2-harmonic p-forms is a closed subspace of the Hilbert space $L^2A^p(\tilde{M})$ of square integrable forms of degree p. The orthogonal projection $P : L^2A^p(\tilde{M}) \longrightarrow \mathcal{H}^p_{(2)}(\tilde{M})$ belongs to the von Neumann algebra $\mathcal{N}_p(\tilde{M})$ of bounded operators on $L^2A^p(\tilde{M})$ that commute with the action of Γ. This algebra has a trace tr_Γ and the L^2-Betti numbers are defined as

$$b^p_{(2)}(\tilde{M}) = \dim_\Gamma \mathcal{H}^p_{(2)}\tilde{M}) = \mathrm{tr}_\Gamma \mathrm{P}.$$

This definition is a part of a more general setup. Let Δ_p denote the Hodge-Laplace operator acting on p-forms. Δ_p is essentially self-adjoint and, for every $\lambda \geq 0$, $\chi_{[0,\lambda]}(\Delta_p)$ belongs to $\mathcal{N}_p(\tilde{M})$. The spectral density function $N_p(\lambda)$ is defined as

$$N_p(\lambda) = \mathrm{tr}_\Gamma \chi_{[0,\lambda]}(\Delta_p).$$

Note that $b^p_{(2)}(\tilde{M}) = N_p(0)$. The operator $\chi_{[0,\lambda]}(\Delta_p)$ has a smooth distributional kernel $k_\lambda(x, y)$ and

$$N_p(\lambda) = \int_{\mathcal{F}} \mathrm{tr}_{\mathbb{C}} k_\lambda(x, x)\, dV$$

where the integration extends over a nice fundamental domain $\mathcal{F}$ for the action of Γ and $\mathrm{tr}_{\mathbb{C}}$ denotes the ordinary matrix trace.

Other L^2-invariants are defined in terms of the spectral density functions. Thus Novikov-Shubin invariants [6], [7] are given by

$$\alpha_p(\tilde{M}) = \limsup_{\lambda \to 0^+} \frac{\ln(N_p(\lambda) - N_p(0))}{\ln \lambda}$$

if $N_p(\lambda) - N_p(0) > 0$ for all $\lambda > 0$. We set $\alpha_p(\tilde{M}) = \infty^+$ otherwise.

The determinant of Δ_p is defined in terms of $N_p(\lambda)$ as

$$\ln \det{}_\Gamma \Delta_p = \int_{0^+}^{\infty} \ln \lambda \, N_p(\lambda).$$

ζ function regularization has to be used in order to handle the integration at infinity and one makes an added assumption that the integral converges at the lower end. Manifolds for which this holds are said to be of determinant class and include all manifolds for which the Novikov-Shubin invariants are positive. Once the determinants are defined, one defines the L^2-torsion as

$$\tau(\tilde{M}) = \frac{1}{2} \sum_p (-1)^{p+1} p \ln \det{}_\Gamma \Delta_p.$$

L^2-Betti numbers and Novikov-Shubin invariants are homotopy invariants. One way of proving this (which paved the way for defining the combinatorial L^2-invariants) is to prove that they are equal to their combinatorial counterparts (cf. [3] and [4]) and then prove the invariance of these combinatorial objects. Equivalence of analytic L^2 and topological L^2-torsions is a deep result of [2].

The most important open problems in this field are what values do the L^2-invariants have.

- **Atiyah's Conjecture** The original question of Atiyah [1] was whether there exists a regular covering $\tilde{M}$ of a compact compact manifold M such that the L^2-Betti numbers are irrational. The answer depends *only* on the group Γ and is negative for a very large class of groups. The problem remains open despite considerable progress. See [5, Chapter 2].
- **Novikov-Shubin invariants** There is a circumstantial evidence that Novikov-Shubin invariants are positive rationals but no clear reason why that might be true. In all cases where the invariants have been computed they are positive rationals or ∞^+.
- **Determinant conjecture** asserts that $\det_\Gamma \Delta_p > 0$ in the context described above. The determinant conjecture follows from the conjecture about Novikov-Shubin invariants above.

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Combinatorial L^2-Betti numbers

Gábor Elek

The goal of the talk is to review the classical results of L^2-theory in the context of dimension theory. Let R be a ring, a *dimension* is a function on the left modules over R satisfying the following properties.

- $dim_R(0) = 0$, $dim_R(R^n) = n$, $dim_R(M) \geq 0$ for any R-module M. The dimension function might take infinite values.
- If $M \sim N$, then $dim_R(M) = dim_R(N)$.
- If $0 \to M \to N \to P \to 0$ is an exact sequence of R-modules, then $dim_R(M) + dim_R(P) = dim(N)$.
- $dim(M) = \sup_{P \subseteq M} dim_R(P)$, where the supremum is taken over all finitely generated projective modules contained in M.

If K is a field or a skewfield, then the usual notion of dimension is a dimension function, taking only integer values. For matrix rings $R = Mat_{n\times n}(K)$ one has $dim_R(M) = \frac{1}{n^2} dim_K(M)$ taking dyadic rational values. In the late thirties John von Neumann gave examples of rings with dimension theories with arbitrary real values. His first example is the rank closure of the direct limit ring [1], his second class of examples are the group von Neumann algebras [2]. Using dimension theory one can define Betti numbers for simplicial coverings [2]. Let $\widetilde{K}$ is a simplicial complex with a free and simplicial Γ-action, where $\widetilde{K}/\Gamma$ is a finite simplicial complex. Let us consider the simplicial cochain complex:

$$C^0(\widetilde{K}) \xrightarrow{d} C^1(\widetilde{K}) \xrightarrow{d} \cdots \xrightarrow{d} C^n(\widetilde{K}) \tag{1}$$

Then it is a complex of $C\Gamma$-modules, hence one can tensor it with the von Neumann algebra $N\Gamma$. The resulting complex is the L^2-cohomology complex:

$$N\Gamma \otimes C^0(\widetilde{K}) \xrightarrow{id\otimes d} N\Gamma \otimes C^1(\widetilde{K}) \xrightarrow{id\otimes d} \cdots \xrightarrow{id\otimes d} N\Gamma \otimes C^n(\widetilde{K}) \tag{2}$$

The dimensions of the cohomologies of the complex above are the L^2-Betti numbers. In the talk, we reviewed the properties of these numbers.

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Lifshitz tails

Frédéric Klopp

On $L^2(\mathbb{R}^d)$, consider a homogeneous ergodic alloy-type random Hamiltonian

$$H_\omega := H_0 + V_\omega$$

where the background operator H_0 is, e.g. a $\mathbb{Z}^d$-periodic Schrödinger operator and the random potential is given by

$$V_\omega(x) = \sum_{\gamma \in \mathbb{Z}^2} \omega_\gamma V(x - \gamma)$$

where V is any non-negative compactly supported single site potential, and the $(\omega_\gamma)_\gamma$ are non trivial, i.i.d. random variables.

One defines the integrated density of states ("IDS") of H_ω in the usual way. Namely, for any energy E, one shows that the limit

$$N(E) := \lim_{|\Lambda| \to +\infty} \frac{1}{|\Lambda|} \sharp\{\text{eigenvalues of } H_{\omega|\Lambda} \leq E\}$$

exists; here, the operator $H_{\omega|\Lambda}$ is H_ω restricted to the cube Λ. It defines a non decreasing, non random function of energy E. The almost sure spectrum of H_ω is the set of growth points of N.

The main purpose of the talk is to present and explain the behavior of the IDS at spectral edges. In many cases, it consists in a very fast decay which goes under the name of "Lifshitz tails".

We concentrate on the heuristics that explain the Lifshitz tails and underline the importance of the local uncertainty principle induced by the background Hamiltonian H_0. We also explain the importance of the various assumptions on the model in particular the sign definiteness of the single site potentials. Finally, we discuss some technical tools (mainly from the theories of partial differential operators and of Schrödinger operators) that can be used to draw rigorous results and proofs along the lines given by the heuristics. We discuss open questions.

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Multi-scale Analysis and Localization

Peter Stollmann

This talk gives an outline of mathematical work concerning a phenomenon prominently encountered in the physics of disordered systems and known under the name of localization. A starting point was the explanation of the suppression of effective transport in disordered media by P.W. Anderson who used random Schrödinger operators in his analysis in the late '50s. Multi-scale analysis here means an inductive scheme to prove exponential off-diagonal decay of the Greens function of the problem. This method has been developed in the early '80s on a rigorous mathematical level, [8, 7] and uses ideas from percolation theory. Of the more recent progress, let me mention [4, 10, 9], where new models are treated and a strengthening of the method has been obtained. A different way of proof is the Aizenman-Molchanov [1] or Fractional Moment Method, for which there is now a version in the continuum as well: see [3, 2, 5]. For monographs on random Schrödinger operators, covering a great deal of the literature up to the year '00 we refer to [6, 11, 12].

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L^2-Betti numbers and measure equivalence

Thomas Schick

This is a survey talk, reporting on work of Damian Gaborieau and Roman Sauer about L^2-Betti numbers and measure equivalence. It shows that even in the geometric theory of L^2-invariants, probability theory enters, albeit here in a rather deterministic way.

The main references are [1], in some sense the original article on the subject, and [3, 4] which introduce the (different) approach we mainly describe in the talk. A recommended gentle introduction (without too many details of proofs) is [2].

Two countable discrete groups G, H are called *orbit equivalent* if they have equivalent standard free Borel actions. Here, a *standard free Borel action* is a measure preserving action on the standard Borel probability space ($[0, 1]$, Lebesgue measure) such that for every group element the fixed set has measure zero. Two actions are orbit equivalent if there is a measure preserving Borel-automorphism α of the measure space such that almost every orbit of the first action is mapped onto an orbit of the second action.

If two groups are orbit equivalent then they are *measure equivalent*, i.e. they admit commuting free Borel actions on a common measure space with Borel fundamental domains of finite positive measure.

The talk was mainly concerned with the following theorem due to Damian Gaboriau:

Theorem: *If G, H are two countable discrete groups which are orbit equivalent, then their L^2-Betti numbers coincide, i.e. $b_k^{(2)}(G) = b_k^{(2)}(H)$ for all $k \geq 0$.*

We present the proof of this theorem as given by Roman Sauer. Its main steps are the following. First, one introduces a couple of algebras for a standard Borel free group action as above. These play the role of the complex numbers, the group algebra, and the von Neumann algebra of the group. It is important that they depend only on the orbit equivalence relation, but not on the group.

Secondly, using homological algebra, one defines L^2-Betti numbers for an orbit equivalence relation; using also a finite trace on the von Neumann algebra introduced. One should note that L^2-Betti numbers of groups can be defined in quite a similar way, using the algebras associated to the group instead of the equivalence relation.

The main step is then to prove that the L^2-Betti numbers of a group and of an orbit equivalence relation generated by a group coincide. This uses the similarity of the definitions, together with more homological algebra, and a great dose of dimension theory for von Neumann algebras.

One should note that this is only an equality on the level of dimensions. Novikov-Shubin invariants are e.g. not invariants of orbit equivalence.

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K-theorie, Gap-Labelling and Quasicrystals

Johannes Kellendonk

I gave a review about the Gap-Labelling for aperiodic systems. Further information including citations can be found in [1]. Quite generally, under gap labelling one may understand an order preserving map from Gap(H), the set of gaps in the spectrum of a Schrödinger operator H with its natural order, into a countable ordered group. This ordered group may have to be more complicated than the integers, in case one has a complicated spectrum (e.g. a Cantor spectrum).

The first approach was that by Johnson and Moser for Schrödinger operators on the real line. Let $\alpha(E)$ be the density of zeros of a real solution ψ of $H\psi = E\psi$. When working with the hull of the potential and assuming that it carries an ergodic probability measure this density can be obtained as an average. α is monotonically increasing and constant on gaps. Thus it defines an order preserving map $\mathrm{Gap}(H) \to \mathbb{R}$ and one is interested in finding a countable discrete subgroup of $\mathbb{R}$ which contains its image. Johnson and Moser calculated this group for almost periodic potentials obtaining the so-called frequency module.

The K-theoretical approach which was proposed by Bellissard works with the spectral of projections P_Δ of H on its states below the gap $\Delta \in \mathrm{Gap}(H)$. If A is a C^*-algebra which has the property that it contains all bounded functions of H vanishing at infinity then these spectral projections belong to A. The ordered K_0-group of A may serve as the target space of the gap-labelling map which assigns to a gap Δ the K_0-class $[P_\Delta]$ of the spectral projection. This approach has two benefits: first, A can be constructed so as to reflect the (aperiodic) order of the solid, and second, $K_0(A)$ is a topological invariant thus insuring a kind of stability of the gap-label under perturbations. This is the abstract K-theoretical gap-labelling which can be completed to a gap-labelling by real numbers if one chooses a trace τ on A. The choice of the trace reflects the physical phase in which one supposes the material to be. The trace induces a state τ_* on the ordered K_0-group and when composed with the map above gives a gap-labelling into the real numbers, $\Delta \mapsto \tau_*[P_\Delta]$. The gap-labelling group is by definition $\tau_*(K_0(A))$. Similar as for the Johnson and Moser gap-labelling ergodicity properties on the trace identify the gap-label of a gap with the value of the integrated density of states (IDS) at the gap.

The choice of A is crucial. Taking A to be the algebra of bounded functions of H vanishing at infinity is useless, as its K-theory is not only uncomputable in the interesting cases but also does not contain much structure. For an aperiodic system the algebra A can be constructed from the aperiodic structure. In fact, for an aperiodic system the Hamiltonian is typically of the form $H = -\Delta + V$ with $V(x) = \sum_{y\in P} v(x-y)$. Here v is a local potential with short enough range and P the aperiodic configuration, for instance, the set of equilibrium positions of the atoms of an aperiodic solid. P is typically uniformly discrete. We suppose it is of finite local complexity, , i.e. for all R the set of R-patches $\{(B_R(x)\cap P)-x | x \in P\}$ is finite. V is an example of a P-equivariant function. In general we call a function f on the euclidean space P-equivariant with range R if $(B_R(x)\cap P)-x = B_R(y)\cap P)-y$ implies $f(x) = f(y)$. Let $C_P(\mathbb{R}^d)$ be the sup-norm closure of all continuous complex valued P-equivariant functions with finite range. It carries an $\mathbb{R}^d$ action by translation. Then the algebra of the aperiodic structure described by P is the corresponding crossed product algebra $A_P = C_P(\mathbb{R}^d) \rtimes \mathbb{R}^d$.

The gap-labelling theorem for this case states that the gap-labelling group $\tau_*(K_0(A_P))$ is the countable subgroup of the real numbers generated by the relative frequencies of the local patches in P [2]. This supposes that the relative frequencies can be defined and, via ergodicity arguements, related with the trace τ on A.

It is not clear how to measure directly a gap label, i.e. the IDS at a gap. Measuring it at a gap would require measuring the density of states for all energies below the gap. There is, however, a quantity which equals the IDS at the Fermi energy, provided the latter lies in a gap, and which has only to be measured near the Fermi energy. This is the pressure per unit energy (a compressibility) on the boundary of the system. Its relation with the IDS is topological, it can be obtained by non-commutative topological methods [3].

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Spectral asymptotics of Laplacians on bond-percolation graphs

Peter Müller

This talk is based on joint work with Werner Kirsch [6] and Peter Stollmann [8].

Spectral theory of random graphs is still a widely open field. The recent contributions [1, 3, 7, 2, 4] take a probabilistic point of view to derive heat-kernel estimates for Laplacians on *supercritical* Bernoulli bond-percolation graphs in the d-dimensional integer lattice $\mathbb{Z}^d$. On the other hand, traditional methods from spectral theory are used in [6] to investigate the integrated density of states of Laplacians on *subcritical* bond-percolation graphs. Depending on the boundary condition that is chosen at cluster borders, two different types of *Lifshits asymptotics* are found [6] at the lower spectral edge $E = 0$ and the upper spectral edge $E = 4d$. For example, the integrated density of states of the Neumann Laplacian behaves as

$$\text{“ } N_{\mathrm{N}}(E) - N_{\mathrm{N}}(0) \sim \exp\{-E^{-1/2}\} \text{ ”} \qquad \text{as } E \downarrow 0 \tag{1}$$

at the lower spectral edge for bond probabilities p below the percolation threshold p_c. The quotation marks in (1) indicate that, strictly speaking, one should take appropriate logarithms on both sides. The Lifshits exponent $1/2$ in (1) is independent of the spatial dimension d. This is explained by the fact that, asymptotically, N_{N} is dominated by the smallest eigenvalues which arise from very long *linear* *clusters* in this case. In contrast, for the Dirichlet Laplacian and $p < p_c$, we show that

$$\text{“ } N_{\mathrm{D}}(E) \sim \exp\{-E^{-d/2}\} \text{ ”} \qquad \text{as } E \downarrow 0\,. \tag{2}$$

Note that $N_{\mathrm{D}}(0) = 0$. The Lifshits exponent in (2) comes out as $d/2$, because the dominating small Dirichlet eigenvalues arise from large *fully connected cube- or sphere-like clusters*. Thus, depending on the boundary condition (and the spectral edge) different geometric graph properties show up in the integrated density of states. The reader is referred to the literature cited in [6] for a discussion of other spectral properties of these and closely related operators, for the history of the problem and what is known in the physics literature. Lifshits asymptotics for a Neumann Laplacian on Erdős–Rényi random graphs are studied in [5].

Let us now turn to the *supercritical phase* of bond-percolation graphs. Clearly, one would not expect the contribution of the finite clusters to alter the picture completely. But for the infinite percolating cluster, the story is different. We prove in [8] that the percolating cluster produces a *van Hove asymptotics*

$$\text{“ } N_{\mathrm{N}}(E) - N_{\mathrm{N}}(0) \sim E^{d/2} \text{ ”} \qquad \text{as } E \downarrow 0 \tag{3}$$

in the Neumann case for $p > p_c$. There is also an additional Lifshits-tail behaviour due to finite clusters, but it is hidden under the dominating asymptotics (3). Loosely speaking, (3) is true because the percolating cluster looks like $\mathbb{Z}^d$ on very large length scales (bigger than the correlation length) for $p > p_c$. On smaller scales its structure is more like that of a jagged fractal. The Neumann Laplacian does not care about these small-scale holes, however. All that is needed for (3) to be true is the existence of a suitable d-dimensional, infinite grid. In contrast, the Dirichlet Laplacian does care about holes at all scales so that (2) continues to hold for $p \geq p_c$ [8]. The reason for this is that low-lying Dirichlet eigenvalues require large fully connected cube- or sphere-like regions, and this is a large-deviation event.

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Spectral invariants of symmetric spaces via representation theory

Martin Olbrich

Let $X = G/K$ be a Riemannian symmetric space of the noncompact type. Here G is a real, connected, linear, semisimple Lie group without compact factors, and $K \subset G$ is a maximal compact subgroup. It is the universal cover of compact locally symmetric spaces of the form $Y = \Gamma\backslash X$, where $\Gamma \cong \pi_1(Y)$ can be identified with a discrete, torsion-free, cocompact subgroup of G. In this talk we are going to discuss spectral invariants of the p-form Laplacians Δ_p acting on $L^2(X, \Lambda^p T^* X)$ (with respect to Γ):

$$
\begin{aligned}
b_p^{(2)}(Y) &= \dim_\Gamma(\ker \Delta_p) \quad (L^2\text{-Betti numbers}) ,\\
\alpha_p(Y) &= \sup\{\beta \mid \mathrm{Tr}_\Gamma e^{-t\Delta'_p} = O(t^{-\frac{\beta}{2}}) \text{ as } t \to \infty\}\\
&\qquad\qquad\qquad\qquad (\text{Novikov-Shubin invariants}) ,\\
\tau^{(2)}(Y) &= \frac{1}{2}\sum (-1)^{p+1} p \log \det\nolimits_\Gamma(\Delta'_p) \quad (L^2\text{-torsion}) .
\end{aligned}
$$

Here Δ'_p denotes the restriction of Δ_p to the orthogonal complement of $\ker \Delta_p$. We write $\alpha_p(Y) = \infty^+$ iff Δ_P has a spectral gap around zero. Since $Y \cong B\Gamma$ these spectral invariants coincide with the corresponding L^2-invariants of Γ.

The fundamental rank $\mathrm{frk}(X)$ is defined algebraically as $\mathrm{rk}_\mathbb{C} G - \mathrm{rk}_\mathbb{C} K$. It can be expressed geometrically as $\mathrm{frk}(X) = \min\{\dim X^k \mid k \in K\}$, where $X^k = \{x \in X \mid kx = x\}$ is the fixed point set of k.

The compact dual X^d of X is defined as follows: Let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebras of G, K. Then we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and $\mathfrak{g}^d := \mathfrak{k} \oplus i\mathfrak{p}$ is another subalgebra of the complexification of $\mathfrak{g}$. Let G^d be the corresponding analytic subgroup of the complexification $G_\mathbb{C}$ of G. Then G^d is a compact group, and $X^d = G^d/K$. We normalize the Riemannian metric on X^d such that multiplication by i becomes an isometry $T_{eK}X \cong \mathfrak{p} \to i\mathfrak{p} \cong T_{eK}X^d$.

Using the Harish-Chandra Plancherel theorem [5] and results on relative Lie algebra cohomology [2] one can compute the L^2-invariants explicitely. It turns out that their behaviour is almost entirely determined by $\mathrm{frk}(X)$. A proof of the following theorem is given in [9]. It completes results previously obtained in [3],[4],[1],[8],[7], and [6].

Theorem 1. Set $n = \dim X$, $m = \mathrm{frk}(X)$. Let $\chi(Y)$ be the Euler characteristic of Y. Then we have

(a) $b_p^{(2)}(Y) \neq 0 \Leftrightarrow m = 0$ and $p = \frac{n}{2}$.
In particular, $b_{\frac{n}{2}}^{(2)}(Y) = (-1)^{\frac{n}{2}}\chi(Y) = \frac{\mathrm{vol}(Y)}{\mathrm{vol}(X^d)}\chi(X^d)$.

(b) $\alpha_p(Y) \neq \infty^+ \Leftrightarrow m > 0$ and $p \in [\frac{n-m}{2}, \frac{n+m}{2}]$.
In this range $\alpha_p(Y) = m$.

(c) $\tau^{(2)}(Y) \neq 0 \Leftrightarrow m = 1$.
In this case $\tau^{(2)}(Y) = (-1)^{\frac{n-1}{2}}\pi\frac{\mathrm{vol}(Y)}{\mathrm{vol}(X^d)}\chi(X_1^d)Q_X$, where $X_1 \subset X$ is a

certain subsymmetric space with $\text{frk}(X) = 0$ and Q_X is an explicitely computable positive rational number.

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Geometric Theory of Lattice Vibrations and Specific Heat

Mikhail Shubin
(joint work with Toshikazu Sunada)

We discuss, from a geometric standpoint, the specific heat of a solid. This is a classical subject in solid state physics which dates back to a pioneering work by Einstein [4] and its refinement by Debye [3]. (See also [2, 5] for quantum theory of crystal lattices, and [1] for a review on the theory of specific heat as it looks from a physical points of view.) Using a special quantization of crystal lattices and calculating the asymptotic of the integrated density of states at the bottom of the spectrum, we obtain a rigorous derivation of the classical Debye T^3 law on the specific heat at low temperatures. The idea and method are taken from discrete geometric analysis which has been recently developed for the spectral geometry of crystal lattices (cf. [6] and references there).

The talk is based on [7].

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The Eigenvalue Distribution of Random Schrödinger Operators and CMV Matrices

Mihai Stoiciu

In addition to the Anderson localization, one can also study the statistical distribution of the eigenvalues of random Schrödinger operators. It turns out that in the regimes where the Anderson localization is expected, the local statistical distribution of these eigenvalues is Poisson (no local correlation).

The first result on the local structure of the spectrum of the one-dimensional Schrödinger operators was obtained by Molchanov in [5]. This was extended to the multi-dimensional Anderson tight-binding model by Minami [4], using the proof of the Anderson localization discovered by Aizenman and Molchanov [1].

A similar result holds for the eigenvalues of random CMV matrices; see [10] and [11]. More precisely, for some classes of random CMV matrices, the local statistical distribution of the eigenvalues converges to the Poisson distribution. The same result holds for the distribution of the zeros of random paraorthogonal polynomials on the unit circle. In [3], Davies and Simon extended these results to zeros of regular random orthogonal polynomials.

The CMV matrices were introduced by Cantero, Moral and Velázquez in [2]. These unitary matrices are intimately connected with the orthogonal polynomials on the unit circle (see the monographs [6] and [7] on the theory of orthogonal polynomials on the unit circle and the review article [9] on CMV matices).

A CMV matrix is a five-diagonal matrix realization for the unitary operator $z \to zf(z)$ on $L^2(\mathbb{T};\mu)$, where μ is a non-trivial probability measure on the unit circle $\mathbb{T}$ (we call a measure non-trivial if it is not supported on finitely many points). For any such measure μ we can apply the Gram-Schmidt procedure to the set of polynomials $\{1, z, z^2, \ldots\} \in L^2(\mathbb{T},\mu)$ and get the set of monic orthogonal polynomials $\{\Phi_0(z,d\mu),\ \Phi_1(z,d\mu),\ \Phi_2(z,d\mu),\ \ldots\} \in L^2(\mathbb{T};\mu)$.

These polynomials obey the recurrence relation

$$\Phi_{k+1}(z,d\mu) = z\Phi_k(z,d\mu) - \overline{\alpha}_k\Phi_k^*(z,d\mu) \qquad k \geq 0 \tag{1}$$

where, for $\Phi_k(z) = \sum_{j=0}^k b_j z^j$, the reversed polynomial $\Phi_k^*(z)$ is defined by $\Phi_k^*(z) = \sum_{j=0}^k \overline{b}_{k-j} z^j$. The recurrence coefficients $\{\alpha_n\}_{n\geq 0}$ are called Verblunsky coefficients; they are complex numbers of absolute value < 1.

If we apply the Gram-Schmidt algorithm to the sequence $\{\ 1,\ z,\ z^{-1},\ z^2,\ z^{-2},\ \ldots\ \}$ we get the set $\{\ \chi_0(z),\ \chi_1(z),\ \chi_2(z),\ \ldots\ \}$, which is a basis of $L^2(\mathbb{T};\mu)$. The CMV matrix associated to the measure μ is the matrix representation of the operator $f(z) \to zf(z)$ on $L^2(\mathbb{T};\mu)$. It has the form:

$$\mathcal{C} = \begin{pmatrix} \bar\alpha_0 & \bar\alpha_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \dots \\ \rho_0 & -\bar\alpha_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \dots \\ 0 & \bar\alpha_2\rho_1 & -\bar\alpha_2\alpha_1 & \bar\alpha_3\rho_2 & \rho_3\rho_2 & \dots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar\alpha_3\alpha_2 & -\rho_3\alpha_2 & \dots \\ 0 & 0 & 0 & \bar\alpha_4\rho_3 & -\bar\alpha_4\alpha_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{2}$$

where $\rho_k = \sqrt{1-|\alpha_k|^2}$.

Note that the Jacobi matrices obtained in a similar way for orthogonal polynomials on the real line are tri-diagonal matrices. As in the case of orthogonal polynomials on the real line, an important connection between CMV matrices and monic orthogonal polynomials is

$$\Phi_n(z) = \det(zI - \mathcal{C}^{(n)}) \tag{3}$$

where $\mathcal{C}^{(n)}$ is the upper left $n \times n$ corner of $\mathcal{C}$.

If $|\alpha_{n-1}| = 1$, then the CMV matrix decouples between $(n-1)$ and n. The upper left corner is an $(n \times n)$ unitary matrix

$$\mathcal{C}^{(n)} = \mathcal{C}^{(n)}_{\{\alpha_0,\alpha_1,\dots,\alpha_{n-1}\}} \tag{4}$$

We will consider random CMV matrices and study the statistical distribution of their eigenvalues. We will randomize the matrix $\mathcal{C}^{(n)}_{\{\alpha_0,\alpha_1,\dots,\alpha_{n-1}\}}$ by taking independent identically distributed random variables α_0, α_1, …, α_{n-2}. The last variable, α_{n-1} will be chosen to be uniformly distributed on the unit circle. As in the case of random Schrödinger operators, for various classes of $n \times n$ random CMV matrices, the local statistical distribution of the eigenvalues of these matrices will converge (as $n \to \infty$) to the Poisson distribution. This fact indicates that, as n gets large, there is no local correlation between these eigenvalues. Following [10] and [11], we have:

Theorem. *Consider the random CMV matrices $\mathcal{C}^{(n)} = \mathcal{C}^{(n)}_{\{\alpha_0,\alpha_1,\dots,\alpha_{n-1}\}}$ where $\alpha_0, \alpha_1, \dots, \alpha_{n-2}$ are i.i.d. random variables distributed uniformly in a disk of radius $r < 1$, and α_{n-1} is another random variable independent of the previous ones and uniformly distributed on the unit circle.*

Consider the space $\Omega_n = \{\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1}) \in D(0,r) \times D(0,r) \times \dots \times D(0,r) \times \mathbb{T}\}$ with the probability measure $\mathbb{P}_n$ obtained by taking the product of the uniform (Lebesgue) measures on each $D(0,r)$ and on $\mathbb{T}$. Fix a point $e^{i\theta_0} \in \mathbb{T}$ and let $\zeta^{(n)}$ be the point process defined by $\zeta^{(n)} = \sum_{k=1}^{n} \delta_{z_k}$, where $\{z_1, z_2, \dots, z_n\}$ are the eigenvalues of the matrix $\mathcal{C}^{(n)}$ (each eigenvalue z_i depends on $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$).

Then, on a fine scale (of order $\frac{1}{n}$) near $e^{i\theta_0}$, the point process $\zeta^{(n)}$ converges to the Poisson point process with intensity measure $n\,\frac{d\theta}{2\pi}$ (where $\frac{d\theta}{2\pi}$ is the normalized Lebesgue measure). This means that for any fixed $a_1 < b_1 \le a_2 < b_2 \le \dots \le a_m <$

b_m and any nonnegative integers $k_1, k_2, \ldots, k_m$, the sequence:

$$\mathbb{P}_n\big(\zeta^{(n)}\left(e^{i(\theta_0+\frac{2\pi a_1}{n})}, e^{i(\theta_0+\frac{2\pi b_1}{n})}\right) = k_1, \ldots, \zeta^{(n)}\left(e^{i(\theta_0+\frac{2\pi a_m}{n})}, e^{i(\theta_0+\frac{2\pi b_m}{n})}\right) = k_m\big)$$

converges to $e^{-(b_1-a_1)}\frac{(b_1-a_1)^{k_1}}{k_1!} \ldots e^{-(b_m-a_m)}\frac{(b_m-a_m)^{k_m}}{k_m!}$ as $n \to \infty$.

The same result holds if we consider singular rotation-invariant distributions: $\Omega_n = \{\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-2}, \alpha_{n-1}) \in C(0,r) \times C(0,r) \times \cdots \times C(0,r) \times \mathbb{T}\}$ with the probability measure $\mathbb{P}_n$ obtained by taking the product of the uniform (one-dimensional Lebesgue) measures on each circle $C(0,r)$ ($r \in (0,1)$) and on $\mathbb{T}$.

Recent work of Simon [8] showed that if the Verblunsky coefficients are independent random variables and with exponentially decaying distributions, then the eigenvalues of the CMV matrices repel each other giving a "clock distribution" on the unit circle. It would be very interesting to discover which distributions of the random Verblunsky coefficients give a transition in the statistical distribution of the eigenvalues from non-correlation (Poisson) to repulsion ("clock").

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Elliptic operators on infinite graphs

Józef Dodziuk

Colin de Verdière [1] defines elliptic operators A on functions $f : V \longrightarrow \mathbb{R}$ from the set V of vertices of a graph $K = (V, E)$ (without loops or multiple edges) as follows. Let $Af(x) = \sum_{x \in V} b_{xy} f(y)$. These operators are required to be local in the sense that $b_{xy} \neq 0$ only when either $x = y$ or $x \sim y$, i.e. x and y span an edge. Such local operators in turn can be written as

$$Af(x) = \sum_{x\sim y} a_{xy}(f(x) - f(y))) + V(x)f(x). \tag{1}$$

A is elliptic if and only if $a_{xy} \neq 0$ whenever $x \sim y$. Note that we do not require that $a_{xy} = a_{yx}$ but we will consider only operators for which $V(x) = 0$ for all $x \in V$ and $a_{xy} > 0$ for all oriented edges $[x, y]$. The main example is the graph Laplacian

$$\Delta f(x) = \sum_{y\sim x}(f(x) - f(y)).$$

We will show that simple analogs of results from PDE yield nontrivial consequences and that even if one is interested only in the graph Laplacian, more general elliptic operators of the form (1) arise naturally.

We have the following two elementary propositions.

Proposition 1. ***Maximum principle****. Suppose $Au(x) < 0$. Then x is* not *a local maximum of u.*

Proposition 2. ***Harnack inequality****. Suppose $u > 0$ and $Au \geq 0$. Then*

$$\frac{a_{xy}}{\sum_{z\sim x} a_{xz}} \leq \frac{u(x)}{u(y)} \leq \frac{\sum_{z\sim y} a_{yz}}{a_{yx}}.$$

We next make two assumptions that are analogs of bounded geometry and uniform ellipticity in the continuous setting. Namely, we require that our graph has a bounded valence, i.e. that the number $m(x)$ of edges emanating from $x \in V$ is uniformly bounded and that there exist a positive constant C such that $C^{-1} \leq a_{xy} \leq C$ for all edges $[x, y]$. The following theorem follows from the maximum principle [2], [3].

Theorem 3. *Under the assumptions stated above, the initial value problem*

$$Au(x) + \frac{\partial u}{\partial t} = 0, \qquad u(x, 0) = u_0(x)$$

admits a unique bounded solution for every bounded function $u_0(x)$.

Now consider the graph Laplacian Δ as a bounded operator on $\ell^2(V)$. In [2] we proved that Δ admits a positive ground state ϕ, i.e. a positive function satisfying $\Delta\phi = \lambda_0\phi$ where $\lambda_0 = \inf \operatorname{spec}\Delta$. The proof of existence and of properties of ϕ depend only on the maximum principle and Harnack inequality.

We want to study the long time behavior of the heat semigroup $e^{-t\Delta} = P_t$ and denote by $p_t(x, y)$ its kernel (i.e. matrix). Consider the renormalized semigroup

$$\tilde{P}_t = e^{\lambda_0 t}[\phi^{-1}]P_t[\phi]$$

where $[f]$ denote the operator of multiplication by f. A calculation shows that $\tilde{P}_t = e^{-tA}$ with A given by the formula

$$Au(x) = \sum_{y\sim x} \frac{\phi(y)}{\phi(x)}(u(x) - u(y))$$

which is of the form (1). Applying Theorem 3 the the operator $A + \partial/\partial t$ one sees (cf. [2]) that $\tilde{P}_t\mathbf{1} = \mathbf{1}$ which is equivalent to $P_t\phi = e^{-\lambda_0 t}\phi$. This has as a consequence [2]

Theorem 4. *Suppose K is a graph acted on freely by a nonamenable group so that the quotient $\Gamma \setminus K$ is finite. Then*

$$p_t(x,x) = O(t^{-1}e^{-\lambda_0 t}).$$

In particular, $\mathrm{tr}_\Gamma(e^{-t\Delta}) = O(t^{-1}e^{-\lambda_0 t})$.

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The Atiyah conjecture for congruence subgroups

Peter A. Linnell

Let G be a discrete group and let $\ell^2(G)$ denote the Hilbert space $\{\sum_{g\in G} a_g g \mid a_g \in \mathbb{C},\ \sum_{g\in G} |a_g|^2 < \infty\}$. There is a well defined multiplication $\ell^2(G)\times\ell^2(G) \to \ell^\infty(G) = \{\sum_{g\in G} \mid a_g \in \mathbb{C},\ \sup_{g\in G}|a_g| < \infty\}$ given by

$$\sum_{g\in G} a_g g \sum_{h\in G} b_h h = \sum_{g,h\in G} a_g b_h gh = \sum_{g\in G}\big(\sum_{x\in G} a_{gx^{-1}} b_x\big) g.$$

Then the group von Neumann algebra $\mathcal{N}(G)$ of G is $\{\alpha \in \ell^2(G) \mid \alpha\ell^2(G) \subseteq \ell^2(G)\}$. Let $\mathcal{B}(\ell^2(G))$ denote the bounded linear operators on $\ell^2(G)$, acting on the left. Then $\mathcal{N}(G)$ can be considered as a $\mathbb{C}$-subalgebra of $\mathcal{B}(\ell^2(G))$, and it can also be described as $\{\alpha \in \mathcal{B}(\ell^2(G)) \mid (\alpha v)g = \alpha(vg)$ for all $g \in G$ and $v \in \ell^2(G)\}$, or alternatively as the weak closure of $\mathbb{C}G$ in $\mathcal{B}(\ell^2(G))$. Let $\mathcal{U}(G)$ denote the closed densely defined unbounded linear operators affiliated to $\mathcal{N}(G)$. Then $\mathcal{U}(G)$ is a quotient ring containing $\mathcal{N}(G)$; more precisely $\mathcal{N}(G)$ is a subring of $\mathcal{U}(G)$ and every element of $\mathcal{U}(G)$ can be written in the form $\alpha\beta^{-1}$, where $\alpha, \beta \in \mathcal{N}(G)$ and β is a non-zerodivisor in $\mathcal{N}(G)$. Algebraically we have

$$\mathbb{C}G \subseteq \mathcal{N}(G) \subseteq \ell^2(G) \subseteq \mathcal{U}(G),$$

where the ring operations are compatible with these inclusions; the only caution is that $\ell^2(G)$ is closed under multiplication if and only if G is finite.

$\mathcal{U}(G)$ is a finite von Neumann regular ring and has a nice dimension function $\dim_{\mathcal{N}(G)}$, which is also defined on $\mathcal{N}(G)$-modules. If $e \in \mathcal{U}(G)$ is an idempotent, i.e. $e = e^2$, then $e \in \mathcal{N}(G)$, so $e \in \ell^2(G)$ and we may write $e = \sum_{g\in G} a_g g$. Then $\dim_{\mathcal{N}(G)} e\mathcal{U}(G) = a_1$. In the case $|G|$ is finite, $\mathbb{C}G = \mathcal{N}(G) = \ell^2(G) = \mathcal{U}(G)$, and $|G|\dim_{\mathcal{N}(G)} M = \dim_{\mathbb{C}} M$.

We also need to describe pro-p groups, where p is a prime number which will be fixed for the rest of this report. These are groups G for which the normal subgroups of p-power index give a basis for the identity which make G into a compact Hausdorff topological group. For any group G, let $d(G)$ denote the minimum elements required to generate G. If G is a pro-p group, then the rank of G is defined to be $\sup d(G/H)$, where H runs through the normal subgroups of finite index. Thus the rank of $\mathbb{Z}_p$, the p-adic integers, is 1. On the other hand, $\mathbb{Z}_p$ contains free abelian subgroups of uncountable rank. A torsion-free pro-p group of finite rank is the same object as a torsion-free compact p-adic analytic (or Lie) group. For positive integers d, u, let $\mathrm{CS}(d,u,p) = \{A \in \mathrm{M}_d(\mathbb{Z}_p) \mid A \equiv I_d \mod p^u\}$ where I_d denotes the identity $d \times d$ matrix; this is a subgroup of finite index in $\mathrm{GL}_d(\mathbb{Z}_p)$, and is often called a congruence subgroup. We shall always assume that $u \geq 2$ in the case $p = 2$. Then $\mathrm{CS}(d,u,p)$ is a torsion-free pro-p group of finite rank (we need $u \geq 2$ in the case $p = 2$ to ensure that the group is torsion-free). Furthermore, any group G which has a normal subgroup of index a power of p isomorphic to $\mathrm{CS}(d,u,p)$ is a pro-p group of finite rank. We can now state

Theorem 1. *Let G be a torsion-free pro-p group of finite rank. Then there is a skew field D such that $\mathbb{C}G \subseteq D \subseteq \mathcal{U}(G)$.*

This is joint work with Dan Farkas, and also uses ideas that Thomas Schick has shared with me. This theorem verifies the strong Atiyah conjecture for such G. In particular if X is a finite CW-complex with fundamental group a torsion-free pro-p group of finite rank, then all L^2-Betti numbers are integers.

Here is a brief sketch of Theorem 1. Let $\overline{\mathbb{Q}}$ denote the field of all algebraic numbers in $\mathbb{C}$. First the following is proven; full details can be found in [2].

Proposition 2. *Let G be a torsion-free pro-p group of finite rank. Then there exists a skew field D such that $\overline{\mathbb{Q}}G \subseteq D \subseteq \mathcal{U}(G)$.*

To prove this version of the Atiyah conjecture, it will be sufficient to prove that if e is a positive integer, $\alpha \in \mathrm{M}_e(\overline{\mathbb{Q}}G)$ and $\hat{\alpha} \colon \ell^2(G)^e \to \ell^2(G)^e$ is the map induced by left multiplication by α, then $\dim_{\mathcal{N}(G)}(\ker \hat{\alpha})$ is an integer. Here we use Lück's approximation results [3, §13] as generalized by [1, Theorem 1.6]. Suppose $G = G_0 \supseteq G_1 \supseteq \cdots$ is a descending chain of subgroups with $\bigcap_{i \geq 0} G_i = 1$, $G_i \lhd G$ and $|G/G_i| < \infty$ for all i. Let $\alpha_i \colon \ell^2(G/G_i)^e \to \ell^2(G/G_i)^e$ be the maps induced by left multiplication by α. Then $\dim_{\mathcal{N}(G)}(\ker \hat{\alpha}) = \lim_{i\to\infty} |G/G_i|^{-1} \dim_{\mathbb{C}}(\ker \alpha_i)$. Therefore we want to prove for an appropriately chosen chain of normal subgroups G_i, that $\lim_{i\to\infty} |G/G_i|^{-1} \dim_{\mathbb{C}}(\ker \alpha_i) \in \mathbb{Z}$.

The key is that G has a "powerful" normal subgroup H of finite index; in the case G is a torsion-free subgroup of finite index in $\mathrm{GL}_d(\mathbb{Z}_p)$, we can take H to be a congruence subgroup $\mathrm{CS}(d,u,p)$, where u is large enough so that this congruence subgroup is contained in G. Let k be a field of characteristic p and let $\omega(H)$ denote the augmentation ideal of kH. Then we can form the associated graded ring $\mathrm{gr}(kH) = \bigoplus_{n=0}^{\infty} \omega(H)^n/\omega(H)^{n+1}$. The important property is that $\mathrm{gr}(kH)$ is a polynomial ring $R := k[X_1, \ldots, X_r]$, where r is the rank of H and R is graded by total degree. Let I_n denote the ideal of R consisting of all polynomials of total degree at least n together with 0, and for $\beta \in R$, let $\beta_n \colon R/I_n \to R/I_n$ denote the R-module map induced by left multiplication by β. Then in the spirit of Lück's approximation theorems, it is not difficult to show that if $\beta \neq 0$, then $\lim_{n\to\infty} \dim_k(\ker \beta_n)/\dim_k(R/I_n) = 0$. Then a few algebraic tricks are required to lift this to the required result that $\lim_{i\to\infty} |G/G_i|^{-1} \dim_{\mathbb{C}}(\ker \alpha_i) \in \mathbb{Z}$. In particular suppose S is a discrete valuation ring with maximal ideal qS where $q \in S$ such that S/qS is a field of characteristic p, and let $J = qSG + \omega(G)$, where $\omega(G)$ denotes the augmentation ideal of SG. Then J is a maximal ideal of SG such that SG/J is a field of characteristic p, and one can form the completed group ring $S[[G]]$ with respect to the powers of J. An important property used in the proof of Proposition 2 is that $S[[G]]$ is a Noetherian domain.

To go from $\overline{\mathbb{Q}}$ to $\mathbb{C}$, ideas which Thomas Schick has shared with me are required. Let $\Delta(G)$ denote the f.c. center of G, that is $\{g \in G \mid g$ has only finitely many conjugates$\}$. If $\Delta(G) = 1$, then certainly G has trivial center and the center of $\mathcal{U}(G)$ is $\mathbb{C}$. The following result is required.

Proposition 3. *Let G be a group with $\Delta(G) = 1$. If there is a skew field E such that $\overline{\mathbb{Q}}G \subseteq E \subseteq \mathcal{U}(G)$, then there is a skew field D such that $\mathbb{C}G \subseteq D \subseteq \mathcal{U}(G)$.*

Remark 4. Proposition 3 ought to be true without the hypothesis $\Delta(G) = 1$. However in practice one can embed G in a suitable group G_0 with $\Delta(G_0) = 1$.

Remark 5. There are several variations of Proposition 3; for example $\mathcal{U}(G)$ can be replaced with an arbitrary finite von Neumann regular ring, and the coefficient fields $\overline{\mathbb{Q}}$ and $\mathbb{C}$ with fields of nonzero characteristic. Furthermore by using comments of Elek and his embedding of the group ring in certain reduced ultraproducts, it ought to be possible to extend Lück's approximation results from $\overline{\mathbb{Q}}G$ to $\mathbb{C}G$ in many situations.

Proof of Theorem 1. We may assume that $G \neq 1$. Then $\times_{i=-n}^{n} G$ is also a pro-p group of finite rank for all positive integers n, hence Proposition 2 is true for this group and we deduce that the conclusion of Proposition 2 is true for the group $B := \times_{i=-\infty}^{\infty} G$. Now the wreath product $H := G \wr \mathbb{Z}$ contains the normal subgroup B for which the Atiyah conjecture (in the form of Proposition 2) is true and $H/B \cong \mathbb{Z}$, and it is known that this implies the Atiyah conjecture for H. In other words there is a skew field E such that $\overline{\mathbb{Q}}H \subseteq E \subseteq \mathcal{U}(H)$. Since H has trivial center (because $G \neq 1$), the result follows from Proposition 3. □

Finally we state the following open problem.

Problem 6. Let G be a torsion-free subgroup of $\mathrm{GL}_d(\mathbb{Z})$. Does there exist a skew field D such that $\mathbb{C}G \subseteq D \subseteq \mathcal{U}(G)$?

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Integrated Density of States of random Schrödinger Operators on Manifolds

Norbert Peyerimhoff
(joint work with Daniel Lenz, Olaf Post, Ivan Veselić)

In this talk we consider *spectral properties* of random Schrödinger operators on a manifold X where the randomness enters via both *the metric* of the manifold and *the potential*.

We first introduce the model **(RSM)**: Let $(X, \tilde{g}_0)$ be a non-compact Riemannian covering of a compact Riemannian manifold (M, g_0) with infinite deck transformation group Γ. Let $(\Omega, \mathbb{P})$ be a probability space which parametrizes a family of metrics $\{g_\omega\}_{\omega\in\Omega}$ on X. These metrics are uniformly bounded w.r.t. $\tilde{g}_0$ and their Ricci curvatures are uniformly bounded from below. We also require that there is an ergodic Γ-action on Ω and that the deck transformations $\gamma : (X, g_\omega) \to (X, g_{\gamma\omega})$ are isometries with corresponding unitary operators $U_{(\omega,\gamma)} : L^2(X, g_{\gamma^{-1}\omega}) \to L^2(X, g_\omega)$. There is a family of Schödinger operators $H_\omega = \Delta_\omega + V_\omega$ with non-negative, jointly measurable potential $V : \Omega \times X \to [0, \infty)$ such that $V_\omega := V(\omega, \cdot)$ is in $L^1_{\mathrm{loc}}(X)$. Finally, we require that these operators satisfy the equivariance condition

$$U_{\omega,\gamma} H_{\gamma^{-1}\omega} U^*_{\omega,\gamma} = H_\omega.$$

Concrete examples of this model are a single periodic operator (in which case $\Omega = \{0\}$ and $H_0 = \Delta_0 + V$) or random operators with alloy type potential **(RAP)** or random Laplacians with alloy type conformal deformation of the metric **(RAM)**. In the model **(RAM)** we choose $H_\omega = \Delta_\omega$ and

$$g_\omega(x) := \left(\sum_{\gamma\in\Gamma} e^{r_\gamma(\omega)} v(\gamma^{-1}x) \right) g_0(x).$$

The non-negative single site function $v \in C_c^\infty(X)$ satisfies $v \geq C\chi_{\mathcal{F}}$ ($\mathcal{F}$ = fundamental domain of the Γ-action on X) for some constant $C > 0$, and the coupling constants $r_\gamma : \Omega \to [a, b]$ are independent, identically distributed random variables on Ω with C^1-density.

Ergodicity yields the following non-randomness properties of the spectrum.

Theorem (see [3]): *Let $\{H_\omega\}_\Omega$ be as in (RSM). Then H_ω is a measurable family of operators. The spectral components (essential, absolute and singular* *continuous, pure point) coincide for almost all* $\{H_\omega\}$ *and the discrete spectrum of almost all* $\{H_\omega\}$ *is empty. For* $f \in L^\infty(\mathbb{R})$ *let*

$$\rho_H(f) := \frac{\mathbb{E}\left(\mathrm{tr}(\chi_{\mathcal{F}} f(H_\bullet))\right)}{\mathbb{E}(\mathrm{vol}_\bullet(\mathcal{F}))},$$

where $\mathbb{E}$ *denotes expectation in* Ω *and* vol_ω *denotes the Riemannian volume of the metric* g_ω*. The measure* ρ_H *is called the* density of states *of* $\{H_\omega\}$*. Its distribution function* $N_H(\lambda) = \rho_H(\chi_{(-\infty,\lambda)})$ *is called the* integrated density of states (IDS)*. The almost sure spectrum of* H_ω *coincides with the points of increase of* $N_H(\lambda)$*.*

In the case of *amenable* groups Γ, the IDS can also be obtained via an exhaustion process (see [6] for the case of almost periodic elliptic operators in Euclidean space).

Theorem (Pastur-Shubin trace formula for **(RSM)**): *Let* $\{H_\omega\}_\Omega$ *be as in (RSM),* $N_H(\lambda)$ *be its IDS, and* Γ *be* amenable. *Any* tempered Følner sequences $I_n \subset \Gamma$ *induces a sequence of domains* $\phi(I_n) := \overline{\bigcup_{\gamma \in I_n} \gamma\mathcal{F}} \subset X$*. Let* H_ω^n *denote the restriction of* H_ω *to* $\phi(I_n)$ *with Dirichlet boundary condition. Then we have for almost all* $\omega \in \Omega$*:*

$$N_\omega^n(\lambda) := \frac{\#\{i : \lambda_i(H_\omega^n) < \lambda\}}{\mathrm{vol}_\omega(\mathcal{F})} \to N_H(\lambda),$$

at all continuity points of $N_H(\lambda)$*.*

This result is derived from appropriate heat kernel estimates and uses an ergodic theorem of Lindenstrauss [5].

We also have Wegner estimates for the models **(RAP)** and **(RAM)** (see [4]). The strategy of proof is similar to [1].

Theorem (Wegner estimate for **(RAM)**): *Let* $\{\Delta_\omega\}_\Omega$ *be as in (RSM). For any finite* $I \subset \Gamma$*, let* Δ_ω^I *denote the restriction of* Δ_ω *to* $\phi(I) \subset X$ *with Dirichlet boundary condition. For every* $p > 1$ *and every interval* $[1/a, a] \subset (0, \infty)$ *there is a constant* $C_{p,a} > 0$ *such that*

$$\mathbb{E}(\mathrm{tr}(E_{[\lambda-\epsilon,\lambda+\epsilon]}(\Delta_\bullet^I))) \leq C_{p,a}\epsilon^{1/p}\#I^+,$$

for all finite $I \subset \Gamma$*,* $\epsilon > 0$*, and* $[\lambda - 2\epsilon, \lambda + 2\epsilon] \subset [1/a, a]$*, where* $I^+ := \{\gamma \in \Gamma : \mathrm{supp} v(\gamma^{-1}\cdot) \cap \phi(I) \neq \emptyset\}$*.*

This result implies Hölder-continuity of the IDS in the amenable case. As a consequence, there are examples of manifolds with abelian deck transformation groups which carry random Laplacians with continuous IDS, whereas the IDS of the corresponding periodic operator has discontinuities. (Discontinuities of the IDS of the periodic operator Δ_0 coincide with L^2-eigenvalues of Δ_0 on X; examples of manifolds with such L^2-eigenvalues can be found in [2].) Hence, this is a situation in which the introduction of randomness *improves the regularity* of the IDS.

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Uniform existence of the integrated density of states for quasicrystals

Daniel Lenz
(joint work with Peter Stollmann)

We discuss results on uniform existence of the integrated density of states for certain models for quasicrystals. The results are taken from [1, 2, 3], to which we refer for details, references and proofs.

Quasicrystals are a special class of solids discovered in 1984 by Shechtman/Blech /Gratias/Cahn. They exhibit a distinctive form of (dis)order, called aperiodic order in the mathematical community. While there is (so far) no axiomatic theory of aperiodic order, common models are based on Delone sets satisfying certain regularity features.

A subset Λ of $\mathbb{R}^d$ is called Delone if it is uniformly discrete and relatively dense. Here, uniformly discrete means that there is a minimal distance between different elements of Λ and relatively dense means that the maximal distance of an arbitrary element of $\mathbb{R}^d$ to Λ is bounded. Note that by a simple Voronoi type construction we can replace Delone sets by tilings of the space with convex polyhedra. We will assume the following regularity features.

Finite local complexity (FLC): For each $R > 0$, the set $\{(\Lambda - x) \cap B_R : x \in \Lambda\}$ is finite, where B_R denotes the closed ball with radius R around the origin in $\mathbb{R}^d$.

Repetitivity: For each finite $P \subset \mathbb{R}^d$, the set $\{x \in \mathbb{R}^d : x + P \subset \Lambda\}$ is either empty or relatively dense.

Uniform patch frequencies (UPF): For every finite $P \subset \mathbb{R}^d$, and all (a_n) in $\mathbb{R}^d$, the limit

$$\lim_{n\to\infty} \frac{\sharp\{x \in a_n + B_n : x + P \subset \Lambda\}}{|B_n|}$$

exists. Here, $\sharp$ denotes cardinality of a set and $|\cdot|$ denotes Lebesgue measure.

All these features can be thought of as order features. FLC is essentially a local requirement, the other two are global requirements. They are trivially satisfied for lattices. However, there is a huge amount of examples, which are not lattices and in fact satisfy the following condition:

Aperiodicity: For $t \in \mathbb{R}^d$ with $t \neq 0$, $t + \Lambda \neq \Lambda$.

Our aim is to study operators associated to Delone sets satisfying the above properties. These operators arise in the quantum mechanical treatment of quasicrystals. They are defined as follows: A finite range operator A associated to a Delone set Λ is a bounded operator

$$A : \ell^2(\Lambda) \longrightarrow \ell^2(\Lambda)$$

such that there exists $R > 0$ with

- $A(x, y) = 0$, if $(x + B_R) \cap (y + B_R) = \emptyset$.
- $A(x, y) = A(t + x, t + y)$ whenever $((t + x + B_R) \cup (t + y + B_R)) \cap \Lambda = t + ((x + B_R) \cup (y + B_R)) \cap \Lambda)$.

We want to study convergence of the normalized eigenvalue counting functions $N_A^n : \mathbb{R} \longrightarrow [0, \infty)$,

$$N_A^n(\lambda) := \frac{\sharp\{\text{Eigenvalues not exceeding } \lambda \text{ of } A \text{ restricted to } \Lambda \cap B_n\ \}}{\sharp B_n \cap \Lambda}$$

for a selfadjoint finite range operator A.

Theorem. [1] *Let A be a finite range operator associated to a Delone set satisfying FLC, Repetitivity and UPF. Then, there exists a measure μ_A on $\mathbb{R}$ with distribution function N_A, i.e. $N_A(\lambda) = \mu_A((-\infty, \lambda])$, such that the spectrum of A equals the topological support of μ_A and $N_A^n \longrightarrow N_A$ with respect to the supremum norm.*

Results on vague convergence of the distribution functions were established by Kellendonk in 1995 using techniques going back to earlier work of Bellissard. Related result were also obtained by Hof in 1993 and 1996. The main point in the theorem is the very uniform convergence result. The proof of this uniform convergence is achieved by considering the distribution functions as elements of the Banach space consisting of all bounded right continuous functions on $\mathbb{R}$ equiped with the supremum norm and appealing to the following result.

Theorem. [1] *Let Λ be a Delone set satisfying FLC, Repetitivity and UPF. Let $(B, \|\cdot\|)$ be a Banach space and*

$$F : \textit{Bounded measurable sets in } \mathbb{R}^d \longrightarrow B$$

be almost additive and Λ-invariant. Then, the limit $\lim_{n\to\infty} \frac{1}{|B_n|} F(B_n)$ exists.

Here, Λ-invariance means that $F(P) = F(t + P)$ whenever $t + (P \cap \Lambda) = (t + P) \cap \Lambda$. Almost additivity means that F is additive up to a boundary term. More precisely, there exist $C > 0$ and

$$b : \text{Bounded measurable sets in } \mathbb{R}^d \longrightarrow [0, \infty)$$

with $\lim b(C_n)/|C_n| = 0$ for every van Hove sequence (C_n) and $b(P) \leq b(P') + b(Q)$ whenever P' is the disjoint union of P and Q, such that

- $\|F(\cup_{j=1}^k P_j) - \sum_{j=1}^k F(P_j)\| \leq \sum_{j=1}^k b(P_j)$, whenever the P_j, $j = 1, \dots, k$, are disjoint up to their boundaries;
- $\|F(P)\| \leq C|P| + b(P)$.

For special Delone sets arising from so called primitive substitution a similar result was proven by Geerse/Hof in 1991. Our proof relies on decomposing the Delone sets in Voronoi cells on increasing length scales. Techniques of this kind were also developed by Priebe in 2000 in her study of self similarity of Delone sets.

So far we have been concerned with a single Delone set and a single operator. In fact, each Delone set Λ gives rise to a dynamical system $(\Omega(\Lambda), T)$. Here, the hull of λ $\Omega(\Lambda)$ is the closure in a certain topology of the set $\{t + \Lambda : t \in \mathbb{R}^d\}$ of translates of Λ. The map T is the action of $\mathbb{R}^d$ on Delone sets by translations. Due to the assumptions on Λ this dynamical system is minimal and uniquely ergodic. The unique T-invariant probability measure on $\Omega(\Lambda)$ will be denoted by m.

Each finite range operator A on Λ can naturally be extended to a family (A_Γ) of operators $A_\Gamma : \ell^2(\Gamma) \longrightarrow \ell^2(\Gamma)$ for $\Gamma \in \Omega(\Lambda)$. These operators belong to a certain C^*-algebra $\mathcal{A}$ which carries a trace given by

$$\tau : \mathcal{A} \longrightarrow \mathbb{R}, \tau(A) := \int \mathrm{tr}_\Gamma(M_f A_\Gamma) dm(\Gamma),$$

where tr_Γ denotes the usual trace on the bounded operators on $\ell^2(\Gamma)$, $f \geq 0$ is an arbitrary function on $\mathbb{R}^d$ with $\int f(x)dx = 1$, and $M_f : \ell^2(\Gamma) \longrightarrow \ell^2(\Gamma)$ is just multiplication by f. The distribution function N_A arising above as limit is given by an explicit Shubin-Pastur type trace formula as follows:

Theorem [1, 2] *With the notation above, and the spectral family E_A of the selfadjoint finite range operator A, the formula $N_A(\lambda) = \tau(E_A((-\infty, \lambda]))$ holds for every $\lambda \in \mathbb{R}$.*

We finish this text with two open questions:

(1) Do similar results on uniform convergence of the distribution functions hold in the context of operators on graphs which are periodic with respect to an action of an amenable group?
(2) Do similar results on uniform convergence of the distribution functions hold in the context of operators on graphs which are periodic with respect to an action of an residually finite group?

Note that in the setting of the questions the situation is more regular than our situation in the sense that the operators are periodic but is also less regular in the sense that the group is not Euclidean space any more.

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Open questions in tiling cohomology

Johannes Kellendonk
(joint work with Ian F. Putnam)

We consider a tiling or a Delone set P in the Euclidean space $\mathbb{R}^d$. We suppose that it is of finite local complexity, i.e. for all R the set of R-patches $\{(B_R(x) \cap P) - x | x \in P\}$ is finite. The tiling defines a topological space Ω_P, also called its hull, which is the completion of the set of its translates w.r.t. the tiling metric. Equivalently, it is the spectrum of the C^*-algebra of P-equivariant functions.

The cohomology of P is by definition the Cech cohomology $\check{H}(\Omega_P, \mathbb{Z})$ of its hull. It is computable for quasi-periodic tilings [2, 3].

Another cohomology group related to the tiling plays a role: the Lie-algebra cohomology $H(\mathbb{R}^d, C^\infty(\Omega_P, \mathbb{R}))$ of $\mathbb{R}^d$ with coefficients in the real functions on Ω_P which are smooth w.r.t. the translation action. An analog of the Cech-de Rham complex provides us with a graded ring homomorphism $\theta : \check{H}(\Omega_P, \mathbb{Z}) \to H(\mathbb{R}^d, C^\infty(\Omega_P, \mathbb{R}))$.

Cohomology captures topological information about the dynamical system. If we are also given an invariant Borel probability measure μ on the hull then we can capture some of the geometric structure which comes from the (euclidean) geometry of the $\mathbb{R}^d$-orbits. We do this by exploiting Ruelle-Sullivan currents associated with sub-actions. These yield linear functionals on dynamical cohomology which we combine into a graded homomorphism $H(\mathbb{R}^d, C^\infty(\Omega_P, \mathbb{R})) \to H(\mathbb{R}^d, \mathbb{R}) \cong \Lambda \mathbb{R}^{d^*}$ into the exterior algebra of the dual of $\mathbb{R}^d$. The Ruelle Sullivan map is the composition of the two $\tau_\mu : \check{H}(\Omega_P, \mathbb{Z}) \to \Lambda \mathbb{R}^{d^*}$. Based on our calculations with quasi-periodic tilings we conjecture [1]:

Conjecture *Let P be a repetitive Delone set of finite local complexity. Suppose that the translation action on its hull is uniquely ergodic. Let μ be the ergodic measure. Then P is aperiodic (i.e. has no periodic direction) if and only if $\tau_\mu(\check{H}^1(\Omega_P, \mathbb{Z}))$ is dense in $\mathbb{R}^{d^*}$.*

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Metric graphs and their smooth approximations

Olaf Post
(joint work with Peter Hislop)

In the first part of the talk we introduce the concept of a metric graph and its relations with the discrete graph Laplacian and the Laplacian on a suitable approximation of the metric graph. In the second part we present a joint work with Peter Hislop on exponential localization on a random radial (metric) tree graph.

A *metric* (or *quantum*) graph is a discrete graph $G = (V, E)$ where we associate a length ℓ_e to each edge $e \in E$. In particular, we identify e with the interval $(0, \ell_e)$. The natural Laplacian on a metric graph is the *differential* operator given by $(\Delta f)_e = -f_e''$ on the interval $e \cong (0, \ell_e)$ with boundary conditions at the vertices turning Δ into a self-adjoint operator Δ_G in $L_2(G) = \bigoplus_{e \in E} L_2(e)$. Therefore, a metric graph is a 1-dimensional space rather than a 0-dimensional space in the discrete graph case (where the natural Laplacian is a *difference* operator). If all lengths ℓ_e are the same, there is a simple relation between the spectrum of the metric graph Laplacian Δ_G and the discrete graph Laplacian (cf. [Cat97]). A general survey on metric graphs can be found in [Kuc04, Kuc05].

In some respects "natural" boundary conditions are given by the so-called *Kirchhoff* boundary conditions: the continuity and flux conservation at each vertex $v \in V$, namely

$$f_{e_1}(v) = f_{e_2}(v) \qquad \text{and} \qquad f'(v) := \sum_{e \in E} f_e'(v) = 0$$

for all e_1, e_2 in E_v (the set of edges adjacent to the vertex $v \in V$) where $f_e'(v)$ is the derivative on the edge e at the vertex v *towards* the vertex. More generally, we can replace the second condition by $f'(v) = q(v)f(v)$ for some real number $q(v)$ at each vertex and obtain a δ-potential of strength $q(v)$ at each vertex v. For a general discussion of boundary conditions we refer to [KS99]. The Kirchhoff boundary conditions are natural in the following sense:

Suppose that X_ε is a smooth approximation of the metric graph, e.g., the (smoothened) ε-neighbourhood of G if G is embedded in $\mathbb{R}^\nu$. One can define more abstract models of approximations (cf. [EP05, Pos06]), e.g. the surface of a tubular neighbourhood of G in $\mathbb{R}^3$. We assume that the metric graph and its approximation is *uniform* in the sense that the vertex degree is uniformly bounded and the edge lengths have a uniform lower positive bound. For the approximation X_ε we assume that the building blocks (the vertex and edge neighbourhoods) do not vary too much (in a sense stated precisely in [Pos06]). If the approximation X_ε has boundary, we impose *Neumann* boundary conditions on Δ_{X_ε}.

The main result of [Pos06] is the following (cf. [RS01, KZ01, EP05] for the compact case and the convergence of the discrete spectrum):

Theorem. *With a suitable identification operator* $J_\varepsilon \colon L_2(G) \to L_2(X_\varepsilon)$ *the convergence* $\|(\Delta_{X_\varepsilon} + 1)^{-1} J_\varepsilon - J_\varepsilon(\Delta_G + 1)^{-1}\| \to 0$ *as* $\varepsilon \to 0$ *holds. In particular, the spectrum of* Δ_{X_ε} *converges to the spectrum of* Δ_G *in every compact spectral* *interval. The same statement is true for the essential and discrete spectrum, respectively.*

In the second part of the talk we consider a *radial metric tree*, i.e., a tree graph where the length $\ell_e = \ell_n$ and the vertex degree $d(v) = d_n \geq 3$ depend only on the generation n (discrete distance from the root vertex o). We also add a δ-potential $q = (q(v))_v \geq 0$ at each vertex (via the boundary condition $f'(v) = q(v)f(v)$) which we assume to depend only on the generation, $q(v) = q_n$. In the *random* case we assume that (q_n) are independent i.i.d. random variables. Using a symmetry reduction result for radial trees (cf. [Sol04]) we can reduce the problem to a problem on the half-line $\mathbb{R}_+$ with generalized point interactions at a vertex, namely $f(v_-) = (d_n - 1)^{-1/2} f(v_+)$ and $f'(v_-) = (d_n - 1)^{1/2} f'(v_+)$ where $v = v_n$ are the vertices with distance ℓ_n from the following one and $v_0 = 0$. In a second step we can reduce the problem to a discrete transfer matrix problem (using the discretization $(f(v_-), f'(v_-))$. Our main result is the following:

Theorem. *Suppose that the probability measure P_0 of a single site potential q_0 has a density ρ_0 with compact support in $[0, \infty)$ (i.e., $\mathrm{d}P_0(q) = \rho_0(q)\mathrm{d}q$). Then the continuous Laplacian with radial random i.i.d. δ-potentials at each vertex has almost sure spectrum $\Sigma = [0, \infty)$. Furthermore, the almost sure spectrum is pure point, and each eigenfunction decays exponentially with decay rate $\gamma(\lambda)$ given by the Lyapunov-exponent of the transfer matrix.*

More general models are possible, e.g., random length (ℓ_n), random vertex degrees (d_n) or even random graph decorations (e.g., attaching a line segment at a vertex). In the *non-radial* case, the behaviour is completely different: one has absolutely continuous (ac) spectrum at weak disorder near the ac-spectrum of the unperturbed tree Laplacian (cf. [ASW05]).

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Spectral properties of Anderson-percolation Hamiltonians

Ivan Veselić

We report on results on spectral properties of Laplacians on percolation graphs and more general Anderson-percolation Hamiltonians. The results are mostly taken from the papers [4, 5, 3], to which we refer for further references, generalisations of the results presented here, and more details.

In this report we will for simplicity restrict ourselves to the following situation. We consider the graph with vertex set $\mathbb{Z}^d$, where two of the vertices are joined by an edge if their ℓ^1 distance is equal to one. The group $\Gamma := \mathbb{Z}^d$ is acting on the graph by translations. Denote by $\{\Lambda_L\}_{L\in 2\mathbb{N}}$ the exhaustion of $\mathbb{Z}^d$ by cubes $\Lambda_L := \mathbb{Z}^d \cap [-L/2, L/2]^d$.

We construct a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ associated to percolation on $\mathbb{Z}^d$. Let $\Omega = \times_{x\in\mathbb{Z}^d}[0,\infty]$ be equipped with the σ-algebra $\mathcal{A}$ generated by finite dimensional cylinders sets. Denote by $\mathbb{P}$ a probability measure on Ω and assume that the measurable shift transformations $\tau_\gamma \colon \Omega \to \Omega$, $(\tau_\gamma\omega)_x = \omega_{x-\gamma}$ are measure preserving. Moreover, let the family $\tau_\gamma, \gamma \in \Gamma$ act ergodically on Ω. By the definition of $\tau_\gamma, \gamma \in \Gamma$ the stochastic field $q\colon \Omega \times \mathbb{Z}^d \to [0,\infty]$ given by $q(\omega, x) = \omega_x, x \in \mathbb{Z}^d$ is *stationary* or *equivariant*, i.e. $q(\tau_\gamma\omega, x) = q(\omega, x-\gamma)$. The mathematical expectation associated to the probability $\mathbb{P}$ will be denoted by $\mathbb{E}$ and the distribution measure of q_0 by μ.

Define for $\omega \in \Omega$ the random vertex set $X(\omega) := \{x \in \mathbb{Z}^d \mid q(\omega, x) < \infty\}$ and denote by the same symbol the induced subgraph of $\mathbb{Z}^d$. In other words, if $q(\omega, x)$ is infinite, we delete the vertex x together with all incident edges, otherwise we retain it in the graph. For each ω let $A_\omega \colon D(\omega) \to D(\omega)$, $D(\omega) := \ell^2(X(\omega))$ be the adjacency operator of the graph $X(\omega)$. Define now an Anderson-percolation Hamiltonian by

$$H_\omega := A_\omega + q(\omega, \cdot)\colon D(\omega) \to D(\omega)$$

where $(q(\omega, \cdot)\psi)(x) = q(\omega, x)\psi(x)$ is a random potential. In the sequel we may or not assume

$$\exists\, p \in\,]0,1[\;\text{ such that } \mu(\{0\}) = p \;\text{ and } \mu(\{\infty\}) = 1 - p \tag{1}$$

$$q(\cdot, x), x \in \mathbb{Z}^d \quad \text{are i.i.d. random variables} \tag{2}$$

Let us consider two examples of random potentials. Set $V(\omega, x) = 2d - \deg_{X(\omega)}(x) \geq 0$. Then the two operators $A_\omega \pm V(\omega, \cdot)$ are sometimes called Neumann, respectively Dirichlet Laplacian on $X(\omega)$ (up to a constant). The fixed sign of the potential V is useful for Dirichlet-Neumann-bracketing estimates.

Remark. More generally, Anderson-percolation Hamiltonians can be defined on covering graphs with a free, coompact group action. For certain results it is required that the group is amenable. The structure of the stochastic process q generalises in an obvious manner to this setting. It is possible to carry out a similar analysis on bond or mixed percolation graphs. Rather than considering the adjacency operator, one can derive similar results for equivariant, hermitian, finite hopping range operators. In this generality, some of our results are new even in the periodic case when Ω contains only one element.

Denote by $\sigma_{disc}, \sigma_{ess}, \sigma_{ac}, \sigma_{sc}, \sigma_{pp}$ the discrete, essential, absolutely continuous, singular continuous, and pure point part of the spectrum, and by σ_{fin} the set of eigenvalues which posses an eigenfunction with finite support. For each Λ_L denote by H_ω^L the truncation of H_ω to $\ell(\Lambda_L(\omega))$, $\Lambda_L(\omega) := \Lambda_L \cap X(\omega)$ and the associated normalised eigenvalue counting function by $N_\omega^L(E) := \frac{1}{L^d}\mathrm{Tr}[\chi_{]-\infty,E]}(H_\omega^L)]$. The distribution function, given by an averaged trace per unit volume, $E \mapsto N(E) = \mathbb{E}\left\{\langle \delta_0, \chi_{]-\infty,E]}(H_\omega)\delta_0\rangle\right\}$ is called *integrated density of states* (IDS). Denote by ν the measure on $\mathbb{R}$ associated to N. The following theorem establishes the non-randomness of basic spectral quantities and a relation between the IDS and the spectrum.

Theorem 1. There exists an $\Omega' \subset \Omega$ of full measure and subsets of the real numbers Σ and $\Sigma_\bullet$, where $\bullet \in \{disc, ess, ac, sc, pp, fin\}$, such that for all $\omega \in \Omega'$

$$\sigma(H_\omega) = \Sigma \quad \text{and} \quad \sigma_\bullet(H_\omega) = \Sigma_\bullet \quad \text{for any } \bullet = disc, ess, ac, sc, pp, fin$$

and

$$\lim_{L\to\infty} N_\omega^L(E) = N(E) \quad \text{at all continuity points of } N \tag{3}$$

Moreover, $\Sigma_{disc} = \emptyset$ and $\Sigma = \operatorname{supp}\nu$. Under assumption (1) the convergence (3) holds for all $E \in \mathbb{R}$.

There is a probability threshold p_c such that if $\mu(\{\infty\}) < 1 - p_c$ and (2) holds

(4) an infinite component $X^\infty(\omega)$ of the graph $X(\omega)$ exists a.s.

In this case one can consider the adjacency operators A_ω^∞ on $X^\infty(\omega)$ and the analogous statements to the above theorem hold. We denote the associated quantities using a superscript $^\infty$, e.g. N^∞ for the IDS of A_ω^∞ .

Theorem 2.

(i) $\Sigma_{fin} = \operatorname{supp}\, \nu_{pp}$.

(ii) Denote by A^G the adjacency operator of the subgraph of $\mathbb{Z}^d$ induced by G. If (1) and (2) hold, then

$$\Sigma_{fin} = \tilde{\Sigma} := \{E \in \mathbb{R} \mid \exists \text{ finite } G \subset \mathbb{Z}^d \text{ and } f \in \ell^2(G) \text{ such that } H^G f = Ef\}$$

(iii) If (4) holds we have $\Sigma_{fin}^\infty = \operatorname{supp}\, \nu_{pp}^\infty$.

(iv) If (4) and (2) hold then $\Sigma_{fin}^\infty = \Sigma_{fin}$.

The set $\tilde{\Sigma}$ is dense in $[-2d, 2d]$ and consists of algebraic integers. The next theorem provides estimates in the spirit of Wegner and Delyon/Souillard for Anderson-percolation Hamiltonians.

Theorem 3. Assume (2), then:

(i) $\Sigma = [-2d, +2d] + \text{supp}\ \mu|_{\mathbb{R}}$.
(ii) $\Sigma_{fin} \supset \tilde{\Sigma} + \text{supp}\ \mu|_{\mathbb{R}}$
(iii) If $\mu = \mu_c + (1-p)\delta_\infty$, i.e. μ has no atoms at finite values, then the IDS of H_ω is continuous.
(iv) Assume that for $a, b \in \mathbb{R}$ the measure μ is absolutely continuous on the interval $]a-2d, b+2d[$, i.e. $\mu|_{]a-2d,b+2d[}(dx) = f(x)dx$, and that $f \in L^\infty$. Then, for every interval I with $\text{dist}(I,]a,b[^c) \geq \delta > 0$ we have

$$\mathbb{E}\{\text{Tr}\,[\chi_I(H_\omega^L)]\} \leq C\,|I|\,L^d \tag{5}$$

where $C = 2^{d+2}\left(\frac{b-a+4d+1}{\delta}\right)^2 \frac{\|f\|_\infty}{\mu(]a-2d,b+2d[)}$.

It follows that the constant C in (5) is an upper bound on the *density of states*, i.e. $\frac{dN(E)}{dE} \leq C$ for all $E \in]a,b[$.

If (2) holds and $\mu = \frac{1}{3}\chi_{[0,1]} + \frac{1}{3}\delta_{4d} + \frac{1}{3}\delta_\infty$, the Wegner estimate (5) is valid for all energies $E < 2d$. Moreover, by [1] Lifshitz asymptotics hold at $\min \Sigma = -2d$. Thus one can prove using a multiscale analysis, see e.g. [2], that for some $\alpha \in]0,1[$ the spectrum of H_ω in $U = [-2d, -2d+\alpha]$ has no continuous component and consists of a dense set of eigenvalues, whose eigenfunctions decay exponentially in space almost surely.

Denote by $\sigma_\epsilon(H_\omega)$ the set of eigenvalues of H_ω. Then

$$\Sigma_{fin} \subsetneq \sigma_\epsilon(H_\omega) \subsetneq \Sigma_{pp} \quad \text{for almost all } \omega \tag{6}$$

Here the sets Σ_{fin} and Σ_{pp} are almost surely non-random, and $\Sigma_{fin} \supset \tilde{\Sigma} + 4d$. The set $\sigma_\epsilon(H_\omega) \cap U$ is non-empty almost surely and disjoint to $\tilde{\Sigma} + 4d$. By the Wegner estimate the IDS is absolutely continuous in U, therefore for any $E \in U$ the probability $\mathbb{P}\{E$ in an eigenvalue of $H_\omega\}$ vanishes and U is even disjoint to Σ_{fin}. Thus we have a highly fluctuating set $\sigma_\epsilon(H_\omega)$ strictly sendwiched between two almost surely non-random sets Σ_{fin} and Σ_{pp}.

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L^2-invariants for manifolds with boundary

Thomas Schick
(joint work with Eckehard Hess, Wolfgang Lück)

In this short and informal talk we present some rather old results about analytic L^2-invariants for manifolds of finite volume, compared to combinatorial L^2-invariants of certain (canonical) compactifications, and present some of the most important open questions.

More precisely, let (M°, g) be a complete Riemannian manifolds of finite volume and with universal covering of bounded geometry. Then, analytic L^2-invariants can be defined using the heat kernel of the universal covering and a fundamental domain, compare [2] for the L^2-Betti number. On the other hand, assume that M° is the interior of a compact manifold M with boundary. Using a finite triangulation of the latter, combinatorial L^2-invariants of M are defined.

We now want to relate these two different sets of invariants. The comparison uses as an intermediate step analytic L^2-invariants of the compact manifold with boundary, which coincide with the combinatorial ones (compare e.g. [2]).

We study in particular the following situation: the interior of M admits a complete metric of constant curvature -1 and of finite volume. Then, by a quite general result of Cheeger and Gromov, the combinatorial L^2-Betti numbers of M are equal to the analytic L^2-Betti numbers of M°. This works much more generally, we only need that the boundary of M, consisting of flat manifolds, has vanishing L^2-Betti numbers, and that the universal covering of M° has bounded geometry.

In [4], the Novikov-Shubin invariants are considered in this case. This is more complicated, because the Novikov-Shubin invariants of the boundary of M are non-vanishing. Nonetheless, Lott and Lück can prove that the Novikov-Shubin invariants are always positive if $\dim(M) = 3$, and give certain estimates.

The case of dimension 3 is particularly interesting, because by the Thurston conjecture, there is a precise decomposition of an arbitrary compact 3-manifold into pieces with nice geometry. Among those pieces one has compact manifolds with boundary whose interior admits a complete hyperbolic metric of finite volume.

The most elaborate work concerning the above question is carried out about the L^2-torsion. Indeed, we have the following theorem of Wolfgang Lück and the author [5]:

Theorem: If M is a manifold of dimension as above, then the L^2-torsion of M is proportional to the volume of the hyperbolic manifold (M°, g), with the same constant of proportionality as for closed hyperbolic manifolds.

The constant of proportionality is shown to be non-zero in all odd dimensions by Hess and the author in [3]. This calculation is extended to all symmetric spaces of fundamental rank 1 by Olbrich in [6].

Despite all these results, many questions are still wide open. We list some of the more important ones.

- Find more precise estimates for the Novikov-Shubin invariants of manifolds whose interior admits a complete hyperbolic metric of finite volume. Led by the general result about L^2-Betti numbers, I conjecture that they are strictly smaller than in the closed case (i.e. strictly smaller than the contribution coming from the complete interior).
- Extend the calculation to other locally symmetric spaces. One should first attack the rank one case, but the case of higher rank is potentially even more interesting (albeit more complicated because of the intricate structure of the ends of a finite volume locally symmetric space of higher rank). The conjecture is that the L^2-Betti numbers and the L^2-torsion should be given by the complete interior. For the Novikov-Shubin invariants, I only expect to be able to get estimates, but hopefully one can show that they are always finite.
- In the case of hyperbolic manifolds, the finite volume condition might be weakenalbe. In particular, one should study convex cocompact hyperbolic manifolds. Of course, here the "analytic" invariant will not be given by an integral over a fundamental domain (this being of infinite volume). Further inspiration is needed to get the correct analytic expression in this case.
- One should extend the calculations to other L^2-invariants. I'd stress in particular the L^2-eta invariant or L^2-rho invariant, respectively. Their geometric significance was worked out in [1, 7]. Here, one of the difficulties will be to give a convincing invariant definition of these invariants for manifolds with boundary to start with.
- The last question I want to mention concerns manifolds with boundary which admit not quite a locally symmetric metric (of finite volume) on the interior, but only asymptotically so. I'd expect that most of the results for the truly locally symmetric case carry over. However, the proofs available so far have to be modified, in particular for the L^2-torsion, and much more care is needed to get uniformity in some of the estimates one needs along the way.

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