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Analysis and Topology in Interaction

Organised by
Ulrich Bunke (Göttingen)
Sebastian Goette (Regensburg)
Kiyoshi Igusa (Waltham)
Thomas Schick (Göttingen)

March 12th – March 18th, 2006

Abstract. This workshop brought together, on the one side, mathematicians working in areas of global analysis and index theory, which are related with problems in algebraic topology, and on the other side, specialists in fields like surgery theory, higher homotopy theory or twisted cohomology theories. Its particular aim was to promote the flow of ideas and techniques between these two areas.

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Introduction by the Organisers

The MFO workshop "Analysis and Topology in Interaction", organised by Ulrich Bunke (Göttingen), Sebastian Goette (Regensburg), Kyoshi Igusa (Brandeis) and Thomas Schick (Göttingen), was held from March 12th through March 18th, 2006.

The aim of this meeting was to reflect on the current state of the interaction between mathematical fields of analysis and topology. It brought together representatives of various mathematical communities ranging from homotopy theory, index theory, global analysis up to mathematical physics. The program of the conference has been a mixture of research and overview talks, sometimes with introductory elements. Topics of the latter kind were the connection of Quantum field theory constructions with twisted K-theory, and motivic groups. There were two more informal introductory lectures on stacks in topological and smooth categories, and the insights given by the study of the Brownian motions into Hodge theory.

The research talks reflected recent developments in the corresponding fields and covered a broad area between topology, geometry, and analysis. The intention of the program was to communicate these developments across the borders of mathematical communities. They stimulated discussions and hopefully future collaborations.

More specifically, the subjects covered by research talks included in particular

- twisted K-theory (explicit constructions and applications, and homotopy theoretic approaches to its calculation);
- index theory on manifolds with boundary or with singularities, and study of corresponding index theorems and eta-invariants; spectral flow;
- algebraic geometry methods in algebraic topology and vice versa (stacks, motivic geometry);
- isomorphism conjectures in K- and L-theory;
- signatures of singular spaces;
- (refined) torsion invariants, in particular for families; signatures for families;
- spectral theory and geometric implications;
- symplectic geometry via topology of the loop space;
- topology of the mapping class group.

Altogether, there were 23 “official” talks and a number of informal presentations (partly extended discussions), and ample free interaction between the participants. One of the evenings was devoted to a problem session; the open problems presented are included in this report.

The conference was attended by 52 participants coming mainly from all over Europe, Northern America and East Asia. Among them, a couple of very young pre-doctoral mathematicians, supported by the EU, had the unique opportunity to participate in such a stimulating event at a very early stage of their career. It is a pleasure to thank the institute for providing a pleasant and stimulating atmosphere.

Workshop: Analysis and Topology in Interaction

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Abstracts

Euler Characteristics and Gysin Sequences for Group Actions on Boundaries

Ralf Meyer
(joint work with Heath Emerson)

The results that I explained in this talk already appeared in [6]. The following summary is very similar to the introduction to [6].

Let G be a locally compact group, let X be a proper G-space, and let $\overline{X}$ be a compact G-space containing X as a G-invariant open subset. Suppose that both X and $\overline{X}$ are H-equivariantly contractible for all compact subgroups H of G; we briefly say that they are *strongly contractible* and call the action of G on $\partial X = \overline{X} \setminus X$ a *boundary action*.

For example, the group $G = \mathrm{PSl}(2, \mathbb{Z})$ admits the following two distinct boundary actions. On the one hand, since G is a free product of finite cyclic groups, it acts properly on a tree X ([16]). Let ∂X be its set of ends, which is a Cantor set, and let $\overline{X}$ be its ends compactification. Then X and $\overline{X}$ are strongly contractible, and the action of G on ∂X is a boundary action. On the other hand, $\mathrm{PSl}(2, \mathbb{Z}) \subseteq \mathrm{PSl}(2, \mathbb{R})$ acts by Möbius transformations on the hyperbolic plane $\mathbb{H}^2$. We compactify $\mathbb{H}^2$, as usual, by a circle at infinity. Again, $\mathbb{H}^2$ and $\overline{\mathbb{H}^2}$ are strongly contractible and the action on the circle $\partial\mathbb{H}^2$ is a boundary action. Other examples are: a word-hyperbolic group acting on its Gromov boundary; a group of isometries of a $\mathrm{CAT}(0)$ space X acting on the visibility boundary of X.

The central problem of this talk is to understand the map on K-theory induced by the obvious inclusion $u\colon C^*_{\mathrm{red}} G \to C(\partial X) \rtimes G$, where $G \times \partial X \to \partial X$ is a boundary action and $C(\partial X) \rtimes G$ is its reduced crossed product C^*-algebra. Our result is analogous to the classical Gysin sequence, which we recall first.

Let X be a locally compact space and let $\pi\colon V \to X$ be a vector bundle over X, say of rank n. Let BV and SV be the (closed) disk and sphere bundles of V, respectively (with respect to some choice of metric on the bundle). Let H^*_c denote cohomology with compact supports. Since the bundle projection $BV \to X$ is a proper homotopy equivalence, we have $H^*_c(BV) \cong H^*_c(X)$ and $\mathrm{K}^*(BV) \cong \mathrm{K}^*(X)$. We assume now that the bundle V is oriented or K-oriented, respectively. Then we get Thom isomorphisms $H^{*-n}_c(X) \cong H^*_c(V)$ or $\mathrm{K}^{*-n}(X) \cong \mathrm{K}^*(V)$, and excision for the pair (BV, SV) provides us with long exact sequences of the form

$$\cdots \to \mathrm{H}^{*-n}_{\mathrm{c}}(X) \xrightarrow{\varepsilon^*} \mathrm{H}^*_{\mathrm{c}}(X) \xrightarrow{\pi^*} \mathrm{H}^*_{\mathrm{c}}(SX) \xrightarrow{\delta} \mathrm{H}^{*-n+1}_{\mathrm{c}}(X) \to \cdots,$$

$$\cdots \to \mathrm{K}^{*-n}(X) \xrightarrow{\varepsilon^*} \mathrm{K}^*(X) \xrightarrow{\pi^*} \mathrm{K}^*(SX) \xrightarrow{\delta} \mathrm{K}^{*-n+1}(X) \to \cdots.$$

These are the classical *Gysin sequences*. In cohomology, the map ε^* is the cup product with the *Euler class* $e_V \in H^n(X)$ of the oriented bundle V. In K-theory, it is the cup product with the *spinor class* (see [7, IV.1.13]).

We are only interested in the case where X is a smooth n-dimensional manifold and $V = TX$ is its tangent bundle. Then the map $\varepsilon^*\colon \mathrm{H}_{\mathrm{c}}^{*-n}(X) \to \mathrm{H}_{\mathrm{c}}^*(X)$ vanishes unless $* = n$, for dimension reasons; if X is not compact, then $\mathrm{H}_{\mathrm{c}}^0(X) = 0$ and hence $\varepsilon^* = 0$. If X is compact, then $e_{TX} \in H^n(X)$ has the property that $\langle e_{TX}, [X]\rangle$ is the *Euler characteristic* $\chi(X)$ of X (see [2]). There is a similar formula for the map $\varepsilon^*\colon \mathrm{K}^{*-n}(X) \to \mathrm{K}^*(X)$: it vanishes on $\mathrm{K}^1(X)$ and is given by $x \mapsto \chi(X)\dim(x) \cdot \mathrm{pnt}!$ on $\mathrm{K}^0(X)$; here the functional $\dim\colon \mathrm{K}^0(X) \to \mathbb{Z}$ sends a vector bundle to its dimension and $\mathrm{pnt}! \in \mathrm{KK}_{-n}\bigl(\mathbb{C}, C_0(X)\bigr) \cong \mathrm{K}^n(X)$ is the wrong way element corresponding to the inclusion of a point in X, which is a K-oriented map. Notice that $\dim = 0$ unless X is compact. Since the map ε^* factors through $\mathbb{Z}$, we can cut the above long exact sequence into a pair of shorter exact sequences.

Now that all the words in the title are explained, we can bring them together. We return to the situation of a boundary action. For expository purposes, we first assume that G is a torsion-free discrete group. If G is torsion-free, then X is a universal free proper G-space, so that $G\backslash X$ is a model for the classifying space BG. We warn the reader that $\mathrm{K}^*(G\backslash X)$ depends on the particular choice of BG because K-theory is only functorial for *proper* maps.

The exact sequences in the following theorem are quite similar to the classical Gysin sequence for the tangent bundle.

Theorem 1. *Let G be a torsion-free discrete group and let $G \times \partial X \to \partial X$ be a boundary action, where X is a finite-dimensional simplicial complex with a simplicial action of G. Assume that G satisfies the Baum-Connes conjecture with coefficients $\mathbb{C}$ and $C(\partial X)$. Let $u\colon C^*_{\mathrm{red}}G \to C(\partial X) \rtimes G$ be the embedding induced by the unit map $\mathbb{C} \to C(\partial X)$.*

If $G\backslash X$ is compact and $\chi(G\backslash X) \neq 0$, then there are exact sequences

$$0 \to \langle \chi(G\backslash X)[1_{C^*_{\mathrm{red}}G}]\rangle \xrightarrow{\subseteq} \mathrm{K}_0(C^*_{\mathrm{red}}G) \xrightarrow{u_*} \mathrm{K}_0(C(\partial X) \rtimes G) \to \mathrm{K}^1(G\backslash X) \to 0,$$

$$0 \to \mathrm{K}_1(C^*_{\mathrm{red}}G) \xrightarrow{u_*} \mathrm{K}_1(C(\partial X) \rtimes G) \to \mathrm{K}^0(G\backslash X) \xrightarrow{\dim} \mathbb{Z} \to 0.$$

Here $\langle \chi(G\backslash X)[1_{C^*_{\mathrm{red}}G}]\rangle$ denotes the free cyclic subgroup of $\mathrm{K}_0(C^*_{\mathrm{red}}G)$ that is generated by $\chi(G\backslash X)[1_{C^*_{\mathrm{red}}G}]$, and dim *maps a vector bundle to its dimension.*

If $G\backslash X$ is not compact or if $\chi(G\backslash X) = 0$, then there are exact sequences

$$0 \to \mathrm{K}_0(C^*_{\mathrm{red}}G) \xrightarrow{u_*} \mathrm{K}_0(C(\partial X) \rtimes G) \to \mathrm{K}^1(G\backslash X) \to 0,$$

$$0 \to \mathrm{K}_1(C^*_{\mathrm{red}}G) \xrightarrow{u_*} \mathrm{K}_1(C(\partial X) \rtimes G) \to \mathrm{K}^0(G\backslash X) \to 0.$$

Corollary 1. *The class of the unit element in $\mathrm{K}_0(C(\partial X) \rtimes G)$ is a torsion element of order $|\chi(G\backslash X)|$ if $G\backslash X$ is compact and $\chi(G\backslash X) \neq 0$, and not a torsion element otherwise.*

Several authors have already noticed various instances of this corollary ([3, 4, 1, 11, 14, 17, 13]): for lattices in $\mathrm{PSl}(2,\mathbb{R})$ and $\mathrm{PSl}(2,\mathbb{C})$, acting on the boundary of hyperbolic 2- or 3-space, respectively; for closed subgroups of $\mathrm{PSl}(2,\mathbb{F})$ for a non-Archimedean local field $\mathbb{F}$ acting on the projective space $\mathbb{P}^1(\mathbb{F})$, where X is the

Bruhat-Tits tree of $\mathrm{PSl}(2,\mathbb{F})$; for free groups acting on their Gromov boundary. Moreover, Mathias Fuchs has simultaneously obtained the assertions of Theorem 1 and Corollary 1 for some subgroups of almost connected Lie groups, by a completely different method.

Comparing the classical and non-commutative Gysin sequences, we see that the inclusion $u\colon C^*_{\mathrm{red}}G \to C(\partial X) \rtimes G$ plays the role of the embedding $C(M) \to C(SM)$ induced by the bundle projection $SM \to M$. Therefore, if we view $C^*_{\mathrm{red}}G$ as the algebra of functions on a non-commutative space $\hat{G}$, then $C(\partial X) \rtimes G$ plays the role of the algebra of functions on the sphere bundle of $\hat{G}$. Such an analogy has already been advanced by Alain Connes and Marc Rieffel in [5, 12] for rather different reasons (and for a different class of boundary actions).

For groups with torsion and, more generally, for locally compact groups, we must use an *equivariant* Euler characteristic in $\mathrm{KK}_0^G(C_0(X),\mathbb{C})$ instead of the Euler characteristic of $G\backslash X$. To define it, we use a general notion of Poincaré duality in bivariant Kasparov theory. An *abstract dual* for a space X consists of a G-C^*-algebra $\mathcal{P}$ and a class $\Theta \in \mathrm{RKK}_n^G(X;\mathbb{C},\mathcal{P})$ for some $n \in \mathbb{Z}$ such that the map

$$\mathrm{PD}\colon \mathrm{RKK}^G_{*-n}(Y; A \otimes \mathcal{P}, B) \to \mathrm{RKK}^G_*(X \times Y; A, B), \quad f \mapsto \Theta \otimes_{\mathcal{P}} f,$$

is an isomorphism for all pairs of G-C^*-algebras A, B and all G-spaces Y. This formalises a result of Gennadi Kasparov ([8, Theorem 4.9]).

Let X be any G-space that has such an abstract dual. The diagonal embedding $X \to X \times X$ yields classes

$$\Delta_X \in \mathrm{RKK}_0^G(X; C_0(X), \mathbb{C}), \qquad \mathrm{PD}^{-1}(\Delta_X) \in \mathrm{KK}^G_{-n}(C_0(X) \otimes \mathcal{P}, \mathbb{C}).$$

Let $\bar{\Theta} \in \mathrm{KK}_n^G(C_0(X), C_0(X)\otimes\mathcal{P})$ be obtained from Θ by forgetting the X-linearity. We define the *abstract equivariant Euler characteristic* by

$$\mathrm{Eul}_X = \bar{\Theta} \otimes_{C_0(X)\otimes\mathcal{P}} \mathrm{PD}^{-1}(\Delta_X) \in \mathrm{KK}_0^G(C_0(X), \mathbb{C}).$$

Examples show that this class deserves to be called an Euler characteristic. We were led to this definition by the consideration of the Gysin sequence.

Gennadi Kasparov constructs such an abstract duality for a smooth Riemannian manifold in [8, Section 4], using for $\mathcal{P}$ the algebra of C_0-sections of the Clifford algebra bundle on X. A fairly simple computation shows that the associated equivariant Euler characteristic is the class in $\mathrm{KK}_0^G(C_0(X),\mathbb{C})$ of the de Rham operator on X, which we call the *equivariant de-Rham-Euler characteristic of* X.

If X is a simplicial complex and G acts simplicially, then a Kasparov dual for X is constructed in [9]. Since the description of Θ in [9] is too indirect for our purposes, we need a slightly different construction. This requires quite a lot of work. In the end, we can compute the abstract Euler characteristic that we get from this duality. It is given by the expected combinatorial formula and therefore called the *equivariant combinatorial Euler characteristic*. An important feature of the (abstract) Euler characteristic is that it does not depend on the choice of the abstract dual. Therefore, if X admits a smooth structure and a triangulation at the same time, then the combinatorial and the de Rham Euler characteristic of X coincide. This result is due to Wolfgang Lück and Jonathan Rosenberg ([10, 15]).

Now we outline the first steps of the proof of Theorem 1 and its analogies for general locally compact groups. The starting point is the extension of G-C^*-algebras

$$0 \to C_0(X) \to C(\overline{X}) \to C(\partial X) \to 0,$$

which yields a six term exact sequence for the functor $\mathrm{K}_*^{\mathrm{top}}(G, \,\cdot\,)$; this denotes the domain of the Baum-Connes assembly map. The strong contractibility of $\overline{X}$ implies that $\mathrm{K}_*^{\mathrm{top}}\big(G, C(\overline{X})\big) \cong \mathrm{K}_*^{\mathrm{top}}(G)$. The resulting map

$$\mathrm{K}_*^{\mathrm{top}}(G) \cong \mathrm{K}_*^{\mathrm{top}}\big(G, C(\overline{X})\big) \to \mathrm{K}_*^{\mathrm{top}}\big(G, C(\partial X)\big)$$

in the exact sequence is induced by the unital inclusion $\mathbb{C} \to C(\partial X)$. A purely formal argument shows that the map

$$\mathrm{K}_*^{\mathrm{top}}\big(G, C_0(X)\big) \to \mathrm{K}_*^{\mathrm{top}}\big(G, C(\overline{X})\big) \cong \mathrm{K}_*^{\mathrm{top}}(G)$$

in the exact sequence is given by the Kasparov product with the equivariant *abstract* Euler characteristic $\mathrm{Eul}_X \in \mathrm{KK}_0^G(C_0(X), \mathbb{C})$.

Our interest in the problem of calculating the K-theory of boundary crossed products was sparked by discussions with Guyan Robertson at a meeting in Oberwolfach in 2004. We would like to thank him for drawing our attention to this question. We also thank Wolfgang Lück for helpful suggestions regarding Euler characteristics.

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Witt Groups and Witt Spaces

Jonathan Woolf

The Witt group of a commutative ring R is the group of stable equivalence classes of non-singular symmetric bilinear forms under direct sum with metabolic forms (see [7] for details). Tensor product makes the Witt group into a ring. If R is a local ring in which 2 is invertible the Witt group together with the rank provide a complete classification of non-singular symmetric bilinear forms. In general the situation is more complicated but the Witt group provides important invariants.

Examples:

(1) $W(\mathbb{Z}) \cong \mathbb{Z}$ (the signature),

(2) For prime p we have

$$W(\mathbb{Z}/p) \cong \begin{cases} \mathbb{Z}/2 & p = 2 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & p = 1 \bmod 4 \\ \mathbb{Z}/4 & p = 3 \bmod 4 \end{cases}$$

(3) There is a split exact sequence

$$0 \to W(\mathbb{Z}) \to W(\mathbb{Q}) \to \bigoplus_p W(\mathbb{Z}/p) \to 0$$

where the right-hand term consists of obstructions to a rational form being Witt-equivalent to an integral form.

The Witt group of $\mathbb{Q}$ has a geometric interpretation as the $4k$-dimensional bordism group of those compact, oriented PL-pseudomanifolds whose rational intersection homology satisfies Poincaré duality *locally.* These pseudomanifolds are called Witt spaces and an characterisation of them can be given in terms of the allowed links of singularities. Any complex variety is a Witt space and these provide an important, but by no means exhaustive, class of examples.

Theorem 1 (Siegel [9])**.**

$$\Omega_*^{Witt} \cong \begin{cases} \mathbb{Z} & * = 0 \\ W(\mathbb{Q}) & * = 4k, k > 0 \\ 0 & \textit{otherwise} \end{cases}$$

A similar interpretation of the Witt group of $\mathbb{Z}$, involving intersection Poincaré (IP) spaces which are the pseudomanifolds whose *integral* intersection homology satisfies Poincaré duality locally, is also possible. Again the allowed links can be explicitly characterised and the bordism groups computed.

Theorem 2 (Pardon [8]).

$$\Omega_*^{IP} \cong \begin{cases} W(\mathbb{Z}) & * = 4k, k > 0 \\ \mathbb{Z}_2 & * = 4k+1, k > 0 \\ 0 & \textit{otherwise} \end{cases}$$

In both cases the isomorphism with the Witt group is obtained by sending a $4k$-dimensional pseudomanifold to the class in the Witt group of the intersection form on the $2k$-dimensional intersection homology group. The right-hand term of

$$0 \to W(\mathbb{Z}) \to W(\mathbb{Q}) \to \bigoplus_p W(\mathbb{Z}/p) \to 0$$

can now be interpreted as the space of obstructions to a Witt space being Witt-bordant to an IP space — the obstructions are local to the singularities and each can be explicitly realised by plumbing (see [6, 9]).

The isomorphism between $W(\mathbb{Q})$ and bordism groups of Witt spaces can be generalised as follows. In [1] Balmer shows that (under mild conditions) Witt groups may be associated to any triangulated category with duality in which 2 is invertible. The groups are indexed by $\mathbb{Z}$ but are always 4-periodic. These Balmer–Witt groups of triangulated categories are generalisations of the Witt group of a ring: elements are represented by self-dual objects considered up to a relation of 'Witt-equivalence'. Furthermore, [2] shows that the zeroth group of the bounded derived category of modules over a local ring R in which 2 is invertible is isomorphic to $W(R)$. [10] shows that the Balmer–Witt groups of the PL-constructible derived category of sheaves form a homology theory on polyhedra isomorphic to the 4-periodic colimit of the bordism groups of Witt spaces. By relating constructible sheaves to combinatorial sheaves this homology theory can also be identified with Ranicki's free rational L-theory. Unfortunately the requirement that 2 be invertible means that an integral analogue of this result, which might be of some utility in surgery theory, is not possible in the context of derived categories of sheaves (it is essential to work with categories of complexes instead).

This correspondence between Witt equivalence classes of self-dual sheaves and bordism classes of Witt spaces suggests geometric interpretations of sheaf-theoretic results. For instance, in [4], Cappell and Shaneson prove an up-to-cobordism analogue of Beilinson, Bernstein and Deligne's decomposition theorem for intersection homology: if $f : X \to Y$ is a stratified map between spaces with only even dimensional strata and $\mathcal{E}$ is a constructible self-dual complex of sheaves on X then the pushforward $Rf_*\mathcal{E}$ is, up to a certain notion of cobordism, the direct sum of 'simple' self-dual complexes supported on the strata of Y. This has interesting consequences for the signatures, and more generally L-classes, of stratified spaces. By relating Cappell and Shaneson's cobordism relation to Balmer's Witt-equivalence this result can be reinterpreted as saying that the bordism class of a Witt space with a stratified map to Y is a sum of classes indexed by the strata of Y.

This geometric version retains torsion information which is lost in Cappell and Shaneson's result. From another perspective, it can be seen as a wide-ranging generalisation of Novikov additivity (shown by Siegel to be a consequence of the existence of a 'pinch cobordism' between certain Witt spaces).

There are other flavours of Balmer–Witt groups of sheaves on stratified spaces, obtained by varying the constructibility condition. In particular the Balmer–Witt group $W_*^{\text{alg-c}}(X)$ of sheaves on a complex algebraic variety X which are constructible with respect to complex algebraic stratifications. These are used in [3] to construct L-classes for singular varieties. These groups have alternative descriptions as the Witt groups of the derived category of perverse sheaves and, via the Riemann–Hilbert correspondence, as the Witt groups of the regular holonomic derived category of D-modules (see e.g. [5]). Using [2] we can identify the zeroth group as the Witt group of the abelian category of perverse sheaves on X.

Conjecture 1 (c.f. Youssin [11]). *The Witt group $W(\mathrm{Perv}(X))$ of perverse sheaves on X decomposes as a direct sum indexed by the simple objects of $\mathrm{Perv}(X)$. Furthermore, if $V = \{f = 0\} \subset X$ is a hypersurface then there is a split surjection*

$$W(\mathrm{Perv}(X)) \to W(\mathrm{Perv}(V))$$

induced by the perverse vanishing cycles functor ${}^p\varphi_f$.

This conjecture holds when X is a curve. Although the Witt group of perverse sheaves will have uncountably many summands (including as it does a summand for each point of the curve) the class of any self-dual perverse sheaf will project to zero in all but finitely many of these. For example, if Y is smooth of complex dimension $2n$ and $f : Y \to \mathbb{C}$ has a single isolated singularity at 0 then the pushforward of the constant sheaf induces a class in $W_{4n}^{\text{alg-c}}(X) \cong W(\mathrm{Perv}(\mathbb{C}))$. This class maps under the perverse vanishing cycles functor to

$$\begin{cases} 0 & z \neq 0 \\ [I_F] & z = 0 \end{cases}$$

in $W(\mathrm{Perv}(z)) \cong W(\mathbb{Q})$ where I_F is the intersection form on the middle homology of the Milnor fibre F of the singularity. The rational Witt class of I_F is sufficient to distinguish between many of the ADE singularities; the first pair we cannot distinguish is A_7 and E_7.

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Additivity for the Parametrised Topological Euler Characteristic and Reidemeister Torsion

Wojciech Dorabiała
(joint work with Bernard Badzioch)

The motivation for the results described here comes from the work of Dwyer, Weiss and Williams on refining the structure of fibre bundles of topological manifolds [4]. One of the main tools used in their work is Waldhausen's algebraic K-theory of spaces [9] which we will denote by $A(-)$. Recall, that for a space X the infinite loop space $A(X)$ can be obtained by applying Waldhausen's $S_\bullet$-construction to the category of homotopy finitely dominated retractive spaces over X. In particular, any such retractive space represents a point in $A(X)$. This justifies the following:

Definition 1. *Let X be a homotopy finitely dominated space. The homotopy characteristic of X is the point $\chi^h(X) \in A(X)$ represented by the retractive space*

$$X \times \{0,1\} \rightleftarrows X\,.$$

The homotopy characteristic $\chi^h(X)$ is a homotopy type invariant of X and can be viewed as a generalisation of the classical Euler characteristic of the space X. Indeed, we have an isomorphism $\pi_0 A(X) \cong \tilde{K}_0(\mathbb{Z}[\pi_1 X]) \oplus \mathbb{Z}$ which sends the connected component of $\chi^h(X)$ to the pair $(\omega(X), \chi(X))$ where $\omega(X)$ is the Wall finiteness obstruction while $\chi(X)$ is the Euler characteristic of X.

One especially appealing aspect of this extension of the notion of the Euler characteristic of a space is that one can modify the above construction to define invariants which reflect not only the homotopy type of a space but also its geometry. Let $A^{\%}(-)$ denote the cohomology theory represented by the spectrum $A(*)$. The functor $A^{\%}(-)$ is not equivalent to $A(-)$, but comes equipped with an assembly map, i.e. a natural transformation $a\colon A^{\%}(X) \to A(X)$, and this assembly map measures the failure of the algebraic K-theory of spaces to be excisive. The homotopy characteristic $\chi^h(X)$ usually does not admit a lift to $A^{\%}(X)$, but a lift does exist if X is of the homotopy type of a compact ENR (See [10], [8]). If, in addition, X itself is a compact ENR (in particular - a compact topological manifold) there is a canonical way to construct a lift of $\chi^h(X)$. The resulting point in $A^{\%}(X)$ is called the topological characteristic of X and is denoted by $\chi^t(X)$. The more rigid nature of $\chi^t(X)$ is revealed by the fact that topological characteristics of ENRs are not invariant under homotopy equivalences. They are, however, invariant with respect to cell-like maps; if $f\colon X \to Y$ is cell-like then it defines a canonical path in $A^{\%}(Y)$ joining $f_*\chi^t(X)$ with $\chi^t(Y)$.

Dwyer, Weiss and Williams showed that the approach sketched above leads to very interesting invariants of fibre bundles. Let $E \xrightarrow{p} B$ be a fibration whose fibres are homotopy finitely dominated spaces. Define

$$A_B(E) := \coprod_{x\in B} A(p^{-1}(x)).$$

This space can be equipped with an appropriate topology such that we obtain a fibration $p^h\colon A_B(E) \to B$. Its fibres are the spaces $A(p^{-1}(x))$ which contain points $\chi^h(p^{-1}(x))$ – the homotopy characteristics of $p^{-1}(x)$ for $x \in B$. Since the structure maps gluing the fibres of p together are homotopy equivalences, the homotopy invariance of χ^h implies that the elements $\chi^h(p^{-1}(x))$ for various $x \in B$ are compatible enough to define a continuous section of p^h:

$$\chi^h(p)\colon B \to A_B(E), \qquad x \mapsto \chi^h(p^{-1}(x)).$$

We take $\chi^h(p)$ to be the homotopy characteristic of the fibration $E \xrightarrow{p} B$. It is an element of $A(p)$ – the space of sections of p^h.

If $E \xrightarrow{p} B$ is a fibration as above and if its fibres are compact ENRs we have defined topological characteristics of the fibres, so one can try to replicate the above constructions on the level of the excisive A-theory $A^{\%}(-)$. Thus, we can take $A^{\%}_B(E) := \coprod_{x\in B} A^{\%}(p^{-1}(x))$ and define $A^{\%}(p)$ to be the space of sections of the quasi-fibration $p^t\colon A^{\%}_B(E) \to B$. Recall, however, that χ^t is invariant only with respect to cell-like maps. As a consequence for an arbitrary fibration p the topological characteristics $\chi^t(p^{-1}(x))$ cannot be combined to give a section of p^t. If we assume that the fibres of p are topological manifolds and that $E \xrightarrow{p} B$ is a bundle of manifolds, then the structure maps of p are given by homeomorphisms and we do get a section $\chi^t(p) \in A^{\%}(p)$ which we call the topological characteristic of p.

Just as in the non-parametrised case, we have the assembly map $a\colon A^{\%}(p) \to A(p)$. If p is a fibre bundle of compact topological manifolds, and so both $\chi^h(p)$ and $\chi^t(p)$ are defined, the topological characteristic $\chi^t(p)$ is a lift of $\chi^h(p)$ (more precisely, there is a canonical path in $A(p)$ joining $a(\chi^t(p))$ with $\chi^h(p)$). A beautiful theorem of Dwyer, Weiss and Williams [4] shows that the converse is also true: for a fibration $E \xrightarrow{p} B$ with homotopy finitely dominated fibres the homotopy characteristic $\chi^h(p)$ admits a lift to $A^{\%}(p)$ if and only if p is fibrewise homotopy equivalent to a bundle of compact topological manifolds.

The process of rectifying the structure of a fibration by means of its characteristics can be pushed one step further. For a fibration $E \xrightarrow{p} B$ let $Q_B(E) := \coprod_{x\in B} Q(p^{-1}(x))$ where $Q(p^{-1}(x))$ is the stable homotopy spectrum of $p^{-1}(x)$. Let $Q(p)$ denote the space of sections of the quasi-fibration $p^s\colon Q_B(E) \to B$; the unit map $Q(S^0) \to A(*)$ induces a map $\eta\colon Q(p) \to A^{\%}(p)$. If p is a bundle of smooth manifolds, then it defines an element $\chi^s(p) \in Q(p)$ and this smooth characteristic of p is a lift of $\chi^t(p)$. Again [4] gives the converse to this last statement: for a bundle $E \xrightarrow{p} B$ of topological manifolds we can lift $\chi^t(p)$ to $Q(p)$ if and only if p is related to a smooth bundle of manifolds via a fibrewise homotopy equivalence.

The results described above show that the existence of lifts of $\chi^h(p)$ and $\chi^t(p)$ provides a great deal of information about the geometry of the bundle p. Unfortunately, for a given bundle p, the elements $\chi^h(p)$ and $\chi^t(p)$ are usually very hard to compute. Going back for a moment to the classical Euler characteristic χ of CW-complexes, recall that the property of this invariant which makes it computable is its additivity; if X_0, X_1, X_2 are subcomplexes of X such that $X = X_1 \cup_{X_0} X_2$ then $\chi(X) = \chi(X_1) + \chi(X_2) - \chi(X_0)$.

As it turns out an analogous property holds for both the homotopy characteristic (see [3]) and the topological characteristic (stated below).

Theorem 1 (Theorem 1 of [1]). *Suppose that the fibre M of a bundle $p : E \to B$ is a compact manifold which splits into a union of submanifolds $M = M_1 \cup M_2$, where $M_0 = M_1 \cap M_2$ is a submanifold of codimension 1. Furthermore, suppose that p admits a splitting into subbundles $p_i\colon E_i \to B$ for $i = 0, 1, 2$ such that M_i is a fibre of p_i, and $E_0 = E_1 \cap E_2$. Then there exists a path in $A^{\%}(p)$ joining $\chi^t(p)$ with $j_{1*}\chi^t(p_1) + j_{2*}\chi^t(p_2) - j_{0*}\chi^t(p_0)$. Here j_{i*} is the map induced by the inclusion $j_i\colon E_i \hookrightarrow E$.*

Another important consequences of the work of Dwyer, Weiss and Williams are their definitions of smooth, homotopy and topological Reidemeister torsion for bundles which, are secondary invariants of the respective characteristics of bundles. Just as the homotopy characteristic of a space generalises the notion of Euler characteristic, homotopy Reidemeister torsion is an extension of the notion of the classical Franz-Reidemeister torsion of spaces. Arguments similar to the ones used to prove Theorem 1 for $\chi^t(p)$ show that the topological Reidemeister torsion is additive (again, the same property for the homotopy torsion has been proved in [3]). Aside from increasing the computability of topological torsion, this fact brings us closer to understanding the relationship of the torsion of Dwyer, Weiss and Williams to the analytic torsion of Bismut and Lott [2] and the torsion of Igusa-Klein [7]. The subject of comparing the higher Franz-Reidemeister torsion of Igusa and Klein with the higher analytic torsion of Bismut and Lott was undertaken by Goette in [5], [6].

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Equivariant Index Theory and the Eta Invariant

Ken Richardson
(joint work with J. Brüning, F. W. Kamber)

1. Introduction

In this note, we exhibit formulas for the multiplicities of the equivariant index, summarising joint work with J. Brüning and F. Kamber ([5],[6]). Also, in work with I. Prokhorenkov ([8]), if a Killing vector field with isolated singularities exists, then the Witten deformation method can be used to obtain formulas for the multiplicities.

2. Notation and preliminaries

Suppose that a compact Lie group G acts by isometries on a compact, connected manifold M, and let $E = E^+ \oplus E^-$ be a graded, G-equivariant vector bundle over M. We consider a first order G-equivariant differential operator $D = D^+ : \Gamma(M, E^+) \to \Gamma(M, E^-)$ which is transversally elliptic, and let D^- be the formal adjoint of D^+.

The group G acts on $\Gamma(M, E^\pm)$ by $(gs)(x) = gs\left(xg^{-1}\right)$, and the (possibly infinite-dimensional) subspaces $\ker(D)$ and $\ker(D^*)$ are G-invariant subspaces. Thus, each of $\Gamma(M, E^\pm)$, $\ker(D)$, and $\ker(D^*)$ decomposes as a direct sum of irreducible representation spaces. Let $\rho : G \to \operatorname{End}(V_\rho)$ be an irreducible unitary representation of G. Let $\Gamma(M, E^\pm)^\rho$ be the subspace of sections that is the direct sum of the irreducible G-representation subspaces of $\Gamma(M, E^\pm)$ that are unitarily equivalent to the ρ representation. It can be shown that the extended operator

$$\overline{D_s} : H^s\left(\Gamma\left(M, E^+\right)^\rho\right) \to H^{s-1}\left(\Gamma\left(M, E^-\right)^\rho\right)$$

is Fredholm and independent of s, so that each irreducible representation of G appears with finite multiplicity in $\ker D^\pm$. Let $a_\rho^\pm \in \mathbb{Z}^+$ be the multiplicity of ρ in $\ker(D^\pm)$.

We define the virtual representation-valued index of D as in [1], as

$$\operatorname{ind}^{G}(D) := \sum_{\rho} \left(a_{\rho}^{+} - a_{\rho}^{-}\right)[\rho],$$

where $[\rho]$ denotes the equivalence class of the irreducible representation ρ. The index multiplicity is

$$\operatorname{ind}^{\rho}(D) := a_{\rho}^{+} - a_{\rho}^{-} = \frac{1}{\dim V_{\rho}} \operatorname{ind}\left(D|_{\Gamma(M,E^{+})^{\rho} \to \Gamma(M,E^{-})^{\rho}}\right).$$

In particular, if ρ_0 is the trivial representation of G, then

$$\operatorname{ind}^{\rho_0}(D) = \operatorname{ind}\left(D|_{\Gamma(M,E^{+})^{G} \to \Gamma(M,E^{-})^{G}}\right),$$

where the superscript G implies restriction to G-invariant sections.

There is a clear relationship between the index multiplicities and Atiyah's equivariant distribution-valued index $\operatorname{ind}_g(D)$; the multiplicities determine the distributional index, and vice versa. Because the operator $D|_{\Gamma(M,E^{+})^{\rho} \to \Gamma(M,E^{-})^{\rho}}$ is Fredholm, all of the indices $\operatorname{ind}^{G}(D)$, $\operatorname{ind}_g(D)$, and $\operatorname{ind}^{\rho}(D)$ depend only on the homotopy class of the principal transverse symbol of D. Incidentally, if a formula for $\operatorname{ind}^{\rho_0}(D)$ in terms of the principal transverse symbol is known, then it follows that $\operatorname{ind}^{\rho}(D)$ can be computed in a similar way, because the invariant index $\operatorname{ind}^{\rho_0}\left(\widetilde{D}\right)$ of the differential operator twisted by the dual representation ρ^* on the bundle $E \otimes V_{\rho}^{*}$ is the same as $\operatorname{ind}^{\rho}(D)$.

A heat kernel analysis shows that $\operatorname{ind}^{\rho_0}(D)$ may be expressed as an integral over $G \times M$ that concentrates near sets of the form

$$\bigcup_{G_x \in [H]} x \times G_x,$$

where the isotropy subgroup G_x is the subgroup of G that fixes $x \in M$, and $[H]$ is a conjugacy class of isotropy subgroups.

A large body of work over the last twenty years has yielded theorems that express $\operatorname{ind}_g(D)$ in terms of topological and geometric quantities (as in the Atiyah-Segal-Singer index theorem for elliptic operators or the Berline-Vergne Theorem for transversally elliptic operators — see [2],[3],[4]). However, until now there has been very little known about the problem of expressing $\operatorname{ind}^{\rho}(D)$ as a sum of topological or geometric quantities which are determined at the different strata

$$\Sigma^{[H]} := \bigcup_{G_x \in [H]} x$$

of the G-manifold M.

3. New formulas for the multiplicities of the equivariant index

With the notation as in Section 2, suppose that there is only one stratum Σ, so that every orbit of G is principal. This case is well-understood, as M/G is a manifold, and the index $\mathrm{ind}^{\rho_0}(D)$ satisfies

$$\mathrm{ind}^{\rho_0}(D) = \mathrm{ind}\left(D^G\right),$$

where D^G is an operator induced on $\Gamma(M/G, \mathcal{E})$, where over each orbit $\mathcal{O} \in M/G$, $\mathcal{E}$ is the set of sections $V \in \Gamma(\mathcal{O}, E|_{\mathcal{O}})$ that are invariant under the G-action. Thus, the equivariant index theorem for this operator is a result of the Atiyah-Singer Index Theorem, applied the operator D^G.

A similar case that has been understood for a long time — the situation where all orbits have the same dimension (which implies that all isotropy subgroups have the same dimension). It turns out that this is equivalent to the problem of computing the index of an elliptic operator on an orbifold, and this problem was solved by Kawasaki [7].

In [5], the more general case, where the orbits have different dimensions, is treated. As an example, we now discuss the case when there are two isotropy types (including the case when the two types of orbits have different dimensions).

In the theorem that follows, we let

$$D_\varepsilon(M, \Sigma) = [M \setminus T_\varepsilon(\Sigma)] \cup [(-\varepsilon, \varepsilon) \times S\Sigma] \cup [M \setminus T_\varepsilon(\Sigma)]$$

be the double of the blowup of M along the singular stratum Σ, with a specific choice of metric (in [5]). Also, let $A_1 * A_2$ denote the product of the two differential operators A_1 and A_2, compatible with the product in K-theory (see [1]).

The theorem that follows is proved using the heat kernel approach.

Theorem 1. *(special case of theorem in* [5]*) Let E be a Hermitian vector bundle over a closed, Riemannian manifold M, such that a compact Lie group G acts on (M, E) by isometries. Suppose that the action of G on M has only two isotropy types. Let M_0 denote the principal stratum, and let Σ denote the singular stratum. Let $D : \Gamma(M, E^+) \to \Gamma(M, E^-)$ be a first order, transversally elliptic, G-equivariant differential operator. We assume that near Σ, D can be written as the product*

$$D = \left\{ Z \left(\nabla^E_{\partial_r} + \frac{1}{r} D^S \right) \right\} * D^\Sigma,$$

where r is the distance from Σ, where Z is a local bundle isomorphism, the map D^S is a purely first order operator that differentiates in directions tangent to the unit normal space $S_x\Sigma$, and D^Σ is a global transversally elliptic, first order operator on the stratum Σ. Then

$$\mathrm{ind}^{\rho_0}(D) = \int_{M_0/G} \overline{\alpha}^G - \left(\frac{\eta\left(D^S\right)^G - h\left(D^S\right)^G}{2} \right) \mathrm{ind}^{\rho_0}\left(D^\Sigma\right),$$

where

(1) $\overline{\alpha}^G$ *is the Atiyah-Singer integrand for the induced operator on the orbit space*
$D_\varepsilon(M,\Sigma)/G$ *of the double of the blowup of* Σ*, with* $\varepsilon \to 0$.
(2) $\eta\left(D^S\right)^G$ *is the eta invariant of the operator* $\left(D^S\right)^G$*, the elliptic operator on the orbit space* $S_x\Sigma/G$ *induced from the equivariant operator* D^S *on* $S_x\Sigma$*, for any fixed* $x \in \Sigma$.
(3) $h\left(D^S\right)^G$ *is the dimension of* $\ker\left(D^S\right)^G$ *on* $S_x\Sigma/G$.

This theorem can be extended to more general situations with more complicated stratifications. Also, an important application of this theorem involves the solution to the problem of finding a formula for the basic index of the transversal Dirac operator on a Riemannian foliation, a problem which has been open for 25 years (see [6]).

4. Equivariant Witten deformations

A completely different approach to calculating the index $\mathrm{ind}^\rho(D)$ has been discovered in joint work with I. Prokhorenkov (see [8]). The idea is to generalise Witten's deformation proof of the Atiyah-Hirzebruch Theorem (which states that the S^1-equivariant index of the spin^c Dirac operator is identically zero) to more general operators and groups. Given a vector field generated by a one-parameter subgroup of isometries, that has isolated fixed points on the manifold M, then one may write the index $\mathrm{ind}^\rho(D)$ as a sum of combinatorial indices, each of which is generated by the local expression of D and the group action near a fixed point.

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Family Hirzebruch Signature Theorem with Converse

E. Bruce Williams
(joint work with Michael Weiss)

Let $\sigma(X) \in \mathbf{Z}$ be the signature of X, where X is a connected, oriented $4k$-dim Poincaré complex.

For smooth manifolds, index theory applied to the signature operator yields:

(1) a characteristic class formula for the signature of a smooth manifold (Hirzebruch Signature Theorem),
(2) a KO-theory Thom class (at odd primes) for vector bundles, and
(3) a family index theorem for the fibrewise signature operator on a smooth bundle with closed manifold fibres.

The work of Sullivan, Kirby, and Siebenmann implies that (1) and (2) extend to topological manifolds and topological euclidean bundles. In fact Sullivan showed that (at odd primes) the theory of stable topological euclidean bundles is equivalent to the theory of stable spherical fibrations equipped with a KO-orientation.

The goal of this talk is to describe a refined version of the signature which we use to study the following question.
Question: Let $p : E \to B$ be a fibration where B is a connected, CW complex and the fibres of p are homotopy equivalent to an n-dim Poincaré complex F. When is p fibre homotopy equivalent to a fibre bundle $q : \mathcal{E} \to B$ with fibres closed n-dim topological manifolds? (I.e. when does p admit manifold- bundle structure?)

We show that the fibration p has a parametrised (refined) signature. If p does admit manifold-bundle structure, then the parametrised (refined) signature satisfies a certain family index theorem we describe below. Suppose F is homotopy equivalent to a closed manifold, and dim(B) is less than the concordance stable range for all the manifolds homotopy equivalent to F. Then p admits manifold-bundle structure iff this family index theorem is satisfied.
The refinement of the signature occurs in 4 stages.

Stage I: Symmetric chain complexes (Ranicki)

Let R be a ring with involution $-$, and $\mathcal{C}$ a left, f.g. free R-chain complex. Let $\mathcal{C}^t$ be $\mathcal{C}$ made into a right R-chain complex via $-$. A n-dim symmetric structure on $\mathcal{C}$ is given by an n-cycle in $(\mathcal{C}^t \otimes_R \mathcal{C})^{h\mathbf{Z}/2}$.

Let $L^n(R)$ be the group of bordism classes of chain complexes equipped with a (nonsingular) n-dim symmetric structure. The group $L^n(R)$ is the nth -homotopy group of the bordism spectrum $\mathbf{L}^\bullet(R)$.

There is an assembly map $\mathbf{H}_\bullet(X; \mathbf{L}^\bullet(\mathbf{Z})) \to \mathbf{L}^\bullet(\mathbf{Z}\pi_1(X))$. If we replace homotopy fixed point with homotopy orbit in the definition of $\mathbf{L}^\bullet(R)$, we get the quadratic L-theory spectrum $\mathbf{L}_\bullet(R)$, and a norm map $\mathbf{L}_\bullet(R) \to \mathbf{L}^\bullet(R)$. Let $\hat{\mathbf{L}}(R)$ be the spectrum which is the cofibre of this norm map. The map $\mathbf{L}^\bullet(\mathbf{Z}) \to \hat{\mathbf{L}}(\mathbf{Z})$ is a map of ring spectrum.

If X^n is a n-dim Poincaré complex, then the geometry which yields Steenrod operations also yields a preferred (nonsingular) n-dim symmetric structure on $C(\tilde{X})$, the cellular chain complex of the universal cover of X.

Ranicki proved the following refinements of (1) and (2).

(1) If X^n is a topological manifold, then $\sigma^*(X)$ has a preferred lifting to a $\mathbf{L}^\bullet(\mathbf{Z})$-fundamental class in the domain of the assembly map.

(2) Any oriented spherical fibration has a preferred $\hat{\mathbf{L}}(\mathbf{Z})$-Thom class. A stable spherical fibration comes (up to fibre homotopy) from removing the zero section of a topological euclidean bundle iff the $\hat{\mathbf{L}}(\mathbf{Z})$-Thom class lifts to a $\mathbf{L}^\bullet(\mathbf{Z})$-Thom class.

The new statement (1) does not have a converse. In order to get a converse we need to go the next stage of refinement.

Stage II: Visible Symmetric Chain Complexes (Weiss and Ranicki)
See Ranicki's book "Algebraic L-theory and topological manifolds)

Visible symmetric theory for $\mathbf{Z}\pi$ is the same as symmetric theory except $(\mathcal{C}^t \otimes_{\mathbf{Z}\pi} \mathcal{C})^{h\mathbf{Z}/2}$ is replaced by $((\mathcal{C}^t \otimes_{\mathbf{Z}} \mathcal{C})^{h\mathbf{Z}/2})_{h\pi}$. If we further replace the discrete ring $\mathbf{Z}\pi$ with the simplicial ring $\mathbf{Z}\Omega X$, we get the functor $\mathbf{VL}^\bullet(\mathbf{Z}\Omega X)$, with assembly map $\mathbf{H}_\bullet(X;\mathbf{L}^\bullet(\mathbf{Z})) \to \mathbf{VL}^\bullet(\mathbf{Z}\Omega X)$. Note: $\mathbf{VL}^\bullet(\mathbf{Z}) = \mathbf{L}^\bullet(\mathbf{Z})$.
Warning: Ranicki uses different notation for $\mathbf{VL}^\bullet(\mathbf{Z}\Omega X)$.

An n-dim Poincaré complex determines an element $\sigma_V^*(X)$ in the n-th homotopy groups of $\mathbf{VL}^\bullet(\mathbf{Z}\Omega X)$. Assume $n > 4$. Then $\sigma_V^*(X)$ lifts to a fundamental class in the domain of the assembly map iff x is homotopy equivalent to a n-dim closed topological manifold.

Suppose $p : E \to B$ is a fibrations such that for any $b \in B$, $F_b = p^{-1}(b)$ is an n-dim Poincaré complex. Then p has a parametrised signature which is a continuous rule which assigns to each $b \in B$ a point in the n-th loop space of $\mathbf{VL}^\bullet(\mathbf{Z}\Omega F_b)$. Modulo issues involving Wh_1 one can show that p is fibre homotopy equivalent to a *block* bundle iff the parametrised signature satisfies a family index theorem, i.e. factors through the fibrewise assembly map to give a $\mathbf{L}^\bullet(\mathbf{Z})$-fundamental class for each fibre. This result is not explicitly already in the literature, but it is closely related to Quinn's thesis and the papers of Lueck and Ranicki on L-theory transfers.

Stage III: Visible symmetric structures on retractive spaces

In order to replace block bundles with fibre bundles in the last paragraph we have to further refine our L-theory in two more stages. For details see [WW3]=Weiss and Williams, Automorphisms of manifolds and algebraic K-theory: Part III which is on Michael Weiss's home page at Aberdeen.

Replace systems of chain complexes with retractive spaces over X. See sections 2 and 3 of [WW3]. This yields the functor $\mathbf{VL}^\bullet(X)$.

Stage IV: Modify using algebraic K-theory of spaces

Recall that given a spectrum $\mathbf{A}$ with an action by $\mathbf{Z}/2$ we get a norm map from the (homotopy orbit spectrum) to the (homotopy fixed spectrum) and the cofibre is called the Tate spectrum. If X is a n-dim Poincaré complex, then the

Spivak fibration can be used to construct an action by $\mathbf{Z}/2$ on $\mathbf{A}(X)$, the algebraic K-theory of X in the sense of Waldhausen, and there is a natural transformation from $\mathbf{VL}^\bullet(X)$ to the Tate spectrum. We "twist" this construction n times, and let $\mathbf{VLA}^\bullet(X)$. be the fibre product of $\mathbf{VL}^\bullet(X)$ and the (homotopy fixed spectrum) over the Tate spectrum. See section 8 of [WW3].

We then get a result which is analogous to the last paragraph of Stage II except we replace $\mathbf{VL}^\bullet(\mathbf{Z}\Omega F_b)$ with $\mathbf{VLA}^\bullet(X)$, block bundle by fibre bundle, and we have to assume that dim(B) is less than the concordance stable range for all the manifolds homotopy equivalent to F.

K^1 Representative, Eta Invariant and Index of Toeplitz Operators

Xianzhe Dai

(joint work with Weiping Zhang)

On an even dimensional compact spin manifold with boundary, the famous Atiyah-Patodi-Singer formula computes the index of the Dirac operator with a global boundary condition, the APS boundary condition. In this magical formula, topology (the index), geometry (the characteristic polynomial in curvature) and analysis (the spectral invariant eta) all come together.

We discuss an odd dimensional analog of the Atiyah-Patodi-Singer index formula. This involves a generalisation of the classical Toeplitz operator. More precisely, let M be an odd dimensional oriented spin manifold with boundary ∂M. We assume that M carries a fixed spin structure. Then ∂M carries the canonically induced orientation and spin structure. Let g^{TM} be a Riemannian metric on TM such that it is of product structure near the boundary ∂M.

Let E be a Hermitian vector bundle over M. Let ∇^E be a Hermitian connection on E. We assume that the Hermitian metric g^E on E and connection ∇^E are of product structure over $[0, 1) \times \partial M$. The canonical (twisted) Dirac operator D^E is defined and, over $[0, 1) \times \partial M$, one has

$$D^E = c\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x} + \pi^* D^E_{\partial M}\right),$$

where $D^E_{\partial M} : \Gamma((S(TM) \otimes E)|_{\partial M}) \to \Gamma((S(TM) \otimes E)|_{\partial M})$ is the induced Dirac operator on ∂M.

As is well known, the APS projection $P_{\partial M}$ is an elliptic global boundary condition for D^E. However, to get self adjoint boundary conditions, we need to modify it by a Lagrangian subspace of $\ker D^E_{\partial M}$, namely, a subspace L of $\ker D^E_{\partial M}$ such that $c(\frac{\partial}{\partial x})L = L^\perp \cap (\ker D^E_{\partial M})$. Since ∂M bounds M, by the cobordism invariance of the index, such Lagrangian subspaces always exist.

Now the modified APS projection is obtained by adding the projection onto the Lagrangian subspace:

$$P_{\partial M}(L) = P_{\partial M} + P_L,$$

where P_L denotes the orthogonal projection from $L^2((S(TM)\otimes E)|_{\partial M})$ to L. Then $(D^E, P^E_{\partial M}(L))$ forms a self-adjoint elliptic boundary problem. We will denote the corresponding elliptic self-adjoint operator by $D^E_{P_{\partial M}(L)}$.

Finally, we can introduce the analog of the Hardy space in this setting. Let $L^{2,+}_{P_{\partial M}(L)}(S(TM)\otimes E))$ be the space of the direct sum of eigenspaces of *non-negative* eigenvalues of $D^E_{P_{\partial M}(L)}$. This will be our analog of the Hardy space. We denote by $P_{P_{\partial M}(L)}$ the orthogonal projection from $L^2(S(TM)\otimes E)$ to $L^{2,+}_{P_{\partial M}(L)}(S(TM)\otimes E))$.

Given $N > 0$ a positive integer, let $\mathbf{C}^N$ be the trivial complex vector bundle over M of rank N, which carries the trivial Hermitian metric and the trivial Hermitian connection. Then all the above construction can be developed in the same way if one replaces E by $E\otimes \mathbf{C}^N$. And all the operators considered here extend to act on $\mathbf{C}^N$ by identity. If there is no confusion we will also denote them by their original notation.

Now let $g : M \to GL(N, \mathbf{C})$ be a smooth automorphism of $\mathbf{C}^N$. With simple deformation, we can assume that g is unitary. That is, $g : M \to U(N)$. Clearly, g extends to an action on $S(TM) \otimes E \otimes \mathbf{C}^N$ by acting as identity on $S(TM) \otimes E$. We still denote this extended action by g.

Since g is unitary, one verifies easily that the operator $gP_{\partial M}(L)g^{-1}$ is again an orthogonal projection on $L^2((S(TM) \otimes E \otimes \mathbf{C}^N)|_{\partial M})$, and that $gP_{\partial M}(L)g^{-1} - P_{\partial M}(L)$ is a pseudodifferential operator of order less than zero. Moreover, the pair $(D^E, gP_{\partial M}(L)g^{-1})$ forms a self-adjoint elliptic boundary problem. We denote its associated elliptic self-adjoint operator by $D^E_{gP_{\partial M}(L)g^{-1}}$. Thus $D^E_{gP_{\partial M}(L)g^{-1}}$ has the boundary condition which is the conjugation by g of the previous APS type condition.

The necessity of the conjugated boundary condition here is from the fact that, if $s \in L^2(S(TM) \otimes E \otimes \mathbf{C}^N)$ verifies $P_{\partial M}(L)(s|_{\partial M}) = 0$, then gs verifies $gP_{\partial M}(L)g^{-1}((gs)|_{\partial M} = 0$.

Thus, consider also the analog of Hardy space for the conjugated boundary value problem, $L^{2,+}_{gP_{\partial M}(L)g^{-1}}(S(TM) \otimes E \otimes \mathbf{C}^N)$ which is the space of the direct sum of eigenspaces of *nonnegative* eigenvalues of $D^E_{gP_{\partial M}(L)g^{-1}}$. Let $P_{gP_{\partial M}(L)g^{-1}}$ denote the orthogonal projection from $L^2(S(TM) \otimes E \otimes \mathbf{C}^N)$ to $L^{2,+}_{gP_{\partial M}(L)g^{-1}}(S(TM) \otimes E \otimes \mathbf{C}^N)$.

Definition 1. *The Toeplitz operator $T^E_g(L)$ is defined by*

$$T^E_g(L) = P_{gP_{\partial M}(L)g^{-1}} \circ g \circ P_{P_{\partial M}(L)} :$$
$$L^{2,+}_{P_{\partial M}(L)}\left(S(TM) \otimes E \otimes \mathbf{C}^N\right) \to L^{2,+}_{gP_{\partial M}(L)g^{-1}}\left(S(TM) \otimes E \otimes \mathbf{C}^N\right).$$

One verifies that $T^E_g(L)$ is a Fredholm operator. We establish an index formula for it in terms of geometric data.

Theorem 1. *We have*

$$\operatorname{ind} T^E_g(L) = -\left(\frac{1}{2\pi\sqrt{-1}}\right)^{(\dim M+1)/2} \int_M \widehat{A}\left(R^{TM}\right) \operatorname{Tr}\left[\exp\left(-R^E\right)\right] \operatorname{ch}(g, d)$$

$$+ \overline{\eta}(\partial M, g) - \tau_\mu \left(P_{\partial M}(L), g P_{\partial M}(L) g^{-1}, \mathcal{P}_M \right).$$

Here $\mathrm{ch}(g,d)$ *is the odd Chern character of* g*:*

$$\mathrm{ch}(g) = \sum_{n=0}^{(\dim X)/2} \frac{n!}{(2n+1)!} \mathrm{Tr}\left[\left(g^{-1} dg \right)^{2n+1} \right],$$

and $\tau_\mu \left(P_{\partial M}(L), g P_{\partial M}(L) g^{-1}, \mathcal{P}_M \right)$ *is a triple Maslov index* [KL] *(an integer). And* $\overline{\eta}(\partial M, g)$ *is an eta invariant defined on the finite cylinder* $[0,1] \times \partial M$*.*

The definition of the eta invariant here involves a perturbation. Let $\psi = \psi(x)$ be a radial cut off function which is identically one in the ϵ-tubular neighborhood of ∂M ($\epsilon > 0$ sufficiently small) and vanishes outside the 2ϵ-tubular neighborhood of ∂M. Consider the Dirac type operator

$$D^{\psi,g} = (1-\psi) g^{-1} D^E g + \psi D^E = D^E + (1-\psi) g^{-1} [D^E, g]$$

on $[0,1] \times \partial M$. We equip it with the boundary condition $P_{\partial M}(L)$ on one of the boundary component $\{0\} \times \partial M$ and the boundary condition $\mathrm{Id} - g P_{\partial M}(L) g^{-1}$ on the other boundary component $\{1\} \times \partial M$. Then $(D^{\psi,g}, P_{\partial M}(L), \mathrm{Id} - g P_{\partial M}(L) g^{-1})$ forms a self-adjoint elliptic boundary problem. For simplicity, we will still denote the corresponding elliptic self-adjoint operator by $D^{\psi,g}_{[0,1]}$. Its η-invariant will be denoted by $\eta(D^{\psi,g}_{[0,1]})$, and the reduced η-invariant by $\overline{\eta}(D^{\psi,g}_{[0,1]})$ [APS1, APS2].

Definition 2. *We define an invariant of* η *type for the even dimensional manifold* ∂M *and the* K^1 *representative* g *by*

$$\eta(\partial M, g) = \overline{\eta}(D^{\psi,g}_{[0,1]}) - \mathrm{sf}\{ D^{\psi,g}_{[0,1]}(s); 0 \leq s \leq 1 \},$$

where $D^{\psi,g}_{[0,1]}(s)$ *is a path connecting* $g^{-1} D^E g$ *with* $D^{\psi,g}_{[0,1]}$ *defined by*

$$D^{\psi,g}(s) = D^E + (1 - s\psi) g^{-1} [D^E, g]$$

on $[0,1] \times \partial M$*, with the boundary condition* $P_{\partial M}(L)$ *on* $\{0\} \times \partial M$ *and the boundary condition* $\mathrm{Id} - g P_{\partial M}(L) g^{-1}$ *at* $\{1\} \times \partial M$*.*

Thus defined, it can be shown that $\overline{\eta}(X, g)$ does not depend on the cut off function ψ.

Corollary 1. *The number*

$$\left(\frac{1}{2\pi\sqrt{-1}} \right)^{(\dim M + 1)/2} \int_M \widehat{A}\left(R^{TM} \right) \mathrm{Tr}\left[\exp\left(-R^E \right) \right] \mathrm{ch}(g,d) + \overline{\eta}(\partial M, g)$$

is an integer.

Remark 1. *Our index formula is closely related to the so called WZW theory in physics* [W]*. When* $\partial M = S^2$ *or a compact Riemann surface and* E *is trivial, the local term in the index formula is precisely the Wess-Zumino term, which allows an integer ambiguity, in the WZW theory. Thus, our eta invariant* $\overline{\eta}(\partial M, g)$ *gives an intrinsic interpretation of the Wess-Zumino term without passing to the bounding* 3*-manifold.*

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Quantum Field Theory Methods in Twisted K-Theory

Jouko Mickelsson

Twisted K-theory can be viewed as a study of local families of Fredholm operators, defined over an open cover of a space X such that on the overlaps of sets in the open cover the Fredholm operators are related by conjugation by a family of projective unitary transformations.

The local families of projective unitary transformations are in cocycle relation, thus defining a principal $PU(H)$ bundle over X. The (equivalence) class of this bundle is known as a gerbe, and its characteristic class (the Dixmier-Douady class) lies in the cohomology group $H^3(X, \mathbb{Z})$.

In quantum field theory a gerbe arises because certain (gauge) symmetry transformations cannot be lifted from the classical setting to the quantised theory; typically, there is an obstruction coming from ill-defined phases. This happens in (chiral) gauge theory, where one studies families of Dirac operators D_A parametrised by connections in a fixed vector bundle over a compact spin manifold M. The quantum Dirac operators are acting in Fock spaces, carrying an irreducible representation of the canonical anticommutation relations. For different gauge connections these representations are in general inequivalent. The group of gauge tranformations is acting through a projective representation in the bundle of Fock spaces parametrised by gauge connections.

The above gerbe construction becomes nontrivial when passing to the moduli space $X = \mathcal{A}/\mathcal{G}_0$ of connections. Here $\mathcal{G}_0$ is the group of based gauge transformations, fixed in some given point on M. The Dixmier-Douady class can be viewed as an obstruction to a fully gauge covariant quantisation. The construction is interesting already in the simplest situation when M is the unit circle S^1 and the gauge group G is a compact simple Lie group. In this case $\mathcal{G}_0$ is the group ΩG of based loops in G and $X = G$ is the group of holonomies around the circle. In this case the Freed-Hopkins-Teleman theorem states that the G-equivariant twisted K-theory $K^*_G(G, [\omega])$ is the Verlinde algebra in conformal field theory at level k, where ω, the Dixmier-Douady class, is multiple of the basic form by $k +$ the dual Coxeter number of G, [FHT].

A simple construction of the twisted equivariant K-theory in the case $X = G$ was given using a supersymmetric Wess-Zumino-Witten model in $1+1$ space-time dimensions, [Mi]. The construction is essentially based on representation theory of loop algebras with an observation that a gauge covariant interaction with an external field $A \in \mathcal{A}$ can be added, producing the required family of Fredholm operators parametrised by points in $\mathcal{A}$ and transforming covariantly with respect to a projective unitary highest weight representation of the smooth loop group LG.

The case of gauge connections over a manifold M when $\dim M > 1$ is more complicated because of usual renormalisation problems in quantum field theory. However, the constructions can be carried out when one takes a homotopy approximation to the moduli space X. Assuming that $M = S^{2n+1}$ then one can show that X is homotopy equivalent to the group $Map(S^{2n}, G)$ of smooth based maps to G. In particular, for $G = U(N)$ in the limit $N \to \infty$ the moduli space becomes, by Bott periodicity, $U(\infty) = Map(S^{2n}, U(\infty))$.

The construction of the supersymmetric Wess-Zumino-Witten model can be generalised to this infinite-dimensional setting. An additional renormalisation is however needed. This is kind of infinite vacuum energy substraction related to the infinite-dimensionality of $U(\infty)$, similar to the normal ordering related to the Fourier modes on the unit circle.

The case of the true moduli space $\mathcal{A}/\mathcal{G}_0$ for $\dim M > 1$ is still unfinished. Besides of the renormalisation problems, the difficulty is that in general there is no natural polarisation like in one dimension, to positive and negative Fourier modes defining the Fock vacuum and a Borel decomposition in the loop algebra. This problem can be solved by going to a light cone formalism. The gauge connections and gauge transformations are defined on the light cone in Minkowski space, with appropriate boundary conditions at infinity. Then the decomposition of fields to positive and negative Fourier modes along the light rays corresponds to decompostion to positive and negative energies of wave operator or the free Dirac operator, with initial values given on the cone. The important aspect of this polarisation is that for Lie algebra valued functions it defines closed subalgebras of positive (resp. negative) Fourier modes along the cone.

Finally, a Fredholm operator theoretic construction of the nonequivariant case on G is still missing, although different other constructions exist [Do], [Br].

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On the Noncommutative Spectral Flow

Charlotte Wahl

The spectral flow, as introduced by Atiyah-Patodi-Singer, assigns to a continuous path of selfadjoint Fredholm operators on a Hilbert space the net number of eigenvalues changing sign from minus to plus along the path. Motivated by geometric applications it has been generalised to paths of unbounded selfadjoint Fredholm operators with continuous resolvents [BLP] and to paths of families of elliptic operators resp. elliptic operators over C^*-algebras [DZ] [LP]. In the latter two cases the values of the spectral flow are in the K-theory of the base space resp. the C^*-algebra.

We present a definition of the spectral flow of a path $(D_t)_{t\in[0,1]}$ of regular selfadjoint Fredholm operators on the standard Hilbert $\mathcal{A}$-module $H_{\mathcal{A}}$, where $\mathcal{A}$ is a unital C^*-algebra, such that $(D_t)_{t\in[0,1]}$ defines a regular selfadjoint Fredholm operator on the Hilbert $C([0,1],\mathcal{A})$-module $C([0,1],H_{\mathcal{A}})$. Our definition encompasses the generalisations mentioned before. Furthermore it is invariant under conjugation by strongly continuous paths of unitary operators. Hence we also obtain a rigorous definition for the classical spectral flow in the case where the (separable) Hilbert space depends on the parameter. The motivating example is the spectral flow of a path of Dirac type operators on a manifold with boundary with generalised Atiyah-Patodi-Singer boundary conditions and the corresponding problem for families resp. over C^*-algebras. The latter has applications in higher index theory.

We give two different definitions of the spectral flow: by generalising the concept of spectral sections introduced by Melrose-Piazza and adapting the definition of [DZ], and in terms of Bott periodicity.
First we define the relative index $\mathrm{ind}(P,Q) \in K_0(\mathcal{A})$ for projections P, Q on $H_{\mathcal{A}}$ differing by a compact operator as the index of the Fredholm operator $QP : P(H_{\mathcal{A}}) \to Q(H_{\mathcal{A}})$. If D is a selfadjoint regular Fredholm operator on $H_{\mathcal{A}}$ and K is a compact selfadjoint operator such that $D+K$ is invertible then we call $1_{\geq 0}(D+K)$ a generalised spectral section of D. The difference of two generalised spectral sections P, Q of D is compact, hence $\mathrm{ind}(P,Q)$ is well defined. Now if $(D_t)_{t\in[0,1]}$ is a regular selfadjoint Fredholm operator on $C([0,1], H_{\mathcal{A}})$ and there is a (global) generalised spectral section $(Q_t)_{t\in[0,1]}$ of $(D_t)_{t\in[0,1]}$, and if P_i are generalised spectral sections of $D_i,\ i=0,1$, then we define

$$\mathrm{sf}((\mathrm{D_t})_{\mathrm{t}\in[0,1]}, \mathrm{P_0}, \mathrm{P_1}) := \mathrm{ind}(\mathrm{Q_1}, \mathrm{P_1}) - \mathrm{ind}(\mathrm{Q_0}, \mathrm{P_0}) \ .$$

For more general paths we can define the spectral flow by cutting a path into pieces and adding up the spectral flows of the pieces. However generalised spectral sections need not exist.

The definition of the spectral flow in terms of Bott periodicity is more general but less intuitive. It applies to any regular selfadjoint Fredholm operator $(D_t)_{t\in[0,1]}$ on the Hilbert $C([0,1],\mathcal{A})$-module $C([0,1], H_{\mathcal{A}})$ such that D_0, D_1 are invertible. There is an odd monotonously increasing function $\chi \in C^\infty(\mathbb{R})$ with $\chi'(0) > 0$ such that $\chi^2 - 1$ has compact support and $\chi^2(D_0) = \chi(D_1)^2 = 1$ and $(\chi^2(D_t)-1)_{t\in[0,1]}$ is compact on $C([0,1], H_{\mathcal{A}})$. Then $[(\chi(D_t))_{t\in[0,1]}] \in KK_1(\mathbb{C}, C_0((0,1),\mathcal{A}))$. We set

$$\mathrm{sf}((\mathrm{D_t})_{\mathrm{t}\in[0,1]}) = \beta[(\chi(\mathrm{D_t}))_{\mathrm{t}\in[0,1]}] \in \mathrm{K_0}(\mathcal{A}) \ ,$$

where $\beta : KK_1(\mathbb{C}, C_0((0,1),\mathcal{A})) \to K_0(\mathcal{A})$ is induced by Bott periodicity. The definition is independent of the choice of χ.

Both definitions coincide in the cases where both apply.

An application is the expression of the pairing of odd K-theory with KK-theory in terms of the noncommutative spectral flow [W2].

Another application is the definition of a noncommutative version of the Maslov index for a pair of paths of Lagrangians [W1]. We proved that the noncommutative Maslov index occurs as a correction term in a splitting formula of the spectral flow of a loop of families of Dirac operators resp. Dirac operators over a C^*-algebra. It would be interesting to know whether such a splitting formula also exists for paths with invertible end points which are not necessarily loops. The proof of the classical splitting formula in [N] in which the general case is reduced to the case of loops seems not to work for families resp. over C^*-algebras.

There is also a generalisation of the spectral flow to Breuer-Fredholm operators (see [BCPRSW] for an overview), which is not immediately related to our notion, but we hope that there will be a fruitful interplay between both concepts.

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Semi-Infinite and Seiberg-Witten Homotopy

Christopher L. Douglas
(joint work with Ciprian Manolescu)

Studying the structure of the moduli spaces of solutions to the Seiberg-Witten equations leads to strong invariants of 3- and 4-dimensional manifolds. The solutions to the Seiberg-Witten equations on a 3-dimensional manifold can be identified with the critical points of the Chern-Simons-Dirac functional. Seiberg-Witten-Floer homology is defined in terms of these critical points and the 0-dimensional moduli spaces of flow lines between critical points. Our goal is to incorporate the topology of higher-dimensional moduli spaces of flow lines into SWF homotopy types.

We combine the framework of semi-infinite homotopy theory from [2] with the analytical results and finite-dimensional approximation constructions from [3], both of which incorporate ideas of Furuta, to define a semi-infinite spectrum encoding the SWF homotopy type of a rational homology 3-sphere. Roughly, a semi-infinite prespectrum E on a polarised Hilbert space $\mathcal{H}$ is an assignment to each positive-energy or semi-infinite subspace $V \subset \mathcal{H}$ a space $E(V)$ together with structure maps $\Sigma^{W-V}E(V) \to E(W)$ for each inclusion $V \subset W$ of semi-infinite subspaces. The SWF semi-infinite spectrum for a rational homology 3-sphere is defined as the semi-infinite suspension spectrum of the assignment

$$V_\lambda^\infty \mapsto \operatorname*{hocolim}_{\mu>0} \mathrm{Con}(\mathrm{CSD}|_{V_\lambda^\mu}).$$

Here CSD is the Chern-Simons-Dirac functional, V_λ^μ is the span of the eigenspaces (with eigenvalues between λ and μ) of the linearisation of CSD, the eigenvalue $\lambda \ll 0$ is fixed, and $\mathrm{Con}(-)$ is a rigid form of the Conley index. The existence of semi-infinite suspension spectra is a consequence of joint work in progress with Andrew Blumberg. Note that here the polarisation of the Seiberg-Witten configuration space is defined by the eigenvalue decomposition of the linearisation of the CSD functional, so V_λ^∞ is indeed a semi-infinite subspace.

The rigid Conley index is defined as follows. Recall that the classical Conley index of a Morse function f on a noncompact, finite-dimensional manifold M is a homotopy type $[C(f)]$ whose reduced homology $\widetilde{H}_\bullet([C(f)])$ is isomorphic to the Morse homology of the pair (M, f). The homotopy type $[C(f)]$ is defined as the homotopy type of the quotient N/L where (N, L) is a pair with N an appropriate neighborhood of the isolated invariant set of the flow of f, and $L \subset N$ an exit set—see for example [1, 4]. There is a topological category $\mathcal{CP}$ of such pairs (N, L); the morphisms in $\mathcal{CP}$ are induced by the flow, as in [4]. The geometric realisation of this category is contractible, and the rigid Conley index is defined as

$$\mathrm{Con}(f) = \operatorname*{hocolim}_{(N,L)\in\mathcal{CP}} N/L$$

That this produces a space rather than merely a homotopy type is useful for constructing invariants of families of flows (for example, families of flows coming from the CSD functional for a family of rational homology 3-spheres).

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Motivic Groups in Enriched Algebraic Geometry

Jack Morava

Spec($\mathbb{Z}$) has two rather orthogonal interesting enrichments, either in terms either of some category of E_∞ ring spectra [2, 4, 6] at **finite** primes or in some none commutative sense [1] at the **Archimedean** primes. The language of motivic Galois groups seems to accommodate both these perspectives in terms of a kind of generalised Sullivan arithmetic square [3], which provides an opportunity to re-examine some old questions about the connected components of the idelé-class group [5, 7].

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Direct Images for Relative and Multiplicative K-Theories

Alain Berthomieu

Consider triples (E, F, f) where E and F are flat complex vector bundles on the same differentiable manifold X and $f: E \to F$ is some C^∞ bundle isomorphism between them. In [3], some "relative K-theory" group $K^0_{\text{rel}}(X)$ is constructed as the free abelian group generated by such objects modulo the following relations: isotopy of f (which means homotopy throw isomorphisms) and nullity if f is a parallel isomorphism (with respect to connections ∇_E on E and ∇_F on F defining the flat structures), or if $F = E' \oplus E''$ and $f = s \oplus p$ where

$$0 \to E' \xrightarrow{i} E \xrightarrow{p} E'' \to 0$$

is an exact sequence in the flat bundles category, and $s: E \to E'$ verifies that $s \circ i$ is the identity of E'.

In such a situation, characteristic classes are constructed using Chern-Simons forms: let $\text{ch}(\nabla_E)$ and $\text{ch}(\nabla_F)$ be the explicit differential form representatives of the Chern characters of E and F obtained throw Chern-Weil theory from the connections ∇_E and ∇_F, then Chern-Simons theory provides an odd differential form $\overline{\text{ch}}(\nabla_E, f^*\nabla_F)$ defined modulo coboundaries which verifies

$$d\overline{\text{ch}}(\nabla_E, f^*\nabla_F) = \text{ch}(\nabla_F) - \text{ch}(\nabla_E)$$

(This of course also works for nonflat connexions). Here the curvatures of ∇_E and ∇_F vanish so that $\overline{\text{ch}}(\nabla_E, f^*\nabla_F)$ is a closed form, and one gets a group morphism $\mathcal{N}_{\text{ch}}: K^0_{\text{rel}}(M) \longrightarrow H^{\text{odd}}(M, \mathbb{C})$. This construction generalises the classes considered by Bismut and Lott in [6] (with different objectives) in the particular case of $E \overset{C^\infty}{=\!=} F$ and ∇_F being the adjoint transpose of ∇_E.

On the other hand, consider triples (E, ∇, α) where E is some smooth complex vector bundle on X, ∇ is some connection on E, and α is an odd differential form on X defined modulo coboundaries such that

$$d\alpha = \text{ch}(\nabla) - \text{rk}E$$

then the relevant Karoubi's multiplicative K-theory group $MK^0(X)$ in this context [8] [9] is the free abelian group generated by such triples modulo direct sums and the following relation:

$$(F, \nabla_F, \beta) = \big(E, \nabla_E, \beta - \overline{\text{ch}}(\nabla_E, f^*\nabla_F)\big)$$

if $(F, \nabla_F, \beta) \in MK^0(X)$, if $f: E \to F$ is some smooth bundle isomorphism, and ∇_E and ∇_F are (not necessarily flat) connections on E and F.

These groups enter in the following commutative diagram:

$$(*)\qquad \begin{array}{ccccccc}
K^1_{\mathrm{top}}(X) & \longrightarrow & K^0_{\mathrm{rel}}(X) & \longrightarrow & K^0_{\mathrm{flat}}(X) & \longrightarrow & K^0_{\mathrm{top}}(X) \\
\| \downarrow & & \downarrow \mathcal{N}_{\mathrm{ch}} & & \downarrow & & \downarrow \| \\
K^1_{\mathrm{top}}(X) & \longrightarrow & H^{\mathrm{odd}}(X,\mathbb{C}) & \longrightarrow & MK^0(X) & \longrightarrow & K^0_{\mathrm{top}}(X)
\end{array}$$

whose lines are exact sequences and where $K^0_{\mathrm{flat}}(X)$ is the "algebraic" K-theory group of flat vector bundles on X modulo exact sequences.

Let now $\pi\colon M \to B$ be some submersion between compact manifolds. To some flat vector bundle E on M, one can associate the formal difference of flat vector bundles $H^{\pm}(M/B,E)$ on B consisting of the even and odd degree de Rham cohomology of the fibres of π with coefficients in E. This provides a direct image morphism $\pi_!\colon K^0_{\mathrm{flat}}(M) \to K^0_{\mathrm{flat}}(B)$.

To construct $\pi_*\colon K^0_{\mathrm{rel}}(M) \to K^0_{\mathrm{rel}}(B)$, one needs for $(E,F,f) \in K^0_{\mathrm{rel}}(M)$ some smooth isomorphism between the flat bundles $H^+(M/B,E) \oplus H^-(M/B,F)$ and $H^-(M/B,E) \oplus H^+(M/B,F)$ on B, which would be canonically associated to f. This is found in [3] by following the kernel of $\big(d^{\nabla_t} + (d^{\nabla_t})^*\big)_{t\in[0,1]}$ which is the deformed fibral Euler-Dirac operator corresponding to the deformation of connections $(\nabla_t)_{t\in[0,1]}$. These kernels are not of constant dimension, so that a regularisation trick from [1] is needed. For proving the compatibility with the exact sequences relation in K^0_{rel}, the correspondance of some cohomology spectral sequence and little eigenvalues of fibral Euler-Dirac operator is studied, in the spirit of [4] and [13]. This π_* is compatible with the preceding $\pi_!$.

The direct image $\pi_*\colon K^*_{\mathrm{top}}(M) \to K^*_{\mathrm{top}}(B)$ is constructed in [1] (and is compatible with $\pi_!$) using the foreseen trick, which can be used for adapting the construction of the η-form of Bismut and Cheeger [5] to the case of fibral Euler-Dirac operators with nonconstant dimension kernels.

Once the geometry of M is fixed, this provides for any complex vector bundles E on M and F^+ and F^- on B with connections ∇_E, ∇^+ and ∇^- and such that $[F^+] - [F^-] = \pi_*[E] \in K^0_{\mathrm{top}}(B)$ (with some additional data related to the regularisation trick of [1]), some differential form defined modulo coboundaries, which was denoted $\tau(\nabla_E, \nabla^{M/B}, \nabla^+, \nabla^-)$ in the talk, with the following properties:

1: transgression formula

$$d\tau(\nabla_E, \nabla^{M/B}, \nabla^+, \nabla^-) = \mathrm{ch}(\nabla^+) - \mathrm{ch}(\nabla^-) - \int_{M/B} e(\nabla^{M/B})\mathrm{ch}(\nabla_E)$$

where $\int_{M/B}$ is integration along the fibres of π, $\nabla^{M/B}$ is the Levi-Civita connection on the bundle TM/B of vertical tangent vectors, and $e(\nabla^{M/B})$ is the Chern-Weil representative of the Euler class of TM/B,

2: additivity for direct sums,

3: functoriality for pull-backs on fibred products (the model of the fibre may not change),

4: nullity if ∇_E is flat and $F^{\pm} = H^{\pm}(M/B,E)$ with corresponding flat connections $\nabla^{\pm}$,

5: universal property: there is a unique choice with preceding properties for E with vanishing rational Chern classes.

The functoriality property used on $[0,1]\times M$ allows to calculate the dependence of τ on its data, and this provides the following result: for $(E,\nabla,\alpha)\in MK^0(M)$, choose some vector bundle F and some trivial vector bundle $\mathbb{C}^N$ on B such that $[F]-[\mathbb{C}^N]=\pi_*[E]\in K^0_{\mathrm{top}}(B)$, endowed with connections ∇_F and $d^{\mathbb{C}^N}$ (the canonical one), then

$$(E,\nabla_E,\alpha)\longmapsto\Big(F,\nabla_F,\tau(\nabla_E,\nabla^{M/B},\nabla_F,d^{\mathbb{C}^N})+\int_{M/B}e(\nabla^{M/B})\alpha\Big)-(\mathbb{C}^N,d^{\mathbb{C}^N},0)$$

provides a morphism $MK^0(M)\to MK^0(B)$, which is compatible with the preceding ones and independent of the geometric data used in its definition.

This is closely related to Lott's analytic family index construction [10] but not exactly the same (for example because no *spin*^{c} assumption is needed here).

The form τ (and its properties) allows one to prove for any $(E,F,f)\in K^0_{\mathrm{rel}}(M)$

$$\mathcal{N}_{\mathrm{ch}}\big(\pi_*(E,F,f)\big)=\int_{M/B}e(\nabla^{M/B})\mathcal{N}_{\mathrm{ch}}(E,F,f)$$

thus completing the direct image for the whole diagram (*). This uses [6] in an essential way and generalises (without totally recovering) some of the results from Ma and Zhang's work [11] [12].

The holomorphic counterpart of these results on relative K-theory were established in [2]. See [7] about direct image for MK^0 under closed immersions.

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Homotopy Theory of Compactified Moduli Space

Søren Galatius
(joint work with Ya. Eliashberg)

Let $\mathcal{M}_g$ denote the moduli space of genus g Riemann surfaces and $\overline{\mathcal{M}}_g$ the Deligne-Mumford compactification of $\mathcal{M}_g$. Very briefly, the goal of this work (in progress) is to do for $\overline{\mathcal{M}}_g$ what Madsen and Weiss did for $\mathcal{M}_g$. I will start by a short account of Madsen-Weiss' approach. $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ will always denotes the *stacks*, rather than the coarse moduli space.

1. $\mathcal{M}$ and surface bundles

By a *surface bundle* over a manifold X^k we mean a proper submersion $f : E^{k+2} \to X^k$ with oriented fibres. Let

$$S(X) = (\text{surface bundles } f : E \to X)/\simeq$$

denote the set of isomorphism classes of surface bundles over X.

Thus defined, S is a set-valued functor (under pullback) from the category of smooth manifolds and homotopy classes of maps. As such, it is represented by the stack

$$\mathcal{M} = \coprod_k \big(\coprod_g \mathcal{M}_g\big)^k / \Sigma_k.$$

Thus $H^*(\mathcal{M})$ is the set of characteristic classes of surface bundles. In particular we have the Miller-Morita-Mumford classes $\kappa_i \in H^{2i}(\mathcal{M})$.

2. Formal surface bundles and Madsen-Weiss

Madsen-Weiss' point of view is to replace "surface bundle" by the corresponding *stable normal bundle condition.* This means that we consider triples (f, L, ϕ) consisting of a proper map $f : E^{k+2} \to X^k$, a complex line bundle $L \to E$, and a stable isomorphism $\phi : TE \oplus \mathbb{R}^j \cong f^*TX \oplus L \oplus \mathbb{R}^j$, defined for some $j \gg 0$. We call such a triple a *formal surface bundle* and define

$$\tilde{S}(X) = (\text{formal surface bundles } E \to X)/\simeq .$$

Here, two formal surface bundles $f_\nu : E_\nu \to X$, $\nu = 0, 1$ (suppressing the L's and the ϕ's from the notation), are equivalent if there exists a formal surface bundle $f : W^{k+3} \to X \times \mathbb{R}$, transversal to $X \times \{0, 1\}$, whose restriction to $f^{-1}(X \times \{\nu\})$ is f_ν.

If $f : E \to X$ is a surface bundle, then the differential of f is an epimorphism $TE \to f^*TX$. If we let L denote its kernel, we have a short exact sequence $0 \to L \to TE \to f^*(TX) \to 0$, and a choice of splitting gives an isomorphism $TE \cong f^*(TX) \oplus L$. This defines a forgetful map $S(X) \to \tilde{S}(X)$.

For many purposes, the notion of formal surface bundle is easier to understand than (honest) surface bundles, even though the definition looks more complicated.

For formal reasons (viz. Pontrjagin-Thom theory), the functor $\tilde{S}$ is part of a *cohomology theory*, which is represented by a *Thom spectrum*, often denoted $\mathbb{C}P^{\infty}_{-1}$. It is the Thom spectrum of the map

$$BU(1) \xrightarrow{-L} \mathbb{Z} \times BO,$$

classifying the virtual inverse of the canonical complex line bundle $L \to BU(1)$. Thus we have a natural isomorphism

$$\tilde{S}(X) \cong [X, \Omega^{\infty}\mathbb{C}P^{\infty}_{-1}],$$

and the forgetful map $S \to \tilde{S}$ is represented by a continuous map

$$\mathcal{M} \to \Omega^{\infty}\mathbb{C}P^{\infty}_{-1}.$$

Also for formal reasons (Thom isomorphism), it is easy to calculate $H^*(\mathbb{C}P^{\infty}_{-1})$. It is $\mathbb{Z}$ in even dimensions and vanishes in odd dimensions. The generators map under the map

$$H^*(\mathbb{C}P^{\infty}_{-1}) \to H^*(\Omega^{\infty}\mathbb{C}P^{\infty}_{-1}) \to H^*(\mathcal{M})$$

to the Miller-Morita-Mumford classes. With rational coefficients, they form polynomial generators of the cohomology ring $H^*(\Omega^{\infty}\mathbb{C}P^{\infty}_{-1})$.

Finally, the statement of Madsen-Weiss can be rephrased as follows: The restriction

$$\mathcal{M}_g \to \Omega^{\infty}\mathbb{C}P^{\infty}_{-1}$$

of the forgetful map, is a homology isomorphism in degrees up to $(g-1)/2$.

3. $\overline{\mathcal{M}}$ and Lefschetz fibrations

We now try to apply a similar analysis to the spaces $\overline{\mathcal{M}}_g$ or, more generally, the stack

$$\overline{\mathcal{M}} = \coprod_k \big(\coprod_g \overline{\mathcal{M}}_g\big)^k / \Sigma_k.$$

Points in $\overline{\mathcal{M}}$ are *nodal curves*, i.e. Riemann surfaces with a certain mild kind of singularities, modelled on $\{(z,w) \in \mathbb{C}^2 \mid zw = 0\}$. In nearby points, a singularity $zw = 0$ can deform into $zw = \epsilon$, $\epsilon \in \mathbb{C}$. The *universal nodal curve* is the map

$$\pi : \overline{\mathcal{C}} \to \overline{\mathcal{M}},$$

where $\overline{\mathcal{C}}$ is the stack of pairs (Σ, p) with $\Sigma \in \overline{\mathcal{M}}$ and $p \in \Sigma$. The subspace of $\overline{\mathcal{C}}$ where $p \in \Sigma$ is a node, is a smooth substack $\Sigma \subseteq \overline{\mathcal{C}}$ of complex codimension 2, and the restriction

$$\pi|\Sigma : \Sigma \to \overline{\mathcal{M}}$$

is an immersion with normal crossings, of complex codimension 1.

If X is a smooth manifold and $g : X \to \overline{\mathcal{M}}$ is smooth and transverse to $\pi : \overline{\mathcal{C}} \to \overline{\mathcal{M}}$, then $E = g^*\overline{\mathcal{C}}$ is a smooth manifold, and we have a pullback square

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\pi"'] & \overline{\mathcal{C}} \arrow[d, "\pi"] \\ X \arrow[r, "g"] & \overline{\mathcal{M}} \end{tikzcd}$$

The map $\pi : E \to X$ is no longer a surface bundle (as it would have been with $\mathcal{M}$ in place of $\overline{\mathcal{M}}$), it is a *Lefschetz fibration*. We recall a definition of this notion.

Imprecisely, a Lefschetz fibration is a smooth proper map $f : E^{k+2} \to X^k$, which locally in E looks like

$$(x_1, \ldots, x_{k-2}, z, w) \mapsto (x_1, \ldots, x_{k-2}, zw),$$

where the x_i are real parameters and z and w are complex parameters.

More precisely, we will by a Lefschetz fibration mean a tuple (f, Σ, U, L, q), where $f : E^{k+2} \to X^k$ is a proper map, $\Sigma^{k-2} \subseteq E$ is a closed submanifold such that $f|\Sigma$ is an immersion with normal crossings. $U \to \Sigma$ is a 2-dimensional complex vector bundle, embedded as a tubular neighborhood $U \subseteq E$. $L \to \Sigma$ is a complex line bundle, immersed as a tubular neighborhood $L \to X$, $q : U \to L$ is a nondegenerate fibrewise quadratic form (i.e. $q(v) = \frac{1}{2}b(v, v)$ for a unique $b \in (S^2U)^* \otimes L$), such that the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "q"] & E \arrow[d, "f"] \\ L \arrow[r] & X \end{tikzcd}$$

commutes near the zero section $\Sigma \subseteq U$. Finally, the restriction $f|(E - \Sigma)$ should be a submersion with oriented fibres, the orientation being compatible with the complex structures of U and L near Σ.

It is not hard to see that the map $\pi : \overline{\mathcal{C}} \to \overline{\mathcal{M}}$ is a Lefschetz fibration in this sense, and that it is universal: Any Lefschetz fibration $f : E \to X$ (suppressing much from the notation) is induced by a smooth map $g : X \to \overline{\mathcal{M}}$, transverse to π. Thus, if we let

$$L(X) = (\text{Lefschetz fibrations } f : E \to X)/\simeq,$$

then L is represented by the space $\overline{\mathcal{M}}$: There is a natural isomorphism $L(X) \cong [X, \overline{\mathcal{M}}]$.

4. Formal Lefschetz Fibrations

Following Madsen-Weiss, we replace "Lefschetz fibration" by the corresponding stable normal bundle condition. This leads to the notion of a *Formal Lefschetz Fibration.*

A formal Lefschetz fibration is a tuple

$$(f, V_S, V_N, U, L, q, L', k, \phi, \psi)$$

where

(i) $f : E^{k+2} \to X^k$ is a proper map

(ii) $E = V_S \cup V_N$ is an open cover (S and N stands for "singular" and "nonsingular")

(iii) $U \to V_S$ and $L \to V_S$ are complex vector bundles of dimension 2 and 1, respectively

(iv) $q : U \to L$ is a fibrewise quadratic, nondegenerate form

(v) $L' \to V_N$ is a complex line bundle

(vi) k is an isomorphism of vector bundles over $V_S \cap V_N$: $L' \oplus L \cong U$

(vii) ϕ is a stable isomorphism of vector bundles over V_S: $\mathbb{R}^j \oplus TE \oplus L \cong \mathbb{R}^j \oplus f^*TX \oplus U$.

(viii) ψ is a stable isomorphism of vector bundles over V_N: $\mathbb{R}^j \oplus TE \cong \mathbb{R}^j \oplus f^*TX \oplus L'$.

ϕ, ψ and k should be compatible over $V_N \cap V_S$, in the sense that $(\psi \oplus \mathrm{id}_L) \circ (\mathrm{id}_{\mathbb{R}^j \oplus f^*TX} \oplus k) = \phi$. (Note that we require these to be *equal* as maps. Alternatively we could require them to be homotopic via a homotopy h which we should then include in the data).

Let

$$\tilde{L}(X) = (\text{Formal Lefschetz Fibrations } f : E \to X)/\simeq$$

where $\simeq$ is the equivalence relation generated by increasing j, and by *homotopy*, i.e. if $W^{k+3} \to X^k \times \mathbb{R}$ is a formal Lefschetz fibration, transverse to $X \times \{0,1\}$, then the restriction to $X \times \{0\}$ and $X \times \{1\}$ are equivalent.

There is a forgetful map $L(X) \to \tilde{L}(X)$, defined as follows. Given a Lefschetz fibration (f, Σ, U, L, q), we let

- $V_N = E - \Sigma$, and $L' \to V_N$ is the kernel of $D(f|V_N))$.
- $V_S = U \subseteq E$.

As in the uncompactified case, the point of considering the corresponding stable normal bundle condition is, that $\tilde{L}(X)$ is for many purposes easier to understand. The "usual" cohomology classes in $\overline{\mathcal{M}}_g$, thought of as natural transformations

$$L(X) \to H^*(X; \mathbb{Q}),$$

factor through $\tilde{L}(X)$. In particular we have the Miller-Morita-Mumford classes κ_i, but also some new classes that I will call $\theta_{i,j}$, $i, j \geq 0$. For a Lefschetz fibration $f : E \to X$ they are defined as

$$\theta_{i,j} = (f|\Sigma)_!(c_1^i c_2^j(U)) \in H^{2+2i+4j}(X)$$

5. Classifying FLFs

Pontrjagin-Thom theory implies that formal Lefschetz fibrations are classified by a Thom spectrum. The general procedure is to translate the stable normal bundle condition into a map $\xi : B \to \mathbb{Z} \times BO$. The stable normal bundle of a proper map $f : E \to X$ is a map

$$Nf : E \to \mathbb{Z} \times BO,$$

whose homotopy class is $[f^*TX] - [TE] \in KO^0(E)$, and $\xi : B \to \mathbb{Z} \times BO$ should be such that the bundle conditions on f are equivalent to a lifting of Nf to a map $l : E \to B$. Then the Thom spectrum B^ξ of ξ will classify $\tilde{L}$ in the sense that there is a natural isomorphism

$$\tilde{L}(X) \cong [X, \Omega^\infty B^\xi].$$

In our case, E is a pushout $V_S \leftarrow V_S \cap V_N \to V_N$, and the space B is most easily described as a homotopy pushout of spaces over $\mathbb{Z} \times BO$:

(1)
$$\begin{tikzcd}
B_{SN} \arrow[r] \arrow[d] & B_S \arrow[d] \arrow[ddr, "L-U"] & \\
B_N \arrow[r] \arrow[drr, "-L'"'] & B & \\
& & \mathbb{Z} \times BO.
\end{tikzcd}$$

Here, $B_N = BU(1)$ models the same bundle condition as in the uncompactified case. More interestingly, B_S is the universal space carrying two complex bundles U, L of dimensions 2 and 1, respectively, equipped with a quadratic nondegenerate map $q : U \to L$. This is

$$B_S = E(U(2) \times U(1)) \times_{U(2)\times U(1)} \mathrm{Quad}(\mathbb{C}^2, \mathbb{C}^1),$$

where $\mathrm{Quad}(\mathbb{C}^2, \mathbb{C}^1)$ denotes the space of quadratic, nondegenerate maps. It turns out that this is homotopy equivalent to the classifying space of the maximal torus normaliser in $U(2)$:

$$B_S = B(\Sigma_2 \textstyle\int U(1)).$$

B_{SN} is the sphere bundle of the canonical bundle $U \to B_S$.

The pushout diagram (1) above leads to a map $\xi : B \to \mathbb{Z} \times BO$, and thus a Thom spectrum B^ξ. By Pontrjagin-Thom theory, this will classify formal Lefschetz fibrations. Therefore we will denote it $\underline{FLF} := B^\xi$. I have sketched a proof that there is a natural isomorphism

$$\tilde{L}(X) \cong [X, \Omega^\infty \underline{FLF}].$$

6. Cohomology of $\underline{\text{FLF}}$

The pushout diagram (1) of spaces over $\mathbb{Z} \times BO$ leads to a pushout diagram of Thom spectra, and in turn a cofibration sequence of spectra

$$\mathbb{C}P^\infty_{-1} \longrightarrow \underline{FLF} \longrightarrow B(\Sigma_2 \textstyle\int U(1))^L.$$

$B(\Sigma_2 \int U(1))^L$ is the Thom spectrum (space, in fact) of the bundle $L \to B(\int U(1))$. It is not hard to calculate the cohomology of these spectra, using Thom isomorphism. I will state the answer with rational coefficients.

As stated earlier, $H^*(\mathbb{C}P^\infty_{-1})$ is one-dimensional in each even degree. The classes correspond to the Miller-Morita-Mumford classes.

The inclusion $B(\Sigma_2 \int U(1)) \to BU(2)$ induces an isomorphism in rational cohomology (actually with coefficients in $\mathbb{Z}[1/2]$). Thus the cohomology has basis $c_1^i c_2^j$, $i, j \geq 0$. This gives rise to the characteristic classes $\theta_{i,j}$ described earlier. It is not hard to see that the κ_i classes together with the $\theta_{i,j}$ classes form a basis for $H^*(\underline{FLF}; \mathbb{Q})$. On the level of infinite loop spaces we get

$$H^*(\Omega^\infty \underline{FLF}; \mathbb{Q}) \cong \mathbb{Q}[\kappa_i | i \geq 0] \otimes \mathbb{Q}[\theta_{i,j} | i, j \geq 0].$$

Thus we know precisely what are characteristic classes of *formal* Lefschetz fibrations.

7. Concluding remarks

The forgetful map from Lefschetz fibrations to formal Lefschetz fibrations is classified by a map

$$\overline{\mathcal{M}} \to \Omega^\infty \underline{FLF}.$$

It seems that many of the cohomology classes in $\overline{\mathcal{M}}_g$ that are "usually" considered, can be pulled back from classes in $\Omega^\infty \underline{FLF}$ (namely precisely the κ_i and the $\theta_{i,j}$ classes).

The question of understanding the intersection theory of $\overline{\mathcal{M}}_g$ can now, at least partly, be rephrased as understanding the bordism (or just homology) class of the map $\overline{\mathcal{M}}_g \to \Omega^\infty \underline{FLF}$.

A slightly weaker question is to understand the class

$$[\overline{\mathcal{M}}] \in H_*(\underline{FLF}; \mathbb{Q}). \tag{2}$$

A goal of this work (in progress) is to give a homotopy theoretic description of the class (2). In the longer term, one should of course consider Gromov-Witten theory of an arbitrary symplectic manifold X (in a way that the above would correspond to the case where X is a point). The analogue of (2) would be $[\overline{\mathcal{M}}(X)] \in H_*(\underline{FLF} \wedge X_+; \mathbb{Q})$.

A question orthogonal to that of understanding $[\overline{\mathcal{M}}]$ is to ask, what the analogue of Madsen-Weiss' theorem would be. The naive guess that $\overline{\mathcal{M}}_g \to \Omega^\infty \underline{FLF}$ might be a homology isomorphism in a range increasing with g, turns out to be wrong. Instead we propose to consider the subspace $\tilde{\mathcal{M}}_g \subseteq \overline{\mathcal{M}}_g$, consisting of *irreducible* curves. Then the composition

$$\tilde{\mathcal{M}}_g \subseteq \overline{\mathcal{M}}_g \to \Omega^\infty \underline{FLF}$$

seems to be a homology isomorphism in a stable range. Thus, in that stable range, the cohomology of $\overline{\mathcal{M}}_g$ will be the direct sum of a *stable* part, which is the polynomial algebra in the κ_i and the $\theta_{i,j}$, and an unstable part, which is the homology of $\overline{\mathcal{M}}_g - \tilde{\mathcal{M}}_g$.

Index Theory on Orbifolds via Formal Deformation Quantisation

Hessel Posthuma
(joint work with M. Pflaum and X. Tang)

Symplectic orbifolds are naturally encountered in mathematical physics and Poisson geometry, e.g. as the result of symplectic reduction with respect to a locally free action of a compact Lie group. It is therefore very natural to try to extend known quantisation schemes on symplectic manifolds to this category. The aim of this talk is to explain how this gives a useful approach to index theory on orbifolds and as such may serve as an example for dealing with more singular spaces. In all this we use the the theory of formal deformation quantisation.

A symplectic orbifold X is, loosely speaking, a topological Hausdorff space which locally is homeomorphic to an open neighbourhood of 0 in $\mathbb{R}^{2n}/\Gamma$, where Γ is a finite group acting by linear symplectic transformations with respect to the standard symplectic form on $\mathbb{R}^{2n}$. Besides being an orbifold in the usual sense, the symplectic structure gives the sheaf of smooth functions on X, i.e., those functions that locally lift to smooth Γ-invariant functions in a chart as above, the structure of a sheaf of Poisson algebras. Instead of considering the deformation problem for this Poisson algebra, denoted A_X, we will do the folowing, in the spirit of noncommutative geometry:

Associated to a symplectic orbifold X is a proper étale groupoid G, with structure maps $s, t : G_1 \to G_0$ and $G_0/G_1 \cong X$, such that G_0 carries an invariant symplectic form ω, i.e., $s^*\omega = t^*\omega$. The convolution algebra of an étale groupoid is defined as $A_G := C_c^\infty(G_1)$ with the product

$$(f_1 * f_2)(g) = \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2),$$

for $g \in G_1$. Notice that the center of A_G equals $A_X := C_c^\infty(X)$, and when X happens to be a manifold, A_G is even Morita equivalent to its center. The symplectic nature of the orbifold amounts to a canonical Hochschild class $\pi \in H^2(A_G, A_G)$ which satisfies $[\pi, \pi] = 0$, cf. [8]. The upshot is that the "classical phase space" is already a noncommutative geometry!

A formal deformation quantisation of the noncommutative Poisson algebra (A_G, π) consists of an associative product $\star_c$ on $A_G[[\hbar]]$, compatible with the $\hbar$-adic filtration, such that in zeroth order one recovers the convolution product above, and the Hochschild class of the first order approximation equals π. Such a deformation, denoted by $A_G^\hbar$, can be constructed as follows: Using Fedosov's method [1], one can construct a G-invariant deformation quantisation of the sheaf of smooth functions on the symplectic manifold G_0, with characteristic class $[\Omega] \in H^2(X, \mathbb{C}[[\hbar]])$. Using a kind of crossed product construction, one obtains a deformation $A_G^\hbar$ of A_G, whereas the invariant section of this sheaf give a deformation quantisation $A_X^\hbar$ of A_X, cf. [6]. In sharp contrast to the classical (i.e., undeformed) theory, it turns out that there is a canonical Morita equivalence

$$A_G^\hbar \overset{M}{\sim} A_X^\hbar, \tag{1}$$

when the orbifold X is reduced, so, in a sense, the quantisation "resolves the singularities".

As a first step towards the index theorem, one computes the cyclic theory of the algebra $A_G^\hbar$. To state the following theorem, let $\tilde{X}$ be the "inertia orbifold" associated to X. This orbifold, canonically associated to X, was first considered in [3], and is referred to in the physics literature as the "twisted sectors" of X.

Theorem 1 ([5])**.** *The Hochschild and cyclic cohomology groups of the deformed groupoid algebra $A_G^\hbar$ are given by*

$$HH^\bullet(A_G^\hbar) \cong H^\bullet\left(\tilde{X}, \mathbb{C}((\hbar))\right)$$
$$HC^\bullet(A_G^\hbar) \cong \bigoplus_{k\geq 0} H^{\bullet-2k}\left(\tilde{X}, \mathbb{C}((\hbar))\right).$$

Similar results hold for Hochschild and cyclic homology [5] in terms of compactly supported cohomology, so that these are Poincaré dual over $\tilde{X}$ to the cohomology as stated above. Of special interest is the above result for HC^0 since cyclic cocycles of degree zero are nothing but traces on the algebra $A_G^\hbar$, i.e., linear maps $\mathrm{tr} : A_G^\hbar \to \mathbb{C}((\hbar))$ satisfying

$$\mathrm{tr}(a \star_c b) = \mathrm{tr}(b \star_c a).$$

Therefore we find that

$$\dim_{\mathbb{C}((\hbar))}\{\text{space of traces}\} = \#\ \text{Components}(\tilde{X}),$$

in particular is not one-dimensional when X is a nontrivial orbifold. This is in sharp contrast with the case of smooth symplectic manifolds where it is well-known that there is a unique trace up to normalisation, cf. [1, 4].

Let $K^0_{\mathrm{orb}}(X)$ be the Grothendieck group of formal differences of orbifold vector bundles, sometimes called "orbifold K-theory". A trace tr in the sense above induces an index map

$$\mathrm{tr}_* : K^0_{\mathrm{orb}}(X) \to \mathbb{C}((\hbar))$$

as follows. By taking the trace of idempotents in matrix algebras over $A_G^\hbar$, one gets a map $\mathrm{tr}_* : K_0(A_G^\hbar) \to \mathbb{C}((\hbar))$. To obtain the index map from this, one uses the isomorphisms

$$K_0(A_G^\hbar) \cong K_0(A_G) \cong K^0_{\mathrm{orb}}(X).$$

Here the first isomorphism states that K-theory is "rigid" under deformation quantisation, whereas the second can be viewed as a kind of Serre–Swan theorem for orbifolds. The algebraic index theorem for orbifolds gives a topological formula for the value of an index map associated to a trace on a K-theory class given by a pair (E, F) of orbifold vector bundles, isomorphic outside a compact subset of X.

Theorem 2 ([7])**.** *Let* tr_α *be a trace corresponding to the connected component* $\tilde{X}_\alpha$ *of* $\tilde{X}_\alpha$*. Let* $E \to X$ *be an orbifold vector bundle. Then, up to a constant,*

$$(\mathrm{tr}_\alpha)_*\left([E]-[F]\right) \propto \int_{\tilde{X}_\alpha} \frac{\mathrm{Ch}_\theta(R_E - R_F)}{\det\left(1-\theta^{-1}\exp(-R^\perp)\right)} \hat{A}(R^T) e^{\iota_\alpha^*\Omega/\hbar}.$$

In the formula above, the right hand side consists of the usual characteristic classes which can be explicitly represented by differential forms on $\tilde{X}$ by choosing a Riemannian metric with curvature R and connections on E and F. The map ι_α is the canonical embedding of the connected component $\tilde{X}_\alpha$ into X, and θ is the canonical automorphism acting on vector bundles normal to this embedding, so that one can twist the Chern character of E and F restricted to $\tilde{X}_\alpha$ and write down the denominator familiar from the equivariant index theorem.

Notice that the formula in the theorem is only given up to a constant, since thus far the trace is only determined up to normalisation by its support on $\tilde{X}_\alpha$. When X is a manifold, i.e., $\tilde{X} = X$ and there is a unique trace, the normalisation can easily be fixed and the theorem above reduces to the algebraic index theorem in [1, 4]. For an orbifold, the normaliation issue is much more nontrivial because of the non-uniqueness of the trace, and the actual statement proved in [7] is much stronger: using the concept of a "twisted trace density", a canonical trace with support on $\tilde{X}_\alpha$ is constructed for which the normalisation can explicitly be fixed. The resulting formula was first conjectured in [2].

Finally, one can obtain the the classical Kawasaki index theorem [3], by considering the deformation quantisation $A^\hbar_{T^*X}$ of of the cotangent bundle T^*X of an orbifold X induced by the asymptotic pseudo-differential calculus. Using the Morita equivalence (1), one can compare the operator trace to the canonical trace above. As can be suspected from the index theorem, this trace has support on all connected components of $\tilde{X}$. Since the index of an elliptic operator on a compact orbifold is determined by its symbol, considered as a class in the (compactly supported) orbifold K-theory of T^*X, one derives the cohomological formula of the Kawasaki index from the algebraic index theorem, cf. [7].

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Harmonic Magnus Expansion on the Universal Family of Riemann Surfaces

Nariya Kawazumi

We introduce a higher analogue of the period matrix of a compact Riemann surface, in order to construct "canonical" differential forms representing (twisted) Morita-Mumford classes [10], [8] and their higher relations on the moduli space $\mathbb{M}_g$ of compact Riemann surfaces of genus $g \geq 1$.

For simplicity, we consider the moduli space $\mathbb{M}_{g,1}$ of *triples* (C, P_0, v) of genus g instead of the space $\mathbb{M}_g$. Here C is a compact Riemann surface of genus g, $P_0 \in C$, and v a non-zero tangent vector of C at P_0. The space $\mathbb{M}_{g,1}$ is an aspherical $(3g-1)$-dimensional complex analytic manifold, and the fundamental group is equal to the mapping class group $\Gamma_{g,1} := \pi_0 \mathrm{Diff}_+(\Sigma_g, p_0, v_0)$, where Σ_g is an oriented closed connected C^∞ 2-manifold of genus g, $p_0 \in \Sigma_g$, and $v_0 \in T_{p_0}\Sigma_g \backslash \{0\}$. The universal covering space is just the Teichmüller space $\mathbb{T}_{g,1}$ for the topological triple (Σ_g, p_0, v_0). For any triple (C, P_0, v) one can define the fundamental group of the complement $C \setminus \{P_0\}$ with the tangential basepoint v, which we denote by $\pi_1(C, P_0, v)$. If we choose a symplectic generator of $\pi_1(C, P_0, v)$, we can identify it with a free group of rank $2g$, F_{2g}. This induces a homomorphism $\Gamma_{g,1} \to \mathrm{Aut}(F_{2g})$, which is known to be an injection from a theorem of Nielsen.

Let $n \geq 2$ be an integer, F_n a free group of rank n with free basis $x_1, x_2, \ldots, x_n$, $F_n = \langle x_1, x_2, \ldots, x_n \rangle$. We denote by $H := H_1(F_n; \mathbb{R})$ the first real homology group of the group F_n, and by $[\gamma] \in H$ the homology class induced by $\gamma \in F_n$. The completed tensor algebra generated by H, $\widehat{T} = \widehat{T}(H) := \prod_{m=0}^{\infty} H^{\otimes m}$, has a decreasing filtration of two-sided ideals $\widehat{T}_p := \prod_{m \geq p} H^{\otimes m}$, $p \geq 1$. The subset $1 + \widehat{T}_1$ is a subgroup of the multiplicative group of the algebra $\widehat{T}$. We define a Magnus expansion of the free group F_n in our generalised sense [5].

Definition 1. *A map $\theta : F_n \to 1 + \widehat{T}_1$ is a (real-valued)* Magnus expansion *of the free group F_n, if $\theta : F_n \to 1 + \widehat{T}_1$ is a group homomorphism, and $\theta(\gamma) \equiv 1 + [\gamma]$ (mod $\widehat{T}_2$) for any $\gamma \in F_n$.*

We denote by Θ_n the set of all the real-valued Magnus expansions. The automorphism group $\mathrm{Aut}(F_n)$ of the group F_n acts on Θ_n in a natural way. Moreover the (projective limit of) Lie group(s) $\mathrm{IA}(\widehat{T})$ of all the $\mathbb{R}$-algebra automorphisms $U : \widehat{T} \to \widehat{T}$, which satisfies $U(\widehat{T}_p) = \widehat{T}_p$, for any $p \geq 1$ and $U = 1_H$ on $\widehat{T}_1/\widehat{T}_2 = H$, acts on Θ_n by $U \cdot \theta := U \circ \theta$, $(U \in \mathrm{IA}(\widehat{T}),\ \theta \in \Theta_n)$. The latter action is *free and transitive*, and induces the Maurer-Cartan form $\eta = (\eta_p) \in \Omega^1(\Theta_n) \widehat{\otimes} \mathrm{LieIA}(\widehat{T}) = \prod_{p=1}^{\infty} \Omega^1(\Theta_n) \otimes H^* \otimes H^{\otimes(p+1)}$. Here it should be remarked that we have a natural bijection of manifolds

$$\mathrm{IA}(\widehat{T}) \cong \prod_{m=1}^{\infty} \mathrm{Hom}(H, H^{\otimes m+1}) \cong \prod_{m=1}^{\infty} H^* \otimes H^{\otimes m+1}, \quad U \mapsto U|_H.$$

From the Maurer-Cartan formula $d\eta = \eta \wedge \eta$ we obtain

$$d\eta_p = \sum_{s=1}^{p-1} \underbrace{(\eta_s \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \eta_s)}_{p-s+1} \circ \eta_{p-s}. \tag{1}$$

Consider the double cochain complex

$$C^{*,*} := C^*(K_{p+1}; \Omega^*(\Theta_n; H^* \otimes H^{\otimes(p+1)})^{\mathrm{Aut}(F_n)}),$$

that is, the cellular cochain complex of the Stasheff associahedrons K_{p+1} [12] with values in the de Rham complex of Θ_n with twisted coefficients in $\mathrm{Aut}(F_n)$-module $H^* \otimes H^{\otimes(p+1)}$. The formula (1) means the Maurer-Cartan forms η_p's induce a p-cocycle $Y_p \in Z^p(C^{*,*})$, whose cohomology class

$$[Y_p] \in H^p(C^{*,*}) \cong H^p(\Omega^*(\Theta_n; H^* \otimes H^{\otimes(p+1)})^{\mathrm{Aut}(F_n)})$$

induces the $(0, p+2)$-twisted Morita-Mumford class $(-1)^{\frac{1}{2}p(p+1)} \frac{1}{(p+2)!} m_{0,p+2}$ on the space $\mathbb{M}_{g,1}$ [4]. From [9], [7] the i-th Morita-Mumford class is obtained by contracting the coefficients of $m_{0,2i+2}$ using the intersection form of the surface for any $i \geq 1$.

The map $H^* = H^1(C; \mathbb{R}) \to \Omega^1(C)$ assigning each cohomology class the harmonic 1-form representing it can be regarded as a H-valued 1-form $\omega_{(1)} \in \Omega^1(C) \otimes H$. We denote by φ' and φ'' the $(1,0)$- and the $(0,1)$-parts of $\varphi \in \Omega^1(C) \otimes \mathbb{C}$, respectively. Then $\omega_{(1)}{}' = \overline{\omega_{(1)}{}''}$ is holomorphic. We have $\int_C \omega_{(1)} \wedge \omega_{(1)} = I \in H^{\otimes 2}$, the intersection form. We denote by $\delta_{P_0} : C^\infty(C) \to \mathbb{R}$, $f \mapsto f(P_0)$, the delta 2-current on C at P_0. Then we have a unique $\widehat{T}$-valued 1-current $\omega = \sum_{p \geq 1} \omega_{(p)}$, $\omega_{(p)} \in \Omega^1(C) \otimes H^{\otimes p}$, satisfying the (modified) integrability condition

$$d\omega = \omega \wedge \omega - I \cdot \delta_{P_0},$$

$\omega_{(p)} = \omega_{(1)}$ for $p = 1$, and the normalisation condition $\int_C \omega_{(p)} \wedge *\varphi = 0$ for any closed 1-form φ and each $p \geq 2$. Here $*$ is the Hodge $*$-operator on $\Omega^1(C)$, which is conformal invariant of the Riemann surface C. Moreover, using Chen's iterated integrals [1], we can define a Magnus expansion

$$\theta = \theta^{(C,P_0,v)} : \pi_1(C, P_0, v) \to 1 + \widehat{T}_1, \quad [\ell] \mapsto 1 + \sum_{m=1}^{\infty} \int_\ell \overbrace{\omega\omega\cdots\omega}^{m}.$$

The Magnus expansions $\theta^{(C,P_0,v)}$ for all the triples (C, P_0, v) define a canonical real analytic map $\theta : \mathbb{T}_{g,1} \to \Theta_{2g}$, which we call *the harmonic Magnus expansion on the universal family of Riemann surfaces.* The pullbacks of the Maurer-Cartan forms η_p's give the canonical differential forms representing the Morita-Mumford classes and their higher relations.

Theorem 1 ([6])**.** *For any $[C, P_0, v] \in \mathbb{M}_{g,1}$ we have*

$$(\theta^*\eta)_{[C,P_0,v]} = 2\Re(N(\omega'\omega') - 2\omega_{(1)}{}'\omega_{(1)}{}') \in T^*_{[C,P_0,v]}\mathbb{M}_{g,1} \otimes \widehat{T}_3.$$

Here $N : \widehat{T}_1 \to \widehat{T}_1$ *is defined by* $N|_{H^{\otimes m}} := \sum_{k=0}^{m-1} \begin{pmatrix} 1 & 2 & \cdots & m-1 & m \\ 2 & 3 & \cdots & m & 1 \end{pmatrix}^k$, *and the meromorphic quadratic differential* $N(\omega'\omega')$ *is regarded as a* $(1,0)$*-cotangent vector at* $[C, P_0, v] \in \mathbb{M}_{g,1}$ *in a natural way.*

The second homogeneous term of $N(\omega'\omega')$ coincides with $2{\omega_{(1)}}'{\omega_{(1)}}'$, which is just the first variation of the period matrices [11].

Let $\mathbb{C}_g$ be the universal family of compact Riemann surfaces over the moduli $\mathbb{M}_g$. Using a recipe of Morita [9] we have two differential forms $e^J \in \Omega^2(\mathbb{C}_g)$ and $e_1^J \in \Omega^2(\mathbb{M}_g)$ from $\theta^*\eta_1$, which represent the first Chern class of the relative tangent bundle $T_{\mathbb{C}_g/\mathbb{M}_g}$ and the first Morita-Mumford class, respectively. The form e^J seems to be related to Arakelov's admissible metric. Let $B := \frac{\sqrt{-1}}{2g}\sum_{i=1}^g \psi_i \wedge \overline{\psi_i}$ be the volume form on a compact Riemann surface C induced by the orthonormal basis $\{\psi_i\}_{i=1}^g$ of the holomorphic 1-forms, $\frac{\sqrt{-1}}{2}\int_C \psi_i \wedge \overline{\psi_j} = \delta_{i,j}$, $(1 \le i, j \le g.)$ Let h be the function on $\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g$ satisfying the conditions $\frac{1}{2\pi\sqrt{-1}}\partial\overline{\partial}h\big|_{C\times\{P_0\}} = B - \delta_{P_0}$ and $\int_C \left(h\big|_{C\times\{P_0\}}\right) B = 0$. Then we obtain

Theorem 2.

$$\left(\frac{1}{2\pi\sqrt{-1}}\partial\overline{\partial}h\right)\bigg|_{diagonal} = e^J + \frac{1}{(2-2g)^2}({e_1}^J - {e_1}^F) \ \in \ \Omega^2(\mathbb{C}_g).$$

Here ${e_1}^F := \int_{\text{fibre}} (e^J)^2 \in \Omega^2(\mathbb{M}_g)$.

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Stable Homotopy Theory and Quasi-Coherent Sheaves on the Projective Line

Thomas Hüttemann
(joint work with Stefan Schwede)

Spectra and S-modules. A *spectrum* is a sequence $X = \{X_0, X_1, \ldots\}$ of pointed spaces (*i.e.*, simplicial sets), equipped with structure maps $\lambda_i \colon S^1 \wedge X_i \longrightarrow X_{1+i}$. (Here $S^1 = \Delta^1/\partial\Delta^1$ is the standard simplicial 1-sphere.) The homotopy groups $\pi_k(X)$ of X are defined as the colimit of the sequence

$$\pi_k(X_0) \xrightarrow{S^1 \wedge \cdot} \pi_{k+1}(S^1 \wedge X_0) \xrightarrow{\pi_k(\lambda_0)} \pi_{k+1}(X_1) \xrightarrow{S^1 \wedge \cdot} \ldots$$

(if $k < 0$ the first few terms of this sequence are not defined, but eventually it will be a sequence of ABELian groups).

A more conceptual approach is the following. We consider the sequence $X = \{X_0, X_1, \ldots\}$ as a *graded space*. A particularly important example is the graded space $S := \{S^0, S^1, S^2, \ldots\}$, the sequence of spheres (where we define inductively $S^n := S^1 \wedge S^{n-1}$).

If X and Y are graded spaces, their *tensor product* is the graded space $X \otimes Y$ defined by

$$(X \otimes Y)_n \ = \ \bigvee_{p+q=n} X_p \ \wedge \ Y_q \ .$$

The tensor product of graded spaces is symmetric monoidal with unit object $S^0 = \{S^0, *, *, \ldots\}$.

It is easily seen that the associativity isomorphisms $S^j \wedge S^k \cong S^{j+k}$ yield a map $S \otimes S \longrightarrow S$ of graded spaces. In fact, they equip S with the structure of a monoid object in the category of graded spaces.

A *left S-module* is a graded space X together with an action $S \otimes X \longrightarrow X$ of S on X (which satisfies the usual associativity and unitality conditions). Since S is the free associative monoid generated by S^1 in degree 1, the category of left S-modules is equivalent to the category of spectra.

Similarly, we can define a category of right S-modules which is equivalent to the category of “spectra” with suspensions acting from the right.

Homotopy groups of S-modules. We can now give a reformulation of the definition of homotopy groups. Given a left S-module X, define a graded ABELian group $\Pi_k(X)$, $k \in \mathbb{Z}$, by

$$\Pi_k(X) := \bigoplus_{j \geq 2-k} \pi_{j+k}(X_j) \ .$$

The structure maps of X (considered as a spectrum) induce a degree 1 self-map of $\Pi_k(X)$. Hence we can consider $\Pi_k(X)$ as a graded $\mathbb{Z}[L]$-module where L is an indeterminate. The homotopy group $\pi_k(X)$ is naturally isomorphic to the degree 0 part of the localised graded module $\Pi_k(X)_{(L)} = \Pi_k(X) \otimes_{\mathbb{Z}[L]} \mathbb{Z}[L, L^{-1}]$.

S-Bimodules and their homotopy sheaves. An *S-bimodule* is a graded space X equipped with compatible structures $S \otimes X \otimes S \longrightarrow X$ as left and right S-module. Explicitly, an S-bimodule is a collection of pointed simplicial sets $X = \{X_0,\ X_1,\ X_2,\ \ldots\}$ and structure maps

$$\lambda_n \colon S^1 \wedge X_n \longrightarrow X_{1+n} \quad \text{and} \quad \rho_n \colon X_n \wedge S^1 \longrightarrow X_{n+1}$$

such that the following diagram commutes for all $n \in \mathbb{N}$:

$$\begin{tikzcd} S^1 \wedge X_n \wedge S^1 \arrow[r, "\mathrm{id}\wedge\rho_n"] \arrow[d, "\lambda_n\wedge\mathrm{id}"'] & S^1 \wedge X_{n+1} \arrow[d, "\lambda_{n+1}"] \\ X_{1+n} \wedge S^1 \arrow[r, "\rho_{1+n}"'] & X_{1+n+1} \end{tikzcd}$$

Informally, an S-bimodule consists of two spectra with the same underlying graded space and compatible structure maps.

Given an S-bimodule X, the graded ABELian group $\Pi_k(X)$ has the structure of a graded $\mathbb{Z}[L, R]$-module; the action of the indeterminates L and R are induced by left and right structure maps λ_n and ρ_n, respectively. We denote by $\tilde{\pi}_k(X)$ the associated quasi-coherent sheaf on $\mathbb{P}^1_{\mathbb{Z}} = \mathrm{Proj}\,(\mathbb{Z}[L, R])$.

Stable model structures. The projective line has four interesting open subsets: The projective line itself, two affine lines $U_L := \{R \neq 0\}$ and $U_R := \{L \neq 0\}$, and the algebraic torus $U_0 := U_L \cap U_R$. We call these sets the *distinguished open subsets.*

Let U be a distinguished open set. A map $f \colon X \longrightarrow Y$ of S-bimodules is called a *U-equivalence* if for all $n \in \mathbb{Z}$ the induced map of restrictions of sheaves

$$\tilde{\pi}_n(f)|_U \colon \tilde{\pi}_n(X)|_U \longrightarrow \tilde{\pi}_n(Y)|_U$$

is an isomorphism. Since the homotopy sheaves are quasi-coherent, we can characterise equivalences for U affine by their effect on sections over U: The map $f \colon X \longrightarrow Y$ is a U-equivalence if and only if it induces isomorphisms of modules $\Gamma(U, \tilde{\pi}_n(X)) \longrightarrow \Gamma(U, \tilde{\pi}_n(Y))$ for all $n \in \mathbb{Z}$. Moreover, f is a $\mathbb{P}^1$-equivalence if and only if it is a U_L-equivalence as well as a U_R-equivalence.

Theorem 1. *Let U be a distinguished open subset of $\mathbb{P}^1$. The category of S-bimodules admits a simplicial closed model structure where a map f is a weak equivalence if and only if it is a U-equivalence. The model structure is stable in the sense that simplicial suspension and loop functors induce mutually inverse equivalences on the homotopy category.*

The idea here is that an S-bimodule is a homotopical analogue of a quasi-coherent sheaf on $\mathbb{P}^1$, and the U-equivalences capture exactly the homotopical information living over the subset U.

The four model structures of the theorem are interrelated: If $U \supseteq V$ are distinguished open subsets of $\mathbb{P}^1$, the identity functor is a left QUILLEN functor from the

U-model structure on the category of S-bimodules to the V-structure. Moreover, morphisms in the $\mathbb{P}^1$-homotopy category of S-bimodules are determined by morphisms in the U-homotopy categories with U ranging over the affine distinguished sets. In other words, we have constructed a "bundle of homotopy categories" on $\mathbb{P}^1$, and morphisms are determined by local data:

Theorem 2. *Let X and Z be S-bimodules. Let $(\underline{S\text{-}\mathbf{mod}\text{-}S})_U(X,\, Z)$ denote a space having the homotopy type of a homotopy function complex of maps from X to Z in the U-model structure of S-bimodules. One can choose these spaces in such a way that there results a commutative diagram which is homotopy cartesian:*

$$\begin{tikzcd}
[eqno = (*)](\underline{S\text{-}\mathbf{mod}\text{-}S})_{\mathbb{P}^1}(X,\, Z) \arrow[r] \arrow[d] & (\underline{S\text{-}\mathbf{mod}\text{-}S})_{U_R}(X,\, Z) \arrow[d] \\
(\underline{S\text{-}\mathbf{mod}\text{-}S})_{U_L}(X,\, Z) \arrow[r] & (\underline{S\text{-}\mathbf{mod}\text{-}S})_{U_0}(X,\, Z)
\end{tikzcd}$$

In particular, the associated long exact Mayer-Vietoris *sequence contains a description of* $\mathrm{Ho}_{\mathbb{P}^1}(X, Z) = \pi_0(\underline{S\text{-}\mathbf{mod}\text{-}S})_{\mathbb{P}^1}(X,\, Z)$ *in terms of the abelian groups* $\mathrm{Ho}_{U_L}(X, Z)$, $\mathrm{Ho}_{U_R}(X, Z)$ *and* $\pi_1(\underline{S\text{-}\mathbf{mod}\text{-}S})_{U_0}(X,\, Z)$.

Abelian spectra. The above can be re-done in a linearised setting. Let A denote a noetherian commutative ring (with unit). An *A-*ABEL*ian spectrum* is a graded simplicial A-module $X = \{X_0,\, X_1,\, \ldots\}$ together with structure maps (maps of simplicial sets) $\lambda_n \colon S^1 \wedge X_n \longrightarrow X_{n+1}$ such that the adjoint maps $X_n \longrightarrow \Omega X_{n+1}$ are homomorphisms of simplicial A-modules. Equivalently, we can prescribe structure maps $\tilde{A}[S^1] \otimes_A X_n \longrightarrow X_{n+1}$ which are homomorphisms of simplicial A-modules (where $\tilde{A}[S^1] = A[S^1]/A[*]$ is the reduce free simplicial A-module generated by S^1, and tensor means level-wise tensor product).

As before, we can give a re-interpretation using graded objects. The category of graded simplicial A-modules has a symmetric monoidal product given by

$$(X \otimes_A Y)_n = \bigoplus_{i+j=n} X_i \otimes_A Y_j \ .$$

The graded simplicial A-module

$$\tilde{A}[S] = \{\tilde{A}[S^0],\, \tilde{A}[S^1],\, \ldots\}$$

is a monoid object with respect to the tensor product; the structure maps are induced by the maps $\tilde{A}[S^i] \otimes_A \tilde{A}[S^j] \longrightarrow \tilde{A}[S^{i+j}]$ given by concatenation of generators; they can also be described as the linearisations of the isomorphisms $S^i \wedge S^j \longrightarrow S^{i+j}$. An ABELian spectrum is then nothing but a left $\tilde{A}[S]$-module, that is, a graded simplicial A-module equipped with a left action of $\tilde{A}[S]$.

It is well known that the category of ABELian spectra admits a stable model structure where a map is a weak equivalence (or fibration) if and only if it is a weak equivalence (or fibration) of underlying (non-ABELian) spectra. Moreover, the homotopy category of ABELian spectra is equivalent to the (unbounded) derived category of A.

Abelian S-bimodules. An A-ABEL*ian* S-*bimodule* is a graded simplicial A-module equipped with compatible structures as a left and right $\tilde{A}[S]$-module. Each ABELian S-bimodule X has an underlying S-bimodule (forget the linear structure), so we know what homotopy sheaves $\tilde{\pi}_n(X)$ and U-equivalences should be. Note that now $\tilde{\pi}_n(X)$ is naturally a quasi-coherent sheaf on $\mathbb{P}^1_A$, the projective line over A.

Theorem 3. *Let U be a distinguished open subset of $\mathbb{P}^1$. The category of A-ABELian S-bimodules admits a simplicial closed model structure where a map is a weak equivalence if and only if it is a U-equivalence. The model structure is stable in the sense that simplicial suspension and loop functors induce mutually inverse equivalences on the homotopy category.*

As in the case of S-bimodules the associated homotopy categories for the various U assemble to a "bundle of homotopy categories", giving rise to a homotopy cartesian sqaure of mapping spaces similar to the diagram ($*$) above. This enables us to identify the homotopy categories of ABELian S-bimodules with well-known objects in algebra:

Theorem 4. *Let U be a distinguished open subset of $\mathbb{P}^1_A$. The homotopy category of A-ABELian S-bimodules with respect to U-equivalences is equivalent, as a triangulated category, to the (unbounded) derived category of quasi-coherent sheaves on U. If U is affine ($U \neq \mathbb{P}^1_A$), the equivalence can be realised by a* QUILLEN *equivalence of model categories.*

Loop Space and Floer Homology

KENJI FUKAYA

In this talk I explained an application of the technique to use an L_∞ structure of loop space homology to obtain a restriction of a Lagrangian submanifold. Especially I discussed the case of the Lagrangian $S^1 \times S^{2n}$ in $\mathbb{C}^{2n+1}$. It is explained that the structure of the free loop space of homotopy groups of spheres is related to such a problem. Especially this implies that that the Maslov index of the generator $\gamma \in \pi_1(S^1 \times S^{2n})$ is 2. Here we take the generator so that there exists $[D^2] \in \pi_2(\mathbb{C}^{2n+1}, L)$ which bounds γ and such that

$$\int_{D^2} \omega > 0.$$

Lie 2-Algebras and the Geometry of Gerbes

Danny Stevenson

Connections and curvature on a principal G-bundle $P \to M$ have a natural interpretation in terms of extensions of Lie algebras: a connection on P is a $C^\infty(M)$-linear splitting A of the short exact sequence

$$0 \to \Gamma(\mathrm{ad}(P)) \to \Gamma(TP/G) \to \Gamma(TM) \to 0$$

of Lie algebras. Here $\Gamma(\mathrm{ad}(P))$ denotes the sections of the adjoint bundle $\mathrm{ad}(P)$ associated to P, or, what is the same thing, invariant vertical vector fields on P. Likewise $\Gamma(TP/G)$ denotes the invariant vector fields on P. This exact sequence is the infinitesimal version of the exact sequence of groups

$$1 \to \mathrm{Gauge}(P) \to \mathrm{Aut}_G(P) \to \mathrm{Diff}(M) \to 1$$

where $\mathrm{Aut}_G(P)$ denotes the bundle automorphisms of P and $\mathrm{Gauge}(P)$ denotes the automorphisms of P covering the identity on M. The curvature F_A of A measures the failure of A to be a homomorphism of Lie algebras:

$$F_A(X,Y) = [A(X), A(Y)] - A[X,Y] \;\; X,Y \in \Gamma(TM)$$

The data of A, F_A and the Bianchi identity for F_A can be neatly encoded as the data of a weak homomorphism of Lie 2-algebras

$$\nabla\colon \Gamma(TM) \to \mathrm{DER}(\mathrm{ad}(P))$$

where $\Gamma(TM)$ is thought of as a discrete Lie 2-algebra and $\mathrm{DER}(\mathrm{ad}(P))$ denotes the Lie 2-algebra associated to the crossed module

$$\Gamma(\mathrm{ad}(P)) \to \mathrm{Der}(\Gamma(\mathrm{ad}(P)))$$

arising from the action of $\Gamma(\mathrm{ad}(P))$ on itself by derivations. Recall [4] that a *Lie 2-algebra* $\mathbb{L}$ is a category internal to **LieAlg**, thus $\mathbb{L}$ consists of

- a Lie algebra of objects L_0
- a Lie algebra of morphisms L_1

such that all of the structure maps of $\mathbb{L}$ are Lie algebra homomorphisms. There exist weakened versions of Lie 2-algebras, so-called *semi-strict* Lie 2-algebras. These are closely related to L_∞-algebras.

Starting with the seminal work of Breen and Messing [3], various authors [1, 2] have developed a theory of connections on *non-abelian* gerbes. The aim of our talk was how to show that this quite complicated theory could be neatly encoded using the language of Lie 2-algebras. A *G-gerbe* on M for G a compact Lie group consists of the following data:

- a **surjective submersion** $\pi\colon \mathbb{P} \to \mathbb{M}$
- a right **action** $\mathbb{P} \times \mathrm{AUT}(G) \to \mathbb{P}$ of the *automorphism 2-group* $\mathrm{AUT}(G)$ on $\mathbb{P}$.
- the natural map $\mathbb{P} \times \mathrm{AUT}(G) \to \mathbb{P} \times_{\mathbb{M}} \mathbb{P}$ is required to be a **diffeomorphism**.

Here $\mathrm{AUT}(G)$ is a certain category internal to **Grp**, the category of Lie groups, closely related to the canonical crossed module $\mathrm{Ad}\colon G \to \mathrm{Aut}(G)$ associated to G. Thus $\mathrm{AUT}(G)$ has a group of objects $\mathrm{Aut}(G)$ and a group of morphisms $\mathrm{Aut}(G) \ltimes G$ such that all structure maps are group homomorphisms. $\mathbb{M}$ is the groupoid $X^{[2]} \rightrightarrows X$ associated to a surjective submersion $\pi\colon X \to M$. The data of a *connection* on such a gerbe can be thought of as a splitting of the exact sequence:

$$0 \to \mathrm{ad}(\mathbb{P}) \to T\mathbb{P}/\mathbb{G} \overset{A}{\underset{\to}{\leftarrow}} T\mathbb{M} \to 0 \tag{1}$$

Here $\mathrm{ad}(\mathbb{P})$, $T\mathbb{P}/\mathbb{G}$ and $T\mathbb{M}$ are all certain groupoids internal to the category **VectBund** of vector bundles. To understand curvature in this setting, one should consider the following extension of Lie 2-algebras

$$0 \to \Gamma(\mathrm{ad}(\mathbb{P})) \to \Gamma(T\mathbb{P}/\mathbb{G}) \overset{A}{\underset{\to}{\leftarrow}} \Gamma(TM) \to 0 \tag{2}$$

associated to (1). We now find that curvature measures the failure of A to be a weak homomorphism of Lie 2-algebras. This data can be neatly encoded in a weak homomorphism

$$\nabla\colon \Gamma(TM) \to \mathrm{DER}(\Gamma(\mathrm{ad}(\mathbb{P})))$$

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Eta-Invariants, Torsion Forms and Flat Vector Bundles

Xiaonan Ma

We present a new proof, as well as a $\mathbf{C}/\mathbf{Q}$ extension (and also certain $\mathbf{C}/\mathbf{Z}$ extension), of the Riemann-Roch-Grothendieck theorem of Bismut-Lott for flat vector bundles. We further show that the Bismut-Lott analytic torsion form can be derived naturally from the transgression of the η-forms appearing in the adiabatic limit computations. Finally, we explain how to identify precisely the imagenary part of the η-invariant associated to a non-unitary connection from a deformation argument. We explain now in more detail way.

Let M be a compact smooth manifold. For any complex flat vector bundle F over M with the flat connection ∇^F. Let k be a positive integer such that kF is a topologically trivial vector bundle. Let ∇_0^{kF} be a trivial connection on kF, which can be determined by choosing a global basis of kF. Let $k\nabla^F$ be the connection on kF obtained from the direct sum of k copies of ∇^F.

The mod $\mathbf{Q}$ version of the Cheeger-Chern-Simons character $CCS(F,\nabla^F)$ is defined as

$$CCS(F,\nabla^F)=\frac{1}{k}CS(\nabla_0^{kF},k\nabla^F), \tag{1}$$

where $CS(\nabla_0^{kF},k\nabla^F)$ is the Chern-Simons class associated to $(kF,k\nabla^F,\nabla_0^{kF})$. It determines a well-defined element in $H^{\mathrm{odd}}(M,\mathbf{C}/\mathbf{Q})$

Let h^F be a Hermitian metric on F. Set

$$\omega(F,h^F)=(h^F)^{-1}(\nabla^F h^F),\quad \nabla^{F,e}=\nabla^F+\frac{1}{2}\omega(F,h^F). \tag{2}$$

For any integer $j\geq 0$, let $c_{2j+1}(F,h^F)$ be the Chern form (cf. [BL, (0.2)]) defined by

$$c_{2j+1}(F,h^F)=(2\pi\sqrt{-1})^{-j}2^{-(2j+1)}\mathrm{Tr}\left[\omega^{2j+1}(F,h^F)\right]. \tag{3}$$

Let $c_{2j+1}(F)$ be the associated cohomology class in $H^{2j+1}(B,\mathbf{R})$, which does not depend on the choice of h^F. Then

$$\begin{aligned}\mathrm{Im}(CCS(F,\nabla^F))&=-\frac{1}{2\pi}\sum_{j=0}^{+\infty}\frac{2^{2j}j!}{(2j+1)!}c_{2j+1}(F),\\ \mathrm{Re}(CCS(F,\nabla^F))&=\frac{1}{q}CS(\nabla_0^{qF},q\nabla^{F,e})\ \text{ in }\ H^{\mathrm{odd}}(B,\mathbf{R}/\mathbf{Q}).\end{aligned} \tag{4}$$

Let $Z\to M\to B$ be a fibred manifold with compact base and fibres. Let $e(TZ)$ be the Euler class of the vertical tangent vector bundle TZ. The flat vector bundle (F,∇^F) over M induces canonically a $\mathbf{Z}$-graded flat vector bundle $H^*(Z,F|_Z)=\oplus_{i=0}^{\dim Z}H^i(Z,F|_Z)$ over B (cf. [BL]). Let $\nabla^{H^*(Z,F|_Z)}=\oplus_{i=0}^{\dim Z}\nabla^{H^i(Z,F|_Z)}$ denote the corresponding flat connection induced from ∇^F.

Theorem 1. *We have the following identity in* $H^{\mathrm{odd}}(B,\mathbf{C}/\mathbf{Q})$,

$$\int_Z e(TZ)CCS(F,\nabla^F)=\sum_{i=0}^{\dim Z}(-1)^iCCS(H^i(Z,F|_Z),\nabla^{H^i(Z,F|_Z)}). \tag{5}$$

The imagenary part of (5) is Bismut-Lott's Riemann-Roch-Grothendieck formula [BL]. It turns out that the real part of (5) has been obtained by Bismut in [B3, Theorem 0.2] under the extra condition that TZ is fibre-wise oriented.

In this talk, we present a new approach to the imagenary part of (5) based on considerations of the adiabatic limits of η-invariants of Atiyah-Patodi-Singer [APS3] associated to the so-called sub-signature operators. Besides giving a new proof of (5), our method also provides the real part of (5).

From another aspect, in view of the $\mathbf{R}/\mathbf{Z}$-index theory developed by Lott [L], one can refine the real part of (5) to an identity in $K^{-1}_{\mathbf{R}/\mathbf{Z}}(B)$ if Z is even dimensional and spin^c. Suppose that Z is even dimensional and spin^c. Let $S(TZ)=S^+(TZ)\oplus S^-(TZ)$ be the spinor bundle of TZ. Lott defined a topological index Ind_{top} mapping from $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ to $K^{-1}_{\mathbf{R}/\mathbf{Z}}(B)$. We denote by $\mathbf{C}$ the trivial complex line bundle carrying the trivial metric and connection. Then $\mathcal{F} = [(F, h^F, \nabla^{F,e}, 0) - \mathrm{rk}(F)\mathbf{C}] \in \mathbf{K^{-1}_{R/Z}(M)}$. Set

$$I(F) = \sum_{i=0}^{n}(-1)^i \Big(H^i(Z, F|_Z), h^{H^i(Z,F|_Z)}, \nabla^{H^i(Z,F|_Z),e}, 0\Big). \tag{6}$$

Theorem 2. *In $K^{-1}_{\mathbf{R/Z}}(B)$, we have*

$$\mathrm{Ind}_{top}((S^+(TZ)^* - S^-(TZ)^*) \otimes \mathcal{F}) = I(F) - \mathrm{rk}(F) I(\mathbf{C}). \tag{7}$$

Finally, assume that M is an odd dimensional oriented closed spin manifold carrying a Riemannian metric g^{TM}. Let $S(TM)$ be the associated Hermitian bundle of spinors. Let E be a Hermitian vector bundle over M carrying a unitary connection ∇^E.

Let $D^{E\otimes F,e} : \Gamma(S(TM) \otimes E \otimes F) \longrightarrow \Gamma(S(TM) \otimes E \otimes F)$ be the Dirac operator associated to the connection $\nabla^{F,e}$ on F and ∇^E on E. Let $D^{E\otimes F}$ be the corresponding Dirac operator associated to the connection ∇^F on F and ∇^E on E. Then $D^{E\otimes F,e}$ is formally self-adjoint, and $D^{E\otimes F}$ is not formally self-adjoint if ∇^F does not preserve h^F.

Let $\overline{\eta}\left(D^{E\otimes F}\right)$, $\overline{\eta}\left(D^{E\otimes F,e}\right)$ be the associated reduced eta invariants. Then we can identify the imagenary part of $\overline{\eta}\left(D^{E\otimes F}\right)$ as following.

$$\begin{aligned} &\mathrm{Re}\left(\overline{\eta}\left(D^{E\otimes F}\right)\right) \equiv \overline{\eta}\left(D^{E\otimes F,e}\right) \mod \mathbf{Z}, \\ &\mathrm{Im}\left(\overline{\eta}\left(D^{E\otimes F}\right)\right) = -\frac{1}{2\pi}\int_M \widehat{A}(TM)\mathrm{ch}(E) \sum_{j=0}^{+\infty} \frac{2^{2j} j!}{(2j+1)!} c_{2j+1}(F). \end{aligned} \tag{8}$$

This is a joint work with Weiping Zhang (Chern Institute of Mathematics, Nankai University, China (weiping@nankai.edu.cn)).

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An Invariant Based on the Conformal Laplacian

Christian Bär
(joint work with M. Dahl)

This talk is based on the paper [1]. Throughout the talk let M be a compact oriented differentiable manifold of dimension $n \geq 3$. Given a Riemannian metric g on M the *Conformal Laplacian* (or *Yamabe operator*) Y_g is defined as

$$Y_g = \Delta_g + \frac{n-2}{4(n-1)} \cdot \mathrm{Scal}_g : C^\infty(M) \to C^\infty(M) \subset L^2(M)$$

where $\Delta_g = d^*d$ is the Laplacian and Scal_g is the scalar curvature of g. The operator Y_g is an elliptic differential operator of second order, essentially self-adjoint in $L^2(M)$. Let $\mu_0(g) \leq \mu_1(g) \leq \mu_2(g) \leq \ldots$ be the spectrum of Y_g, the eigenvalues being repeated according to their multiplicities. Let f be a positive smooth function on M. The Conformal Laplacian of the conformally related metric $\overline{g} = f^{\frac{4}{n-2}} g$ is given by

$$Y_{\overline{g}} u = f^{-\frac{n+2}{n-2}} Y_g(fu). \tag{1}$$

Applying (1) to the function $u \equiv 1$ gives the formula

$$\mathrm{Scal}_{\overline{g}} = \frac{4(n-1)}{n-2} f^{-\frac{n+2}{n-2}} Y_g f \tag{2}$$

for the scalar curvature of $\overline{g}$.

We now introduce a differential topological invariant of a compact manifold by counting the number of small eigenvalues of the Conformal Laplacian.

Definition 1. Let M be a compact differentiable manifold. The *κ-invariant* $\kappa(M)$ is defined to be the smallest integer k such that for every $\varepsilon > 0$ there is a Riemannian metric g_ε on M for which

$$\begin{cases} \mu_k(g_\varepsilon) = 1, \\ |\mu_i(g_\varepsilon)| < \varepsilon, \quad 0 \leq i < k. \end{cases}$$

If no such integer exists set $\kappa(M) := \infty$.

Heuristically, $\kappa(M)$ is the dimension of the "almost-kernel" of the Conformal Laplace operator.

By rescaling the metrics g_ε accordingly one sees that $\kappa(M)$ is also the smallest integer k such that for each constant $C > 0$ there exists a Riemannian metric g_C for which

$$\begin{cases} \mu_k(g_C) > C, \\ |\mu_i(g_C)| \leq 1, \quad 0 \leq i < k. \end{cases}$$

Hence $\kappa(M)$ tells us which is the first eigenvalue that can be made arbitrarily large for appropriate metrics while keeping the preceeding ones bounded.

If we made this definition using the Laplace operator acting on p-forms instead of the Conformal Laplacian, then by Hodge theory the resulting invariant would be nothing but the p^{th} Betti number $b_p(M)$.

¿From the fact that the spectrum of the disjoint union $M_1 \amalg M_2$ is the disjoint union of the spectra of M_1 and of M_2 it follows that

$$\kappa(M_1 \amalg M_2) = \kappa(M_1) + \kappa(M_2). \tag{3}$$

The next proposition concerns the relation between $\kappa(M)$ and scalar curvature.

Proposition 1. *Let M be a compact differentiable manifold of dimension $n \geq 3$. Then*

(1) $\kappa(M) = 0$ if and only if there is a metric of positive scalar curvature on M.

(2) If M has a metric with $\mathrm{Scal} \geq 0$, then $\kappa(M) \leq b_0(M)$.

Proof. If $\kappa(M) = 0$, then there is a metric with $\mu_0 = 1$. The corresponding eigenfunction f_0 can chosen to be positive. From Equation (2) it follows that $\overline{g} = f_0^{\frac{4}{n-2}} g$ has positive scalar curvature. Conversely, if g is a metric of positive scalar curvature on M, then $Y_g > 0$ and we can rescale so that $\mu_0 = 1$. Hence $\kappa(M) = 0$.

For a metric g with $\mathrm{Scal} \geq 0$ on M we have $Y_g = \Delta_g \geq 0$ and the zero eigenspace consists of the locally constant functions. Hence $\mu_0(g) = \ldots = \mu_{b_0(M)-1}(g) = 0$ and $\mu_{b_0(M)}(g) > 0$. □

The following theorem controls the spectrum of Y_g under surgeries of codimension at least three. This enables us to examine the behavior of $\kappa(M)$ under such surgeries.

Theorem 1. *Let (M, g) be a closed Riemannian manifold. Let $\widetilde{M}$ be obtained from M by surgery in codimension at least three. Then for each $k \in \mathbb{N}$ and for each $\varepsilon > 0$ there exists a Riemannian metric $\widetilde{g}$ on $\widetilde{M}$ such that the first $k+1$ eigenvalues of the operators Y_g and $Y_{\widetilde{g}}$ are ε-close, that is*

$$|\mu_j(g) - \mu_j(\widetilde{g})| < \varepsilon$$

for $j = 0, \ldots, k$.

As an immediate consequence we obtain

Corollary 1. *Let M be a compact differentiable manifold of dimension $n \geq 3$. Suppose $\widetilde{M}$ is obtained from M by surgery of codimension ≥ 3. Then*

$$\kappa(\widetilde{M}) \leq \kappa(M).$$

Hence for any $\kappa_0 \in \mathbb{N}_0$ the property of having $\kappa \leq \kappa_0$ is preserved under surgery of codimension at least three. For $\kappa_0 = 0$ this means that the property of admitting a metric of positive scalar curvature is preserved under such surgeries. This is a famous by now classical result of Gromov and Lawson [3]. We do not give a new proof of this fact since we use the work of Gromov and Lawson when we prove Theorem 1.

As to the case $\kappa_0 = 1$ it is interesting to note that the property of allowing a scalar flat metric is not preserved under such surgeries. It follows that the converse of statement (2) in Proposition 1 does not hold. For example, the n-dimensional torus T^n has a flat metric but no metric of positive scalar curvature [4]. Thus $\kappa(T^n) = 1$. Performing surgery in codimension at least three on T^n yields a manifold M^n not admitting metrics with positive or zero scalar curvature. Yet we have $\kappa(M^n) = 1$.

Also note that the condition $\kappa = 0$ is not preserved under surgery of codimension 2. Like any compact connected 3-manifold the 3-torus T^3 can be obtained from S^3 by a sequence of surgeries in codimension 2. But we have $\kappa(T^3) = 1 > \kappa(S^3) = 0$. This also shows that Theorem 1 cannot hold for surgeries in codimension less than three.

The κ-invariant measures how close Y can come to being a positive operator for some Riemannian metric on M. Since Y is positive if and only if M allows a metric of positive scalar curvature one can also view κ as a measure of how close one can get to having positive scalar curvature. Therefore it is not unreasonable to suspect that κ is related to the $\hat{A}$ or α-genus of M, the primary obstruction to allowing metrics of positive scalar curvature. We show that this indeed is the case. On the one hand we have

Theorem 2. *Let M be a compact spin manifold of dimension $n = 4m$. Then*

$$|\hat{A}(M)| \leq 2^{2m-1}\kappa(M).$$

This together with a classical eigenvalue estimate by Cheeger [2] implies the following isoperimetric result.

Corollary 2. *Let M be a compact spin manifold of dimension $n = 4m$ with $|\hat{A}(M)| > 2^{2m-1}$. Then there exists a constant $C = C(M)$ such that for each Riemannian metric with $|\,\mathrm{Scal}\,| \leq 1$ there exists a hypersurface $S \subset M$ dividing M into two connected components M_1 and M_2 such that*

$$\mathrm{vol}_{n-1}(S) \leq C \cdot \min\{\mathrm{vol}_n(M_1), \mathrm{vol}_n(M_2)\}.$$

On the other hand, we can bound $\kappa(M)$ from above in terms of the dimension and the α-genus, at least for simply connected manifolds of dimension $n \geq 5$. First we make the following

Observation. *Let M be a simply connected compact differentiable manifold of dimension $n \geq 5$. If M is non-spin or if $n \equiv 3, 5, 6, 7 \mod 8$ then*

$$\kappa(M) = 0.$$

This comes from the fact that in these cases M is well-known to carry a metric of positive scalar curvature, see [3], [5].

In dimensions $n \equiv 0 \mod 4$ the α-genus of a spin manifold is integer-valued and it essentially coincides with the $\hat{A}$-genus. More precisely, if $n = 8l$ then $\alpha(M) = \hat{A}(M)$ while if $n = 8l + 4$ then $\alpha(M) = \frac{1}{2}\hat{A}(M)$.

Theorem 3. *Let M be a simply connected differentiable manifold of dimension $n \equiv 0 \mod 4$. Write $n = 8l$ or $n = 8l + 4$ with $l \geq 1$ and let $|\alpha(M)| = 4^l p + q$, $p \geq 0$, $0 \leq q < 4^l$. Then*

$$\kappa(M) \leq p + \min\{q, l\}.$$

As a special case we see that for spin manifolds as in the Theorem we have $\kappa = 1$ if $\alpha = 1$. In dimensions $n \equiv 1, 2 \mod 8$ this and the converse is true. In those dimensions we have $\alpha(M) \in \mathrm{KO}^{-n}(\mathrm{pt}) \cong \mathbb{Z}/2\mathbb{Z}$. By $|\alpha(M)| \in \mathbb{Z}$ we mean 0 if $\alpha(M)$ is trivial and 1 otherwise.

Theorem 4. *Let M be a simply connected spin manifold of dimension $n = 8l + 1$ or $8l + 2$, $l \geq 1$. Then*

$$\kappa(M) = |\alpha(M)|.$$

This shows that $\kappa(M)$ can distinguish certain exotic spheres. In particular, $\kappa(M)$ is *not* invariant under homeomorphisms, only under diffeomorphisms.

Even though Theorem 2 shows that $\kappa(M)$ can become arbitrarily large it turns out that in a stable sense it takes only the values 0 and 1. More precisely, let B be a compact simply connected 8-dimensional spin manifold with $\hat{A}(B) = 1$. Then $\alpha(M \times B) = \alpha(M)$ for all spin manifolds M.

Theorem 5. *Let M be a simply connected spin manifold. Then*

$$\kappa(M \times B^p) \leq 1$$

for all sufficiently large p.

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The Universal Functorial Equivariant Lefschetz Invariant

Julia Weber

The Lefschetz number is a classical invariant in algebraic topology. It is an integer $L(f) \in \mathbb{Z}$ associated to an endomorphism $f\colon X \to X$ and contains fixed point information: If $L(f) \neq 0$, then f has a fixed point. Generalisations which give more precise lower bounds on the number of fixed points (generalised Lefschetz invariant $\lambda(f)$ [5, 8]) or which capture all iterates of f at the same time (Lefschetz zeta function [1, 2]) have been developed.

An invariant which maps to all of these generalisations and still has the characteristic properties of the Lefschetz number, namely homotopy invariance and additivity, has been constructed by Lück [4]. We generalise this construction to the equivariant setting, for all discrete groups G.

On the one hand, this gives finer invariants. The extra structure given by a G-action on a space is taken into account. On the other hand, this generalisation enlarges the scope of the invariant. If we have an infinite discrete group G acting properly on a finite G-CW-complex X, the space X seen as a CW-complex is infinite. This situation cannot be treated by the classical theory, but the equivariant version can be applied.

1. Construction of the invariant

We recall the construction of the universal functorial Lefschetz invariant [4]. Given an endomorphism $f\colon X \to X$ of a finite connected CW-complex X, we lift it to an endomorphism $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ of the universal covering space $\widetilde{X}$ of X. We then consider the map induced on the cellular chain complex,

$$C_*(\widetilde{f})\colon C_*(\widetilde{X}) \to C_*(\widetilde{X}).$$

This endomorphism of a $\mathbb{Z}\pi_1(X)$-chain complex is not quite equivariant, it is twisted by the map $\phi\colon \pi_1(X) \to \pi_1(X)$ induced by f. For $y \in C_*(\widetilde{X})$ and $\gamma \in \pi_1(X)$ we have $C_*(\widetilde{f})(y\gamma) = C_*(\widetilde{f})(y)\phi(\gamma)$. (We have to choose a basepoint $x \in X$ and a path from $f(x)$ to x, but since the construction turns out to be independent of these choices we neglect them in this presentation.)

Definition 1. *Let $\mathcal{D}_{X,f}$ be the additive category whose objects are the ϕ-twisted endomorphims of finitely generated free $\mathbb{Z}\pi_1(X)$-modules. We set*

$$\begin{aligned} U(X,f) &:= K_0(\mathcal{D}_{X,f}) \\ u(f) &:= [C_*(\widetilde{f})] = \sum (-1)^i [C_i(\widetilde{f})] \in K_0(\mathcal{D}_{X,f}). \end{aligned}$$

The assignment U extends to a functor on the category of endomorphisms of finite CW-complexes, and for every endomorphism $f\colon X \to X$ the function u picks an element in the corresponding abelian group. They satisfy homotopy invariance (U is a homotopy invariant functor, and u is compatible with homotopy equivalences) and additivity. Thus the pair (U, u) is a functorial Lefschetz invariant [4, Definition 2.3].

Theorem 1. [4, Theorem 2.5] *The pair (U, u) is the universal functorial Lefschetz invariant for endomorphisms of finite CW-complexes.*

We extend this result to the equivariant setting. For discrete groups G, we consider G-equivariant endomorphisms of finite proper G-CW-complexes X. (These are defined like CW-complexes, cells being of the form $G/H \times D^i$ for finite subgroups $H \leq G$.)

We need a generalisation of the fundamental group to G-spaces, and the right notion is the fundamental category $\Pi(G, X)$ [3, Definition 8.15]. It is a mixture of the fundamental groupoid of X and the orbit category of G.

A $\mathbb{Z}\Pi(G, X)$-module is defined to be a functor $\Pi(G, X) \to \mathcal{A}b$, and morphisms are defined to be natural transformations. (Note the analogy: If we view the group $\pi_1(X)$ as a category with one object, then a $\mathbb{Z}\pi_1(X)$-module becomes a functor $\pi_1(X) \to \mathcal{A}b$.)

There is a functor $\widetilde{X}\colon \Pi(G, X) \to \mathcal{T}op$ which generalises the notion of universal covering space to G-spaces [3, Definition 8.22]. It incorporates information about all fixed point sets X^H, their components and the respective universal coverings. We also have an endomorphism $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$. The equivariant endomorphism $f\colon X \to X$ induces $\phi\colon \Pi(G, X) \to \Pi(G, X)$, and on the cellular chain complex we obtain a ϕ-twisted endomorphism of $\mathbb{Z}\Pi(G, X)$-modules

$$C_*(\widetilde{f})\colon C_*(\widetilde{X}) \to C_*(\widetilde{X}).$$

Definition 2. *Let $\mathcal{E}_{G,X,f}$ be the additive category consisting of all ϕ-twisted endomorphims of finitely generated free $\mathbb{Z}\Pi(G, X)$-modules. We set*

$$\begin{aligned} U_G(X, f) &:= K_0(\mathcal{E}_{G,X,f}) \\ u_G(f) &:= \big[C_*(\widetilde{f})\big] = \sum (-1)^i \big[C_i(\widetilde{f})\big] \in K_0(\mathcal{E}_{G,X,f}). \end{aligned}$$

The pair (U_G, u_G) satisfies G-homotopy invariance and additivity for diagrams of G-equivariant maps. In addition, we have an induction structure in G. We call a pair satisfying all these properties a functorial equivariant Lefschetz invariant [6, Definition 2.3]. The pair constructed here has a universal initial property among all functorial equivariant Lefschetz invariants.

Theorem 2. [6, Theorem 0.1] *The pair (U_G, u_G) is the universal functorial equivariant Lefschetz invariant for equivariant endomorphisms of finite proper G-CW-complexes, for discrete groups G.*

2. Splitting result and applications

The invariant (U_G, u_G) contains much information, and $U_G(X, f)$ can be quite large. We see that the abelian group $U_G(X, f)$ has a direct sum decomposition into summands corresponding to the fixed point sets X^H, for subgroups $H \leq G$. This not only gives structural insight but is also helpful for concrete calculations. The information contained in $u_G(f)$ splits up into the information given by the restrictions of f to the pairs $(X^H, X^{>H})$, where $X^{>H}$ is the subset of X^H of points with larger isotropy group than H.

The decomposition is obtained from a K-theoretic splitting theorem. This is valid for all K-groups, not only for K_0. For the sake of clarity of presentation, we formulate it here under the assumption that all fixed point sets X^H are connected. Otherwise we have consider the components of the fixed point sets.

Theorem 3. [6, Theorem 4.3] *Let all fixed point sets X^H be connected. Then for all $n \in \mathbb{Z}$ we have*

$$K_n(\mathcal{E}_{G,X,f}) \cong \bigoplus_{(H)\in \mathrm{consub}(G)} K_n(\mathcal{E}^H_{G,X,f}).$$

Here (H) runs through the set of conjugacy classes of subgroups of G. The category $\mathcal{E}^H_{G,X,f}$ consists of twisted endomorphisms of $\mathbb{Z}A^H$-modules, where A^H is the group extension $1 \to \pi_1(X^H) \to A^H \to WH \to 1$, with $WH := N_G H/H$ denoting the Weyl group of H in G.

The study of Lefschetz invariants was motivated by interest in fixed points. In order to extract fixed point information, we define a trace map.

Definition 3. *There is a trace map*

$$\begin{aligned} \mathrm{tr_G}\colon \mathrm{K_0}(\mathcal{E}_{\mathrm{G,X,f}}) \quad &\to \quad \bigoplus_{(H)\in \mathrm{consub}(G)} \mathbb{Z}\pi_1(X^H)/\sim \quad =: \Lambda_G(X,f) \\ u_G(f) \quad &\mapsto \quad \lambda_G(f). \end{aligned}$$

We call the pair (Λ_G, λ_G) the generalised equivariant Lefschetz invariant.

The idea of the trace map is as follows: An element in $K_0(\mathcal{E}^H_{G,X,f})$ can be represented by a matrix B with entries in $\mathbb{Z}A^H$. We define the trace map on B by taking the sum of the $\pi_1(X^H)$-part of the diagonal elements modulo a twisted conjugacy relation, $\mathrm{tr_G(B)} := \sum \overline{\mathrm{b_{ii}}} \in \mathbb{Z}\pi_1(\mathrm{X^H})/\sim$. This trace map is a variation of the trace map from K-theory to Hochschild homology.

The claim that $\lambda_G(f)$ contains fixed point information is made precise by the refined equivariant Lefschetz fixed point theorem. It shows that the generalised equivariant Lefschetz invariant $\lambda_G(f)$ is equal to the sum of all fixed point contributions.

Theorem 4. [6, Theorem 0.2] *Let G be a discrete group, let M be a cocompact proper smooth G-manifold and let $f\colon M \to M$ be a G-equivariant endomorphism such that* $\mathrm{Fix}(f) \cap \partial M = \emptyset$ *and such that for every $x \in \mathrm{Fix}(f)$ the determinant of the map* $(\mathrm{id_{T_xM}} - \mathrm{T_xf})$ *is nonzero. Then*

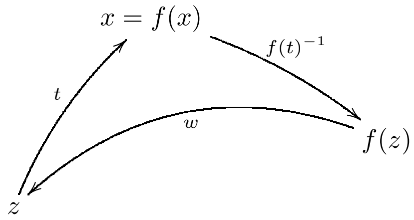
$$\lambda_G(f) = \sum_{Gx \in G\backslash \mathrm{Fix}(f)} b_x \quad \in \Lambda_G(M,f),$$

where b_x is the contribution of the fixed point orbit of x.

If all isotropy groups G_x are of odd order, then

$$b_x \quad = \quad \frac{\det\bigl(\mathrm{id_{T_xM}} - \mathrm{T_x(f)}\bigr)}{\bigl|\det\bigl(\mathrm{id_{T_xM}} - \mathrm{T_x(f)}\bigr)\bigr|} \cdot \beta_x \in \mathbb{Z}\pi_1\bigl(M^{G_x}\bigr)/\sim \quad \subseteq \Lambda_G(M,f),$$

where β_x is the equivalence class of the loop $[t * f(t)^{-1} * w] \in \pi_1(M^{G_x}, z)$. (Due to the equivalence relation, the element b_x is independent of the choices involved.)



Generalisations of the Lefschetz number can be used to obtain more precise lower bounds on the number of fixed points of an endomorphism. These statements use the Nielsen number $N(f)$, which is defined using the generalised Lefschetz invariant $\lambda(f)$.

Based on the generalised equivariant Lefschetz invariant $\lambda_G(f)$, we can introduce equivariant Nielsen invariants. These give lower bounds for the number of fixed point orbits in the G-homotopy class of f. Under mild hypotheses, these bounds are sharp. These results generalise results of Wong [9] (for compact Lie groups) to all discrete groups G.

We use them to prove the converse of the equivariant Lefschetz fixed point theorem.

Theorem 5. [7, Theorem 6.2] *Let G be a discrete group, and let M be a cocompact proper smooth G-manifold satisfying the standard gap hypotheses, i.e.,* $\dim M^H \geq 3$ *and* $\dim M^H - \dim M^{>H} \geq 2$ *for all $H \leq G$. Let $f\colon M \to M$ be a G-equivariant endomorphism. Then*

$$\lambda_G(f) = 0 \quad \Longrightarrow \quad f \text{ is } G\text{-homotopic to a fixed point free } G\text{-map.}$$

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Refined Analytic Torsion

Maxim Braverman
(joint work with Thomas Kappeler)

We construct a canonical element, called the *refined analytic torsion*, of the determinant line of the cohomology of a closed oriented odd-dimensional manifold M with coefficients in a flat complex vector bundle E. The refined analytic torsion depends holomorphically on the flat connection on E and, hence, defines a holomorphic section of the the determinant line bundle over the space of complex representations of the fundamental group of M. We compute the Ray-Singer norm of the refined analytic torsion. In particular, if there exists a flat Hermitian metric on E, we show that this norm is equal to 1. The refined analytic torsion also encodes the information about the η-invariant of the Atiyah-Patodi-Singer odd signature operator. In particular, when the bundle E is acyclic, the refined analytic torsion is a non-zero complex number, whose absolute value is equal (up to an explicit correction term) to the Ray-Singer torsion and whose phase is expressed in terms of the η-invariant. The fact that the Ray-Singer torsion and the η-invariant can be combined into one holomorphic function allows to use the methods of complex analysis to study both invariants. We present several applications of these methods. In particular, we compute the ratio of the refined analytic torsion and the Turaev refinement of the combinatorial torsion.

Definition of the refined analytic torsion. Let M be a closed oriented odd dimensional manifold. Denote by $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$ the space of n-dimensional complex representations of the fundamental group $\pi_1(M)$ of M. For $\alpha \in \mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$ we denote by E_α the flat vector bundle over M whose monodromy is equal to α. Let ∇_α be the flat connection on E_α. We defined a canonical non-zero element

$$\rho_{\mathrm{an}}(\alpha) \;=\; \rho_{\mathrm{an}}(\nabla_\alpha) \;\in\; \mathrm{Det}\big(\, H^\bullet(M,E_\alpha)\,\big)$$

of the determinant line $\mathrm{Det}\big(H^\bullet(M,E_\alpha)\big)$ of the cohomology $H^\bullet(M,E_\alpha)$ of M with coefficients in E_α. This element, called the *refined analytic torsion*, carries information about the Ray-Singer metric and about the η-invariant of the Atiyah-Patodi-Singer odd signature operator. In particular, if α is a unitary connection, then the Ray-Singer norm of $\rho_{\mathrm{an}}(\alpha)$ is equal to 1. If, in addition, the representation α is acyclic, then $\mathrm{Det}\big(H^\bullet(M,E_\alpha)\big)$ is canonically isomorphic to $\mathbb{C}$ and $\rho_{\mathrm{an}}(\alpha)$ can be viewed as a non-zero complex number. For unitary acyclic representation the absolute value of this number is equal to the Ray-Singer torsion, while its phase is equal, up to an explicitly calculated correction term, to the η-invariant.

The construction of the refined analytic torsion is based on the study of the graded determinant of the Atiyah-Patodi-Singer odd signature operator. If the representation α is not unitary, this operator is not self-adjoint. To carry out the construction of the refined analytic torsion we proved several new results about determinants of non-self-adjoint operators, which have an independent interest.

Analyticity of the refined analytic torsion. The disjoint union of the lines $\mathrm{Det}\big(H^\bullet(M,E_\alpha)\big)$, ($\alpha\in\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$), forms a line bundle

$$\mathcal{D}et\to\mathrm{Rep}(\pi_1(M),\mathbb{C}^n),$$

called the *determinant line bundle*. It admits a nowhere vanishing section, given by the Turaev torsion, and, hence, has a natural structure of a trivialisable holomorphic bundle.

We prove that $\rho_{\mathrm{an}}(\alpha)$ is a nowhere vanishing *holomorphic* section of the bundle $\mathcal{D}et$. It means that the ratio of the refined analytic and the Turaev torsions is a holomorphic function on $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$. For an acyclic representation α, the determinant line $\mathrm{Det}\big(H^\bullet(M,E_\alpha)\big)$ is canonically isomorphic to $\mathbb{C}$ and $\rho_{\mathrm{an}}(\alpha)$ can be viewed as a non-zero complex number. We show that $\rho_{\mathrm{an}}(\alpha)$ is a holomorphic function on the open set $\mathrm{Rep}_0(\pi_1(M),\mathbb{C}^n)\subset\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$ of acyclic representations.

Recently, Burghelea and Haller [5, 6] constructed another holomorphic function on the space of acyclic representations, whose absolute value is related to the Ray-Singer torsion. Their function is different from ours and is not related to the η-invariant.

Comparison with the Turaev torsion. In [10, 11], Turaev constructed a refined version of the combinatorial torsion associated to a representation α, which depends on additional combinatorial data, denoted by ϵ and called the *Euler structure*, as well as on the *cohomological orientation* of M, i.e., on the orientation $\mathfrak{o}$ of the determinant line of the cohomology $H^\bullet(M,\mathbb{R})$ of M. In [8], the Turaev torsion was redefined as a non-zero element $\rho_{\epsilon,\mathfrak{o}}(\alpha)$ of the determinant line $\mathrm{Det}\big(H^\bullet(M,E_\alpha)\big)$.

One of our main results states that, for each connected component $\mathcal{C}$ of the space $\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$, there exists a constant $\theta\in\mathbb{R}$, such that

$$\frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\epsilon,\mathfrak{o}}(\alpha)} \;=\; e^{i\theta}\cdot f_{\epsilon,\mathfrak{o}}(\alpha), \tag{1}$$

where $f_{\epsilon,\mathfrak{o}}(\alpha)$ is a holomorphic function of $\alpha\in\mathrm{Rep}(\pi_1(M),\mathbb{C}^n)$, given by an explicit local expression.

Recently, Rung-Tzung Hunag [9] showed by an explicit calculation for lens spaces that the constant θ might depend on the connected component $\mathcal{C}$. He also proved that θ is independent of the Euler structure ϵ.

Sketch of the proof of formula (1). Using the calculation of the Ray-Singer norm of the Turaev torsion, given in Theorem 10.2 of [8] and the formula for the Ray-Singer norm of the refined analytic torsion [2, Th. 11.3], we obtain that

$$\left|\frac{\rho_{\mathrm{an}}(\alpha)}{\rho_{\epsilon,\mathfrak{o}}(\alpha)}\right| \;=\; |f_{\epsilon,\mathfrak{o}}(\alpha)|. \tag{2}$$

Both, the left and the right hand side of this equality, are absolute values of holomorphic functions. If the absolute values of two holomorphic functions are equal, then the two functions are equal up to a multiplication by a locally constant function, whose absolute value is equal to one. Hence, (1) follows from (2).

Application: relation of the η-invariant with the phase of the Turaev torsion. If $\alpha \in \mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n)$ is an acyclic unitary representation, then the refined analytic torsion $\rho_{\mathrm{an}}(\alpha)$ is a non-zero complex number, whose phase is equal, up to a correction term, to the η-invariant η_α of the odd signature operator corresponding to the flat connection on E_α. Hence, if α_1 and α_2 are two acyclic unitary representations which lie in the same connected component of $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$, equality (1) allows to compute the difference $\eta_{\alpha_1} - \eta_{\alpha_2}$ in terms of the phases of the Turaev torsions $\rho_{\epsilon,\mathfrak{o}}(\alpha_1)$ and $\rho_{\epsilon,\mathfrak{o}}(\alpha_2)$. The significance of this computation is that it allows to study the spectral invariant η_α by the methods of combinatorial topology. With some additional assumptions on the manifold M and on the representations α_1 and α_2 a similar result was established in [7] by a completely different method.

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T-duality and Generalised Geometry

Peter Bouwknegt

(joint work with J. Evslin, K. Hannabuss and V. Mathai)

Introduction

References and motivation. My talk [1] was mostly based on the papers [2, 3]. I discussed global aspects of T-duality, and how it leads to various notions of generalised geometry. In this abstract I will review the case of principal circle bundles, and how this fits in with the notion of Hitchin's generalised geometry [4, 5] (see also [6] for a brief overview of generalised geometry). The analogous story for principal torus bundles is more involved [3] and requires a generalisation of generalised geometry [7].

Closed strings on $\mathbf{M} \times \mathbf{S}^1$. The spectrum of a closed bosonic string on a spacetime manifold $M \times S^1_R$ is invariant under $R \to 1/R$, where R denotes the radius of the 'compactification circle' S^1_R. In fact this, so-called, T-duality (or 'Target space duality') extends to a symmetry of the full string theory. A closed bosonic string cannot tell the difference between a compactification on a circle of radius R or a circle of radius $1/R$. Generalising this to superstring theories, T-duality can become an equivalence between different superstring theories (e.g. type IIA on S^1_R and IIB on $S^1_{1/R}$). T-duality can be generalised to manifolds which are only locally a product with a circle, i.e. principal circle bundles, to higher rank cases, i.e. principal torus bundles, and even to torus fibrations (the latter case being closely related to mirror symmetry through the Strominger-Yau-Zaslow conjecture).

The Buscher rules. While for general manifolds we cannot solve the string theory exactly, the transformation rules of the massless fields under T-duality can be derived from an effective nonlinear sigma model by a procedure called 'gauging of the isometry', which leads to a sigma model on the correspondence space of the T-duality transformation (see below). In particular this leads to a set of transformation rules between the metric $g_{\mu\nu}$ and a locally defined antisymmetric rank-2 tensor (2-form) $B_{\mu\nu}$ (whose globally defined 3-form curvature $H = dB$ is known as the H-flux), known as the Buscher rules.

Global aspects of T-duality

Principal circle bundles. Analysing the Buscher rules leads to the following global picture of T-duality. Given a pair (E, H), consisting of (an isomorphism class of) a principal circle bundle over a base manifold M, or equivalently an element $F \in H^2(M, \mathbb{Z})$, and an H-flux $H \in H^3(E, \mathbb{Z})$, the T-dual pair $(\widehat{E}, \widehat{H})$ satisfies the relations $\widehat{F} = \pi_* H,\ F = \widehat{\pi}_* \widehat{H}$

$$\begin{tikzcd} S^1 \arrow[r] & E \arrow[d, "\pi"'] \\ & M \end{tikzcd} \qquad\qquad \begin{tikzcd} S^1 \arrow[r] & \widehat{E} \arrow[d, "\widehat{\pi}"'] \\ & M \end{tikzcd}$$

and is completely determined by requiring the relation

$$p^* H - \widehat{p}^* \widehat{H} = 0\,,$$

on the correspondence space $E \times_M \widehat{E}$

$$\begin{tikzcd} & E \times_M \widehat{E} \arrow[dl, "p=1\times\hat{\pi}"'] \arrow[dr, "\hat{p}=\pi\times 1"] & \\ E \arrow[dr, "\pi"'] & & \widehat{E} \arrow[dl, "\hat{\pi}"] \\ & M & \end{tikzcd}$$

The Gysin sequence. There is a nice interpretation of the global T-duality rules above by diagram chasing through the Gysin sequence of the principal circle bundle E and the corresponding one for the T-dual $\widehat{E}$ [2]

$$\longrightarrow H^3(M,\mathbb{Z}) \xrightarrow{\pi^*} H^3(E,\mathbb{Z}) \xrightarrow{\pi_*} H^2(M,\mathbb{Z}) \xrightarrow{F\cup} H^4(M,\mathbb{Z}) \longrightarrow$$

Twisted K-theory. Physically, T-dual string theories are equivalent, and therefore should have the same D-brane spectrum. Since D-brane charges are supposedly classified by twisted K-theory this is suggestive of the following theorem

Theorem 1 ([2]). *Let (E,H) and $(\widehat{E},\widehat{H})$ be T-dual pairs of principal circle bundles and H-fluxes. There exists an isomorphism between the twisted K-theories (and similarly for the twisted cohomologies) of E and $\widehat{E}$, i.e. $K^i(E,H) \cong K^{i+1}(\widehat{E},\widehat{H})$.*

An interesting example, worked out in [8], is $\mathbb{RP}^3 \times \mathbb{RP}^7$ (considered as an $S^1 \times S^1$ principal bundle over $\mathbb{CP}^1 \times \mathbb{CP}^3$) with torsion flux of order two (recall $H^3(\mathbb{RP}^3 \times \mathbb{RP}^7,\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$), which turns out to be T-dual (by performing two consecutive S^1 T-dualities) to $S^3 \times S^7$ without flux.

Generalisations of geometry

Hitchin's Generalised geometry. We will now show how T-duality, as discussed above, fits in with (Hitchin's) generalised geometry. The idea behind generalised geometry is to replace structures defined on the tangent bundle TE by structures on $TE \oplus T^*E$. In particular we have

- A bilinear form on sections $(X,\Xi) \in \Gamma(TE \oplus T^*E)$:

$$\langle (X_1,\Xi_1),(X_2,\Xi_2)\rangle = \tfrac{1}{2}(\imath_{X_1}\Xi_2 + \imath_{X_2}\Xi_1)$$

- A (twisted) Courant bracket:

$$\begin{aligned}[(X_1,\Xi_1),(X_2,\Xi_2)]_H =& ([X_1,X_2], \mathcal{L}_{X_1}\Xi_2 - \mathcal{L}_{X_2}\Xi_1 \\ & - \tfrac{1}{2} d\,(\imath_{X_1}\Xi_2 - \imath_{X_2}\Xi_1) + \imath_{X_1}\imath_{X_2}H)\end{aligned}$$

- A Clifford algebra: $\{\gamma_{(X_1,\Xi_1)},\gamma_{(X_2,\Xi_2)}\} = 2\langle (X_1,\Xi_1),(X_2,\Xi_2)\rangle$
- A Clifford module $\Omega^\bullet(E)$: $\gamma_{(X,\Xi)} \cdot \Omega = \imath_X\Omega + \Xi \wedge \Omega$
- A (twisted) differential on $\Omega^\bullet(E)$: $d_H\Omega = d\Omega + H \wedge \Omega$

If we choose a connection A on the principal circle bundle $\pi : E \to M$, and 'dimensionally reduce' $\Omega \in \Omega^k(E)_{S^1}$ and $(X,\Xi) \in \Gamma(TE \oplus T^*E)_{S^1}$ as in

$$\Omega = \Omega_{(k)} + A \wedge \Omega_{(k-1)}\,, \qquad X = x + f\partial_A\,, \qquad \Xi = \xi + gA\,,$$

where $\Omega_{(k)},\Omega_{(k-1)} \in \Omega^\bullet(M)$, $x \in \Gamma(TM)$, $\xi \in \Omega^1(M)$, $f,g \in C^\infty(M)$, and similarly for $\widehat{E}$, then we have isomorphisms

$$\begin{aligned}&\tau : \Omega^\bullet(E)_{S^1} \to \Omega^\bullet(\widehat{E})_{S^1}\,, && \tau(\Omega_{(k)} + A \wedge \Omega_{(k-1)}) = -\Omega_{(k-1)} + \widehat{A} \wedge \Omega_{(k)} \\ &\phi : \Gamma(TE \oplus T^*E)_{S^1} \to \Gamma(T\widehat{E} \oplus T^*\widehat{E})_{S^1}\,, && \phi(x,f;\xi,g) = (x,g;\xi,f)\end{aligned}$$

Theorem 2 ([5]). *We have*

(a) *The map τ induces a chain map on the differential complexes*

$$(\Omega^\bullet(E)_{S^1}, d_H) \to (\Omega^\bullet(\widehat{E})_{S^1}, d_{\widehat{H}})$$

, i.e. $\tau \circ d_H = -d_{\widehat{H}} \circ \tau$, and hence an isomorphism on twisted cohomology.

(b) *The map ϕ is orthogonal with respect to the pairing on $TE \oplus T^*E$, hence induces an isomorphism on the Clifford algebras.*

(c) *For $v \in \Gamma(TE \oplus T^*E)_{S^1}$ we have $\tau(\gamma_v \cdot \Omega) = \gamma_{\phi(v)} \cdot \tau(\Omega)$, hence τ induces an isomorphism of Clifford modules $\tau : \Omega^\bullet(E)_{S^1} \to \Omega^\bullet(\widehat{E})_{S^1}$*

(d) *ϕ preserves the twisted Courant bracket.*

The theorem shows that T-duality preserves the important structures present in generalised geometry (such as generalised complex structures, generalised Kähler structures, etc) and is therefore a convenient framework for T-duality.

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The Farrell-Jones Conjecture for Algebraic K-Theory for Word-Hyperbolic Groups

Wolfgang Lück
(joint work with Arthur Bartels, Holger Reich)

This is a joint project with Arthur Bartels and Holger Reich. Our main result is

Theorem 1. *Let R be an (associative) ring (with unit). Let G be a word-hyperbolic group. Then the Farrell-Jones Conjecture for algebraic K-theory with coefficients in R is true for G, i.e. the assembly map*

$$H_n(E_{\mathcal{V}Cyc}(G);\mathbf{K}_R) \xrightarrow{\cong} K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

The Farrell-Jones Conjecture was formulated by Farrell-Jones in [2]. For a survey about the Farrell-Jones Conjecture and the Baum-Connes Conjecture and their status we refer for instance to [4]. The special case, where G is the fundamental group of a closed Riemannian manifold with strictly negative sectional curvature is already treated in [1].

Theorem 2. *Let $\mathcal{F}$ be the family of subgroups for which the Farrell-Jones Conjecture for algebraic K-theory with coefficients with arbitrary rings R as coefficients is true for G. Then:*

(1) If $G \in \mathcal{F}$ and $H \subseteq G$ is a subgroup of G, then $G \in \mathcal{F}$;
(2) If G_1 and G_2 belong to $\mathcal{F}$, then $G_1 \times G_2$ belongs to $\mathcal{F}$;
(3) Word hyperbolic belong to $\mathcal{F}$;
(4) Nilpotent groups belong to $\mathcal{F}$;
(5) Let $\{G_i \mid i \in I\}$ be a directed system of groups $G_i \in \mathcal{F}$. (We do not require the structure maps $G_i \to G_j$ to be injective.) Then $\operatorname{colim}_{i\in I} G_i$ belongs to $\mathcal{F}$;
(6) Suppose that R is regular with $\mathbb{Q} \subseteq R$. Let $1 \to H \to G \to Q \to 1$ be an extension of groups. Suppose that K is either a word hyperbolic group or an elementary amenable group and that the same is true for Q. Then Farrell-Jones Conjecture for algebraic K-theory with coefficients in R is true for G.

We mention some consequences of the Farrell-Jones Conjecture.

- Let R be a principal ideal domain and $G \in \mathcal{F}$. Then the reduced projective class group $\widetilde{K}_0(RG)$, the Whitehead group $\mathrm{Wh}(G,R)$ and $K_n(RG)$ for $n \leq -1$ all vanish if G is torsionfree.
- Suppose that $G \in \mathcal{F}$. Then the following version of the Bass Conjecture is true:

  Let F be a field of characteristic zero and let G be a group. The Hattori-Stallings homomorphism induces an isomorphism

$$K_0(FG) \otimes_{\mathbb{Z}} F \to \mathrm{class}_F(G)_f,$$

where $\text{class}_F(G)_f$ consists of functions $f\colon G \to F$ which vanish on elements of infinite order and are constant on F-conjugacy classes of elements of finite order.

- Suppose that $G \in \mathcal{F}$. Then the following version of the Bass Conjecture is true:

  Let R be a commutative integral domain and let G be a group. Let $g \in G$ be an element in G. Suppose that either the order $|g|$ is infinite or that the order $|g|$ is finite and not invertible in R. Then for every finitely generated projective RG-module the value of its Hattori-Stallings rank $\text{HS}_{RG}(P)$ at (g) is trivial.
- Moody's induction result which he proved for virtually poly-cyclic groups in [5] holds for all groups in $\mathcal{F}$, i.e. the canonical map
$$\text{colim}_{H\subseteq G, |H|<\infty} K_0(RH) \to K_0(RG)$$
  is bijective for all regular rings R with $\mathbb{Q} \subseteq R$.
- Higson-Lafforgue-Skandalis [3] give groups G for which the Baum-Connes Conjecture with coefficients is not true. However, as a consequence of our result these groups G belong to $\mathcal{F}$.

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Problem Session

Notes taken by Sebastian Goette and Thomas Schick

- **Bruce Williams**: Let $F \to E \xrightarrow{p} B$ be a smooth fibre bundle with n-dimensional closed fibre. The bundle projection p induces a transfer map $p^!\colon L_i(\mathbb{Z}\pi_1 B) \to L_{i+n}(\mathbb{Z}\pi_1 E)$ in quadratic L-theory (the corresponding map doesn't exist in symmetric L-theory). There are natural maps from quadratic L-theory to the (topological) K-theory of the (reduced) group C^*-algebra, so we get a diagram
$$\begin{array}{ccc} L_i(\mathbb{Z}\pi_1 B) & \longrightarrow & K_i(C^*_r\pi_1 B) \\ \big\downarrow{\scriptstyle p^!} & & \\ L_{i+n}(\mathbb{Z}\pi_1 E) & \longrightarrow & K_{i+n}(C^*_r\pi_1 E). \end{array}$$

The question now is whether a corresponding transfer map in K-theory exists which makes the diagram commutative?

One can look at a corresponding transfer map in topological K-homology, the left hand side of the Baum-Connes assembly map. It is obtained by Kasparov product with the class of the fibrewise signature operator in $KK(C(E), C(B))$. Is there a corresponding transfer on the right hand side that makes the following diagram commute? If the Baum-Connes conjecture is true, this implies the existence of the asked transfer map for C^*-algebra K-theory.

$$\begin{tikzcd}
K_\bullet(E) \arrow[r] & K^{top}(C_r\pi_1 E) \\
K_\bullet(B) \arrow[r] \arrow[u, "\mathrm{sig}"] & K^{top}(C_r\pi_1 B) \arrow[u, "?"]
\end{tikzcd} .$$

Wolfgang Lück suggests to regard pairings with Witt groups.

- **Wolfgang Lück**: Let G be a loop group. Applying standard spectral sequence calculations in equivariant K-theory, calculations of Kitchloo show that in $K_n^G(\underline{E}G)$ the positive energy representations of G should play an important role. Unfortunately, there is no theory of equivariant K-theory for such groups (because they are not locally compact). Ongoing work of Lück-Joachim should provide a satisfactionary definition.

  The question now is, whether there is a corresponding right hand side "$K_n(C_r^* G)$" of the Baum-Connes conjecture for such groups, i.e. a suitable reduced C^*-algebra (or possibly l^1-algebra or other associated Banach algebras). This right hand side again should be related to (positive energy) representations of loop groups (or Kac-Moody groups). Once the right hand side is defined, an assembly map should be constructed and shown to be an isomorphism.

  Ralf Meyer suggests to study, instead of a hard to define C^*-algebra, the C^*-category of (unitary) representations. Mike Hopkins suggests to replace $L^2(G)$ by the right hand side of a Peter-Weyl formula. He also points out that positive energy representations should play a role in the definition of the wanted C^*-algebra of G, since they are also present in the definition of $K_n^G(\underline{E}G)$. Moreover, he suggests to look at C^*-algebras of 2-groups or 2-groupoids.

- **Ulrich Bunke**: Let X be a smooth manifold. Then there exists a lift of the Chern character to smooth versions of K-theory as defined e.g. by Bunke-Schick or by Hopkins-Singer, and with values in Deligne cohomology or equivalently, in Cheeger-Simons differential characters:

$$\begin{tikzcd} \hat{K}^\bullet(X) \arrow[r, "\widehat{\mathrm{ch}}"] \arrow[d, equal] & \hat{H}^\bullet(X,\mathbb{Q}) \arrow[d] \\ \hat{K}^\bullet(X) \arrow[r, "(\mathrm{ch}_{\mathrm{top}},\mathrm{ch}_{\mathrm{dR}})"] & H^\bullet(X,\mathbb{Q}) \times \Omega^\bullet(X)\,. \end{tikzcd}$$

  Question: Can Chern class be lifted in the same way, but with values in integral Deligne cohomology $\hat{H}^\bullet(X,\mathbb{Z})$?

  Mike Hopkins explains that he expects that this can be solved, if necessary by adding additional data in the definition of smooth K-theory cycles.

- **Thomas Schick**: If one twists the signature operator with a vector bundle with sufficiently small curvature, then the index is a homotopy invariant. Is the kernel a homotopy invariant as well?

  More precisely, fix smooth Riemannian closed manifolds (M,g) and (M',g') and a homotopy equivalence $f\colon M' \to M$. Is there a constant $\delta > 0$ (depending on the data which we just fixed) such that for every bundle (E,∇) with curvature (in sup-norm) bounded by δ the kernels of D_E and D'_{f^*E} are isomorphic, where D_E is the signature operator on M twisted by (E,∇)?

  Goette, Braverman express there expectation that this is quite unlikely.

- **Maxim Braverman**: Given a flat bundle $E \to M$ and an analytic map $f\colon \mathrm{Flat}(E) \to \mathbb{C}$ from the set of flat connections, which is gauge equivariant. As example regard regularised determinants of Dirac operators twisted by flat bundles. Does this induce an analytic map from the variety R of representations of $\pi_1(M)$ to $\mathbb{C}$?

  Problem: every representation yields a flat bundle; in the neighborhood of a given representation all these bundles are isomorphic (with non-canonical isomorphism). This way one can construct a map from R to $\mathrm{Flat}(E)$ that is natural only modulo gauge transformations. But such a map has not yet been constructed analytically.

- **Mike Hopkins and Kenji Fukaya**: If M is a closed 3-manifold, can one find a Lagrangian embedding of M into $\mathbb{C}^3$? It is known that, if M is hyperbolic, then after connected sum with sufficiently many copies of $S^1 \times S^2$ such an embedding exists; but it is complete open how many factors are needed. It is known that M can not be embedded, but what about $M\#M$?

- **Kenji Fukaya**: Find a set of axioms that characterise Gromov-Witten invariants uniquely; of course as "small" as possible. Note that the axioms by Kontsevich-Manin are not sufficient. A possible application is the following: If M is Kähler, use an embedding of M into $\mathbb{C}P^n$ (or other N), and then ideas like in the Atiyah-Singer proof of the index theorem to compute relative Gromov-Witten invariants. Problem: one does not just want to compute a number, but products in cohomology; even the target groups vary too much. Another problem: not all classes in M come from the ambient space N.

- **Mike Hopkins**: Freed-Hopkins-Teleman show that if G is a compact Lie group, then the positive energy representations of the loop group of G at level τ are isomorphic to the twisted equivariant K-theory $K_G^{\tau+h}(G)$, where h is the dual Coxeter number. If G is simple, simply connected, simply laced then these representations are also isomorphic to the representations of the quantum group G_κ, with $\kappa = \exp(2\pi i/(\tau + h))$.

  The problem now is to find a direct relation between the representations of the quantum group and the twisted equivariant K-theory. This is interesting and should be possible, because the Coxeter number h mysteriously shows up on both sides.

Reporter: Ansgar Schneider

Participants

Sara Azzali
Dipartimento di Matematica
Universita di Roma "La Sapienza"
Istituto "Guido Castelnuovo"
Piazzale Aldo Moro, 2
I-00185 Roma

Prof. Dr. Bernard Badzioch
Department of Mathematics
State University of New York at
Buffalo
Buffalo NY 14260-2900
USA

Prof. Dr. Christian Bär
Institut für Mathematik
Universität Potsdam
Postfach 601553
14415 Potsdam

Prof. Dr. Alain Berthomieu
CNRS UMR 5580
Laboratoire Emile Picard
Universite Paul Sabatier
F-31062 Toulouse

Prof. Dr. Jean-Michel Bismut
Mathematiques
Universite Paris Sud (Paris XI)
Centre d'Orsay, Batiment 425
F-91405 Orsay Cedex

Prof. Dr. Peter G. Bouwknegt
Mathematical Sciences Institute
Australian National University
Canberra ACT 0200
Australia

Prof. Dr. Maxim Braverman
Northeastern University
Dept. of Mathematics
360 Huntingon Avenue
Boston MA 02115
USA

Prof. Dr. Ulrich Bunke
Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Prof. Dr. Alan Carey
Mathematical Sciences Institute
Australian National University
Canberra ACT 0200
Australia

Prof. Dr. Xianzhe Dai
Department of Mathematics
University of California at
Santa Barbara
Santa Barbara, CA 93106
USA

Wojciech Dorabiala
Department of Mathematics
Penn State-Altoona
3000 Ivyside Drive
Altoona, PA 16601
USA

Dr. Christopher Douglas
Dept. of Mathematics
Stanford University
Stanford, CA 94305
USA

Prof. Dr. Johan L. Dupont
Matematisk Institut
Aarhus Universitetet
Ny Munkegade
DK-8000 Aarhus C

Prof. Dr. Kenji Fukaya
Dept. of Mathematics
Faculty of Science
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto 606-8502
JAPAN

Prof. Dr. Soren Galatius
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
USA

Prof. Dr. Sebastian Goette
Naturwissenschaftliche Fakultät I
Mathematik
Universität Regensburg
93040 Regensburg

Jochen Heinloth
Fachbereich Mathematik
Universität Duisburg-Essen
Campus Essen
Universitätsstr. 3
45117 Essen

Prof. Dr. Mike J. Hopkins
Department of Mathematics
Massachusetts Institute of
Technology
Cambridge, MA 02139-4307
USA

Dr. Thomas Hüttemann
Department of Mathematics
University of Leicester
University Road
GB-Leicester, LE1 7RH

Prof. Dr. Kiyoshi Igusa
Department of Mathematics
Brandeis University
Waltham, MA 02454-9110
USA

Dr. Michael Joachim
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Dr. Branislav Jurco
Fachbereich Physik
Universität München
Theresienstraße 37
80333 München

Prof. Dr. Nariya Kawazumi
Graduate School of
Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Prof. Dr. Kai Köhler
Mathematisches Institut
Heinrich-Heine-Universität
Gebäude 25.22
Universitätsstraße 1
40225 Düsseldorf

Andrew Korzeniewski
Institut Mittag-Leffler
The Royal Swedish Academy
of Sciences
Auravägen 17
S-182-60 Djursholm

Sebastian Krause
Fakultät für Mathematik
Universität Regensburg
Universitätsstr. 31
93053 Regensburg

Dr. Xiang Dong Li
Mathematiques
Universite Paris Sud (Paris XI)
Centre d'Orsay, Batiment 425
F-91405 Orsay Cedex

Clara Löh
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Wolfgang Lück
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Xiaonan Ma
Centre de Mathematiques
Ecole Polytechnique
Plateau de Palaiseau
F-91128 Palaiseau Cedex

Dr. Ralf Meyer
Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Prof. Dr. Jouko Mickelsson
Department of Theoretical Physics
Royal Institute of Technology
SCFAB
S-10691 Stockholm

Dr. Paul Mitchener
Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Prof. Dr. Jack Morava
Department of Mathematics
Johns Hopkins University
Baltimore, MD 21218-2689
USA

Prof. Dr. Ryszard Nest
Matematisk Afdeling
Kobenhavns Universitet
Universitetsparken 5
DK-2100 Kobenhavn

Prof. Dr. Paolo Piazza
Dipartimento di Matematica
Universita di Roma "La Sapienza"
Istituto "Guido Castelnuovo"
Piazzale Aldo Moro, 2
I-00185 Roma

Dr. Hessel Posthuma
Mathematisch Instituut
Radboud Universiteit Nijmegen
Toernooiveld 1
NL-6525 Nijmegen

Prof. Dr. Ken Richardson
Department of Mathematics
Texas Christian University
Box 298900
Ft Worth, TX 76129
USA

Dr. Thomas Schick
Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Jan Schlüter
c/o Sebastian Goette,NWF1-Math
Universität Regensburg
93040 Regensburg

Ansgar Schneider
Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Markus Spitzweck
Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Dr. Danny Stevenson
Dept. of Mathematics
University of California
Riverside, CA 92521-0135
USA

Dr. Charlotte Wahl
Department of Mathematics
Virginia Polytechnic Institute and
State University
Blacksburg, VA 24061-0123
USA

Dr. Nathalie Wahl
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
USA

Robert Waldmüller
Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Dr. Julia Weber
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn

Moritz Wiethaup
Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Prof. Dr. E. Bruce Williams
Dept. of Mathematics
University of Notre Dame
Mail Distribution Center
Notre Dame, IN 46556-5683
USA

Christoph Wockel
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt

Dr. Jonathan Woolf
Dept. of Mathematical Sciences
University of Liverpool
P. O. Box 147
GB-Liverpool L69 7ZL

Prof. Dr. Wei-Ping Zhang
Chern Institute of Mathematics
Nankai University
Weijn Road 94
Tianjin 300071
P.R. CHINA