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Representations of Finite Groups

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Abstract. The workshop "Representations of finite groups" was organized by A. Kleshchev (Eugene), M. Linckelmann (Aberdeen), G. Malle (Kaiserslautern) and J. Rickard (Bristol). It covered a wide variety of aspects of the representation theory of finite groups and related objects like Hecke algebras.

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Introduction by the Organisers

The meeting was organized by A. Kleshchev (Eugene), M. Linckelmann (Aberdeen), G. Malle (Kaiserslautern) and J. Rickard (Bristol). This meeting was attended by over 50 participants with broad geographic representation. It covered a wide variety of aspects of the representation theory of finite groups and related objects like Hecke algebras. This workshop was sponsored by a project of the European Union which allowed us to invite in addition to established researchers also a couple of young people working on a PhD in representation theory. In eleven longer lectures of 40 minutes each and twentytwo shorter contributions of 30 minutes each, recent progress in representation theory was presented and interesting new research directions were proposed. Besides the lectures, there was plenty of time for informal discussion between the participants, either continuing ongoing research cooperation or starting new projects.

The topics of the talks came roughly from two major areas: on the one hand side, the investigation of representation theoretic properties of general finite groups and related objects, on the other hand the determination and detailed analysis of representations of special classes of finite groups and related objects like Hecke algebras.

One of the main topics touched upon in several talks was the investigation of the various open conjectures on characters and blocks of finite groups, like Alperin's, Broué's and Dade's conjecture. A major breakthrough presented at this meeting by G. Navarro was the reduction (jointly with M. Isaacs and G. Malle) of the McKay conjecture on character degrees to a statement purely about simple groups, and the verification of this condition for certain families of finite simple groups. In the same direction, Puig announced some reduction statements for Alperin's weight conjecture.

Workshop: Representations of Finite Groups

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Abstracts

Kazhdan–Lusztig cells and the Murphy basis

Meinolf Geck

Let W be a finite or affine Weyl group and H be the associated generic Iwahori–Hecke algebra. By definition, H is equipped with a standard basis usually denoted by $\{T_w \mid w \in W\}$. In a fundamental paper, Kazhdan and Lusztig (1979) constructed a new basis $\{C_w \mid w \in W\}$ and used this, among other applications, to define representations of H endowed with canonical bases. While that construction is completely elementary, it has deep connections with the geometry of flag varieties and the representation theory of Lie algebras and groups of Lie type.

One of the main consequences of the geometric interpretation are certain "positivity properties" for which no elementary proofs have ever been found. In turn, these positivity properties allowed Lusztig to establish a number of fundamental properties of the basis $\{C_w\}$, which are concisely summarized in a list of 15 items (P1)–(P15) in Lusztig's book on "Hecke algebras with unequal parameters" (2003). For example, one application of these properties is the construction of a "canonical" isomorphism from H onto the group algebra of W (when W is finite).

My talk is a report on a recent paper of mine (with the same title, to appear in Proc. London Math. Soc.) In this paper, I give elementary, purely algebraic proofs for (P1)–(P15) in the case where W is the symmetric group Σ_n. I can also prove a tiny piece of "positivity" in this case, namely, the fact that the structure constants of Lusztig's ring J are 0 or 1. The key idea of our approach is to relate the basis $\{C_w\}$ to the basis constructed by Murphy (J. Algebra 1992, 1995). The main problem in establishing that relation is that the Murphy basis elements are not directly indexed by the elements of $W = \Sigma_n$. Indeed, those elements are written as x_{ST} where (S, T) runs over all pairs of standard λ-tableaux, for various partitions λ of n. Now, the Robinson–Schensted correspondence does associate to each element of Σ_n a pair of standard tableaux of the same shape, but this works on a purely combinatorial level; we need to relate this to basis elements of H. It turns out that the "leading matrix coefficients" introduced by myself (in Represent. Theory, 2002) provide a bridge to pass from the Kazhdan–Lusztig basis to the Murphy basis of H. While the explicit form of the base change seems to be rather complicated, our results are sufficiently fine to enable us to translate combinatorial properties of the Murphy basis into properties of the Kazhdan–Lusztig basis.

These results also give the first elementary proof of the fact that the Kazhdan–Lusztig basis of H (where $W = \Sigma_n$) is a "cellular basis" in the sense of Graham–Lehrer (Invent. Math. 1996).

We just remark that similar results have been obtained for H of type B_n in the "asymptotic case" studied by Bonnafé–Iancu. See my paper "Relative Kazhdan–Lusztig cells", available at `http://arXiv.org/math.RT/0504216`, and the references there.

Toward nonrecursive definition of Kleshchev multipartitions

Susumu Ariki
(joint work with Victor Kreiman, Shunsuke Tsuchioka)

A result by Geck, Hiss and Malle showed that classification of simple modules in Harish-Chandra series of the groups $GU_n(q), CSp_{4n}(q), SO_{2n+1}(q)$, for q odd, etc. is reduced to that of Hecke algebras of type B [3]. This motivated the study of modular representations of Hecke algebras of type B, and we have a good progress in the last decade. Let $\mathcal{H}_n(q,Q)$ be the Hecke algebras of type B defined by $(T_0 - Q)(T_0 + 1) = 0$, $(T_i - q)(T_i + 1) = 0$, for $1 \leq i < n$, and the type B braid relations. One focus of the study was the classification of simple $\mathcal{H}_n(q,Q)$-modules. In Geck-Rouquier theory, which was developed by Geck, Jacon and others, they use Lusztig's a-values and the Kazhdan-Lusztig basis. Our approach was based on Specht module theory for Hecke algebras of type B, which was given by Dipper, James and Murphy. Let $q \neq 1$ and e the multiplicative order of q. If $-Q$ is not a power of q, then simple $\mathcal{H}_n(q,Q)$-modules are labelled by pairs of e-restricted partitions, by Dipper-James' Morita equivalence theorem. More precisely, $D^{(\mu,\lambda)} \neq 0$ if and only if both λ and μ are e-restricted. When $-Q$ is a power of q, my previous result [2] applied to $\mathcal{H}_n(q,Q)$ gives a labelling of simple $\mathcal{H}_n(q,Q)$-modules. To explain this, let $\mathfrak{g}$ be the Kac-Moody Lie algebra of type $A^{(1)}_{e-1}$, and let Λ_i, for $i \in \mathbb{Z}/e\mathbb{Z}$, be the fundamental weights. We denote by $B(\Lambda_i)$ the Kashiwara crystal associated with Λ_i which is realized on the set of e-restricted partitions. Then we have the following.

Theorem 1. $D^{(\mu,\lambda)} \neq 0$ *if and only if* $\lambda \otimes \mu \in B(\Lambda_0 + \Lambda_m) \in B(\Lambda_0) \otimes B(\Lambda_m)$.

We say that (μ,λ) is a Kleshchev bipartition when $D^{(\mu,\lambda)} \neq 0$. This result gives a recursive characterization of the complete set of simple $\mathcal{H}_n(q,-q^m)$-modules $\{D^{(\mu,\lambda)} \neq 0\}$. The aim of this talk is to give a nonrecursive characterization of Kleshchev bipartitions. Recall that Littelmann realized Kashiwara crystals $B(\Lambda)$ on the set of LS paths of type Λ. One of his result [5] gives a criterion for $\pi \otimes \pi' \in B(\Lambda) \otimes B(\Lambda')$ to lie in $B(\Lambda + \Lambda')$. Hence, if we succeed in interpreting this result into the language of partitions, we obtain nonrecursive characterization of Kleshchev bipartitions. This is in fact possible, and by using very explicit operations τ, base and roof on the abacus displays of λ and μ, we can state the following

Theorem 2. $\lambda \otimes \mu \in B(\Lambda_0 + \Lambda_m)$ *if and only if* $\mathrm{roof}(\mu) \subset \tau_m(\mathrm{base}(\lambda))$.

When λ is an e-core, the theorem was proved by the first and the last authors. The proof of the general case is now in checking process. To prove the theorem, we consider the last and the initial direction of the LS path corresponding to λ and μ, respectively. We can compute them by Kashiwara's treatment of LS paths [4]. The initial direction was obtained by Kreiman, Lakshmibai, Magyar and Weyman, when they studied the Demazure crystal of $B(\Lambda_i)$. We have obtained description of the last direction in the same spirit as theirs, which allowed us to remove the restriction on λ. The relationship of the last and the initial directions with Demazure crystals is not very necessary for the proof of the theorem, but it gives us a very satisfactory picture. The relationship is also explained in [1], using various results of Kashiwara.

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Decomposition matrices for Cyclotomic Hecke algebras of type $\mathbf{G(r,p,n)}$ and Clifford theory

Nicolas Jacon

(joint work with Gwenaëlle Genet)

Let r, p, $n > 2$ be integers such that p divides r. Let $d = r/p$. In the classification of [8] of finite groups generated by complex reflections, there is a single family of groupe $G(r,p,n)$ and 34 others "exceptionnal" complex reflection groups. The group $G(r,p,n)$ is defined to be the group of $n \times n$ permutation matrices such that:

(1) there is exactly one nonzero element in each row and each column,
(2) the entries are either 0 or r^{th}-root of unity,
(3) the d^{th} power of the product of the non zero entries is 1.

Let x_1, x_2, ..., x_d and q be indeterminates and let $A := \mathbb{C}[x_1^{\pm 1}, ..., x_d^{\pm 1}, q^{\pm 1}]$ and $K := \text{Frac}(A)$. In [1] and [3], Ariki and Broué-Malle have independently defined a Hecke algebra $H(K) := H^{\mathbf{x};q}_{r,p,n}(K)$ associated to $G(r,p,n)$ (which can be seen as a deformation of the group algebra by the indeterminates) over K called *Cyclotomic Hecke algebra of type* $G(r,p,n)$. Ariki has shown that a classification of the simple modules for these algebras can be deduced from the classification of the simple modules for a Cyclotomic Hecke algebra $H'(K)$ of type $G(r,1,n)$ (also known as Ariki-Koike algebra) by using Clifford theory. Such a classification for Cyclotomic Hecke algebras of type $G(r,1,n)$ has been obtained by Ariki and Dipper-James-Mathas (see [2]). They have shown the existence of "Specht modules" $S^{\underline{\lambda}}_K$ indexed by the set Π^n_r of r-multipartitions of rank n such that:

$$\text{Irr}(H'(K)) = \left\{ S^{\underline{\lambda}}_K \mid \underline{\lambda} \in \Pi^n_r \right\}.$$

Now, $H(K)$ can be seen as a subalgebra of $H'(K)$ and we have a functor of restriction from the set of $H'(K)$-modules to the set of $H(K)$-modules. The restriction of the $H'(K)$-module $S^{\underline{\lambda}}_K$ with $\underline{\lambda} \in \Pi^n_r$ splits into a direct sum of $c(\underline{\lambda})$

simple modules for $H(K)$ which are denoted by $S(\underline{\lambda}, i)_K$, $1 \leq i \leq c(\underline{\lambda})$ (the integer $c(\underline{\lambda})$ is explicitely determined in [1] and is called the index of $\underline{\lambda}$). Hence, we have:

$$\mathrm{Irr}(H(K)) = \{S(\underline{\lambda}, i)_K \mid \underline{\lambda} \in \Pi_r^n,\ 1 \leq i \leq c(\underline{\lambda})\}.$$

Assume now that we have a specialization $\theta : A \to \mathbb{C}$. Let $H(\mathbb{C}) := H_{r,p,n}^{\theta(\mathbf{x}),\theta(q)}(\mathbb{C})$ be the specialized Cyclotomic Hecke algebra over $\mathbb{C}$. A natural problem is to find a classification for the simple modules of this algebra. In this case, Clifford theory can still be applied but the restrictions of the simple modules of the Cyclotomic Hecke algebra $H'(\mathbb{C})$ of type $G(r, 1, n)$ is much more complicated to describe (even in the case $r = p = 2$ that is when $H(\mathbb{C})$ is a Hecke algebra of type D_n). The main result of [5] gives a solution to this problem. The main idea is to use a certain set $\mathcal{B}$ of simple modules for $H'(K)$ which is in natural bijection with $\mathrm{Irr}(H'(\mathbb{C}))$. This set is defined in [7] and is parametrized by a certain set of r-multipartitions Λ_r^n, called FLOTW multipartitions and coming from the crystal graph theory from quantum groups:

$$\mathcal{B} = \left\{ S_K^{\underline{\lambda}} \mid \underline{\lambda} \in \Lambda_r^n \right\}.$$

Note that in the case of Hecke algebras of finite Weyl groups, the existence of such a set is linked with Kahzdan-Lusztig theory (see [4]). Then, using remarkable properties of $\mathcal{B}$ and properties of the decomposition maps, we can show that the simple modules appearing in the restrictions of the modules of $\mathcal{B}$ are in natural bijection with $\mathrm{Irr}(H(\mathbb{C}))$. As a consequence, the following set is in natural bijection with the set of simple $H(\mathbb{C})$-modules:

$$\mathcal{B}_p = \{S(\underline{\lambda}, i)_K \mid 1 \leq i \leq c(\underline{\lambda}),\ \underline{\lambda} \in \Lambda_r^n\}.$$

Moreover the associated decomposition matrix is unitriangular. All details can be found in [5]. Note that another approach of this problem has been recently given by Jun Hu [6] using the above results.

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Decomposition matrices for weight three blocks of symmetric groups and Iwahori–Hecke algebras

Matthew Fayers

0. Introduction

The representations of the symmetric group $\mathfrak{S}_n$ have long been associated with partitions of n. If $\mathbb{F}$ is a field of prime characteristic p, then one may naturally associate to each partition a smaller partition called the p-core, and a non-negative integer called the (p-)weight. The p-core of a partition determines the block of $\mathbb{F}\mathfrak{S}_n$ in which the corresponding Specht module S^λ lies, and it is easy to deduce that weight is also a block invariant; the weight of a block of $\mathbb{F}\mathfrak{S}_n$ gives a useful measure of the complexity of the representation theory of that block, and a common approach to the representation theory of the symmetric groups is to tackle blocks of a given (small) weight.

1. The abelian defect case

The main theorem of the talk is the following.

Theorem 1 ([2]). *Suppose $p \geqslant 5$, and that B is a block of $\mathbb{F}\mathfrak{S}_n$ of weight three. Then the decomposition numbers for B are bounded above by* 1.

This result builds on corresponding results for blocks of weight less than three; the result for weight two blocks was proved by Richards [5], together with a deep understanding of the combinatorics of the partitions of weight two. A flawed proof of Theorem 1 was published by Martin and Russell. Their technique was to use induction on n, with the initial case being the principal block of $\mathfrak{S}_{3p}$, whose decomposition numbers are known explicitly. Many of their inductive arguments rest on the analysis of Scopes's $[3:k]$*-pairs* [6]: two blocks of weight three form a $[3:k]$-pair if a certain criterion in terms of the abacus displays for those blocks is satisfied, and a great deal of representation-theoretic information passes from one block to the other; in particular, if $k \geqslant 3$, then the two blocks are Morita equivalent.

For those decomposition numbers which could not be deduced by examining $[3:k]$-pairs, Martin and Russell used ad hoc techniques, such as row and column removal theorems and the Jantzen–Schaper formula. Unfortunately, there are gaps in their proof. My proof required the addition of two new ingredients.

1.1. **New ingredient 1: Iwahori–Hecke algebras.** Let $\mathcal{H}_n$ denote the Iwahori–Hecke algebra of $\mathfrak{S}_n$ at a pth root of unity in $\mathbb{C}$. Then the representation theory of $\mathcal{H}_n$ is remarkably similar to that of $\mathbb{F}\mathfrak{S}_n$. In fact, the decomposition matrix for a block B of $\mathbb{F}\mathfrak{S}_n$ can be factorised as DJ, where D denotes the decomposition matrix for the block of $\mathcal{H}_n$ with the same p-core, and J is an 'adjustment matrix', which is known to be square and unitriangular with non-negative integer entries. James's Conjecture asserts that if the defect group of B is abelian (i.e. if the weight of B is strictly less than p), then J should be the identity matrix. So my strategy was to prove Theorem 1 in two stages:

I. prove the corresponding theorem for weight three blocks of $\mathcal{H}_n$ (where p need no longer be prime);
II. prove James's Conjecture for weight three blocks.

It should be noted that this was not quite how I originally proved the theorem; the original proof was considerably less tidy, with a less clear separation between (I) and (II). The present proof uses some of the results proved with Kai Meng Tan in the work described below.

To do (I), we use the fact that there is a recursive algorithm for computing the decomposition numbers for $\mathcal{H}_n$, the *LLT algorithm* (whose validity was proved by Ariki). In fact we do not use this algorithm explicitly, but a corollary due to James and Mathas which says that the decomposition numbers for $\mathcal{H}_n$ are unchanged under the operation of 'removing an empty runner from the abacus'. This facilitated an approach to (I) by induction on p.

1.2. **New ingredient 2: downwards induction.** The second new method was to use downwards induction on n. This is possible because there are only finitely many 'Scopes equivalence classes' of weight three blocks, and there is a 'top' class, namely the class of *Rouquier blocks*. These have been extensively studied in recent years, and their decomposition numbers are known. The advantage of working down from the Rouquier blocks is that the James–Mathas 'runner removal theorem' seems to be much more powerful, especially in tandem with other inductive arguments. The disadvantage is that it requires more careful formulation of the induction – one needs to analyse Richards's 'pyramids' which describe Scopes classes.

I did (II) by conventional induction on n, and this required an analysis of how the adjustment matrices of two blocks forming a $[3:k]$-pair are related. I was able to prove a lemma which states this quite explicitly. It was then a case of listing the cases which are not dealt with by the lemma, and using ad hoc arguments, such as inducing a simple module to a Rouquier block, where James's Conjecture is known to hold [4]. It should be noted that this is the argument that deals with the famously difficult decomposition number $[S^{(8^2,4,1)} : D^{(12,9)}]$ in characteristic 5.

2. The non-abelian defect case (joint work with Kai Meng Tan)

Using the techniques of the previous section, it was possible to compute the adjustment matrices for weight three blocks in cases not addressed by James's Conjecture, i.e. in characteristic 2 or 3 [3]. This built on my earlier work for blocks of weight two in characteristic 2 [1]. This result does not tell us anything new about the symmetric group (since all the decomposition matrices in this case are published anyway), but is really about the Hecke algebra at an eth root of unity (for arbitrary e) in a field of characteristic 2 or 3. The statement of the result is too complicated to give here, but essentially the substance is that non-zero off-diagonal entries in the adjustment matrix for a weight three block can be 'traced back' to the Rouquier blocks by induction. (The adjustment matrix for a Rouquier block of the symmetric group is given in terms of decomposition numbers for Schur algebras by Turner [7], and a corresponding result is conjectured (and easy to prove in the weight three case) for Iwahori–Hecke algebras.)

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Cohomology and presentations

Robert M. Guralnick
(joint work with William M. Kantor, Martin Kassabov and Alex Lubtozky)

We want to address the question of how large $H^j(G,M)$ can be for G a finite group and M a faithful irreducible G-module particularly for $j = 1, 2$. We shall see that for $j = 2$ this is closely related to presentations for the finite group.

We first recall some results about H^1. These results are given in [2] and [1]. See [3] and [4] for more results and references.

(1) If M is a faithful irreducible G-module, then $\dim H^1(G,M) \leq (1/2) \dim M$.
(2) If M is a faithful (not necessarily irreducible) G-module of characteristic p, $O_p(G) = 1$ and $\dim M < p - 2$, then $H^1(G,M) = 0$.

In a certain sense the two results are best possible (i.e. there exist modules for (1) where equality occurs and $p-2$ cannot be replaced by $p-1$ in (2)). However, it is still unknown whether $\dim H^1(G,M)$ is absolutely bounded for M a faithful absolutely irreducible G-module. The largest examples known are 3-dimensional and most have dimension at most 1. In this talk, we consider $H^2(G,M)$ for M a faithful irreducible module over a field of characteristic p.

Holt proved that under these hypotheses, $\dim H^2(G,M) \leq 2e_p(G) \dim M$, where $p^{e_p(G)}$ is the order of a Sylow p-subgroup of G and he conjectured that $e_p(G)$ can be replaced by a constant.

In fact, Holt's conjecture is obtained in [3] as a consequence of a result on presentations. In a subsequent paper, we in fact show:

Theorem A. $\dim H^2(G,M) \leq 23.5 \dim M$.

It is likely that 23.5 can be replaced by 1/2 (and again it is not known whether there is an absolute bound if M is absolutely irreducible). We came to this problem via considerations of presentations. Indeed, in [3], we proved that every finite simple group can be presented with two generators and C relations for a fixed $C < 500$ (with the possible exceptions of ${}^2G_2(3^{2k+1})$. Even more suprisingly (given that the conjectures had been that one needs about $\log |G|$ relations) was that the total length of the relations can be taken to be $O(\log(qn))$ for G a Chevalley group of rank n over the field of q elements (consider alternating groups to have $q = 1$).

It is well known that if a group can be presented with C relations, then $\dim H^2(G,M) \leq C \dim M$ for any FG-module for F a field. In order to handle the case of Ree groups and to improve the bounds, we consider profinite presentations. The connection here was noted by Lubotzky.

Let G be a finite group with $d(G)$ the minimal size of a generating set for G. Let F be a free profinite group on $d(G)$ generators. Write $G = F/N$. Let $\hat{r}(G)$ be the minimal number of generators needed for N as a normal closed subgroup of F (it turns out this depends only on $d(G)$ not on the particular epimorphism from F to G). Then

$$\hat{r}(G) = \sup_p \sup_M \Big(\Big\lceil \frac{\dim H^2(G,M) - \dim H^1(G,M)}{\dim M} \Big\rceil + d(G) - \xi_M \Big),$$

where $d(G)$ is the minimum number of generators for G, p runs over all primes, M runs over all irreducible F_pG-modules, and $\xi_M = 0$ if M is the trivial module and 1 if not. In fact this quantity is precisely the number of generators of the G-relation module $N/[N,N]$. We use a mixture of cohomological techniques as well as presentations to prove Theorem A. It seems very likely that $\hat{r}(S) = 2 + c \leq 4$ where c is the rank of the Schur multiplier for the finite simple group S and indeed that $\hat{r}(H) = 2$ for every quasisimple group with trivial Schur multiplier.

We prove that one can reduce the bound for faithful irreducible modules to a similar result for simple groups.

A major open question in the area is whether one needs the same number of relations $r(G)$ for a discrete presentation for G as for a profinite presentation. Clearly $r(G) \geq \hat{r}(G)$, but there is known example where these quantities are unequal (on the other hand, the best method for getting a lower bound for the number of relations required in a presentation is to bound $\hat{r}(G)$).

In work in progress with Tiep, we are developing more representation theoretic ideas to get better bounds on H^2.

We close by noting that an example in [4] shows that for $k > 4$, there exist groups G and faithful irreducible G-modules M such that $\dim H^k(G,M)/\dim M$ is arbitrarily large. However, it is still plausible that this cannot happen for G simple.

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The McKay conjecture and two other topics on characters

Gabriel Navarro

Today, I have chosen to talk about three different topics.

I. The McKay Conjecture. (Joint work with M. Isaacs and G. Malle.)

As we all know, first it was the McKay conjecture for simple groups and the prime $p = 2$. McKay noticed that for many simple groups G, the number of odd degree irreducible characters of G, coincided with the number of odd degree irreducible characters of $\mathbf{N}_G(P)$, where $P \in \mathrm{Syl}_p(G)$. Then Isaacs proved for groups of odd order the statement known today as the McKay conjecture, that is, if $|\mathrm{Irr}_{p'}(G)|$ is the number of irreducible characters of G of degree not divisible by p, then

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|$$

for every prime p and every finite group G. Then it came Alperin-McKay, when Jon Alperin introduced blocks in the conjecture. Afterwards, it was the Alperin Weight Conjecture (AWC). Then its Knörr-Robinson formulation, and finally Dade's conjectures.

The simplest of Dade's conjectures generalized AWC. The next one (the projective version) generalized both AWC and McKay. I had the privilege of being in MSRI, Berkeley, 1990, when Dade announced his conjectures together with the fact that some refined version of those could be reduced to simple groups. Therefore, there was a hope of proving all of them by using the classification.

Our project here is more modest. After 14 years from Dade's announcement, we wanted to see how close we could get from proving McKay. So we wanted a reduction of McKay to simple groups, but a reduction of practical value: one that could be carried out by the specialists in simple groups. Specifically, we knew that groups of Lie type were going to be the key case; and this is why Gunter Malle was in our team.

We met in Valencia in June 2004 and try to state a theorem on simple groups that Malle could prove (or be able to prove in some future).

As we know there are several recent refinements of McKay: congruences of degrees mod p (Isaacs-Navarro), Galois actions (Navarro), or even local fields and Schur indices (Turull). We have not been really worried about them in this present work; although our reduction respects congruences of degrees; certainly we have ignored the Galois action. Our methods simply cannot handle that. But there is a refinement of McKay which is the first step in our reduction, and that we do not ignore. If $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$, then $\mathrm{Irr}_{p'}(G|\theta)$ denotes the set of irreducible characters of G of degree not divisible by p having θ as an irreducible constituent.

CONJECTURE. (Relative McKay). *If $L \lhd G$, $P \in \mathrm{Syl}_p(G)$ and $\lambda \in \mathrm{Irr}(L)$, then*

$$|\mathrm{Irr}_{p'}(G\,|\,\lambda)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P)L\,|\,\lambda)|.$$

Our main theorem is the following.

THEOREM A. *Let G be a finite group, and fix a prime p. If every non-abelian simple group divisible by p involved in G is "good" for p, then*

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|\,,$$

where $P \in \mathrm{Syl}_p(G)$.

The exact definition for a simple group S to be good consists of eight parts, which you can check in our paper [**1**]. Right now, Malle has checked that many simple groups of Lie type are good. He has also checked that alternating and sporadic simple groups are good. Any questions on what to expect in the near future should be addressed to him.

By using our main theorem, we can right now prove the following.

THEOREM B. *Suppose that the Sylow 2-subgroups of G are abelian. If p is any prime and $P \in \mathrm{Syl}_p(G)$, then*

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|\,.$$

II. Rational Characters. (Joint work with P. H. Tiep)

One of the important questions in character theory is to analyze fields of values of characters. Here we are focused on the rational valued characters of G, probably the most important case. Write $\mathrm{Irr}_{\mathrm{rat}}(G)$ for the set of complex irreducible characters of G with values in $\mathbb{Q}$. The relationship between the structure of G and the set $\mathrm{Irr}_{\mathrm{rat}}(G)$ is not fully understood. It is known, for instance, that G has no non-trivial rational valued irreducible characters if and only if G has odd order. This result already requires the classification of finite simple groups. On the other hand, all irreducible characters of G are rational valued if and only if all conjugacy classes of G are rational; however, the structure of such a group G is not completely determined. (After W. Feit and G. Seitz classified the non-solvable composition factors of those rational groups, J. Thompson is working on classifying its solvable composition factors.) All this indicates, in our opinion, that rationality questions are of deep nature.

We considered a natural next step to study groups with two rational characters. Minimal situations are usually important in the theory of finite groups. Our first main result was really hard: groups with two rational characters had a unique class of involutions. But the converse was not true; so we needed something even stronger. We were very happy when we found the following.

THEOREM C. *Let G be a finite group. Then $|\mathrm{Irr}_{\mathrm{rat}}(G)| = 2$ if and only G has two conjugacy classes of rational elements.*

In order to prove this theorem, many difficult natural problems arise. Perhaps one of the biggest has been to prove a conjecture by Rod Gow, which we needed to complete our work.

THEOREM D. *If G has even order, then G has a non-trivial odd degree irreducible rational character.*

We do not know of any general method to prove Theorem C but to classify all those groups, solvable and non-solvable. Right now, we are also working in the case where there are exactly three irreducible rational characters, where we believe an analogous theorem can be proved. (For four rational characters this is already false.)

Theorem D has started a series of related results: we have proved that if p is any prime dividing $|G|$, then G has a non-trivial p'-degree character with values in the cyclotomic field $\mathbb{Q}_p$ (**[3]**). We also have turned our attention to Brauer characters, and we have proved the following.

THEOREM E. *If p is odd and G has even order, then G has a non- trivial p-Brauer irreducible rational character.*

In this case $p = 2$, Theorem E is not true; but A. Turull has joined us to prove that groups that do not have non-trivial rational 2-Brauer characters have non-abelian composition factors of type $L_2(3^{2f+1})$.

As we all know, it is not true that the number of irreducible rational valued characters and the number of rational conjugacy classes of G coincides in general. There are many examples (solvable and non-solvable) illustrating this. However, we wonder why there are so many situations when there is equality. For instance, right now we do not know of any simple group S (except the Tits group!!) for which the equality

$$|\mathrm{Irr}_{\mathrm{rat}}(S)| = |\mathrm{cl}_{\mathrm{rat}}(S)|$$

fails; and we do know that equality is true for alternating groups, sporadic groups, $\mathrm{GL}(n, q)$ and several other groups. Moreover, we believe that for all these groups the Galois actions on characters and classes are permutation isomorphic. If finally, we are right about this, we believe that some general explanation might be of value.

III. Inequalities for characters in certain blocks. (Joint work with G. Malle)

If B is a p-block of a finite group G with defect group D, then we have that

$$k(B)/k_0(B) \leq k(D') \quad \text{and} \quad k(B)/l(B) \leq k(D)$$

for certain type of blocks (where $k(D)$ is the number of conjugacy classes of D, $k_0(B)$ is the number of height zero irreducible characters in B, $k(B)$ is the number of irreducible characters in B, and $l(B)$ is the number of irreducible Brauer characters in B). For instance, we have proved the above inequalities when D is normal in G or when B is a block of a symmetric group or of an exceptional group of Lie type in defining characteristic. We find interesting to speculate to what extent these inequalities are true in general. We do not know of any counterexample (yet), and we should mention that right now we think that these inequalities might be too optimistic for solvable groups.

Finally, we do notice that the second inequality implies the following: if $l(B) = 1$, is it true that $k(B) \leq k(D)$? This seems like a natural question to ask.

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On block identities and block inclusions

Jørn B. Olsson

(joint work with D. Stanton, C. Bessenrodt, G. Navarro, P.H. Tiep)

1. The Navarro-Willems conjecture

Let G be a finite group. We consider for a prime p and a p-block B_p of G the set $\mathrm{Irr}(B_p)$ of irreducible complex characters of B_p. It was conjectured by Navarro and Willems [4], that if for different primes p, q we have a *block equality* $\mathrm{Irr}(B_p) = \mathrm{Irr}(B_q)$ then $|\mathrm{Irr}(B_p)| = 1$. Thus both blocks should be of defect 0. We call such an equality *trivial*.

The Navarro-Willems conjecture holds for all blocks in solvable groups [4], for all blocks blocks in the symmetric groups [5] and their covering groups [3]. Also in [2] the conjecture was verified for *principal blocks* in all finite groups by reducing the question to simple groups.

It has however been noticed by C. Bessenrodt that the extension group $6.A_7$ of the alternating group A_7 provides a counterexample to the conjecture for non-principal blocks $(p, q = 5, 7)$. Such counterexamples are expected to be rare. It would be interesting to have more.

2. Block identities

The fact that all block identities in the symmetric groups and their covering groups are trivial follows from the classification of the nontrivial block inclusions in these groups, as described below. There is an interesting explicit description of all block identities for different primes p and q ([5], [3]).

In the case of blocks of the symmetric groups, the number of such block identities is finite and in fact equal to $\frac{1}{p+q}\binom{p+q}{q}$. The maximum n for which an equality occurs is $n = \frac{(p^2-1)(q^2-1)}{24}$. ([1],[5]).

The above results of course involve the classification of partitions which are cores for two primes simultaneously, due to the Nakayama conjecture. Actually for the classification p and q need not be prime numbers, only relatively prime positive integers. There is a unique partition κ_{pq} of the maximum number $\frac{(p^2-1)(q^2-1)}{24}$, which is a p-core and a q-core. It is still an open question whether the Young diagram of any other partition which is a p-core and a q-core is contained in that of κ_{pq}. See [5] for details.

In the case of spin blocks of the covering groups the number of block identities is again finite. In this case we have to classify partitions into distinct parts which are bar cores for p and q simultaneously. Again p and q need only be relatively prime *odd* positive integers, not necessarily primes. The total number of spin block identities is $\binom{s+t}{t}$ where $s = \frac{p-1}{2}$, $t = \frac{q-1}{2}$. In this case it can be shown that there is a maximal partition $\hat{\kappa}_{pq}$, whose Young diagram contains those of all the others.

In both cases the possibilites are descibed by paths in certain diagrams of integers. As an example the shown diagram is the so-called (7,17)-*Yin-Yang diagram.* It has been taken from [3] and shows the possible parts of all partitions into distinct parts which are bar cores for the primes 7 and 17. (In this diagram the numbers 10,3,27,20,13... below the dividing line are exactly the parts in the partition $\hat{\kappa}_{7\ 17}$.)

Let us finally mention that if $p_1, p_2, ..., p_k$ are distinct odd primes then there exists another prime number q such that $p_1 p_2 ... p_k \mid q + 1$. This shows that the group $\mathrm{GL}(3, q)$ contains a unipotent irreducible character which is of defect 0 for all the primes $p_1, p_2, ..., p_k$. Thus we may have simultaneous (trivial) block identites for arbitrarily many primes.

3. Block inclusions

More generally nontrivial *block inclusions* $\mathrm{Irr}(B_p) \subseteq \mathrm{Irr}(B_q)$ in a finite group G may be studied. We call the inclusion *trivial* if $|\mathrm{Irr}(B_p)| = 1$, i.e., if the smaller block has defect 0. Nontrivial block inclusions occur frequently, for instance if G has a selfcentralizing normal q-subgroup; then G has only one q-block and for any p-block of G of positive defect we get a nontrivial inclusion.

10	3	4	11	18	25	32	39
27	20	13	6	1	8	15	22
44	37	30	23	16	9	2	5

It is possible to classify all nontrivial block inclusions $\mathrm{Irr}(B_p) \subseteq \mathrm{Irr}(B_q)$ in the the symmetric groups and their covering groups. It can only happen when the "small" block B_p has defect 1 and the core of the block has a very special property. However the number of occurrencies is infinite.

This classification easily implies that all block identities in these groups are trivial, as mentioned above.

In the case of blocks of symmetric groups the partition κ_{pq} plays a special rôle as a kind of treshold: It is the smallest p-core which can occur as the core for a block B_p in a non-trivial block inclusion $\mathrm{Irr}(B_p) \subseteq \mathrm{Irr}(B_q)$. Thus when the block identities stop, then the non-trivial block inclusions start! A similar statement holds for spin blocks when κ_{pq} is replaced by $\hat{\kappa}_{pq}$.

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Strengthened McKay Conjecture for p-solvable groups

Alexandre Turull [1]

Let G be a finite group and let p be a prime number. Then we denote by $\mathrm{Irr}(G)$ the set of all the irreducible (complex) characters of G, and we use the notation

$$\mathrm{Irr}_{p'}(G) = \{\chi \in \mathrm{Irr}(G) : p \nmid \chi(1)\}$$

for the set of all the irreducible characters of degree not divisible by p.

The McKay Conjecture asserts that, if G is any finite group, p any prime, and N is the normalizer of a Sylow p-subgroup of G, then $|\,\mathrm{Irr}_{p'}(G)| = |\,\mathrm{Irr}_{p'}(N)|$.

The conjecture, of course, involves local information, in the sense of finite group theory. The author [5] has recently proposed a strengthening of this conjecture that places it over *local fields*, i.e. making it local in the sense of number theory as well, see below. This conjecture is known to be true for symmetric groups, general

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linear groups, special linear groups, and, in an appropriate block interpretation of the conjecture, also for all blocks of cyclic p-defect. In addition a much stronger statement than the conjecture is known [6] to be true for all solvable groups. Indeed, for them one can work over a field which is much smaller than $\mathbf{Q}_p$ the p-adic numbers, and one can actually prove nice correspondences over the intersection of $\mathbf{Q}_p$ and the field of 2 -power roots of unity over $\mathbf{Q}$.

Here, we discuss the proof of the strengthened conjecture for all finite p-solvable groups. This adds new evidence in favor of the strengthened conjecture.

The proof is independent of the Classification of Finite Simple Groups, using, instead, Dade's Classification of Endopermutation Modules. The method of proof relies also on the abstract study of Clifford Classes, i.e. a precise natural definition of what it means for two characters to have the same Clifford Theory. The relationship between Clifford Classes and the Clifford Theory of characters is discussed in [3]. If two characters have the same Clifford class over some field, then their Clifford Theory (including fields of values and Schur indices) is the same. Furthermore, we can use group theoretical reductions to calculate the Clifford classes, see [4]. In our proof of the Conjecture for all p-solvable groups, we obtain an equality between Clifford Classes over local fields for certain pairs of characters related by the Glauberman Correspondence. The proof suggests a possible framework for an attack on the general strengthened McKay Conjecture by the study of the possible Clifford Classes through the composition factors of the group.

For any prime p, we denote by $\mathbf{Q}_p$ the field of p-adic numbers. Let F be some field of characteristic zero. We denote by $\bar{F}$ the algebraic closure of F. Given a character χ of some finite group G, we denote by $F(\chi)$ the field obtained by extending F by all the values of χ on G. We denote by $\mathrm{m}_F(\chi)$ the Schur index of χ with respect to the field F. We denote by $\mathrm{m}_p(\chi)$, the local Schur index of χ, that is, the Schur index of χ with respect to $\mathbf{Q}_p$. Hence, $\mathrm{m}_p(\chi) = \mathrm{m}_{\mathbf{Q}_p}(\chi)$. Our conjecture can then be formulated as:

Conjecture A. *Let p be a prime number, let G be any finite group, and let P be a Sylow p-subgroup of G. Then there exists a bijection*

$$f : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathrm{N}_G(P))$$

satisfying all of the following conditions:

(1) *f commutes with the action of $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$, so, in particular, $\mathbf{Q}_p(\chi) = \mathbf{Q}_p(f(\chi))$ for every $\chi \in \mathrm{Irr}_{p'}(G)$.*

(2) *For every $\chi \in \mathrm{Irr}_{p'}(G)$, we have $\mathrm{m}_p(f(\chi)) = \mathrm{m}_p(\chi)$, so that $f(\chi)$ and χ have the same p-local Schur index.*

Earlier, Isaacs and Navarro [1] had proposed a strengthened McKay's conjecture which claims that there should be a bijection between $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(N)$ which preserves $\pm$ the character degree modulo p. Navarro [2] had also strengthened the McKay Conjecture to preserve some Galois action. Our proof proves these conjectures as well for p-solvable groups. In fact, we prove that, for p-solvable groups, the bijection of the conjecture can be taken to also preserve $\pm$ the degree modulo p.

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Lie superalgebras and finite groups: Geometry and Cohomology

Daniel K. Nakano
(joint work with Brian D. Boe, Jonathan R. Kujawa)

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a simple classical Lie superalgebra over the complex numbers as classified by Kac [3]. The classical Lie superalgebras are the simple Lie superalgebras whose $\mathfrak{g}_0$-component is a reductive Lie algebra. Let G_0 be the reductive algebraic group such that Lie $G_0 = \mathfrak{g}_0$.

This project entails developing a theory for Lie superalgebras much like the theory for finite groups representations in prime characteristic. We first construct "elementary abelian" subalgebras of $\mathfrak{g}$ and show that these subalgebras arise naturally by using results from invariant theory of reductive groups by Luna and Richardson [4]. In particular, if $R = \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0, \mathbb{C})$ is the relative cohomology for the Lie superalgebra $\mathfrak{g}$ relative to $\mathfrak{g}_0$ then there exists a sub Lie superalgebra $\mathfrak{e} = \mathfrak{e}_0 \oplus \mathfrak{e}_1$ such that

$$R \cong S^\bullet(\mathfrak{g}_1^*)^{G_0} \cong S^\bullet(\mathfrak{e}_1^*)^W \cong \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_0, \mathbb{C})^W$$

where W is a finite reflection group. From this isomorphism one sees that R a finitely generated algebra and in fact a polynomial algebra. Relative cohomology in general behaves very badly. There are many examples for finite groups and Lie algebras where the relative cohomology is infinitely generated.

By using the finite generation of R we develop a theory of support varieties for modules over the Lie superalgebra (cf. [2]). Given a $\mathfrak{g}$-module, one can consider the support variety $V_{\mathfrak{e},\mathfrak{e}_0}(M)$ and $V_{\mathfrak{g},\mathfrak{g}_0}(M)$. We prove that $V_{\mathfrak{e},\mathfrak{e}_0}(M)$ can be identified via a "rank variety" description as a certain subvariety of $\mathfrak{e}_1$. This description allows us to conclude that the representation theory for the superalgebra over $\mathbb{C}$ has similar features to looking at modular representations over fields of characteristic two. We also formulate several interesting conjectures relating the relative cohomology theories of $(\mathfrak{g}, \mathfrak{g}_0)$ and $(\mathfrak{e}, \mathfrak{e}_0)$. These conjectures are related to earlier invariant theory results due to Panushev [5].

One of the main goals of this project was to uncover deeper results about combinatorics of the blocks for finite-dimensional representations of the Lie superalgebra $\mathfrak{g}$. The "atypicality" of a block and a simple module (due to Kac and Serganova) are combinatorial invariants used to give a rough measure of the complications involved in the block structure. We show how the atypicality is related to our support variety constructions and give evidence that these constructions naturally allow one to extend the notion of atypicality to all $\mathfrak{g}$-modules in a functorial way.

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The Low-Dimensional Cohomology of Categories

Peter Webb

1. A Resolution of $\underline{R}$

The homology and cohomology of a small category $\mathcal{C}$ over a commutative ring R with 1 are defined as $H_*(\mathcal{C},B) = \mathrm{Tor}_*^{R\mathcal{C}}(B,\underline{R})$ and $H^*(\mathcal{C},A) = \mathrm{Ext}^*_{R\mathcal{C}}(\underline{R},A)$ where $\underline{R}$ is the constant functor, B is a right $R\mathcal{C}$-module and A is a left $R\mathcal{C}$-module, $R\mathcal{C}$ being the category algebra. For these definitions we refer to [14] and a discussion of the use of representations of categories can be found there as well as in many of the other references we give, such as [1, 2, 3, 4, 6, 9, 10, 13]. The category algebra $R\mathcal{C}$ is the free R-module with the morphisms of $\mathcal{C}$ as a basis, the multiplication being composition of morphisms when defined, and otherwise zero.

Homology and cohomology of $\mathcal{C}$ may be computed by taking a projective resolution of $\underline{R}$, and a resolution analogous to the bar resolution in group cohomology was constructed in [12]. We observe here that a resolution may also be associated to any surjection from a free category (see [11] for the definition) to $\mathcal{C}$ in the same way that in group cohomology one associates the Gruenberg resolution to a presentation of a group. Let $\mathcal{F} \to \mathcal{C}$ be a functor where $\mathcal{F}$ is a free category with the same objects as $\mathcal{C}$, and which is surjective on morphisms. This gives rise to an R-algebra homomorphism $R\mathcal{F} \to R\mathcal{C}$ whose kernel N is a 2-sided ideal in $R\mathcal{F}$. There is a left $R\mathcal{C}$-module homomorphism $R\mathcal{C} \to \underline{R}$ specified on the basis morphisms of $R\mathcal{C}$ by $\alpha \mapsto 1_{\mathrm{cod}\,\alpha}$ where $\mathrm{cod}\,\alpha$ is the codomain of α and we define the *left augmentation ideal* of $R\mathcal{C}$ to be its kernel ${}^LI\mathcal{C}$. Similarly there is a morphism of right modules $R\mathcal{C} \to \underline{R}$ whose kernel is the right augmentation ideal $I\mathcal{C}^R$. By the same arguments as those which establish the Gruenberg resolution [5, p. 34] we have the following.

Theorem 1.1. *With the above notation we have a free resolution of $R\mathcal{C}$-modules*
$$\cdots \to N^2/N^3 \to (N{\cdot}{}^LI\mathcal{F})/(N^2{\cdot}{}^LI\mathcal{F}) \to N/N^2 \to {}^LI\mathcal{F}/(N{\cdot}{}^LI\mathcal{F}) \to R\mathcal{C} \to \underline{R} \to 0.$$

A crucial fact which makes this work is that as left $R\mathcal{F}$-modules, N and ${}^{L}I\mathcal{F}$ are both free.

We obtain formulas for homology

$$H_{2n}(\mathcal{C},\underline{R}) \cong (N^n \cap I\mathcal{F}^R \cdot N^{n-1} \cdot {}^{L}I\mathcal{F})/(I\mathcal{F}^R \cdot N^n + N^n \cdot {}^{L}I\mathcal{F})$$

and

$$H_{2n+1}(\mathcal{C},\underline{R}) \cong (I\mathcal{F}^R \cdot N^n \cap N^n \cdot {}^{L}I\mathcal{F})/(N^{n+1} + I\mathcal{F}^R \cdot N^n \cdot {}^{L}I\mathcal{F})$$

which fit into a picture exactly exactly like that on [5, p. 48]. In the case of first homology the formula simplifies to give

$$H_1(\mathcal{C},\underline{R}) \cong (I\mathcal{C}^R \cap {}^{L}I\mathcal{C})/(I\mathcal{C}^R \cdot {}^{L}I\mathcal{C})$$

thereby extending the result for groups that the abelianization of the group is isomorphic to the augmentation ideal (over the integers) modulo its square.

2. Relation Modules and Projective Extensions

We will use the notion of an extension of a category which appears in Hoff [8], bearing in mind that there are also other approaches (see [1, 4, 6, 8, 14]).

Starting with any surjective functor $\mathcal{F} \to \mathcal{C}$ where $\mathcal{F}$ is a free category with the same objects as $\mathcal{C}$ we construct a relation module with similar properties to relation modules coming from presentations of groups. With categories we do not have a satisfactory notion of the kernel of the functor, and it is hard to see how to proceed by taking the abelianization of the kernel as in group theory. Instead we define the *relation module* to be the $R\mathcal{C}$-module $N/(N \cdot {}^{L}I\mathcal{F})$, and it appears in a short exact sequence of $R\mathcal{C}$-modules $0 \to N/(N \cdot {}^{L}I\mathcal{F}) \to R\mathcal{C}^d \to {}^{L}I\mathcal{C} \to 0$, which is part of the resolution we have constructed. It has the property that it occurs as the left hand term in an extension of the category $\mathcal{C}$ which is projective in a certain category of such extensions, in the manner of [5, p. 197]. To construct this extension observe that $H^2(\mathcal{C}, A) \cong \mathrm{Ext}^1_{R\mathcal{C}}({}^{L}I\mathcal{C}, A)$ for all modules A, so that the short exact sequence of $R\mathcal{C}$-modules just mentioned corresponds to a category extension of $\mathcal{C}$ by $N/(N \cdot {}^{L}I\mathcal{F})$, and this has the projective property among extensions of $\mathcal{C}$ by an $R\mathcal{C}$-module because the short exact sequence of $R\mathcal{C}$-modules is projective among extensions of ${}^{L}I\mathcal{C}$.

3. Five term exact sequences

By following the proof of the analogous result for group cohomology on [7, p. 202] we obtain the following.

Theorem 3.1. *Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be an extension of categories, let B be a right $R\mathcal{C}$-module and let A a left $R\mathcal{C}$-module. There are exact sequences*

$$H_2(\mathcal{E}, B) \to H_2(\mathcal{C}, B) \to B \otimes_{R\mathcal{C}} H_1(\mathcal{K}) \to H_1(\mathcal{E}, B) \to H_1(\mathcal{C}, B) \to 0$$

and

$$H^2(\mathcal{E}, A) \leftarrow H^2(\mathcal{C}, A) \leftarrow \mathrm{Hom}_{R\mathcal{C}}(H_1(\mathcal{K}), A) \leftarrow H^1(\mathcal{E}, A) \leftarrow H^1(\mathcal{C}, A) \leftarrow 0.$$

In this result $H_1(\mathcal{K}) = H_1(\mathcal{K}, \underline{\mathbb{Z}}) \cong H_1(|\mathcal{K}|)$ is the product of the abelianizations of the groups $K(x)$ which are the automorphism groups of the objects x of $\mathcal{K}$, and because we have an extension of categories it becomes a left $R\mathcal{C}$-module.

We may define the *Schur multiplier* of $\mathcal{C}$ to be $H_2(\mathcal{C}, \underline{\mathbb{Z}})$, and it has a property which generalizes a corresponding result for groups.

Theorem 3.2. *Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be an extension of categories and suppose that the induced homomorphism $H_1(\mathcal{E}) \to H_1(\mathcal{C})$ is an isomorphism. Then $\varinjlim H_1(\mathcal{K}) = \underline{\mathbb{Z}} \otimes_{\mathbb{Z}\mathcal{C}} H_1(\mathcal{K})$ is a homomorphic image of $H_2(\mathcal{C}, \mathbb{Z})$.*

For groups the corresponding theorem is sometimes stated for central extensions with the property that the normal subgroup is contained in the derived subgroup of the extension group, such extensions being called *stem extensions*. With categories we replace this condition by the stated hypothesis on H_1.

4. Lyndon-Hochschild-Serre Spectral Sequences

In group cohomology the five term sequences may be viewed as coming from the Lyndon-Hochschild-Serre spectral sequences, and we ask if the same is true for categories in general. This is not entirely clear, and we report on a result of Fei Xu [15].

Theorem 4.1. *Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be such that the corresponding functors between the opposite categories are an extension of the opposite category $\mathcal{C}^{\mathrm{op}}$. Let B be a right $R\mathcal{E}$-module and A a left $R\mathcal{E}$-module. There are spectral sequences whose second pages are*

$$E^2_{p,q} = H_p(\mathcal{C}, H_q(\mathcal{K}, B)) \Rightarrow H_{p+q}(\mathcal{E}, B)$$

and

$$E_2^{p,q} = H^p(\mathcal{C}, H^q(\mathcal{K}, A)) \Rightarrow H^{p+q}(\mathcal{E}, A).$$

The fact that we have an extension of $\mathcal{C}^{\mathrm{op}}$ is here used to make $H_q(\mathcal{K}, B)$ into a right $R\mathcal{C}$-module, and $H^q(\mathcal{K}, A)$ into a left $R\mathcal{C}$-module, and this is not the same dependence as the one which appeared in the five term exact sequences presented in the previous section.

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Beyond the Jantzen Region

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(joint work with Ed Cline)

1. Preliminaries. Let G be a semisimple, simply connected algebraic group defined over an algebraically closed field k of positive characteristic p. We will generally follow standard notation, some of which is indicated in the next several paragraphs.

Fix a maximal torus T and a Borel subgroup $B \supseteq T$ which we make correspond to the set $\Phi^- = -\Phi^+$ of negative roots for the root system Φ of T in G. Let $\Pi = \{\alpha_1, \cdots, \alpha_n\} \subseteq \Phi^+$ be the simple roots, and put α_0 equal to the maximal short root. Let $X = X(T)$ be the character group on T and let $X^+ = \{\lambda \in |, (\lambda, \alpha^\vee) \in \mathbb{N}, \forall \alpha \in \Pi\} \subseteq X$ be the set of dominant weights. The set X_1^+ of restricted weights consists of those $\lambda \in X^+$ satisfying $(\lambda, \alpha^\vee) < p$ for all $\alpha \in \Pi$. Let $\{\varpi_1, \cdots, \varpi_n\}$ be the fundamental dominant weights defined by $(\varpi_i, \alpha_j^\vee\rangle = \delta_{i,j}$, $i, j = 1, \cdots, n$. Put $\rho = \varpi_1 + \cdots + \varpi_n$ (the Weyl weight).

Let W be the Weyl group of G. For $\alpha \in \Phi$, $s_\alpha \in W$ denotes the associated reflection. So (W, S) is a Coxeter system if $S = \{s_\alpha \,|\, \alpha \in \Pi\}$. Let $w_0 \in W$ be the longest word. Put $W_p = W \ltimes p\mathbb{Z}\Phi$, the affine Weyl group. Then (W_p, S_p) is a Coxeter system if $S_p = S \cup \{s_{\alpha_{0,-1}}\}$. (For $\alpha \in \Phi$, $n \in \mathbb{Z}$, $s_{\alpha,n} \in W_p$ is the affine reflection defined on X by $s_{\alpha,n}(\lambda) = \lambda - (\langle \lambda, \alpha^\vee) - pn)\alpha$. It is useful to consider the extended affine Weyl group $\widetilde{W}_p := W \ltimes pX$, which is generally not a Coxeter group. Given $\lambda \in X$, $w \in \widetilde{W}_p$, put $w \cdot \lambda = w(\lambda + \rho) - \rho$.

For $\lambda \in X^+$, let $\nabla(\lambda) = H^0(\lambda) := H^0(G/B, \mathcal{L}_\lambda)$ be the costandard module of high weight λ, and $\Delta(\lambda) = H^0(-w_0\lambda)^*$ be the standard (Weyl) module. So $\operatorname{ch} \nabla(\lambda) = \operatorname{ch} \Delta(\lambda) = \chi(\lambda)$ (Weyl character formula).

Let $C^- = \{\mu \mid -p < (\mu+\rho, \alpha_i^\vee) < p, i = 0, \cdots, n\}$. Let $\lambda \in X^+$ and write $\lambda = w \cdot \lambda^-$, where $\lambda^- \in \overline{C^-}$ and w has minimal length among all elements $w' \in W_p$ s.t. $w' \cdot \lambda^- = \lambda$. Define

$$\chi_{\mathrm{KL}}(\lambda) = \sum_{y \in W_p, y \cdot \lambda^- \in X^+} (-1)^{l(w)-l(y)} P_{y,w}(-1)\chi(y \cdot \lambda^-),$$

where $P_{y,w}(q)$ is a Kazhdan-Lusztig polynomial in $q = t^2$. Let X^+_{reg} be the set of regular weights in X, i. e., those weights lying in some $w \cdot C^-$, $w \in W_p$.

Remark 1.1. Assume $p \geq h$ and $\lambda = \lambda_0 + p\lambda \in X^+_{\mathrm{reg}}$, $\lambda_0 \in X_1^+$, $\lambda_1 \in X^+$. Then

(1) (Kato's formula) $\chi_{\mathrm{KL}}(\lambda) = \chi_{\mathrm{KL}}(\lambda_0)\chi(\lambda_1)^{(1)}$.

If $\chi(\lambda) = \sum a_\mu e^\mu$, we put $\chi(\lambda)^{(1)} := \sum a_\mu e^{p\mu}$. Formula (1) is a "combinatorial Steinberg tensor product theorem," and could be proved from that result for quantum enveloping algebras. For a combinatorial proof of (1), see [5].

We say $\lambda \in X^+_{\mathrm{reg}}$ satisfies the **Lusztig character formula** (LCF) provided

(2) $$\mathrm{ch}\, L(\lambda) = \chi_{\mathrm{KL}}(\lambda).$$

The **Lusztig conjecture** assets that the LCF holds for all regular weights lying in the ideal $\Gamma_{\mathrm{Jan}} = \{\lambda \in X^+ \mid (\lambda+\rho, \alpha_o^\vee) \leq p(p-h+2)\}$ (Jantzen region). Here h is the Coxeter number of G. Kato has strengthened the conjecture slightly: He asserts that the LCF holds for all regular weights in the restricted region X_1^+. The two conjectures are the same if $p \geq 2h-3$.

Finally, $\lambda \in X^+_{\mathrm{reg}}$ satisfies the **homological LCF** (hLCF) if, for all $y \in W_p$ such that $y \cdot \lambda^- \in X^+$,

(3) $$t^{l(w)-l(y)}\overline{P}_{y,w} = \sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_G(L(w \cdot \lambda^-), \nabla(y \cdot \lambda^-))t^n.$$

2. Some homological algebra. Let G–mod be the category of finite dimensional rational G-modules, regarded as fully embedded in the bounded derived category $\mathcal{D} := D^b(G\text{–mod})$. We work with the full subcategories $\mathcal{E}^L$ and $\mathcal{E}^R$ of $D^b(G\text{–mod})$, defined in [1]. Put $\widehat{\mathcal{E}^L} = \mathcal{E}^L \oplus \mathcal{E}^L[1]$ and $\widehat{\mathcal{E}^R} = \mathcal{E}^R \oplus \mathcal{E}^R[1]$. Given $\xi \in X^+$, it is easy to see that $L(\xi) \in \widehat{\mathcal{E}^L}$ if and only if $L(\xi)[-l(\xi)] \in \mathcal{E}^L$.

Theorem 2.1. *([1], [2]) (a) For $\xi \in X^+_{\mathrm{reg}}$, $L(\xi) \in \widehat{\mathcal{E}^L} \iff$ for all $\mu \in X^+$, $\mathrm{Ext}^n_G(L(\xi), \nabla(\mu)) \neq 0 \implies n \equiv l(\xi) - l(\mu) \bmod 2$, $\forall \mu \in X^+$. More generally, for any $M \in D^b(G-mod)$, $M \in \mathcal{E}^L \iff$ for all $\mu \in X^+$,*

$$\mathrm{Hom}^n_{D^b(G\text{–mod})}(M, \nabla(\mu)) \neq 0 \implies n \equiv l(\mu) \mod 2.$$

(A dual result holds for $\mathcal{E}^R$, replacing ∇'s by Δ's.)

(b) Let $\xi \in X^+ \cap W_p \cdot 0$ and suppose there is a path $0 = \xi_0 < \xi_1 < \cdots < \xi_m = \xi$ in $X^+ \cap W_p \cdot 0$ with ξ_{i-1} adjacent to ξ_i for $0 < i \leq m$ and with each $\mathrm{Ext}^1_G(L(\xi_{i-1}), L(\xi_i)) \neq 0$. Then $L(\xi) \in \widehat{\mathcal{E}^L}$.

(c) Let Γ be a finite ideal in the poset $(W_p \cdot 0, \uparrow)$ (see [4]). The condition (b) holds for every path ending in Γ if and only the LCF holds for all $\gamma \in \Gamma$.

Here regular weights are **adjacent** if they are mirror images of each other in adjacent chambers. The importance of the categories $\widehat{\mathcal{E}^L}$ and $\widehat{\mathcal{E}^R}$ for cohomology arises from the following: For rational G-modules $M \in \widehat{\mathcal{E}^L}$, $N \in \widehat{\mathcal{E}^R}$, $\dim \mathrm{Ext}^n_G(M,N) = \sum_{\nu\in X^+}\sum_{i+j=n} \dim \mathrm{Ext}^i_G(M,\nabla(\nu)) \cdot \dim \mathrm{Ext}^j_G(\Delta(\nu),N)$. If M and N have composition factors in an ideal of dominant weights, then it is only necessary to take ν to lie in that ideal. Assuming the hypotheses of part (c), for $\lambda- w\cdot(-2\rho)$ and $\nu = y\cdot(-2\rho)$ in Γ, $t^{l(w)-l(y)}\overline{P}_{y,w} = \sum_{n=0}^\infty \dim \mathrm{Ext}^n_G(L(\lambda),\nabla(\nu))t^n$, so $\mathrm{Ext}^\bullet_G(L(\lambda),L(\mu))$ is determined for $\lambda,\mu\in\Gamma$ in terms of Kazhdan-Lusztig polynomials. For more discussion of these results, see also [1]. The paper [8] contains some applications and examples.

Theorem 2.2. *([2]) For $\lambda \in X^+$, $hKLF \iff KLF \quad \& \quad L(\lambda)\in\widehat{\mathcal{E}^L}$.*

3. Conjectures. Assume that $p > 2$ is odd, and, if G has a component of type G_2, assume $p > 3$. Let $\mathcal{A} = \mathbb{Z}[v]_\mathfrak{m} \subset \mathbb{Q}(v)$, where $\mathfrak{m} = (v-1,p)$. Let $U' = \langle E_1,\cdots,E_n,F_1,\cdots,F_n,K_1^{\pm1},\cdots,K_n^{\pm1}\rangle$ be the quantum enveloping algebra over $\mathbb{Q}(v)$. Let U be the $\mathcal{A}$-subalgebra of U' generated by the divided powers $E_i^{(N)}$, $F_i^{(N)}$, $N\geq 1$, together with the elements $K_i^{\pm1}$, $1\leq i\leq n$. Put $\mathcal{O} = \mathcal{A}/(\phi_p)$, $\phi_p = 1+v+\cdots+v^{p-1}\in\mathfrak{m}$, so $\mathcal{O}$ is a DVR with maximal ideal $\mathfrak{n}=(p)$, quotient field K and residue field $\mathbb{F}_p$. Set $\zeta := v+(\phi_p)\in\mathcal{O}$. We put $\widetilde{U}_\zeta = \mathcal{O}\otimes U$ and $U_\zeta = K\otimes U$. Put $\overline{U}_\zeta = \widetilde{U}_\zeta/p\widetilde{U}_\zeta$, and let I be the ideal in $\overline{U}_\zeta$ generated by the elements K_i-1, $1\leq i\leq n$. A result of Lusztig says that $\overline{U}_\zeta/I \cong \mathrm{hy}_0(G) = \mathbb{F}_p\otimes U_\mathbb{Z}(\mathfrak{g})$, the distribution algebra of G over $\mathbb{F}_p$.

The category U_ζ–mod of finite dimensional integrable type 1 modules for U_ζ has irreducible (resp. standard, costandard) modules $L_\zeta(\lambda)$ (resp., $\Delta_\zeta(\lambda)$, $\nabla_\zeta(\lambda)$), $\lambda\in X^+$. For $\mu\in X^+$, $\mathrm{ch}\,\Delta_\zeta(\mu) = \mathrm{ch}\,\nabla_\zeta(\mu) = \chi(\mu)$. Furthermore, subject to some possible restrictions (which we ignore), for $\lambda\in X_1^+$, the LCF holds for λ (by work of Kazhdan-Lusztig, Kashiwara-Tanisaki; see [9] for complete references).

Given $L_\zeta(\lambda)$, fix a high weight vector $v^+\in L_\zeta(\lambda)$. Then there is a unique admissible lattice $\widetilde{L}_\zeta^{\min}(\lambda)$ (resp., $\widetilde{L}_\zeta^{\max}(\lambda)$) of $L_\zeta(\lambda)$ which is minimal (resp., maximal) with respect to all admissible lattices $\widetilde{L}$ such that $\widetilde{L}\cap L_\zeta(\lambda)_\lambda = \mathcal{O}v^+$. For example, $\widetilde{L}_\zeta^{\min}(\lambda) = \widetilde{U}_\zeta\cdot v^+$.

For $\lambda\in X^+$, let $\Delta^{\mathrm{red}}(\lambda) = \widetilde{L}_\zeta^{\min}(\lambda)/\pi\widetilde{L}_\zeta^{\min}(\lambda)$ and $\nabla_{\mathrm{red}}(\lambda) = \widetilde{L}_\zeta^{\max}(\lambda)/\pi\widetilde{L}_\zeta^{\max}(\lambda)$.

Theorem 3.1. *([6]) For $\lambda\in X^+$, write $\lambda = \lambda_0+p\lambda_1$, with $\lambda_0\in X_1^+$ and $\lambda_1\in X^+$. Then $\Delta^{\mathrm{red}}(\lambda) = \Delta^{\mathrm{red}}(\lambda_0)\otimes\Delta(\lambda_1)^{(1)}$ and $\nabla_{\mathrm{red}}(\lambda) = \nabla_{\mathrm{red}}(\lambda_0)\otimes\nabla(\lambda_1)^{(1)}$.*

The following conjecture explains the title of this paper: the idea is provide a family of modules indexed by *all* dominant weights λ (at least the regular ones), agreeing with $L(\lambda)$ in the Jantzen region Γ_{Jan}, and retaining the favorable homological properties of $L(\lambda)$ in general.

Conjecture 3.2. ([2]) Assume that $p > h$. For each $\lambda\in X^+_{\mathrm{reg}}$, $\Delta^{\mathrm{red}}(\lambda)[-l(\lambda)]\in\mathcal{E}^L$ and $\nabla_{\mathrm{red}}(\lambda)[-l(\lambda)]\in\mathcal{E}^R$.

It is a nontrivial theorem in [2] that Conj. 3.2 is true whenever the (Kato version of) the Lusztig conjecture is true, even though the latter ostensibly provides homological information only in the Jantzen region Γ_{Jan}. One key to our extension beyond Γ_{Jan} is the following special case of Conj. 3.2, which we are able to prove without assuming the validity of the Lusztig conjecture.

Theorem 3.3. *([2]) Assume that $p > h$. Let $\lambda \in X^+$. Then $\Delta(\lambda)^{(1)}[-l(p\lambda)] \in \mathcal{E}^L$ and $\nabla(\lambda)^{(1)}[-l(p\lambda)] \in \mathcal{E}^R$. (Here $l(p\lambda) := l(t_{p\lambda}) = \sum_{\alpha\in\Phi^+}(\lambda, \alpha^\vee\rangle$.)*

Remark 3.4. Conj. 3.2 can be formulated as follows: For $\lambda \in X^+_{\mathrm{reg}}$, the $\mathcal{O}$-modules

$$\mathrm{Ext}^\bullet_{\widetilde{U}_\zeta}(\widetilde{L}^{\min}_\zeta(\lambda), \widetilde{\nabla}_\zeta(\mu)) \quad \& \quad \mathrm{Ext}^\bullet_{\widetilde{U}_\zeta}(\widetilde{\Delta}(\lambda), \widetilde{L}^{\max}_\zeta(\lambda))$$

are torsion free for all $\mu \in X^+_{\mathrm{reg}}$. This plays a role in the proof of the following result. Here $\mathfrak{p}$ denotes the Kostant partition function.

Corollary 3.5. *([2]) Assume $\lambda \in X^+$ lies in the root lattice and write $x \cdot (-2\rho) = p\lambda$, $x \in W_p$. Then $\Delta(\lambda)^{(1)}$ satisfies the hKLF, i. e., for any $y \in W_p$ such that $y \cdot (-2\rho) \in X^+$,*

$$t^{l(x)-l(y)}\overline{P}_{y,x} = \sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_G(\Delta(\lambda)^{(1)}, \nabla(y \cdot (-2\rho)))t^n.$$

Also, writing $\mu = y \cdot (-2\rho) = w \cdot 0 + p\xi$, $w \in W$, $\xi \in X^+$, then

$$\sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_G(\Delta(\lambda)^{(1)}, \nabla(\mu))t^n = \sum_{n=0}^{\infty}\sum_{z\in W}(-1)^{l(z)}\mathfrak{p}_{\frac{n-l(w)}{2}}(z\cdot\lambda - \xi)t^n.$$

Finally,

$$\mathrm{ch}\,\Delta(\lambda)^{(1)} = \sum_{\mu\in X^+}(-1)^{l(x)-l(y)}P_{y,x}(-1)\,\mathrm{ch}\,\Delta(\mu).$$

When Conj. 3.2 holds, the first formula in Cor. 3.5 extends from $\Delta(\lambda)^{(1)}$ to any $\Delta^{\mathrm{red}}(\lambda)$, $\lambda \in X^+_{\mathrm{reg}}$. We take this opportunity to note that, because of [3], results like the corollary have application to the cohomology of finite groups of Lie type. We will discuss this in [2].

Conjecture 3.6. ([2]) Assume that $p > h$. For $\lambda \in X^+_{\mathrm{reg}}$, write $\lambda = \lambda_0 + p\lambda_1$, with $\lambda_0 \in X^+_1$ and $\lambda_1 \in X^+$, and set $\Delta^{\mathrm{red}}(\lambda)' := L(\lambda_0) \otimes \Delta(\lambda_1)^{(1)}$ and, $\nabla_{\mathrm{red}}(\lambda)' := L(\lambda_0) \otimes \nabla(\lambda_1)^{(1)}$. Then $\Delta^{\mathrm{red}}(\lambda)'[-l(\lambda)] \in \mathcal{E}^L$ and $\nabla_{\mathrm{red}}(\lambda)'[-l(\lambda)] \in \mathcal{E}^R$.

Conjecture 3.7. ([2]) For $\lambda \in X^+_{\mathrm{reg}}$, $\Delta(\lambda)$ (resp., $\nabla(\lambda)$) has a Δ^{red}-filtration, i. e., a filtration as a G-module with sections of the form $\Delta^{\mathrm{red}}(\nu)$ (resp., $\nabla_{\mathrm{red}}(\nu)$), $\nu \in X^+_{\mathrm{reg}}$.

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The local Grothendieck group conjecture: reduction to the quasi-simple groups

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In the 2003 Oberwolfach meeting on Representations of Finite Groups, we introduced the *Local Grothendieck Group* for a block algebra kGb — as usual, k is an algebraically closed field of characteristic p — and arose the question on the possible coincidence with the ordinary Grothendieck Group of the block, as a refinement of Alperin's Conjecture. Actually, we consider the extension of the Grothendieck Group to a discrete valuation ring $\mathcal{O}$ of characteristic zero with residual k.

As a matter of fact, our definition depended on the possible existence of a k^*-*lifting* of the functor $\mathfrak{aut}_b\colon \mathfrak{ch}^*(\mathcal{F}_{b,\mathrm{sc}}) \to \mathfrak{Gr}$; here, $\mathcal{F}_b$ is the Brauer-Frobenius category of b, $\mathcal{F}_{b,\mathrm{sc}}$ is the full subcategory of $\mathcal{F}_b$ over the *selfcentralizing Brauer b-pairs,* $\mathfrak{ch}^*(\mathcal{F}_{b,\mathrm{sc}})$ is the *category of chains* of $\mathcal{F}_{b,\mathrm{sc}}$, $\mathfrak{Gr}$ is the category of (finite) groups and $\mathfrak{aut}_b$ maps any *chain* $\mathfrak{q}$ of $\mathcal{F}_{b,\mathrm{sc}}$ on its automorphism group $\mathcal{F}(\mathfrak{q})$ — the stabilizer of $\mathfrak{q}$ in the automorphism group of maximal term of the chain. Indeed, Alperin's Conjecture depends on a canonical k^*-central extension $\hat{\mathcal{F}}(\mathfrak{q})$ of $\mathcal{F}(\mathfrak{q})$, which is split for the principal block. Soon after, we obtained a proof of this existence, namely:

Theorem 1. *The functor $\mathfrak{aut}_b$ can be lifted to a functor $\widehat{\mathfrak{aut}}_b\colon \mathfrak{ch}^*(\mathcal{F}_{b,\mathrm{sc}}) \to k^*\text{-}\mathfrak{Gr}$ mapping any chain $\mathfrak{q}$ on $\hat{\mathcal{F}}(\mathfrak{q})$, in an essentially unique way.*

Here, k^*-$\mathfrak{Gr}$ denotes the category of k^*-central extensions that we call k^**-groups.* Then, denoting by $\mathcal{G}_k(\hat{F})$ the extension to $\mathcal{O}$ of the Grothendieck group of a k^*-group $\hat{F}$ — the Grothendieck group of the category of $k_*\hat{F}$-modules — we define the *local Grothendieck group* $\mathcal{LG}_k(G,b)$ of b as the *inverse limit* of the $\mathcal{O}$-modules $\mathcal{G}_k\big(\hat{\mathcal{F}}(\mathfrak{q})\big)$ when $\mathfrak{q}$ runs over the chains of $\mathcal{F}_{b,\mathrm{sc}}$.

The relationship between $\mathcal{LG}_k(G,b)$ and Alperin's Conjecture, *via* Knörr & Robinson's formulation, is given by the following result, also announced in 2003:

Theorem 2. *We have* $\mathrm{rank}_{\mathcal{O}}\big(\mathcal{LG}_k(G,b)\big) = \sum_{\mathfrak{q}}(-1)^{\ell(\mathfrak{q})}\mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k\big(\hat{\mathcal{F}}(\mathfrak{q})\big)\big)$ *where* $\mathfrak{q}$ *runs over a set of representatives for the isomorphism classes of (strict) chains of* $\mathcal{F}_{b,\mathrm{sc}}$ *and* $\ell(\mathfrak{q})$ *denotes the length of* $\mathfrak{q}$.

The proof of this result depends on the following general result where $\tilde{\mathcal{F}}_{b,\mathrm{sc}}$ denotes the *exterior quotient* of $\mathcal{F}_{b,\mathrm{sc}}$ — the quotient of $\mathcal{F}_{b,\mathrm{sc}}$ by the *inner automorphisms* of the objects — and, for any p'-group K, $\tilde{\mathcal{F}}^{(K)}_{b,\mathrm{sc}}$ denotes the category formed by the *faithful K-objects* of $\tilde{\mathcal{F}}_{b,\mathrm{sc}}$ — the objects Q of $\tilde{\mathcal{F}}_{b,\mathrm{sc}}$ endowed with an injective group homomorphism $K \to \tilde{\mathcal{F}}(Q)$ — and the obvious *K-morphisms.*

Theorem 3. *Let* $\mathfrak{m}\colon \tilde{\mathcal{F}}^{(K)}_{b,\mathrm{sc}} \to \mathcal{O}\text{-}\mathfrak{mod}$ *be a contravariant functor which maps any morphism on an isomorphism. Then,* $\mathbb{H}^n(\tilde{\mathcal{F}}^{(K)}_{b,\mathrm{sc}},\mathfrak{m}) = \{0\}$ *for any* $n \geq 1$.

Then, the stabilizer $\widetilde{\mathrm{Aut}}(G)_b$ of b in the outer automorphisms of G clearly acts on both $\mathcal{G}_k(G,b)$ and $\mathcal{LG}_k(G,b)$, and in 2003 we proposed the following conjecture:
(C) *For any block b of G there is an* $\mathcal{O}\widetilde{\mathrm{Aut}}(G)_b$*-module isomorphism*

$$\mathcal{G}_k(G,b) \cong \mathcal{LG}_k(G,b).$$

Actually, the induction arguments force to replace the finite group G by a k^*-central extension $\hat{G}$ of G; let us say that the k^*-group $\hat{G}$ is *quasi-simple* if G contains a normal simple group S with trivial centralizer. The main purpose of our talk is to provide evidence for the following statement:

Theorem 4. *If any quasi-simple k^*-group fulfills* (C) *then any k^*-group fulfills* (C).

Obviously, the tools to carry out our purpose are "reduction theorems" for both members of the conjectural isomorphism $\mathcal{G}_k(\hat{G},b) \cong \mathcal{LG}_k(\hat{G},b)$, which actually have their own interest independently of the conjecture. Passing over the "induction" from smaller blocks and the "division" by normal blocks of defect zero — responsible for the consideration of k^*-groups — which are well-known, let us summarize the remaining reduction results.

Let $\hat{H}$ be a normal k^*-subgroup of $\hat{G}$ and c a $\hat{G}$-stable block of $\hat{H}$ fulfilling $cb = b$; denote by Q_δ be a defect pointed group and by $\mathcal{H}_c$ the Brauer-Frobenius category of c. The main difficulty in finding a relationship between $\mathcal{LG}_k(\hat{H},c)$ and $\mathcal{LG}_k(\hat{G},b)$ comes from the "distance" between $\mathcal{H}_{c,\mathrm{sc}}$ and $\mathcal{F}_{b,\mathrm{sc}}$. To solve the problem, we have to consider the *nilcentralized Brauer b-pairs* — namely the Brauer b-pairs R_g such that g is a *nilpotent* block of $C_{\hat{G}}(R)$ — we have to extend Theorem 1 to the full subcategory $\mathcal{F}_{b,\mathrm{nc}}$ of $\mathcal{F}_b$ over the nilcentralized Brauer b-pairs and we have to prove that the corresponding inverse limit coincides with $\mathcal{LG}_k(\hat{G},b)$. Then, we consider the following situation:

Proposition 5. *Assume that* $\hat{H}$ *contains* $C_{\hat{G}}(Q_\delta)$. *Then, we have* $c = b$ *and any selfcentralizing Brauer* $(\hat{H},b)$*-pair* R_g *determines a unique nilcentralized Brauer* $(\hat{G},b)$*-pair* $R_{\hat{g}}$.

In this situation, the restriction from $\hat{G}$ to $\hat{H}$ defines $\mathcal{O}$-module homomorphisms

$$\mathcal{G}_k(\hat{G}, b) \longrightarrow \mathcal{G}_k(\hat{H}, b) \quad \text{and} \quad \mathcal{L}\mathcal{G}_k(\hat{G}, b) \longrightarrow \mathcal{L}\mathcal{G}_k(\hat{H}, b)$$

and, for any cyclic p'-subgroup C of $\hat{G}/\hat{H}$, we still can replace $\hat{H}$ by the converse image $\hat{H}^C$ of C in $\hat{G}$. At this point, it is handy to introduce the *residue $\mathcal{O}$-submodules* $\mathcal{R}_{\hat{H}}\mathcal{G}_k(\hat{H}^C, b)$ and $\mathcal{R}_{\hat{H}}\mathcal{L}\mathcal{G}_k(\hat{H}^C, b)$ as the intersection of the kernels of the respective restrictions to $\mathcal{G}_k(\hat{H}^D, b)$ and to $\mathcal{L}\mathcal{G}_k(\hat{H}^D, b)$ for any proper subgroup D of C. Denote by $\mathcal{C}$ the set of cyclic p'-subgroups of $\hat{G}/\hat{H}$.

Theorem 6. *Assume that $\hat{H}$ contains $C_{\hat{G}}(Q_\delta)$. Then, the restriction induces $\mathcal{O}$-module isomorphisms*

$$\mathcal{G}_k(\hat{G}, b) \cong \Big(\prod_{C\in\mathcal{C}} \mathcal{R}_{\hat{H}}\mathcal{G}_k(\hat{H}^C, b)\Big)^{\hat{G}/\hat{H}} \quad \text{and} \quad \mathcal{L}\mathcal{G}_k(\hat{G}, b) \cong \Big(\prod_{C\in\mathcal{C}} \mathcal{R}_{\hat{H}}\mathcal{L}\mathcal{G}_k(\hat{H}^C, b)\Big)^{\hat{G}/\hat{H}}.$$

Fortunately enough, in our induction argument the gap between $\hat{H}$ and $\hat{H}.C_{\hat{G}}(Q_\delta)$ is *solvable* since the quotient of these k^*-groups maps into a direct product of outer automorphisms of simple groups, and thus the next result handle the situation:

Proposition 7. *Assume that $\hat{G} = \hat{H}.C_{\hat{G}}(Q_\delta)$. If $\hat{G}/\hat{H}$ is either a p-group or a p'-group then any selfcentralizing Brauer $(\hat{H}, c)$-pair R_g determines a unique nilcentralized Brauer $(\hat{G}, b)$-pair $R_{\hat{g}}$ and the restriction induces $\mathcal{O}$-module isomorphisms $\mathcal{G}_k(\hat{G}, b) \cong \mathcal{G}_k(\hat{H}, c)$ and $\mathcal{L}\mathcal{G}_k(\hat{G}, b) \cong \mathcal{L}\mathcal{G}_k(\hat{H}, c)$.*

Consequently, in the general situation, we consider a normal k^*-subgroup $\hat{K}$ of $\hat{G}$ and a $\hat{G}$-stable block d of $\hat{K}$ fulfilling $db = b$ such that $\hat{H} \subset \hat{K} \subset \hat{H}.C_{\hat{G}}(Q_\delta)$ and that $\hat{K}/\hat{H}$ is p-solvable. If C is a cyclic p'-subgroup C of $\hat{G}/\hat{H}$, denote by $\bar{C}$ its image in $\mathcal{F}_b(Q)/\mathcal{H}_c(Q)$, by C_c the corresponding kernel, by D the image of C in $\hat{G}/\hat{K}$, and by $\bar{D}$ and D_d the corresponding items; then, given two blocks c^C of $\hat{H}^C$ and d^D of $\hat{K}^D$ fulfilling $cc^C = c^C$, $dd^D = d^D$ and $c^C d^D \neq 0$, the restriction throughout $k_*\hat{H}^C c^C \to k_*\hat{K}^D d^D$ still induces $\mathcal{O}$-module isomorphisms

$$\mathcal{R}_{\hat{K}^{D_d}}\mathcal{G}_k(\hat{K}^D, d^D) \cong \mathcal{R}\mathcal{G}_k(\bar{D}) \otimes_{\mathcal{R}\mathcal{G}_k(\bar{C})} \mathcal{R}_{\hat{H}^{C_c}}\mathcal{G}_k(\hat{H}^C, c^C)$$
$$\mathcal{R}_{\hat{K}^{D_d}}\mathcal{L}\mathcal{G}_k(\hat{K}, d^D) \cong \mathcal{R}\mathcal{G}_k(\bar{D}) \otimes_{\mathcal{R}\mathcal{G}_k(\bar{C})} \mathcal{R}_{\hat{H}^{C_c}}\mathcal{L}\mathcal{G}_k(\hat{H}^C, c^C).$$

In the last step, we have a $\hat{G}$-stable k^*-direct decomposition $\hat{H} = \hat{\prod}_{i\in I}\hat{H}_i$, where the k^*-quotient H_i of $\hat{H}_i$ is simple for any $i \in I$, and we still have $c = \otimes_{i\in I} c_i$, where c_i is a block of $\hat{H}_i$. Then, any cyclic p'-subgroup C of $\hat{G}/\hat{H}$ acts on the set I and, if $J \subset I$, setting $\hat{H}_J = \hat{\prod}_{j\in J}\hat{H}_j$ and denoting by C_J the image in $\widetilde{\mathrm{Aut}}(\hat{H}_J)$ of the stabilizer of J in C, it can be constructed extensions $(\hat{H}_J)^{C_J}$ of C_J by $\hat{H}_J$, and $(\prod_{j\in J} \hat{H}_j)^{C_J}$ of C_J by $\prod_{j\in J} \hat{H}_j$; in particular, $\hat{H}^C$ becomes a normal k^*-subgroup of $\hat{\prod}_O(\hat{H}_O)^{C_O}$, where O runs over the set of C-orbits of I, and, for such an O and some $i \in O$, we consider the *tensor induction* from $(\prod_{j\in O} \hat{H}_j)^{C_i}$ to $(\prod_{j\in O} \hat{H}_j)^{C_O}$. Finally, the following two results allow us to complete the reduction.

Theorem 8. *With the same notation, similarly consider* $\hat{G}'$, b', $\hat{H}'$, c' *and* $Q'_{\delta'}$. *Assume that* $C = \hat{G}/\hat{H} \cong \hat{G}'/\hat{H}'$ *is a cyclic* p'*-group and that* $\hat{H}$ *and* $\hat{H}'$ *respectively contain* $C_{\hat{G}}(Q_\delta)$ *and* $C_{\hat{G}'}(Q'_{\delta'})$. *Then, the restriction induces* $\mathcal{O}$*-module isomorphisms*

$$\mathcal{R}_{\hat{H}}\mathcal{G}_k(\hat{G}, b) \otimes_{\mathcal{R}\mathcal{G}_k(C)} \mathcal{R}_{\hat{H}'}\mathcal{G}_k(\hat{G}', b') \cong \mathcal{R}_{\hat{H} \hat{\times} \hat{H}'}\mathcal{G}_k(\hat{G} \hat{\times}_C \hat{G}', b \otimes b')$$
$$\mathcal{R}_{\hat{H}}\mathcal{L}\mathcal{G}_k(\hat{G}, b) \otimes_{\mathcal{R}\mathcal{G}_k(C)} \mathcal{R}_{\hat{H}'}\mathcal{L}\mathcal{G}_k(\hat{G}', b') \cong \mathcal{R}_{\hat{H} \hat{\times} \hat{H}'}\mathcal{L}\mathcal{G}_k(\hat{G} \hat{\times}_C \hat{G}', b \otimes b').$$

Theorem 9. *With the same notation, assume that* $C = \hat{G}/\hat{H}$ *is a cyclic* p'*-group, that we have a* C*-stable* k^**-direct decomposition* $\hat{H} = \hat{\prod}_{i \in I} \hat{H}_i$ *and that* C *acts transitively on* I. *Fix* $i \in I$, *denote by* D *the stabilizer of* i *in* C *and by* d *the block of* $\hat{K} = (\hat{H}_i)^D$ *associated with* b, *and consider the* $|I|$*-power homomorphism from* C *to* D. *Then, the tensor induction determines* $\mathcal{O}$*-module isomorphisms*

$$\mathcal{R}\mathcal{G}_k(C) \otimes_{\mathcal{R}\mathcal{G}_k(D)} \mathcal{R}_{\hat{H}}\mathcal{G}_k(\hat{K}, d) \cong \mathcal{R}_{\hat{H}}\mathcal{G}_k(\hat{G}, b)$$
$$\mathcal{R}\mathcal{G}_k(C) \otimes_{\mathcal{R}\mathcal{G}_k(D)} \mathcal{R}_{\hat{H}}\mathcal{L}\mathcal{G}_k(\hat{K}, d) \cong \mathcal{R}_{\hat{H}}\mathcal{L}\mathcal{G}_k(\hat{G}, b).$$

Path models and Chevalley groups

Arun Ram

Motivation 1. Langlands = McKay

Motivation 2. character of a simple module $L(\lambda) = \sum_{p \in B(\lambda)} \mathrm{wt}(p)$,

where $B(\lambda)$ is a path model of $L(\lambda)$.

Historically, the source of the path model is

$$\text{path model} = \text{crystal} = \text{quantum group at } q = 0.$$

Later it was realised that

$$\text{path model} = \text{affine Hecke algebra at } q = 0 = \text{Schubert calculus},$$

where the Schubert calculus here appears naturally as the K-theory of the flag variety. The first equality was recently extended to a q-version,

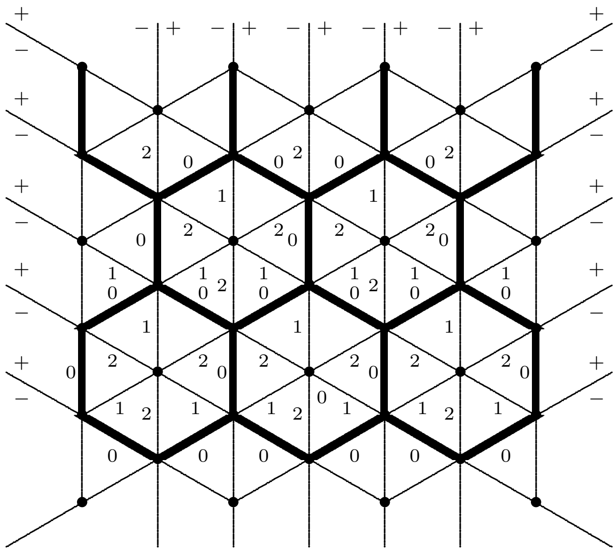
$$q\text{-path model} = \text{affine Hecke algebra}$$

which gives a formula for the Hall-Littlewood polynomial P_λ in the same form as that for the character of the simple module $L(\lambda)$. and specialises to the classical path model at $q = 0$ (see [Sc] and [Ra]). Iwahori's original construction of the affine Hecke algebra is as the double coset algebra of a p-adic group and this leads to the idea that there may be a refined version of the path model such that

$$\text{labeled path model} = p\text{-adic group}.$$

The data of a general Chevalley group is

(1)



R^+ is an index set for the hyperplanes H_α through 0

$\widetilde{W} = \{\text{alcoves}\}$	$P^\vee = \{(\text{centers of})\ \text{hexagons}\}$,
$W = \{\text{alcoves in the 0-hexagon}\}$	$(P^\vee)^+ = P^\vee \cap C$

Let $x_{ij}(f)$ with $1 \le i < j \le n$, $f \in \mathbb{F}$ be the elementary matrices in $GL_n(\mathbb{F})$ and let $h_\lambda(g) = \mathrm{diag}(g^{\lambda_1}, \ldots, g^{\lambda_n})$ for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$. Then, in the same way that $GL_n(\mathbb{F})$ is generated by

$$\begin{array}{lll} x_{ij}(f) \quad \text{and} \quad x_{ji}(f), & & \text{for } 1 \le i < j \le n,\ f \in \mathbb{F}, \text{ and} \\ h_\lambda(g), & & \text{for } \lambda \in \mathbb{Z}^n,\ g \in \mathbb{F}^*, \end{array}$$

the *Chevalley group* $G(\mathbb{F})$ is generated by

$$\begin{array}{lll} x_\alpha(f) \quad \text{and} \quad x_{-\alpha}(f), & & \text{for } \alpha \in R^+,\ f \in \mathbb{F}, \\ h_\lambda(g), & & \text{for } \lambda \in P^\vee,\ g \in \mathbb{F}^*, \end{array}$$

with relations which one can find in Steinberg's Yale lecture notes [St].

The diagram

$$\begin{array}{lllllll} \mathbb{F} & = & \mathbb{C}((t)) & & & & \\ \cup\vert & & & & & & \\ \mathfrak{o} & = & \mathbb{C}[[t]] & \longrightarrow & k & = & \mathbb{C} \end{array}$$

gives

$$\begin{array}{cccccc}
t\text{-adic group} & G & = & G(\mathbb{F}) & & \\
 & \cup| & & \cup| & & \\
\begin{array}{c}\text{maximal compact}\\ \text{subgroup}\end{array} & K & = & G(\mathfrak{o}) & \xrightarrow{\Phi} & G(k) \\
 & \cup| & & \cup| & & \cup| \\
\begin{array}{c}\text{Iwahori}\\ \text{subgroup}\end{array} & I & = & \Phi^{-1}(B) & \longrightarrow & B
\end{array}$$

where B is the subgroup of $G(k)$ generated by $x_\alpha(c)$ and $h_\lambda(d)$, for $\alpha \in R^+$, $\lambda \in P^\vee$, $c \in k$, $d \in k^*$. A row reduction argument (as in linear algebra) gives that

$$G(k) = \bigsqcup_{w \in W} BwB \qquad \begin{array}{c}\text{Bruhat}\\ \text{decomposition}\end{array}$$

$$\begin{array}{c}\text{Cartan}\\ \text{decomposition}\end{array} \qquad G = \bigsqcup_{\lambda \in (P^\vee)^+} Kt_\lambda K \qquad G = \bigsqcup_{\mu \in P^\vee} U^- t_\mu K \qquad \begin{array}{c}\text{Iwasawa}\\ \text{decomposition}\end{array}$$

$$G = \bigsqcup_{w \in \tilde{W}} IwI \qquad G = \bigsqcup_{v \in \tilde{W}} U^- vI,$$

where U^- is the subgroup of $G(\mathbb{F})$ generated by $x_{-\alpha}(f)$ for $\alpha \in R^+$, $f \in \mathbb{F}$.

$G(k)/B$ is the *flag variety*,
G/K is the *loop Grassmanian*, and
G/I is the *affine flag variety*,

The *Mirković-Vilonen cycles* of shape λ and weight μ are the elements of

$$B(\lambda)_\mu = \left\{ \text{ irreducible components of } \overline{U^- t_\mu K \cap K t_\lambda K} \right\}.$$

Theorem 1. (Geometric Langlands I, see [MV]) *Let $G^\vee$ be the Langlands dual group. Let $L(\lambda)$ be the simple $G^\vee$-module of highest weight λ. Then*

$$\dim(L(\lambda)_\mu) = \mathrm{Card}(B(\lambda)_\mu),$$

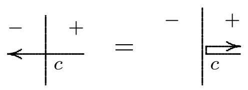
where $L(\lambda)_\mu$ is the μ-weight space of $L(\lambda)$.

Let $w \in \tilde{W}$, and let $w = s_{i_1} \cdots s_{i_\ell}$ be a minimal length walk to w. From Steinberg's Yale lecture notes, the points of G/I in the double coset IwI are given by

$$IwI = \{x_{\alpha_{i_1}}(c_1)s_{i_1} \cdots x_{\alpha_{i_\ell}}(c_\ell)s_{i_\ell} I \mid c_1, \ldots, c_\ell \in \mathbb{C}\}.$$

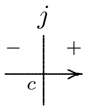
We view these points as labelings of the walk $s_{i_1} \cdots s_{i_\ell}$ to w so that the points of IwI are labeled nonfolded walks to w.

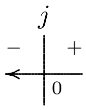
To see which of these are in $U^- vI$ use the *periodic orientation* (the signs in the picture (0.1)), and the straightening law

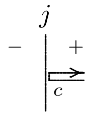


for $c \in \mathbb{C}^*$,

to rewrite $\quad x_{\beta_1}(c_1)\cdots x_{\beta_\ell}(c_\ell)wI = x_{\beta'_1}(c'_1)\cdots x_{\beta'_\ell}(c'_\ell)vI, \quad$ so that the nonfolded labeled walk is rewritten as a positively folded labeled path p of type $(i_1,\ldots,i_\ell)$ where a step of type j is

(2) 

with $c\in\mathbb{C}$, or 

or 

with $c\in\mathbb{C}^*$.

Here v is the ending alcove of p, and

$$x_{\beta'_k}(c'_k) \quad \text{is equal to} \quad x_{z\alpha_j}(c) \quad \text{or} \quad x_{-z\alpha_j}(0), \quad \text{or} \quad x_{-z\alpha_j}(c),$$

if the kth step of p is the first or the second or the third of the cases in (0.2). So

$$U^-vI\cap IwI = \left\{ x_{\beta'_1}(c'_1)\cdots x_{\beta'_\ell}(c'_\ell)vI \;\middle|\; \begin{array}{l} p \text{ is a labeled positively folded} \\ \text{walk of type } (i_1,\ldots,i_\ell) \text{ ending at } v \end{array} \right\}.$$

The original Littelmann paths are the positively folded walks (with labels ignored) which have no steps of the second type in (0.2).

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Complexes of injective kG-modules

David Benson

(joint work with Henning Krause)

I spent the autumn of 2005 in Paderborn as a guest of Henning Krause, using a Forschungspreis from the Alexander von Humboldt-Stiftung. The joint work I describe here grew out of our attempt to understand the work of Dwyer, Greenlees and Iyengar [6]. We begin with some history. Let G be a finite group, and let k be an algebraically closed field of characteristic p.

- 1990 In joint work with Jon Carlson [1], we developed a duality for the cohomology ring, expressed in terms of a spectral sequence. In particular, we showed that if $H^*(G,k)$ is Cohen–Macaulay then it is Gorenstein.
- 1995 Greenlees [8] reformulated our spectral sequence in terms of local cohomology, $H^{*,*}_{\mathfrak{m}}H^*(G,k)\Rightarrow H_*(G,k)$, where $\mathfrak{m}$ is the maximal ideal of elements of positive degree.
- 1997 In joint work with John Greenlees [3, 4], we generalised the spectral sequence to compact Lie groups and to virtual duality groups.

2000 Greenlees and Lyubeznik [9] found a way to apply Grothendieck's dual localisation to the spectral sequence, with respect to a homogeneous prime ideal in cohomology.

2002 Dwyer, Greenlees and Iyengar [6] gave a derived category formulation in $D(C^*(BG;k))$ as a statement of Gorenstein duality.

2003 In joint work with John Greenlees [5], we showed that this derived duality localises well with respect to homogeneous prime ideals in group cohomology, thereby giving a derived category interpretation to the spectral sequence of Greenlees and Lyubeznik.

We write $C\mathsf{Inj}\,kG$ for the category whose objects are complexes of injective kG-modules and whose arrows are degree preserving chain maps. Recall that projective kG-modules are the same as injective kG-modules; but for a more general ring it is more appropriate to use injective modules.

We write $K\mathsf{Inj}\,kG$ for the category with the same objects, but where the maps are homotopy classes of chain maps. This is a triangulated category, where the triangles come from the mapping cone construction. It has a well defined symmetric monoidal tensor product $-\otimes_k -$ coming from taking then total complex of the tensor product of chain complexes, and using the diagonal action of the group G.

Let $C^*(BG) = C^*(BG;k)$ denote the cochains on the classifying space. This is a differential graded algebra (DGA) which is associative, but not commutative. Instead, it is E_∞, which means that it is commutative up to all higher homotopies. Using one of the categories of spectra with a commutative and associative smash product introduced in the 1990s by for example Elmendorf, Kříž, Mandell and May [7], this allows us to make a left derived tensor product $- \overset{L}{\otimes}_{C^*(BG)} -$ on $D(C^*(BG;k))$.

Theorem 1. *If G is a finite p-group, then there is a natural equivalence of triangulated categories*

$$K\mathsf{Inj}\,(kG) \simeq D(C^*(BG))$$

that makes $-\otimes_k -$ in $K\mathsf{Inj}\,kG$ correspond to the E_∞ tensor product $- \overset{L}{\otimes}_{C^*(BG)} -$ on $D(C^*(BG))$.

If G is not a p-group then $C^*(BG)$ only "sees" the trivial kG-module k, so $D(C^*(BG))$ is equivalent to the localising subcategory of $K\mathsf{Inj}\,(kG)$ generated by an injective resolution ik of the trivial module.

We now describe the relationship between $K\mathsf{Inj}\,kG$ and the more familiar stable module category $\mathsf{StMod}\,kG$ and derived category $D(\mathsf{Mod}\,kG)$. Taking Tate resolutions gives an equivalence of categories

$$\mathsf{StMod}\,kG \simeq K_{ac}\mathsf{Inj}\,kG$$

where $K_{ac}\mathsf{Inj}\,kG$ is the full subcategory of $K\mathsf{Inj}\,kG$ consisting of the acyclic complexes of injective kG-modules. There is a recollement

$$\mathsf{StMod}\,kG \simeq K_{ac}\mathsf{Inj}\,kG \underset{\underset{-\otimes_k tk}{\longleftarrow}}{\overset{\overset{\mathsf{Hom}_k(tk,-)}{\longleftarrow}}{\longrightarrow}} K\mathsf{Inj}\,kG \underset{\underset{-\otimes_k pk}{\longleftarrow}}{\overset{\overset{\mathsf{Hom}_k(pk,-)}{\longleftarrow}}{\longrightarrow}} D(\mathsf{Mod}\,kG)$$

where pk denotes a projective resolution of k and tk denotes a Tate resolution of k as a kG-module. The compact objects in these subcategories give the more familiar sequence of categories and functors

$$\mathsf{stmod}\, kG \longleftarrow D^b(\mathsf{mod}\, kG) \longleftarrow D^b(\mathsf{proj}\, kG).$$

This explains one of the advantages of $K\mathsf{Inj}\, kG$ over $D(\mathsf{Mod}\, kG)$: namely the trivial module k, regarded as a complex concentrated in degree zero, is not a compact object in $D(\mathsf{Mod}\, kG)$; this causes all sorts of problems. The corresponding object ik in $K\mathsf{Inj}\, kG$ is compact.

The advantage of $K\mathsf{Inj}\, kG$ over $\mathsf{StMod}\, kG$ is the following. The graded endomorphism ring of k in $\mathsf{StMod}\, kG$ is the Tate cohomology ring $\hat{H}^*(G,k)$, which is usually not Noetherian. In contrast, the endomorphism ring of ik in $K\mathsf{Inj}\, kG$ is the ordinary cohomology ring $H^*(G,k)$, which is Noetherian. When working in $\mathsf{StMod}\, kG$, one always has to go back and forth between ordinary and Tate cohomology, which is a considerable annoyance.

Varieties for Modules. In joint work with Iyengar and Krause from autumn 2005, we used the category $K\mathsf{Inj}\, kG$ to solve a problem about varieties for modules, which I now describe.

If X is an object in $K\mathsf{Inj}\, kG$, we define

$$H^*(G,X) = \mathsf{Hom}^*_{K\mathsf{Inj}\, kG}(ik, X).$$

If I is an injective graded $H^*(G,k)$-module, the functor sending X to the abelian group $\mathsf{Hom}_{H^*(G,k)}(H^*(G,X), I)$ takes triangles to long exact sequences and direct sums to direct products. These are the conditions for applying the Brown representability theorem, which states that this functor has a representing object which we denote $T(I)$ in $K\mathsf{Inj}\,(kG)$. So we obtain

$$\mathsf{Hom}_{H^*(G,k)}(H^*(G,X), I) \cong \mathsf{Hom}_{K\mathsf{Inj}\, kG}(X, T(I)).$$

The indecomposable injective graded $H^*(G,k)$-modules are of the form $I_\mathfrak{p}[n]$, where $\mathfrak{p}$ is a homogeneous prime ideal in $H^*(G,k)$, $I_\mathfrak{p}$ is the injective hull of $H^*(G,k)/\mathfrak{p}$ in the category of graded $H^*(G,k)$-modules, and $[n]$ denotes a degree shift. So the objects $T(I_\mathfrak{p})$ in $K\mathsf{Inj}\, kG$ are of particular interest. In some sense, they isolate the layer of the stable module category corresponding to $\mathfrak{p}$.

If X is an object in $K\mathsf{Inj}\, kG$, we define the "variety of X" to be

$$\mathcal{V}_G(X) = \{\, \mathfrak{p} \mid X \otimes_k T(I_\mathfrak{p}) \ncong 0\}.$$

This extends the usual definition of variety as defined by Benson, Carlson and Rickard [2] from $\mathsf{StMod}\, kG \simeq K_{ac}\mathsf{Inj}\, kG$ to $K\mathsf{Inj}\, kG$.

The problem is to describe the variety in terms of the cohomology. We describe the answer in the case of a p-group for simplicity, but there is a similar answer for a general finite group.

Theorem 2 (Benson, Iyengar and Krause)**.** *If G is a finite p-group, X is an object in $K\mathsf{Inj}\, kG$ and $\mathfrak{p}$ is a homogeneous prime ideal in $H^*(G,k)$ then the following are equivalent:*

(i) $X \otimes T(I_{\mathfrak{p}}) \ncong 0$.

(ii) $\mathsf{Ext}^{**}_{H^*(G,k)_{\mathfrak{p}}}(H^*(G,X)_{\mathfrak{p}}, k(\mathfrak{p})) \neq 0$.

(iii) $H^{**}_{\mathfrak{p}}\, H^*(G,X)_{\mathfrak{p}} \neq 0$.

(iv) *For some* $n \in \mathbb{Z}$, $T(I_{\mathfrak{p}}[n])$ *appears in a minimal injective resolution of* $H^*(G,X)$ *as a graded* $H^*(G,k)$*-module.*

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Combinatorial description of isolated blocks in finite classical groups

Bhama Srinivasan

Let $\mathbf{G}$ be a connected, reductive algebraic group defined over $\mathbf{F}_q$ and G the finite reductive group of $\mathbf{F}_q$-rational points of $\mathbf{G}$. Let ℓ be a prime not dividing q. Each ℓ-block of G determines a conjugacy class (s) in a dual group G^* of G, where $s \in G^*$ is an ℓ'-semi simple element. The block is said to be isolated if $C_{\mathbf{G}^*}(s)$ has the same semisimple rank as $\mathbf{G}^*$. If a block is not isolated, the characters in the block can be obtained by Lusztig induction from a Levi subgroup of G which is a dual of $C_{G^*}(s)$. A recent theorem of Bonnafé and Rouquier also shows that a non-isolated block is Morita equivalent to a block of a Levi subgroup (see e.g. [1]). Thus it is important to classify the isolated blocks of G. For example, if $G = Sp(2n,q)$ then the isolated blocks are parametrized by elements whose eigenvalues are 1 and -1 in the natural representation over $\mathbf{F}_q$ of the dual group $SO(2n+1,q)$.

If $\mathbf{G}$ is a classical group with a connected center, a description of all the ℓ-blocks of G (ℓ odd, q odd) was given in [2] in combinatorial terms using the language of symbols. On the other hand, Cabanes and Enguehard [1] have given descriptions in terms of Lusztig induction of the blocks of arbitrary finite reductive groups, with some restrictions on ℓ. In an earlier paper [3] the isolated blocks (ℓ odd, q odd) of $Sp(2n, q)$ or $SO^{\pm}(2n, q)$ were described via Lusztig induction. A combinatorial description of the blocks was, however, not given in [1] or [3] for these groups since such a description of Lusztig induction was not available. However, a combinatorial description of Lusztig induction in the cases of $Sp(2n, q)$ and $O^{\pm}(2n, q)$ was recently given by Waldspurger [4], provided q is large, as a consequence of his proof of the Lusztig conjecture for these groups. Using this description we give a parameterization of the isolated blocks of $Sp(2n, q)$ and $O^{\pm}(2n, q)$ (ℓ odd, q odd) by means of pairs of symbols which are e-cores, where e is the order of q^2 mod ℓ. We also give a description of the characters in a block which lie in the Lusztig rational series $E(G, (s))$ (see [1] 8.23), where (s) is the ℓ'-semisimple class corresponding to the block, again by pairs of symbols. Thus this result represents a completion of the project started in [2].

We now discuss our results in more detail. If $\mathbf{G}$ is a symplectic or orthogonal group the unipotent characters of G are parametrized by symbols. Here a symbol Λ is a pair (S, T) of subsets of $\mathbf{N}$. If e is a positive integer there is a notion of removing an e-hook (resp. e-cohook) from Λ to obtain another symbol. If it is not possible to remove any e-hooks (resp. e-cohooks) from Λ we say Λ is an e-core (resp. e-cocore). (A discussion of these concepts can be found in [2].)

The isolated ℓ-blocks of G (G as above, ℓ odd) are parametrized by classes (s) in the dual group where s has eigenvalues ± 1. The characters of G in the Lusztig series $E(G, (s))$ are then parametrized by pairs of symbols (Λ_1, Λ_2), and we denote the character corresponding to (Λ_1, Λ_2) by $\rho_{(\Lambda_1,\Lambda_2)}$. We now state the theorem proved in [2] when $\mathbf{G}$ has a connected center. For the definition of an e-split Levi subgroup we refer to ([1], p.190).

Theorem 1. Let $G = SO(2n+1, q)$, $CSp(2n, q)$, $CSO^{\pm}(2n, q)$, where q is odd. Let B be an isolated ℓ-block of G where ℓ is odd. Let e be the order of q^2 mod ℓ. The block B then determines an e-split Levi subgroup L and a character $\rho_{(\mu_1,\mu_2)}$ in $E(L, (s))$, where (μ_1, μ_2) is an e-core or e-cocore according as ℓ divides $q^e - 1$ or $q^e + 1$. Then the characters in $B \cap E(G, (s))$ are the irreducible constituents of $R_L^G(\rho_{(\mu_1,\mu_2)})$, and are of the form $\rho_{(\Lambda_1,\Lambda_2)}$ where (Λ_1, Λ_2) is obtained from (μ_1, μ_2) by adding e-hooks or e-cohooks according as ℓ divides $q^e - 1$ or $q^e + 1$.

The proof of this theorem uses a combinatorial description of the Lusztig map R_L^G in the case of these groups due to Asai and Shoji (see [2]). However, such a description was not then available for the groups $\mathbf{G} = Sp_{2n}, SO_{2n}$ which have a disconnected center, the reason being that the groups $C_{\mathbf{G}^*}(s)$ are disconnected. A uniform parametrization of characters in all $E(M, (s))$ where M runs over the e-split Levi subgroups of G, and a description of the R_M^G map in terms of such a parametrization, could not be given. A theorem classifying the isolated blocks of G in these cases was proved in [3] and is stated below.

Theorem 2. Let $G = Sp(2n, q),\ SO^{\pm}(2n, q)$, where q is odd. Let B be an isolated ℓ-block of G where ℓ is odd. Let e be the order of q^2 mod ℓ. The block B then determines an e-split Levi subgroup L and a character $\rho_{(\mu_1,\mu_2)}$ in $E(L,(s))$, such that the characters in $B \cap E(G,(s))$ are the irreducible constituents of $R_L^G(\rho_{(\mu_1,\mu_2)})$.

Recently Waldspurger [4] considered the groups $G = Sp(2n, q),\ O^{\pm}(2n, q)$, where $q > 2n$. If (s) is an isolated class in the dual group, he defined a character sheaf (in the sense of Lusztig) corresponding to a pair of symbols (Λ_1, Λ_2) as above, leading to a class function $\chi_{(\Lambda_1,\Lambda_2)}$ on G and proved Lusztig's conjecture which relates $\chi_{(\Lambda_1,\Lambda_2)}$ and $\rho_{(\Lambda_1,\Lambda_2)}$. As a consequence, he describes the R_M^G map for e-split Levi subgroups of G and characters in $E(M,(s))$ in combinatorial terms. Using these results we can prove the following theorem, which refines Theorem 2. Here an isolated block of $O^{\pm}(2n, q)$ is a block which covers an isolated block of $SO^{\pm}(2n, q)$.

Theorem 3. Let $G = Sp(2n, q),\ O^{\pm}(2n, q)$, where q is odd, and $q > 2n$. Let B be an isolated ℓ-block of G where ℓ is odd. Let e be the order of q^2 mod ℓ. The block B then determines an e-split Levi subgroup L and a character $\rho_{(\mu_1,\mu_2)}$ in $E(L,(s))$, where (μ_1, μ_2) is an e-core or e-cocore according as ℓ divides $q^e - 1$ or $q^e + 1$. Then the characters in $B \cap E(G,(s))$ are the irreducible constituents of $R_L^G(\rho_{(\mu_1,\mu_2)})$, and are of the form $\rho_{(\Lambda_1,\Lambda_2)}$ where (Λ_1, Λ_2) is obtained from (μ_1, μ_2) by adding e-hooks or e-cohooks according as ℓ divides $q^e - 1$ or $q^e + 1$.

As mentioned earlier, for large q this result represents a completion of the project started in [2]. We note that the case of $SO(2n + 1, q)$ was treated in [2]. Finally we remark that a combinatorial description of blocks of this kind fits in with a philosophy of Broué, by which representation-theoretic data on a finite reductive group should be determined by data independent of q.

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Trivial source modules in blocks with cyclic defect groups

Shigeo Koshitani

(joint work with Naoko Kunugi, Raphaël Rouquier)

In modular representation theory of finite groups, one of the most important and interesting questions is Broué's Abelian Defect Group Conjecture (ADGC). It is still open and there are only several cases where the conjecture is checked. In recent papers of the author with Kunugi and Waki we prove that Broué's ADGC is true for several cases for a specific defect group which is elementary abelian of order 9 and which is almost the smallest one in wild-representation type. Actually, one of the most efficient and essential methods there is looking at trivial source (p-permutation) modules, where p is a prime. This means that we could prove Broué's ADGC for some unknown cases provided we get many (enough) trivial source modules. We usually find out trivial source modules by direct calculation, sometimes relying on computers, say GAP for example if the finite groups are large. Therefore, if we know a more general and theoretical method to obtain trivial source modules, it should be nice and meaningful.

We shall present a method to get trivial source modules easily just by looking at values of ordinary characters at p-elements in finite groups instead of doing huge calculation. The method is only for a case where defect groups are cyclic. Nevertheless, it works well at least when we want to prove Broué's ADGC for blocks which have elementary abelian defect groups of order p^2 because we often can reduce the task of getting trivial source modules to the case where the defect groups have central subgroups of order p. The proof is standard and based on celebrated work on blocks with cyclic defect groups, due to E.C. Dade, and also work of R. Rouquier. The result is, however, useful and convenient. Actually, our result is used to prove Broué's ADGC for the Janko simple group J_4 by the author, Kunugi and Waki. Furthermore, nobody has remarked it before as far as we know. Therefore, we believe that it would be worthwhile to present it here.

Theorem. *Let k be an algebraically closed field of characteristic $p > 0$, G a finite group and kG the group algebra. Suppose that A is a block algebra of kG with a defect group P and B is the Brauer correspondent of A in $N_G(P)$, namely, B is a block algebra of $kN_G(P)$. Assume that P is cyclic. Then, the following* (1) $\sim$ (3) *are equivalent:*

(1) *For any non-exceptional character χ of G in A, it holds that $\chi(v) > 0$ for any element $v \in P$.*
(2) *The Brauer tree of A is a star with exceptional vertex in the center (we consider that this condition is satisfied if $P = 1$), and there exists a non-exceptional character χ in A such that $\chi(v) > 0$ for any element $v \in P$.*
(3) *The block algebras A and B are Puig equivalent, that is, there exists an (A, B)-bimodule M such that M is a ΔP-projective p-permutation (trivial source) right $k[G \times N_G(P)]$-module and M induces a Morita equivalence between A and B.*

Corollary. *Let k be an algebraically closed field of characteristic $p > 0$, G a finite group and kG the group algebra. Suppose that A is a block algebra of kG with a defect group P and B is the Brauer correspondent of A in $N_G(P)$, namely, B is a block algebra of $kN_G(P)$. Assume that P is cyclic and $P \neq 1$. Let P_1 be a subgroup of P of order p, and set $N_1 = N_G(P_1)$ and $N = N_G(P)$. Since $N \subseteq N_1$, there is a block algebra B_1 of kN_1 with $B^{N_1} = B_1$ and ${B_1}^G = A$ (block inductions),* *namely, B_1 is the Brauer correspondent of A in N_1. Then, the following* (1) *and* (2) *are equivalent:*

(1) *The block algebras B_1 and B are Puig equivalent, that is, there exists a (B_1, B)-bimodule M such that M is a ΔP-projective p-permutation (trivial source) right $k[N_1 \times N]$-module and M induces a Morita equivalence between B_1 and B.*

(2) *There exists a non-exceptional chracter χ of G in A such that $\chi(v) > 0$ for any element $v \in P - \{1\}$; or $\chi(v) < 0$ for any element $v \in P - \{1\}$.*

"Tilting" modules for symmetric groups?

David J. Hemmer

1. Introduction

Let G be a reductive algebraic group. A rational G-module is said to have a good filtration if it has a filtration with successive quotients isomorphic to induced modules $\nabla(\lambda)$. There is a simple cohomological criterion for having a good filtration, and a formula for the multiplicities, which are independent of the choice of filtration. Indecomposable modules which have both a good and a Weyl filtration are called tilting modules, and are labelled by dominant weights.

Until recently it was believed no such theory could exist for Specht and dual Specht filtrations of symmetric group modules, since well-known examples in characteristic two and three demonstrated that filtration multiplicities are not even well-defined. Nevertheless, in [6] it was shown that the multiplicities are well-defined as long as $\operatorname{char} k > 3$. Necessary and sufficient conditions for determining if a module has a Specht and/or dual Specht filtration were obtained. However the conditions are not in terms of symmetric group cohomology, but rather are stated in terms of GL_n cohomology and the adjoint Schur functor.

Our talk will describe a first attempt at developing a filtration theory for the symmetric group Σ_d in characteristic ≥ 5.

2. A symmetric group filtration condition

We begin by proving two different sufficient conditions for a $k\Sigma_d$ module to have a Specht (or dual Specht) filtration.

Theorem 1. *Let $M \in$ mod-$k\Sigma_d$.*

(i) *If $\operatorname{Ext}^1_{k\Sigma_d}(M, S^\lambda) = 0$ $\forall \lambda \vdash d$ then M has a dual Specht filtration. The multiplicity of S_τ in any such filtration is given by $\dim_k \operatorname{Hom}_{k\Sigma_d}(M, S^\tau)$*

(ii) *If $\operatorname{Ext}^1_{k\Sigma_d}(S^\lambda, M) = 0$ $\forall \lambda \vdash d$ then M has a dual Specht filtration. The multiplicity of S_τ in any such filtration is given by $\dim_k \operatorname{Hom}_{k\Sigma_d}(S^\tau, M)$*

(iii) *If $\operatorname{Ext}^1_{k\Sigma_d}(M, S_\lambda) = 0$ $\forall \lambda \vdash d$ then M has a Specht filtration. The multiplicity of S^τ in any such filtration is given by $\dim_k \operatorname{Hom}_{k\Sigma_d}(M, S_\tau)$*

(iv) *If* $\mathrm{Ext}^1_{k\Sigma_d}(S_\lambda, M) = 0$ $\forall \lambda \vdash d$ *then* M *has a Specht filtration. The multiplicity of* S^τ *in any such filtration is given by* $\dim_k \mathrm{Hom}_{k\Sigma_d}(S_\tau, M)$.

Although the conditions are not necessary, they have the advantage of being stated entirely in terms of the symmetric group theory. The multiplicity formulas in Theorem 1 generalize a known formula when M is a Young module.

3. Modules with Specht and dual Specht filtrations

We next consider symmetric group modules which have both Specht and dual-Specht filtrations. We demonstrate that they are not as well-behaved as tilting modules. Unlike tilting modules, they need not be self-dual. The tensor product of two modules which have both Specht and dual Specht filtrations may have neither! We believe however that a classification of the indecomposable *self-dual* modules with both filtrations is possible, and we conjecture these are exactly the signed Young modules.

Signed Young modules are summands of modules induced from one-dimensional representations of Young subgroups $\Sigma_\lambda \leq \Sigma_d$. They are a simultaneous generalization of Young modules Y^λ and twisted Young modules $Y^\lambda \otimes \mathrm{sgn}$. Donkin recently [2] showed that isomorphism types of indecomposable signed Young modules can be labelled by pairs of partitions λ, τ such that $|\lambda| + p|\tau| = d$. The signed Young module $Y(\lambda|p\tau)$ is the image of a "listing module" under the Schur functor. The listing modules are simultaneous generalizations of tilting modules and injective modules for $GL_n(k)$. We conjecture:

Conjecture 2. *Suppose* $U \in \text{mod-}\,k\Sigma_d$ *is indecomposable, self-dual with a Specht (and hence dual Specht) filtration. Then* $U \cong Y(\lambda|p\mu)$ *for some* λ, μ.

Of course an obvious choice for U in Conjecture 2 is an irreducible Specht module. These were only recently classified by Fayers [4]. As evidence for the conjecture we proved [5] that irreducible Specht modules are indeed signed Young modules. The difficult case, which Fayers handled, was for S^μ when μ is neither p-regular nor p-restricted. Curiously, these are precisely the irreducible Specht modules which are "proper" signed Young modules, i.e. neither Young nor twisted Young.

Next we relate Conjecture 2 to tilting functors for the Schur algebra $S(n, d)$ when $n \geq d$. We prove it is equivalent to the statement that the *listing modules* defined by Donkin in [2] can be characterized as those modules satisfying a certain nice property under the tilting functor.

Suppose V is the natural $GL_n(k)$ module and let T be a full tilting module. The *Ringel dual* of $S(n, d)$ is defined as:

$$S'(n, d) \cong \mathrm{End}_{S(n,d)}(T).$$

The *tilting functor* $\mathcal{T} : \text{mod-}\,S(n, d) \to \text{mod-}\,S'(n, d)$ is given by:

$$\mathcal{T}(U) = \mathrm{Hom}_S(T, U).$$

This setup is more thoroughly described in [3]. When $n \geq d$ then Donkin proved [3, sect. 3] that $S(n, d) \cong S'(n, d)$. We let $\tilde{\mathcal{T}} : \text{mod-}\,S(n, d) \to \text{mod-}\,S(n, d)$

denote the composition of $\mathcal{T}$ with the functor mod- $S'(n,d) \to$ mod- $S(n,d)$ arising from the isomorphism. This is the functor denoted $\tilde{\mathcal{F}}$ in [3], and it is essentially the tilting functor. It is well-known that the tilting functor takes modules with good filtrations to modules with Weyl filtrations and interchanges projective and tilting modules. We can show that symmetric group modules with both filtrations correspond to modules having certain nice properties under $\tilde{\mathcal{T}}$:

Theorem 3. *Let $M \in$ mod- $k\Sigma_d$ be indecomposable and let $U = \mathcal{G}(M)$, where $\mathcal{G}$ is the adjoint Schur functor. Then*

(i) *M has both a Specht and dual Specht filtration if and only if both U and $\tilde{\mathcal{T}}(U)$ have Weyl filtrations.*
(ii) *M is self dual with a Specht (and hence dual Specht) filtration if and only if U has a Weyl filtration and $\tilde{\mathcal{T}}(U) \cong \tilde{\mathcal{T}}(U^\tau)$.*

4. Related problems

There are many properties of signed Young modules in characteristic $p > 3$ which we cannot prove hold for arbitrary indecomposable self-dual $k\Sigma_d$ modules with Specht filtrations. Proving these would provide evidence for Conjecture 2. In the following let M and N denote indecomposable self-dual $k\Sigma_d$-modules with both Specht and dual Specht filtrations.

Problem 4. *The tensor product of two signed Young modules is a direct sum of signed Young modules. Is $M \otimes N$ is a direct sum of such modules?*

The same Mackey theorem argument used in [7, Thm. 6.5] shows if $p > 3$ then $\mathrm{Ext}^i_{k\Sigma_d}(Y(\lambda|p\mu), Y(\sigma|p\tau)) = 0$ for $1 \le i \le p-2$.

Problem 5. *Is $\mathrm{Ext}^i_{k\Sigma_d}(M,N) = 0$ for $1 \le i \le p-2$?*

The indecomposable modules we have found which have both Specht and dual Specht filtrations but are not self dual do not lift to characteristic zero. The signed Young modules do, however, lift.

Problem 6. *Do M and N lift to characteristic zero?*

If, for example, Problem 5 were answered affirmatively, it would in particular show that $\mathrm{Ext}^2_{k\Sigma_d}(M,M) = 0$. This Ext^2-vanishing would [1, 3.7.7] imply that M lifts to characteristic zero.

We know of no examples of indecomposable modules with Specht and dual Specht filtrations that self-extend.

Problem 7. *Suppose $U \in$ mod- $k\Sigma_d$ is indecomposable with both Specht and dual Specht filtrations. Must $\mathrm{Ext}^1_{k\Sigma_d}(U,U) = 0$?*

Finally we recall from [2] that the set of labels $(\lambda|p\mu)$ also label the irreducible modules for the Schur superalgebra $S(m|n,d)$ when $m \ge d$. When $m \ge d$ there is a Schur functor from mod- $S(m|n,d) \to$ mod- $k\Sigma_d$ with a right adjoint functor, $\tilde{\mathcal{G}}$, just as in the ordinary Schur functor setting.

Remark 8. *Conjecture 2 is equivalent to the statement if $M \in \text{mod-}k\Sigma_d$ is indecomposable, self-dual with a Specht filtration then $\tilde{\mathcal{G}}(M)$ is a projective $S(d|d,d)$-module.*

Indecomposable, self-dual $S(n,d)$-modules with good filtrations are in one-to-one correspondence with projective $S(n,d)$ modules (both indexed by dominant weights). Thus if our conjecture is true then indecomposable, self-dual $k\Sigma_d$-modules with Specht filtrations will be in correspondence with projective $S(d|d,d)$-modules.

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Around Mantaci-Reutenauer algebra

Cédric Bonnafé

If (W,S) is a finite Coxeter group, Solomon has constructed a subalgebra $\Sigma(W)$ of the group algebra $\mathbb{Z}W$ together with a morphism of algebras $\theta : \Sigma(W) \to \mathbb{Z}\mathrm{Irr}(W)$. The algebra $\Sigma(W)$ is called the *Solomon algebra*, or the *descent algebra* or the *Solomon descent algebra*.

This algebra has very nice properties: for instance, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Ker}(\theta)$ is the radical of $\mathbb{Q} \otimes_{\mathbb{Z}} \Sigma(W)$ and the Cartan matrix of $\mathbb{Q} \otimes_{\mathbb{Z}} \Sigma(W)$ is unitriangular. Note however that it is not hereditary in general. With G. Pfeiffer [BP], we have computed the Loewy length of $\Sigma(W)$ in most cases (the only remaining irreducible case is when W is of type D_{2n+1}).

However, the morphism θ is surjective if and only if W is a product of symmetric groups. If $W = W_n$ is of type B_n, C. Hohlweg and myself [BH] have constructed a subalgebra $\Sigma'(W_n)$ of $\mathbb{Z}W_n$ containing $\Sigma(W_n)$ and an extension $\theta' : \Sigma'(W_n) \to \mathbb{Z}\mathrm{Irr}(W_n)$ of Solomon homomorphism θ. It turns out that θ' is surjective [BH] and that $\Sigma'(W_n)$ shares many properties with $\Sigma(W_n)$: for instance, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Ker}(\theta')$ is the radical of $\mathbb{Q} \otimes_{\mathbb{Z}} \Sigma'(W_n)$ (see [BH]). Using the surjectivity of θ', we have proposed a construction of irreducible characters of W_n similar to Jöllenbeck's construction for the symmetric group. This involves the Kazhdan-Lusztig cells of W_n for the *asymptotic* choice of parameters (which L. Iancu and myself have computed in [BL]).

In [B], I have continued the study of the algebra $\mathbb{Q} \otimes_{\mathbb{Z}} \Sigma'(W_n)$ and its representation theory. I have proved that its Cartan matrix is unitriangular Using `CHEVIE`, I have computed this Cartan matrix for $n \leq 5$: it shows that $\mathbb{Q} \otimes_{\mathbb{Z}} \Sigma'(W_n)$ is not hereditary for $n \geq 4$ (note that there is a surjective morphism of algebras $\mathbb{Q} \otimes_{\mathbb{Z}} \Sigma'(W_n) \to \mathbb{Q} \otimes_{\mathbb{Z}} \Sigma'(W_{n-1})$). I have computed the Loewy length of $K \otimes \Sigma'(W_n)$, where K is any field (it is equal to n is the characteristic of K is different from 2 and it is equal to $2n-1$ if this characteristic is 2). Note also that it is possible to obtain a description of the Loewy series of the algebra $K \otimes_{\mathbb{Z}} \mathbb{Z}\mathrm{Irr}(W_n)$ whenever K has characteristic $p > 0$ (the description is similar to the description obtained for the Loewy series of $K \otimes_{\mathbb{Z}} \mathbb{Z}\mathrm{Irr}(S_n)$ in [B1]).

I shall now describe a conjecture which has been suggested to me by a conjecture of G. Pfeiffer on Solomon descent algebras. Let $\mathrm{Bip}(n)$ denote the set of bipartitions of n. If $\lambda \in \mathrm{Bip}(n)$, let C_λ denote the conjugacy class of W_n associated to λ. Then there exists a family $(E_\lambda)_{\lambda \in \mathrm{Bip}(n)}$ of orthogonal primitive idempotents of $\mathbb{Q} \otimes_{\mathbb{Z}} \Sigma'(W_n)$ sum that $\sum E_\lambda = 1$ and such that $\dim \mathbb{Q}W_n E_\lambda = |C_\lambda|$ (see [B]).

Question. *Let $w_\lambda \in C_\lambda$. Does there exist a linear character ζ_λ of $C_{W_n}(w_\lambda)$ such that the $\mathbb{Q}W_n$-module $\mathbb{Q}W_n E_\lambda$ affords the character $\mathrm{Ind}_{C_{W_n}(w_\lambda)}^{W_n} \zeta_\lambda$?*

This conjecture implies G. Pfeiffer's conjecture for the Solomon descent algebra of type B. It has been checked using `CHEVIE` for $n \leq 5$.

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Cyclotomic Solomon algebras

Andrew Mathas

(joint work with Rosa Orellana)

In 1976 Louis Solomon introduced the *Solomon algebra* $S(W)$, or *descent algebra*, of a finite Coxeter group W. This algebra is the subalgebra of the group algebra of W which is spanned by the sum of the distinguished coset representatives of the parabolic subgroups of W. We have defined an analogue of Solomon's algebra for the complex reflection groups of type $G(m,1,n)$ which is again spanned by the sum of the "distinguished" coset representatives for the reflection subgroups. Like Solomon's algebras, the cyclotomic algebras have many nice properties. For example, we can explicitly describe the structure constants for our distinguished bases and we can give a basis for the radical. Interestingly, these algebras are all specializations of a "generic" algebra which depends only on n.

p-permutation equivalences

Robert Boltje
(joint work with Bangteng Xu)

Let G and H be finite groups and let $(K,\mathcal{O},k)$ be a p-modular system such that $\mathcal{O}$ contains roots of unity of order $\exp(G)$ and $\exp(H)$. Furthermore, let $A=\mathcal{O}Ge$ and $B=\mathcal{O}Hf$ be blocks with a common abelian defect group D. We will always identify (G,H)-bimodules M with left $G\times H$-modules via $(g,h)m=gmh^{-1}$.

1. Motivation

In [1], Broué introduced the notion of a *perfect isometry* between A and B. This is an element μ of the character group $R(KGe,KHf)$ of (KGe,KHf)-bimodules inducing an isometry

$$I\colon R(KHf)\to R(KGe)\,,\quad \chi\mapsto \mu \underset{H}{\cdot}\chi\,,$$

with respect to the Schur inner product which satisfies additional arithmetical properties. If, for instance, μ is the character of an $(\mathcal{O}G,\mathcal{O}H)$-bimodule which is projective as left $\mathcal{O}G$-module and as right $\mathcal{O}H$-module, then μ has these properties. Broué showed that a derived equivalence $\mathcal{D}^{\mathrm{b}}(B)\to\mathcal{D}^{\mathrm{b}}(A)$ induces a perfect isometry.

Broué also noticed that these perfect isometries usually come in families. Such a family he called an *isotypy* between A and B. To define them, one first assumes that $\zeta\colon\mathcal{B}_f^{\le D}(H)\to\mathcal{B}_e^{\le D}(G)$ is an equivalence between the Brauer categories. The objects of $\mathcal{B}_f^{\le D}(H)$ are f-Brauer pairs (P,φ) with $P\le D$. It is assumed that $\zeta(P,\varphi)$ is a Brauer pair of the form (P,ε). Now, an isotypy between A and B with respect to ζ is a family of perfect isometries $\mu_{(P,\varphi)}\in R(KC_G(P)\varepsilon,KC_H(P)\varphi)$, $(P,\varphi)\in\mathcal{B}_f^{\le D}(H)$, with $(P,\varepsilon)=\zeta(P,\varphi)$, which satisfies certain compatibility conditions.

While general derived equivalences could not explain the existence of isotypies, it was a breakthrough insight of Rickard in [2] that if C_* is a finite chain complex of ΔD-projective p-permutation $(\mathcal{O}Ge,\mathcal{O}Hf)$-bimodules satisfying homotopy equivalences

$$C_*\otimes_{\mathcal{O}H}\check{C}_*\simeq A\quad\text{and}\quad \check{C}_*\otimes_{\mathcal{O}G}C_*\simeq B$$

in the categories of chain complexes of (A,A)-bimodules (resp. (B,B)-bimodules), then the Lefschetz characters of the chain complexes $C_*(\Delta P)$ give rise to an isotypy between A and B. Here, $\Delta P := \{(x,x) \mid x \in P\}$, $\check{C}_*$ denotes the $\mathcal{O}$-dual of C_* and $-(\Delta P)$ denotes the Brauer construction with respect to ΔP. Rickard called such complexes *splendid*. They are nowadays also called *Rickard complexes* or *Rickard equivalences*.

Therefore, we obtain a diagram of implications

$$\begin{array}{ccc} \text{Rickard equivalence} & \Rightarrow & \text{derived equivalence} \\ \Downarrow & & \Downarrow \\ \text{isotypy} & \Rightarrow & \text{perfect isometry} \end{array}$$

with the top row on a category level and the bottom row on a Grothendieck group level.

In this context one should also mention Broué's abelian defect group conjecture which predicts that, in the case where $H = N_G(D)$ and B is the Brauer correspondent of A, all these equivalences exist.

2. p-permutation equivalences

In hindsight, after having Rickard's categorical lift of an isotypy, it seems natural to introduce a notion of equivalence on the Grothendieck group level that works with p-permutation modules.

Definition. A *p-permutation equivalence* between A and B is an element γ of the representation group $T^{\Delta D}(\mathcal{O}Ge, \mathcal{O}Hf)$ of ΔD-projective p-permutation $(\mathcal{O}Ge, \mathcal{O}Hf)$-bimodules which satisfies

$$\gamma \underset{H}{\cdot} \check{\gamma} = [\mathcal{O}Ge] \quad \text{and} \quad \check{\gamma} \underset{G}{\cdot} \gamma = [\mathcal{O}Hf]$$

in the trivial source ring $T(\mathcal{O}[G \times G])$, resp. $T(\mathcal{O}[H \times H])$.

It is easy to see that if C_* is a Rickard equivalence between A and B, then the element $\gamma := \sum_{n \in \mathbb{Z}} (-1)^n [C_n]$ is a p-permutation equivalence between A and B.

Rickard equivalences have two important properties, namely that they imply isotypies and that, by applying the Brauer construction with respect to ΔP for $P \le D$, one obtains again a Rickard equivalence between appropriate blocks of the centralizers $C_G(P)$ and $C_H(P)$. The following theorem shows that p-permutation equivalences share these properties.

Theorem. *Assume that the Frobenius categories $\mathcal{F}^{\le D}(G)$ and $\mathcal{F}^{\le D}(H)$ are equal and let γ be a p-permutation equivalence between A and B. For $P \le D$, consider the element $\gamma(\Delta P) \in T(\mathcal{O}C_G(P)\mathrm{br}_P(e), \mathcal{O}C_H(P)\mathrm{br}_P(f))$ and its character $\mu_P \in R(KC_G(P)\mathrm{br}_P(e), KC_H(P)\mathrm{br}_P(f))$, where br_P denotes the Brauer morphism with respect to P. Then the following hold:*

(a) For every $(P,\varphi) \in \mathcal{B}_f^{\le D}(H)$ there exists a unique $(P,\varepsilon) \in \mathcal{B}_e^{\le D}(G)$ such that $\varepsilon\gamma(\Delta P)\varphi \ne 0$. This defines an equivalence $\zeta\colon \mathcal{B}_f^{\le D}(H) \to \mathcal{B}_e^{\le D}(G)$, $(P,\varphi) \mapsto (P,\varepsilon)$. Moreover, for all $P \le D$ and all $(P,\varphi) \in \mathcal{B}_f^{\le D}(H)$ and $(P,\varepsilon) \in \mathcal{B}_e^{\le D}(G)$ one has: $\varepsilon\gamma(\Delta P)\varphi \ne 0$ if and only if $\varepsilon\mu_P\varphi \ne 0$.

(b) The elements $\varepsilon\mu_P\varphi$, $(P,\varphi) \in \mathcal{B}_f^{\le D}(H)$, with $(P,\varepsilon) = \zeta(P,\varphi)$, form an isotypy between A and B.

(c) Let $P \le D$, let $(P,\varphi) \in \mathcal{B}_f^{\le D}(H)$ and set $(P,\varepsilon) := \zeta(P,\varphi)$. Then the element $\varepsilon\gamma(\Delta P)\varphi$ is a p-permutation equivalence between $\mathcal{O}C_P(G)\varepsilon$ and $\mathcal{O}C_P(H)\varphi$.

Remark. (a) Using the theory of species for p-permutation modules, one sees that p-permutation equivalences are not so far from isotypies. One might hope that it is possible to lift known isotypies to p-permutation equvialences in many instances.

(b) It should be possible to determine all possible p-permutation equivalences γ between A and B with a computer program, even for large groups. A Rickard complex which induces γ then must contain the indecomposable p-permutation modules that occur in γ. This might shed more light on how to construct Rickard complexes.

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Minimal polynomials of semisimple elements of finite groups of Lie type in cross characteristic representations

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(joint work with A. E. Zalesskiĭ)

Let G be a finite group and $\mathbb{F}$ an algebraically closed field of characteristic ℓ. If Θ is any irreducible $\mathbb{F}G$-representation (of degree > 1) and $g \in G$, let $d_\Theta(g)$ denote the degree of the minimal polynomial of $\Theta(g)$. In a number of applications, it is important to know what are the possible values of $d_\Theta(g)$ for $g \in G$, in the first instance when g has prime-power order p^a. Clearly, $d_\Theta(g)$ does not exceed the order of $gZ(G)$ in $G/Z(G)$. The *Minimal Polynomial Problem* aims to show that either $d_\Theta(g)$ satisfies some lower bound, good enough for applications in mind, or else (G,Θ,g) satisfy some very strong constraints. The celebrated Hall-Higman theorem [4] considered the case where $p = \ell$ and G is p-solvable, whereas the *Quadratic Pairs Problem*, investigated by Thompson [8] and others, address the case when $p = \ell$, $G = \langle g^G \rangle$ and $d_\Theta(g) = 2$.

Let us turn our attention to the case where G is an almost quasisimple group and let $S := \mathrm{soc}(G/Z(G))$. If S is a sporadic group, it seems possible to solve the problem using the ordinary and modular Atlases as well as new Brauer character tables available in GAP or online. It is probably a surprise that the case S is an alternating group is still incomplete. In particular, there is nothing known about the case $G = \mathrm{Alt}_p$, g is a p-cycle, and $0 < \ell \le (p-1)/2$, except for the trivial bound that $d_\Theta(g) \ge (p-1)/2$. From now one, assume that $S \in \mathit{Lie}(r)$ and $\ell = r$.

The case $p = r$ has been studied by Suprunenko [7]. On the other hand, the case $p \neq r$ can be handled in principle using the representation theory of algebraic groups and Premet's theorem [6].

Thus we focus on the case G is a finite group of Lie type in characteristic r and $\ell \neq r$. If furthermore G is a classical group and g is a unipotent element (or more generally g belongs to a parabolic subgroup), then the problem has been solved by recent work of DiMartino-Zalesskiĭ [2] and Guralnick-Magaard-Saxl-Tiep [3]. However, until very recently there was hardly anything known about the case where g is a semisimple element (not belonging to any parabolic subgroup). For instance, if g is a Singer cycle of order dividing $(q^n - 1)/(q - 1)$ in $GL_n(q)$, all currently available results yield only the (very weak) lower bound $d_\Theta(g) \geq n$.

The aim of this talk is to announce our recent results which yield *a Hall-Higman type theorem for semisimple elements in cross characteristic representations of finite classical groups in any characteristic, as well as for exceptional groups of Lie type in the case* $p > 3$. In particular, if $g \in GL_n(q)$ is a Singer cycle of order p^a dividing $(q^n - 1)/(q - 1)$, our lower bound is $d_\Theta(g) \geq p^{a-1}(p - 1) - 1$.

Theorem 1. *Let $\mathcal{G}$ be a simple algebraic group defined over a field of characteristic r, $G := \mathcal{G}^F$ for a Frobenius map F, and let $S := \mathrm{soc}(G/Z(G))$. Let Θ be an irreducible representation of G over a field of characteristic $\ell \neq r$, of degree > 1. Let $g \in G$ be a semisimple element of prime power order p^a modulo $Z(G)$. If $\mathcal{G}$ is exceptional, assume in addition that $p > 3$. Then one of the following holds.*

(i) $p^a \geq d_\Theta(g) > p^{a-1}(p - 1)$.

(ii) $p > 2$, $d_\Theta(g) = p^{a-1}(p - 1)$ *and Sylow p-subgroups of $G/Z(G)$ are cyclic. Furthermore, either $a = 1$, or $\ell \neq p$.*

(iii) $S = PSU_n(q)$, $o(g) = p = q + 1$ *is a Fermat prime, g belongs to $GU_1(q)^n$, Θ is a Weil representation, and $d_\Theta(g) = p - 1$. Furthermore, either $n \leq 3$, or g is a pseudoreflection in $GU_n(q)$.*

(iv) $S = PSU_n(q)$ *with* $n \equiv 1 (\mathrm{mod}\, p^b)$ *for some $b \geq 1$, $q + 1 = p$ is a Fermat prime, $o(g) = p^{b+1}$, g^{p^b} is a pseudoreflection in $GU_n(q)$, and Θ is a Weil representation. Furthermore, either $d_\Theta(g) = p^b(p - 1)$, or $(n, p^b, q) = (4, 3, 2)$ and $d_\Theta(g) = p^b(p - 1) - 1$.*

(v) $o(g) = (q^n - 1)/(q - 1)$, $S = PSL_n(q)$, *n a prime, Θ is a Weil representation of degree $o(g) - 1$ or $o(g) - 2$, and $\dim(\Theta) = d_\Theta(g)$. Furthermore, either $n > 2$ and $\ell = o(g) = p$, or $n = 2$ and q is even.*

In the case g has prime order, our result is as follows:

Corollary 2. *Under the assumptions of Theorem 1, assume that G is quasisimple and $g \in G \setminus Z(G)$ is a semisimple element of prime order p. Then one of the following holds.*

(i) $d_\Theta(g) = p$.

(ii) $p > 2$, $d_\Theta(g) = p - 1$ *and Sylow p-subgroups of $G/Z(G)$ are cyclic.*

(iii) $p = q + 1$ *is a Fermat prime, $G/Z(G) = PSU_n(q)$, $g^{q+1} = 1$, Θ is a Weil representation, and $d_\Theta(g) = p - 1$. Furthermore, either $n \leq 3$, or g corresponds to a pseudoreflection in $GU_n(q)$.*

(iv) *$o(g) = p = (q^n - 1)/(q - 1)$, $G/Z(G) = PSL_n(q)$, n a prime, Θ is a Weil representation of degree $p - 1$ or $p - 2$, and $\dim(\Theta) = d_\Theta(g)$. Furthermore, either $n > 2$ and $\ell = p$, or $n = 2$ and q is even.*

Our lower bound is in fact optimal, as shown in the following statement:

Corollary 3. *Keep the assumptions of Theorem 1. Then $d_\Theta(g) \geq p^{a-1}(p-1) - 1$, and this bound is best possible (if one runs over all semisimple elements and all irreducible cross characteristic representations of finite groups of Lie type).*

Let us say a few words about the outline of the proofs. Assume that G is a classical group. Then one can consider the action of g on the minimal module for G. In the case this action is irreducible, our analysis relies on Theorems 4 and 5 below, and on the p-modular representation theory of finite groups with cyclic Sylow p-subgroups (if $\ell = p$, cf. [9]). In turn, the proof of the crucial Theorem 4 relies on the Deligne-Lusztig theory [5] and a fundamental result of Broué-Michel [1]. The general case naturally reduces to the irreducible case. Finally, if G is an exceptional group, our arguments proceed by induction on rank($\mathcal{G}$) and reduction to the classical case.

Theorem 4. *Let $\mathbb{F}$ be an algebraically closed field of characteristic ℓ, $\mathcal{G}$ a connected reductive algebraic group in characteristic $r \neq \ell$, F a Frobenius map on $\mathcal{G}$, and let $G := \mathcal{G}^F$. Let $\mathcal{G}^*$ be an algebraic group with a Frobenius map F^* such that $(\mathcal{G}^*, F^*)$ is dual to $(\mathcal{G}, F)$. Let $p \neq \ell$ be a prime with the property that any nontrivial p-element in $(\mathcal{G}^*)^{F^*}$ is regular semisimple in $\mathcal{G}^*$ and that $(p, |Z(\mathcal{G})/Z(\mathcal{G})^\circ|) = 1$. Consider any irreducible $\mathbb{F}G$-representation Θ, with Brauer character ψ and any nontrivial p-element $g \in G$.*

(A) *Then one of the following holds.*

(i) *$\psi(g) \in \mathbb{Z}$ and $d_\Theta(g) \geq \varphi(|g|)$.*

(ii) *Θ lifts to characteristic 0.*

(B) *Assume in addition that $|T|^3 \leq |G|_{r'}$ for every maximal torus T of G with order divisible by p, that any nontrivial p-element in $\mathcal{G}^F$ is regular semisimple in $\mathcal{G}$, and that $(p, |Z(\mathcal{G}^*)/Z(\mathcal{G}^*)^\circ|) = 1$. Then $d_\Theta(g) = |g|$ in the case of* (ii).

Theorem 5. *Let $SL^\epsilon_p(q) \leq G \leq GL^\epsilon_p(q)$ and let $g \in G$ be an irreducible p-element with $p > 2$. Let $\Theta \in \mathrm{IBr}_\ell(G)$ with $\dim(\Theta) > 1$ and $(\ell, q) = 1$. Then $d_\Theta(g) = o(g) = p$, except when $(p, q, \epsilon) = (3, 2, -)$.*

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Spectra of unipotent elements in cross characteristic representations of finite classical groups

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(joint work with Lino Di Martino)

We study the eigenvalues of unipotent elements in cross-characteristic representations of finite classical groups. Let H be a classical group and let g be a unipotent element of H of order s. We show that for almost every irreducible representation θ of H all s-roots of unity occur as eigenvalues of $\theta(g)$, and we classify all the triples (H, g, θ) for which some s-root of unity does not occur as an eigenvalue of $\theta(g)$. Below p always denotes the defining characteristic of H and P is an algebraically closed field of characteristic $\ell \neq p$. All representations are P-representations. The case $\ell = 0$ is not excluded.

The work is a part of a larger project for the study of the minimum polynomials of elements in group representations. It origins in classical results of Hall and Higman (1959) obtained for p-soluble groups. Most essential recent works are [6], [1], [2] and [7].

For a square matrix M, we denote by $\deg M$ the degree of the minimum polynomial of M, and by $\operatorname{Spec} M$ the spectrum of M, respectively. Note that the spectrum is defined as the *set* of all eigenvalues, *disregarding multiplicities*. For a matrix M, we denote by $\operatorname{Jord} M$ the Jordan canonical form of M; a Jordan block of size h is denoted by J_h.

The study of eigenvalues of unipotent elements in representations of groups of Lie type begins with the following result obtained in my paper [4] of 1986. For p odd, let $\Delta_1(p)$ (resp., $\Delta_2(p)$) denote the set $1 \cup \left\{\varepsilon^j\right\}$, where j runs over the non-squares (resp.: the squares) residues of integers modulo p.

Theorem 1. *Let H be a quasi-simple group of Lie type in characteristic p, such that $(p, |Z(H)|) = 1$, and let $g \in H$ be an element of order p. Let θ be a faithful irreducible representation of H. Then* $\operatorname{Spec}\theta(g)$ *consists of all p-roots of unity, unless p is odd and one of the following holds:*

(1) $H = PSU(3,p)$, $\dim\theta = p(p-1)$ *and g is a transvection;*

(2) $H = SL(2,p^2)$, $\dim\theta = (p^2-1)/2$*;*

(3) $H = Sp(4,p)$, $\dim\theta = (p^2-1)/2$, $\deg\theta(g) = p-1$ *and g is not a transvection;*

(4) $H = PSp(4,p)$, $\dim\theta = p(p-1)^2/2$ *and g is a transvection;*

(5) $H = Sp(2n,p)$ *or* $PSp(2n,p)$, $n > 1$, $\dim\theta = (p^n \pm 1)/2$, *$g$ is a transvection and* Spec $\theta(g) = \Delta_1(p)$ *or* $\Delta_2(p)$. *If $\ell = 2$, then only the minus sign has to be chosen for* $\dim\theta$. *Furthermore $H = Sp(2n,p)$ if $\dim\theta$ is even, while $H = PSp(2n,p)$ if $\dim\theta$ is odd.*

(6) $H = Sp(2,p)$ *or* $PSp(2,p)$, *and either* $\dim\theta = (p+1)/2$ *with* $\operatorname{Spec}\theta(g) = \Delta_1(p)$ *or* $\Delta_2(p)$, *or* $\dim\theta = (p-1)/2$ *with* $\operatorname{Spec}\theta(g) = \Delta_1(p)\setminus\{1\}$ *or* $\Delta_2(p)\setminus\{1\}$. *If $\ell = 2$ then $\dim\theta = (p-1)/2$ only.*

In cases (1) - (4) $\operatorname{Spec}\theta(g)$ *consists of all non-trivial p-roots of unity. In case* (2) *the eigenvalue* 1 *does not occur for g belonging to one of the two unipotent conjugacy classes of H.*

In fact, the claims on dimensions in case (5) has been obtained in [3] for $\ell = 0$ and [2] for $\ell > 0$, together with the fact that the representations in question are Weil.

The case of $H = SL(m,q)$ for arbitrary unipotent elements was examined in [5]:

Theorem 2. *For $m > 2$, if θ is a non-trivial representation of $G = SL(m,q)$ then every unipotent element g has exactly $|g|$ distinct eigenvalues except when $H = SL(3,2)$, $|g| = 4$ and $\dim\theta = 3$.*

New results concern classical groups other than $SL(n,q)$.

Theorem 3. *Let H be one of the following groups: $Sp(m,q)$, $m > 2$ and $(m,q) \neq (4,2)$; $SU(m,q)$, $m > 2$ and $(m,q) \neq (3,2)$;* $\mathrm{Spin}(m,q)$, *m odd, $m > 5$;* $\mathrm{Spin}^{\pm}(m,q)$, *m even, $m > 6$. Let $g \in H$ be an element of order $p^\alpha > p$, and set $t = g^{p^{\alpha-1}}$. Let θ be an ireducible P-representation of H with $\dim\theta > 1$. Then* Spec $\theta(g)$ *contains all the $|g|$-roots of* 1*, unless one of the following holds:*

(1) $H = Sp(m,p)$ *with p odd, t is a transvection and θ is a Weil representation;*

(2) $H = SU(m,2)$, Jord $g = \mathrm{diag}(J_k, Id_{m-k})$ *with $k = 3$ or 5, and θ is a Weil representation;*

(3) $H = Sp(6,3)$, $\mathrm{Jord}\, g = \mathrm{diag}(J_4,1,1)$ *and* $\dim\theta = 78$*;*

(4) $H = Sp(4,9)$ *and* $\dim\theta = 40$*;*

(5) $H = SU(4,3)$ *and* $\dim\theta = 20$*;*

(6) $H = Sp(8,3)$, $\mathrm{Jord}\, g = \mathrm{diag}(J_4,J_4)$ *and* $\dim\theta = 40$*;*

(7) $H = Sp(6,2)$, $\mathrm{Jord}\, g = J_6$ *or* $\mathrm{diag}(J_4,1,1)$ *and* $\dim\theta = 7$*;*

(8) $H = SU(5,2)$, $\mathrm{Jord}\, g = \mathrm{diag}(J_3,J_2)$ *and* $\dim\theta = 10$, *or* $H = SU(7,2)$, $\mathrm{Jord}\, g = \mathrm{diag}(J_5,J_2)$ *and* $\dim\theta = 42$.

(9) $H = Sp(4,3)$ *and* $\dim\theta = 6, 10$ *or* 20.
For each representation of G whose dimension is in the above list there is a unipotent element g such that $\operatorname{Spec}\theta(g)$ *contains less than* $|g|$ *elements.*

Remark: We assume $\alpha > 1$ in Theorem 3, as the case $\alpha = 1$ is contained in Theorem 1.

The spectrum of $\theta(g)$ in Theorem 3 are known as well. The relevant information for case (1) is provided in Theorem 4 and for cases (2), (8) in Theorem 5. For other cases the spectra of unipotent elements can be computed from the atlas of Brauer characters.

In Theorem 4, η is a 3-root of ε, where ε is a primitive 3-root of unity in P.

Theorem 4. *Let $H = Sp(m,p)$ for $p > 2$. Let $g \in H$ be an element of order $p^\alpha > p$ such that $t = g^{p^{\alpha-1}}$ is a transvection, and let θ be a Weil representation of H. Then* $\operatorname{Spec}\theta(g)$ *contains all the $p^{\alpha-1}$-roots of the elements of $\Delta_1(p)$ or $\Delta_2(p)$ unless $p = 3$ and one of the following holds:*
(1) *$m > 4$,* $\operatorname{Jord} g = \operatorname{diag}(J_4, \mathrm{Id}_{m-4})$ *and* $\operatorname{Spec}\theta(g) = \{1, \eta^3, \eta, \eta^4, \eta^7\}$ *or* $\{1, \eta^6, \eta^2, \eta^5, \eta^8\}$*;*
(2) *$m = 6$,* $\operatorname{Jord} g = \operatorname{diag}(J_4, J_2)$*,* $\dim\theta = 13$ *and* $\operatorname{Spec}\theta(g) = \{\eta^i \mid i \in \{1,4,7,3,6\}$ *or* $i \in \{2,5,8,3,6\}\}$*;*
(3) *$m = 4$, and either* $\dim\theta = 4$ *and* $\operatorname{Spec}\theta(g) = \left\{\eta, \eta^4, \eta^7, \eta^6\right\}$ *or* $\left\{\eta^2, \eta^5, \eta^8, \eta^3\right\}$ *or $\ell \neq 2$,* $\dim\theta = 5$ *and* $\operatorname{Spec}\theta(g) = \left\{1, \eta, \eta^4, \eta^7, \eta^6\right\}$ *or* $\left\{1, \eta^2, \eta^5, \eta^8, \eta^3\right\}$.

In Theorem 5, ζ is a primitive 8-root of unity in P.

Theorem 5. *Let $H = SU(m,2)$, $m > 3$. Let $g \in H$ be an element of order $s = 2^\alpha > 2$ such that $t = g^{2^{\alpha-1}}$ is a transvection, and let θ be a Weil representation of H. Then* Spec $\theta(g)$ *contains all $|g|$-roots of unity, unless one of the following holds:*
(1) $\operatorname{Jord} g = \operatorname{diag}(J_3, \mathrm{Id}_{m-3})$ *and* $\operatorname{Spec}\theta(g) = \{\zeta^{2i} : i = 0,1,3\}$*;*
(2) *$m > 5$,* $\operatorname{Jord} g = \operatorname{diag}(J_5, \mathrm{Id}_{m-5})$ *and* $\operatorname{Spec}\theta(g) = \{\zeta^i : i \neq 4,\ 0 \leq i < 8\}$*;*
(3) *$m = 5$,* $\operatorname{Jord} g = J_5$ *and either* $\dim\theta = 10$ *and* $\operatorname{Spec}\theta(g) = \{\zeta^i : i \neq 4, 0 < i < 8\}$ *or* $\dim\theta = 11$ *and* $\operatorname{Spec}\theta(g) = \{\zeta^i : i \neq 4,\ 0 \leq i < 8\}$*;*
(4) *$m = 5$,* $\operatorname{Jord} g = \operatorname{diag}(J_3, J_2)$, $\dim\theta = 10$ *and* $\operatorname{Spec}\theta(g) = \{\zeta^{2i} : 0 < i < 4\}$;
(5) *$m = 7$,* Jord $g = \operatorname{diag}(J_5, J_2)$, $\dim\theta = 42$ *and* $\operatorname{Spec}\theta(g) = \{\zeta^i : 0 < i < 8\}$.

Theorem 6. *Under assumptions of Theorem* 3 *assume that* 1 *is not an eigenvalue of $\theta(g)$. Then one of the following holds:*
(1) $G = Sp(4,3)$ *and* $\dim\theta = 4, 6, 10, 20$*;*
(2) $G = Sp(6,3)$*,* $\operatorname{Jord} g = \operatorname{diag}(J_4, J_2)$ *and* $\dim\theta = 13$*;*
(3) $G = Sp(6,3)$*,* $\operatorname{Jord} g = \operatorname{diag}(J_4, 1, 1)$ *and* $\dim\theta = 78$*;*
(4) $G = Sp(4,9)$ *and* $\dim\theta = 40$*;*
(5) $G = Sp(8,3)$*,* $\operatorname{Jord} g = \operatorname{diag}(J_4, J_4)$ *and* $\dim\theta = 40$*;*
(6) $G = SU(4,3)$ *and* $\dim\theta = 20$*;*
(7) $G = SU(5,2)$*,* $\operatorname{Jord} g = \operatorname{diag}(J_3, J_2)$ *or* $\operatorname{Jord} g = J_5$ *and* $\dim\theta = 10$*;*
(8) $G = SU(7,2)$*,* $\operatorname{Jord} g = \operatorname{diag}(J_5, J_2)$ *and* $\dim\theta = 42$.

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Modules of Constant Jordan Type

Jon F. Carlson [1]

(joint work with Eric M. Friedlander, Julia Pevtsova)

We introduce a curious class of finite dimensional modules, modules of constant Jordan type, which generalizes the class of endotrivial modules (for finite groups). A complete classification of the endotrivial modules over p-groups was recently completed by the author and Jacques Thévenaz. The fact that the condition of constant Jordan type is well formulated for general group schemes is a consequence of a recent paper by Friedlander, Pevtsova, and A. Suslin [3].

Recall that a finite group scheme G (over a field k of characteristic $p > 0$) is a group scheme over k whose coordinate algebra $k[G]$ is finite dimensional over k. We denote the linear dual of $k[G]$ by kG and call this the group algebra of G. A (rational) G-module is a comodule for $k[G]$ or equivalently a kG-module.

A π-point for a finite group scheme G is a left flat map of K-algebras $\alpha_K : K[t]/t^p \to KG$ for some field extension K/k which factors through the group algebra $KC_K \subset KG_K$ of some unipotent abelian subgroup scheme $C_K \subset G_K$ (see [2]). If M is a finite dimensional kG-module, the Jordan type of M at the π-point α_K is the isomorphism class of the $K[t]/t^p$-module $\alpha_K^*(M_K)$ (where $M_K = K \otimes_k M$). Let $\alpha_K : K[t]/t^p \to KG$, $\beta_L : L[t]/t^p \to LG$ be π-points of G. Then α_K is said to be a *specialization* of β_L (written $\beta_L \downarrow \alpha_K$) if for every finite dimensional kG-module M the $K[t]/t^p$-module $\alpha_K^*(M_K)$ is projective whenever the $L[t]/t^p$-module $\beta_L^*(M_L)$ is projective. We say that α_K, β_L are *equivalent* and write $\alpha_K \sim \beta_L$ provided that $\alpha_K \downarrow \beta_L$ and $\beta_L \downarrow \alpha_K$.

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The set of equivalence classes of π-points of a finite group scheme G, written $\Pi(G)$, admits a scheme structure determined by the stable module category $stmod(G)$. With this structure, $\Pi(G)$ is isomorphic to the scheme $Proj H^\bullet(G, k)$. In particular, the closed subsets of $\Pi(G)$ are of the form $\Pi(G)_M$ where M is a finite dimensional kG-module and $\Pi(G)_M$ is the subset of those equivalence classes of π-points $\alpha_K : K[t]/t^p \to KG$ such that $\alpha_K^*(M_K)$ is not projective. This is called the support variety of M. The non-maximal support variety, $\Gamma(G)_M \subset \Pi(G)$ associated to a finite dimensional kG-module M, is defined to be the (closed) subspace of those points $x \in \Pi(G)$ with the property that for some (and thus any) representative α_K of x the Jordan type of $\alpha_K^*(M_K)$ is not maximal [3].

We call the isomorphism type of a finite dimensional $k[t]/t^p$-module M the *Jordan type* of M. We denote the Jordan type of M by $a_p[p] + \cdots a_1[1]$, where a_i denotes the number of blocks of size i in the partition of N associated to M. For any finite dimensional $k[t]/t^p$-module M, the *stable Jordan type* of M is the "stable equivalence" class of Jordan types, where two Jordan types $a_p[p]+\cdots a_1[1]$ and $b_p[p] + \cdots b_1[1]$ are stably equivalent if $a_i = b_i, i < p$. The finite dimensional kG-module M is said to be of constant Jordan type if the Jordan type of $\alpha_K^*(M_K)$ is independent of the choice of π-point $\alpha_K : K[t]/t^p \to KG$. In general, the Jordan type of $\alpha_K^*(M_K)$ for a finite dimensional kG-module M at a π-point $\alpha_K : K[t]/t^p \to KG$ typically depends not only upon the equivalence class $[\alpha_K] \in \Pi(G)$ but also upon the representative of this equivalence class. However, this is not the case if the Jordan type is either generic or maximal. Hence we can prove that a finite dimensional kG-module is of constant Jordan type $a_p[p] + \cdots + a_1[1]$ if and only if for each equivalence class $[\alpha_K] \in \Pi(G)$ there exists some representative $\alpha_K : K[t]/t^p \to KG$ with the property $\alpha_K^*(M_K)$ has type $a_p[p] + \cdots + a_1[1]$.

In addition we can show that modules of constant Jordan type are invariant under taking

- Direct sums,
- Direct summands,
- Tensor products,
- Heller shifts and
- Duals

The proof of these facts depend on showing that the maximal Jordan type of a module is well behaved with respect to the given operations. For example we show that if $\Pi(G)$ is irreducible, then

$$\Gamma(G)_{M\otimes N} = (\Gamma(G)_M \cup \Gamma(G)_N) \cap (\Pi(G)_M \cap \Pi(G)_N).$$

But note that we have counterexamples if the irreducibility assumption is removed.

Examples of modules of constant Jordan type include the trivial module k and $\Omega^n(k)$ for all n. Indeed, a module has constant Jordan type $[1]+n[p]$ or $[p-1]+n[p]$ for some n if and only if it is an endotrivial module. If G is the group algebra of an elementary abelian p-group then the powers of the radical of kG are modules of constant Jordan type. Other modules can be constructed by methods such as the following.

Suppose that G is a finite group scheme over k. Let M and N be kG-modules of constant Jordan type, and suppose that

$$E: \qquad 0 \longrightarrow M \longrightarrow B \longrightarrow N \longrightarrow 0$$

is an exact sequence. Let $\zeta \in \mathrm{Ext}_{kG}(N, M)$ be the class of the exact sequence E. If the for every π-point $\alpha_K : K[t]/t^p \to KG$, we have that the restriction $\alpha_K^*(\zeta)$ is zero, then B has constant Jordan type. Moreover, if the Jordan types of M and N are $\sum_{i=1}^p m_i[i]$ and $\sum_{i=1}^p n_i[i]$, then the Jordan type of B is $\sum_{i=1}^p (m_i + n_i)[i]$.

For group algebras this can be used to prove that the Auslander-Reiten (see [1]) component of a module of constant Jordan type has constant Jordan type. Hence, if kG has wild representation type, then for any n there must exist an indecomposable module with stable constant Jordan type $n[1]$.

There are many questions left open by the study. For example we do not know if there is a module of stable constant Jordan type [2] for any group scheme in which every π-point factors through an abelian unipotent subgroup scheme of rank at least two. We know that the lattice of Jordan types that are realizable by modules of constant Jordan type is a proper subgroup of the lattice of all Jordan types. But we do not know if the quotient is finite.

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On v-decomposition numbers

Kai Meng Tan

Let v be an indeterminate, and let e be an integer greater than 1. The Fock space representation $\mathcal{F}$ of $U_v(\widehat{\mathfrak{sl}}_e)$, as a $\mathbb{C}(v)$-vector space, has two distinguished bases, the standard basis $\{s(\lambda) \mid \lambda \in \mathcal{P}\}$ and the canonical basis $\{G(\lambda) \mid \lambda \in \mathcal{P}\}$, both being indexed by the set $\mathcal{P}$ of all partitions of non-negative integers. The v-decomposition number $d_{\lambda\mu}(v) \in \mathbb{C}(v)$ is the coefficient of $s(\lambda)$ when the canonical basis element $G(\mu)$ is expressed in terms of the standard basis elements, i.e.

$$G(\mu) = \sum_{\lambda \in \mathcal{P}} d_{\lambda\mu}(v) s(\lambda).$$

These v-decomposition numbers are found to enjoy remarkable properties. For example, they are parabolic Kazhdan-Lusztig's polynomials, and when evaluate at $v = 1$, they give the corresponding decomposition numbers for q-Schur algebra in characteristic zero, where q is a primitive e-th root of unity [11]. They thus have a direct relationship with the decomposition numbers of classical Schur algebras and symmetric groups.

1. Parities

One of the consequence of $d_{\lambda\mu}(v)$ being a parabolic Kazhdan-Lusztig's polynomial is that it is an even or odd polynomial in v with non-negative integer coefficients, i.e. $d_{\lambda\mu}(v) \in \mathbb{N}_0[v^2]$ or $d_{\lambda\mu}(v) \in v\mathbb{N}_0[v^2]$. Whether $d_{\lambda\mu}(v)$ is even or odd can be determined easily by a combinatorial description of the partitions λ and μ involved, as follows. The e-core of λ is obtained from λ by successively removing e-hooks from it until there is no e-hook left. In [8, Proposition (2.2) and Corollary (2.3)], Morris and Olsson showed that if we define $\delta_e(\lambda)$ as $(-1)^{l_e}$, where l_e is the sum of the leg-lengths of the e-hooks of λ removed to obtained its e-core, then this is well-defined even though l_e may not be.

Theorem 1 ([10])**.** *Suppose* $d_{\lambda\mu}(v) \neq 0$. *Then* $d_{\lambda\mu}(v) \in \mathbb{N}_0[v^2]$ *if and only if* $\delta_e(\lambda) = \delta_e(\mu)$.

This result appears to be related to the bipartite Ext-quiver conjecture for symmetric group blocks with Abelian defect groups.

Conjecture 2. *Let* D^λ *and* D^μ *be simple modules of a symmetric group block with Abelian defect group. Then* $\mathrm{Ext}^1(D^\lambda, D^\mu) = 0$ *if* $\delta_e(\lambda) = \delta_e(\mu)$.

Remark.

(1) Conjecture 2 has been recently verified for all weight 3 blocks of symmetric groups [4].
(2) Theorem 1 is used in the proof that the projective modules of weight 3 blocks of symmetric groups having Abelian defect group have a common radical length 7 [10].

2. Beyond Rouquier partitions

In a joint work with J. Chuang, we presented closed formulas for $d_{\lambda\mu}(v)$ for μ belonging to a class $\mathcal{P}_\kappa$ of e-regular partitions, which includes the important class of e-regular Rouquier partitions. Here, κ denotes an e-core partition which has an abacus display in which the number of beads on each runner is non-decreasing as we go from left to right, and $\mathcal{P}_\kappa$ is a class of partitions having e-core κ satisfying certain conditions. These formulas are found independently to hold for the entire class of Rouquier partitions, and not just the e-regular ones, by B. Leclerc and H. Miyachi. The author extends $\mathcal{P}_\kappa$ to a larger class $\mathcal{P}^*_\kappa$ which includes e-singular partitions, and in particular, the entire class of Rouquier partitions, and shows that the same formulas hold for this larger class of partitions. The class $\mathcal{P}^*_\kappa$ consists of partitions μ having e-core κ, with the following conditions. Display μ on an abacus in which the number of beads on each runner is non-decreasing as we go from left to right, and read off its e-quotient $(\mu^0, \dots, \mu^{e-1})$ from such a display (the runners are labelled 0 to $e-1$ from left to right). For each $1 \leq i < e$, let d_i be the number of beads on runner i minus the number of beads on runner $(i-1)$. Then the conditions on μ are:

- $|\mu^{i-1}| + |\mu^i| + |\mu^{i+1}| \leq d_i + 1$ for all $1 \leq i < e$ (where $|\mu^e| = 0$);

- whenever $|\mu^{i-1}| + |\mu^i| = d_i + 1$ and $|\mu^j| + |\mu^{j+1}| = d_j + 1$ for some $i < j$, there exists $i < k < j$ such that $d_k > 0$.

The precise statement is as follows:

Theorem 2 ([9, Theorem 3.7])**.** *Let $\mu \in \mathcal{P}^*_\kappa$, with e-quotient $(\mu^0, \dots, \mu^{e-1})$ as above. Then*

$$G(\mu) = \sum_{\lambda} \sum_{\substack{\alpha^0,\dots,\alpha^{e-1} \\ \beta^0,\dots,\beta^{e-2}}} v^{\sum_{j=1}^{e-1} j(|\mu^j|-|\lambda^j|)} \left(\prod_{j=0}^{e-1} c^{\lambda^j}_{\alpha^j \beta^j} c^{\mu^j}_{(\beta^{j-1})'\alpha^j} \right) s(\lambda),$$

where $\beta^{-1} = \beta^{e-1} = \emptyset$, $c^{\rho}_{\sigma\tau}$'s denote the usual Littlewood-Richardson coefficients and λ runs over all partitions having e-core κ whose e-quotient is denoted by $(\lambda^0, \dots, \lambda^{e-1})$.

We are also able to obtain the decomposition numbers $d^l_{\lambda\mu}$ of q-Schur algebras in characteristic l, where q is an element of the underlying field with e as the least positive integer such that $1 + q + \cdots + q^{e-1} = 0$, when $\mu \in \mathcal{P}^*$ as long as l is large enough:

Theorem 3 ([9, Theorem 4.7])**.** *Let $\mu \in \mathcal{P}^*_\kappa$ with e-quotient $(\mu^0, \dots, \mu^{e-1})$. Then $d_{\lambda\mu}(1) = d^l_{\lambda\mu}$ as long as $l > \max_i(|\mu^i|)$.*

Remark.

(1) Theorem 3 includes some cases which are not of 'Abelian defect'.
(2) Theorems 2 and 3 use techniques found in [5].

3. Analogue of Kleshchev's decomposition numbers

In another joint work with J. Chuang and H. Miyachi, we proved that the analogues of (generalised) row and column removal theorems for decomposition numbers of symmetric groups and Hecke algebras of type A holds for v-decomposition numbers [1, Theorem 1]. We then used this result to obtain a combinatorial description of the v-decomposition number $d_{\lambda\mu}(v)$ where the Young diagram of μ can be obtained from that of λ by moving one node. This description is in terms of latticed subsets of sign sequences first defined by Kleshchev in his determination of the corresponding decomposition numbers for the symmetric groups [6]. The precise statement is not easy to state in view of the lengthy technical definition of latticed subsets. We thus refer the interested reader to Theorem 4.2 of [2].

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Homological algebras

William B. Turner

Let R be a commutative ring. Let Q be a quiver, whose underlying graph is a tree. We reveal derived equivalences of increasing sophistication, between:

I. The path algebra RQ, and its Koszul dual.

II. The trivial extension algebra of RQ, and the trivial extension algebra of its Koszul dual.

III. The Schur algebra of RQ, and the Schur algebra of its Koszul dual.

IV. A double of the Schur algebra of RQ, and a double of the Schur algebra of its Koszul dual.

Let Q be a Dynkin quiver, of type A. We lift the derived equivalences of IV to equivalences between:

V. A deformation of the double of the Schur algebra of RQ, and a deformation of the double of the Schur algebra of its Koszul dual.

VI. A quotient of the deformation of the double of the Schur algebra of RQ, and a quotient of the deformation of the double of the Schur algebra of its Koszul dual.

Let p be a prime number, and $(K, \mathcal{O}, k)$ a p-modular system. Let Q be a Dynkin quiver, of type A_{p-1}. We conjecture that any block of a symmetric group over $\mathcal{O}$, is equivalent to a quotient of the deformation of the double of the Schur algebra of $\mathcal{O}Q$, and equivalent to a quotient of the deformation of the double of the Schur algebra of the Koszul dual of $\mathcal{O}Q$.

Runner Removal Morita Equivalences

Hyohe Miyachi
(joint work with Joseph Chuang)

The following decomposition matrices of symmetric groups $\mathfrak{S}_n$ are taken from p.600 at [JM02], which is credited to F. Lübeck and J. Müller.

$10,1$	1							
$9,2$	1	1						
$7,4$		1	1					
$7,2^2$	1	1	1	1				
$6,5$			1		1			
$6,2^2,1$	1	1	1	1	1	1		
$4^2,3$			1	1	1		1	
$4^2,2,1$		1	1	1	**2**	1	1	1

$18,3$	1							
$17,4$	1	1						
$13,8$		1	1					
$13,4^2$	1	1	1	1				
$12,9$			1		1			
$12,4^2,1$	1	1	1	1	1	1		
$8^2,5$			1	1	1		1	
$8^2,4,1$		1	1	1	**1**	1	1	1

Parts of decomposition matrices for $\mathbb{F}_3\mathfrak{S}_{11}$ and $\mathbb{F}_5\mathfrak{S}_{21}$

In their paper [JM02] G.D. James and A. Mathas explained why these two decomposition matrices look so similar to each other in much more general setting. Their explanation is given in terms of of canonical/global crystal bases in the level 1 Fock spaces over quantum algebras of affine type A and is based on Ariki[Ari96]/Varagnolo-Vasserot[VV99] 's solution of Lascoux-Leclerc-Thibon conjecture [LLT96][LT96] (cf. [Lec02]).

In my talk, from the context of module categories I shall explain why these two matrices look so similar in more general setting than James-Mathas'.

To describe more details, we need some more notation.

Let $\mathbf{k}$ be a field which is large enough for the algebra with which we deal.

We write q_m for a primitive m-th root of unity in $\mathbf{k}$ or the identity element in $\mathbf{k}$, in which case $\mathbf{k}$ has characteristic m. We denote by $\mathcal{H}_{\mathbf{k},q}(\mathfrak{S}_n)$ the Iwahori-Hecke algebra of $\mathfrak{S}_n$ over $\mathbf{k}$ with parameter q.

Let B_e be a block ideal of $H_e := \mathcal{H}_{\mathbf{k},q_e}(\mathfrak{S}_r)$.

Let $\Lambda(B_e)$ be the set of partitions λ of r such that $B_e \cdot S^\lambda \neq 0$.

Let $\Lambda^{\mathrm{reg}}(B_e)$ be the e-regular partitions in $\Lambda(B_e)$.

Let $\pi_{k,\alpha}$be the map of $\Lambda(B_e)$ into partitions defined by James-Mathas[JM02, p.601]:

$$\pi_{k,\alpha}(\lambda) = \lambda^+.$$

From now on, we assume that

$$k \geq r, \text{ the parameter } q_e, q_{e+1} \in \mathbb{C} \text{ or } \mathbb{F}_\ell \tag{1}$$

Recall that $0 \leq \alpha \leq e$ is the position where we insert the empty runner, k is the number of beads which we use in abaci.

It is actually sufficient to assume that every $\lambda \in \Lambda(B_e)$ has at most k parts, or equivalently, that the abacus for B_e with k beads contains at least w beads on each runner, where w is the e-weight of B_e.

There exists $\tilde{r}$ and a block $B_{e+1} = B_{e+1}^{k,\alpha}$ of $H_{e+1} := \mathcal{H}_{\mathbf{k},q_{e+1}}(\mathfrak{S}_{\tilde{r}})$ such that for all $\lambda \in \Lambda(B_e)$ we have $|\lambda^+| = \tilde{r}$ and $B_{e+1} \cdot S^{\lambda^+} \neq 0$.

Define $\Lambda(B_{e+1})$ and $\Lambda^{\mathrm{reg}}(B_{e+1})$ in the obvious way, and let $\Lambda^+(B_{e+1}) = \{\lambda^+ \mid \lambda \in \Lambda(B_e)\} \subset \Lambda(B_{e+1})$ and $\Lambda^{+,\mathrm{reg}}(B_{e+1}) = \{\lambda^+ \mid \lambda \in \Lambda^{\mathrm{reg}}(B_e)\} \subset \Lambda^{\mathrm{reg}}(B_{e+1})$.

Main Theorem :

Let f be an idempotent of B_{e+1} such that for all $\nu \in \Lambda^{\mathrm{reg}}(B_{e+1})$, $fD^\nu \neq 0$ if and only if $\nu \in \Lambda^{+,\mathrm{reg}}(B_{e+1})$. Then if $ch(\mathbf{k}) \geq \max\{e+1, w+1\}$,

- there exists a Morita equivalence
$$F : B_e \,\mathrm{mod} \xrightarrow{\sim} fB_{e+1}f \,\mathrm{mod}$$
such that
$$F(D^\lambda) \cong fD^{\lambda^+}$$
for all $\lambda \in \Lambda^{\mathrm{reg}}(B_e)$, and
$$F([S^\lambda]) \cong [fS^{\lambda^+}]$$
for all $\lambda \in \Lambda(B_e)$.
- we have
$$[S^\mu : D^\nu] = 0,$$
for all $\nu \in \Lambda^{+,\mathrm{reg}}(B_{e+1})$ and $\mu \in \Lambda(B_{e+1}) \setminus \Lambda^+(B_{e+1})$.

The proof is based on the following ingredients:

(1) the concrete identification via Morita equivalences at condensed Rouquier blocks [1] $f\mathrm{Rouq}f$, which is an immediate consequence of works :
   (a) For $\mathfrak{S}_n$: Chuang-Kessar[CK02], Chuang-Tan[CT03],
   (b) For $GL_n(\mathbb{F}_q)$: Turner[Tur02], Hida-Miyachi[HM00], Miyachi[Miy01].
(2) (When we deal with characteristic zero Hecke algebras: Lifting Morita equivalence of Rouquier block of Hecke algebra in postive characteristic to characteristic zero.)
(3) Chuang-Rouquier's $\mathfrak{sl}_2$-categorifications for symmetric groups and Iwahori-Hecke algebras.[CRb]
(4) Chuang-Rouquier's perverse equivalences. [CRa]

There are variants of this theorem for Schur algebras. By applying some column removal theorem, we can obtain
$$[S^\lambda : D^\mu]_{H_e} = [S^{\lambda^+} : D^{\mu^+}]_{H_{e+1}}$$
which cover all the decomposition numbers that James and Mathas discussed in [JM02]. One of the merits of our approach is that our theorem works also at abelian defect blocks in postive characteristics. There is yet another meaning of Removal in the title. I shall mention this meaning in the talk.

[1] Here, we mean by "Rouquier block" the one which was predicted in [Rou98] and their variants. There is a generalization of Lusztig's family of Weyl groups to complex reflection groups. This generalized family is also called "Rouquier family or Rouquier block" which agree with Gyoja's homological definition/ interpretation of Lusztig's family in the case of Weyl group (one parameter, crystallographic) .

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On Generalized Characters of Nilpotent Groups, with applications to bounding numbers of conjugacy classes and numbers of characters in blocks

Geoffrey R. Robinson

NOTATION AND TERMINOLOGY: If A is a finite group, $k(A)$ denotes the number of conjugacy classes of A. If A normalizes another group B, then $k_A(B)$ is the number of A-conjugacy classes of B. If A is a subgroup of a group B, then $k_+(B, A)$ is the number of irreducible characters of B which don't vanish identically on $A^\#$. (For example, if P is a Sylow p-subgroup of a finite group G, then $k_+(G, P)$ is the number of irreducible characters of G which don't lie in p-blocks of defect 0, which indicates one reason for our interest in this quantity). Notice also that $k_+(B, A) = k(B)$ if $\cap_{b\in B} A^b \neq 1$ (by Clifford's theorem). By a rational conjugacy class $\mathcal{C}$ of the finite group A, we mean a union of conjugacy classes which is minimal subject to the requirement that if $a \in \mathcal{C}$ and $\langle b\rangle = \langle a\rangle$, then also $b \in \mathcal{C}$. We recall that if A is a finite group with $C_A(F(A)) = Z(F(A))$, where $F(A)$ is the largest nilpotent normal subgroup of A, then a nilpotent injector of A is a maximal nilpotent subgroup of A containing $F(A)$.

INTRODUCTION: For some time, I have been interested in the following invariant of a finite group H :
$\sigma(H^{\#}) = \min\{\sum_{h\in H^{\#}} |\theta(h)|^2 : \theta$ is a generalized character of H which is not a multiple of the regular character.$\}$. One reason for this interest is that when H is embedded in another finite group G, the orthogonality relations yield:

$$k_+(G,H) \leq \frac{\sum_{x\in H^{\#}} |C_G(x)|}{\sigma(H^{\#})} \leq (\max_{x\in H^{\#}}[C_G(x) : C_H(x)])(k(H)-1)\frac{|H|}{\sigma(H^{\#})}.$$

It is conceivable that this invariant could be useful in other contexts: there seems to be quite subtle relationship between $\sigma H^{\#}$ and the group-theoretic structure of H. For example, if there is a proper set of prime divisors π of H such that no non-identity π-element of G commutes with a non-identity π'-element, then $\sigma(H^{\#})$ is at most the number of non-identity π-elements of H.

In [2], I showed that $\sigma(H^{\#}) \geq k(H) - 1$ for every H. Equality is clearly attained for Abelian H, but is also attained when H is a Frobenius group of order $p^n(p^n - 1)$ with cyclic complement of order $p^n - 1$ (for any prime p and positive integer n.)However, the conditions under which this bound can be attained are very restrictive. Notice that for the above Frobenius groups,

$$(\sqrt{|H|} - 1) < \sigma(H^{\#}) < \sqrt{|H|}.$$

It may be of some interest to note that (as made explicit in [2] and [4]) there are two essentially distinct possibilities for a generalized character θ of H which realises the minimal value $\sigma(H^{\#})$:
i) There is a sign ε such that θ agrees with $\varepsilon\chi$ on non-identity elements of H.
ii) (Up to multiplying by ± 1 and adding integer multiples of the regular character) θ is a rational-valued character of H with $0 < \theta(1) \leq \frac{|H|}{2}$.

Another bound, noted in [4], which can be useful is that $\sigma(H^{\#})$ is at least the size of the smallest non-trivial rational conjugacy class of H. Equality is attained for $H = \mathrm{SL}(2, 2^n)$, which has $\sigma(H^{\#}) = 2^{2n} - 1$).

In [4], I was able to prove the following:
THEOREM 1: *Let H be a finite p-group for some prime p. Then $\sigma(H^{\#}) = |H| - \chi(1)^2$, where χ is an irreducible character of maximal degree of H. Hence if H is a Sylow subgroup of a finite group G, then*

$$k_+(G,H) \leq (k(H)-1)\frac{|H|}{|H| - \chi(1)^2}(\max_{x\in H^{\#}}[C_G(x) : C_H(x)]).$$

An extension of this result to nilpotent groups might be relevant to questions posed by several mathematicians. For example, in [6], J.G. Thompson asked whether $k(G) \leq |I|$ when G is solvable and I is a nilpotent injector of G. The same question could be posed whenever $C_G(F(G)) \leq F(G)$ (and Brauer's $k(B)$-problem for p-constrained groups (which is open at present, though resolved for p-solvable groups) would be a particular case). Also, in [1], Liebeck and Pyber asked whether a finite group G always has a nilpotent subgroup N with $k(G) \leq |N|$.

Recently, we have been successful in extending the result to a general nilpotent group H :

THEOREM 2 ([5]): *Let N be a finite nilpotent group. Then $\sigma(N^{\#}) = |N| - \chi(1)^2$, where χ is an irreducible character of N of maximal degree.*

COROLLARY 3: ([5]) *Let N be a nilpotent subgroup of the finite group G and let χ be an irreducible character of maximal degree of N and let $C = \cap_{g\in G} N^g$. Then:*
i)

$$k_+(G,N) \le \frac{|N|}{|N|-\chi(1)^2}(k_N(G) - [G:N]) \le \frac{|N|(k(N)-1)}{|N|-\chi(1)^2}\max_{x\in N^{\#}}[C_G(x):C_N(x)].$$

Furthermore,

$$k_+(G,N) \le \frac{|N|-1}{|N|-\chi(1)^2}\max_{x\in N^{\#}}|C_G(x)|.$$

ii) *If $C \ne 1_G$, then $k(G) \le |C_G(z)|$ for some $z \in Z(C)^{\#}$ and*

$$k(G) \le \frac{|N|}{|N|-\chi(1)^2}(k_N(G) - [G:N]) \le \frac{|N|(k(N)-1)}{|N|-\chi(1)^2}\max_{x\in N^{\#}}[C_G(x):C_N(x)].$$

We also mention that the result of [4] for p-groups had a more subtle block-theoretic version which stated that if D was a non-trivial defect group for a p-block B of G, then for some primitive idempotent e of $RC_G(D)$ (in a root of B, R being the "usual" complete discrete valuation ring), we have

$$k(B)(|D| - \mu(1)^2) \le \sum_{x\in D^{\#}} \operatorname{rank}_R(eB(x)e),$$

where $B(x)$ is the sum of the Brauer correspondent blocks of B for $RC_G(x)$ and μ being an irreducible character of D of maximal degree.

With the aid of the main result of [3], this can be made more palatable (but generally weaker), by noting that

$$\operatorname{rank}_R(eB(x)e) \le [C_G(x):C_G(D)]|Z(D)|$$

(since e is in a block with defect group $Z(D)$).

This yields:

THEOREM 4:([5])*Let G be a finite group, p be a prime, and let B be a p-block of G with defect group $D > 1$. Let μ be an irreducible character of maximal degree of D. Then we have*

$$k(B) \le \frac{|D|(k(D)-1)\max_{x\in D^{\#}}[C_G(x):C_D(x)]}{(|D|-\mu(1)^2)[C_G(D):Z(D)]}.$$

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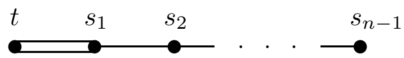
Kazhdan-Lusztig cells with unequal parameters

Lacrimioara Iancu

We are interested in the theory of Kazhdan–Lusztig cells for finite Coxeter groups, especially in the case of "unequal parameters". In his recent book [5], Lusztig has formulated a number of precise conjectures, concerning the a-function, distinguished involutions, constructible characters and so on. Our (long-term) aim is to settle (at least some of) these conjectures for a Coxeter group W of type B_n, with diagram given as follows.

B_n



The parameters are specified by a weight function $L\colon W \to \mathbb{Z}$ such that

$$b := L(t) > 0 \qquad \text{and} \qquad a := L(s_1) = L(s_2) = \cdots = L(s_{n-1}) > 0.$$

In a joint paper with C. Bonnafé (see [1]), we gave an explicit combinatorial description of the left cells in the case where b is large with respect to a, in terms of a generalised Robinson–Schensted correspondence. More recently, in a joint paper with M. Geck (see [4]), and based on results of Bonnafé [3], we have determined the a-function and proved several of Lusztig's conjectures for this case.

We now turn to the general case, where b is not necessarily large with respect to a. This is a joint project and work in progress with C. Bonnafé, M. Geck and T. Lam. It turns out that there are three essentially different cases.

(ASY) This is the "asymptotic case" already mentioned above, where $b/a > n-1$.

(FRAC) This is the case where $r-1 < b/a < r$ for some $r \in \{1, \ldots, n-1\}$. We conjecture that the partition into left cells is given by the "domino insertion with respect to an r sized 2-core", as studied by Shimozono–White and Lam; in particular, the partition into left cells should only depend on r. Let us call these cells "r-cells". Furthermore, all representations carried by left cells will be irreducible.

(INT) This is the case where $b/a = r \in \{1, \ldots, n-1\}$. Then the left cells should be the minimal subsets of W which are at the same time unions of r-cells and of $(r+1)$-cells.

These conjectures are supported by empirical evidence from computations with GAP. Our hope is that, once these conjectures are proven, they will enable us to make some progress concerning a proof of Lusztig's conjectures mentioned above. (Note, on the other hand, that Lusztig's conjectures wouldn't imply our conjectured combinatorial description of the left cells.)

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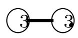
Spetses for primitive reflection groups

Jean Michel

(joint work with M. Broué, G. Malle)

Spetses. Spetses is a joint idea of Michel Broué, Gunter Malle, J. M. which we had during a nice conference in Greece in august 1993. We build for some complex reflection groups called *spetsial* data such as

Unipotent characters for G_4



Character	Degree	FakeDegree	Eigenvalue	Family
$\phi_{1,0}$	1	1	1	C_1
$\phi_{2,1}$	$\frac{3-\sqrt{-3}}{6}q\Phi_3'\Phi_4\Phi_6''$	$q\Phi_4$	1	$X_3.01$
$\phi_{2,3}$	$\frac{3+\sqrt{-3}}{6}q\Phi_3''\Phi_4\Phi_6'$	$q^3\Phi_4$	1	$X_3.02$
$Z_3:2$	$\frac{\sqrt{-3}}{3}q\Phi_1\Phi_2\Phi_4$	0	ζ_3^2	$X_3.12$
$\phi_{3,2}$	$q^2\Phi_3\Phi_6$	$q^2\Phi_3\Phi_6$	1	C_1
$\phi_{1,4}$	$\frac{-\sqrt{-3}}{6}q^4\Phi_3''\Phi_4\Phi_6''$	q^4	1	$X_5.1$
$\phi_{1,8}$	$\frac{\sqrt{-3}}{6}q^4\Phi_3'\Phi_4\Phi_6'$	q^8	1	$X_5.2$
$\phi_{2,5}$	$\frac{1}{2}q^4\Phi_2^2\Phi_6$	$q^5\Phi_4$	1	$X_5.3$
$Z_3:11$	$\frac{\sqrt{-3}}{3}q^4\Phi_1\Phi_2\Phi_4$	0	ζ_3^2	$X_5.4$
G_4	$\frac{1}{2}q^4\Phi_1^2\Phi_3$	0	-1	$X_5.5$

Φ_3', Φ_3'' (resp. Φ_6', Φ_6'') are factors of Φ_3 (resp Φ_6) in $\mathbb{Q}(\zeta_3)$.

The Fourier matrix for G_4.

		01	02	12		1	2	3	4	5
	1	.	.	.	.	.	.	.	.	.
01	.	$\frac{3-\sqrt{-3}}{6}$	$\frac{3+\sqrt{-3}}{6}$	$\frac{\sqrt{-3}}{3}$	.	.	.	.	.	.
02	.	$\frac{3+\sqrt{-3}}{6}$	$\frac{3-\sqrt{-3}}{6}$	$-\frac{\sqrt{-3}}{3}$	.	.	.	.	.	.
12	.	$\frac{\sqrt{-3}}{3}$	$-\frac{\sqrt{-3}}{3}$	$\frac{\sqrt{-3}}{3}$	.	.	.	.	.	.
	.	.	.	.	1	.	.	.	.	.
1	.	.	.	.	.	$-\frac{\sqrt{-3}}{6}$	$\frac{\sqrt{-3}}{6}$	$\frac{1}{2}$	$\frac{\sqrt{-3}}{3}$	$\frac{1}{2}$
2	.	.	.	.	.	$\frac{\sqrt{-3}}{6}$	$-\frac{\sqrt{-3}}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{-3}}{3}$	$\frac{1}{2}$
3	.	.	.	.	.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	.	$-\frac{1}{2}$
4	.	.	.	.	.	$\frac{\sqrt{-3}}{3}$	$-\frac{\sqrt{-3}}{3}$	.	$\frac{\sqrt{-3}}{3}$	.
5	.	.	.	.	.	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	.	$\frac{1}{2}$

Finite reductive groups. Let $\mathbf{G}$ be a reductive group over $\overline{\mathbb{F}}_q$, and F an isogeny such that $\mathbf{G}^F$ is finite (a finite group of Lie type). In this situation the *unipotent characters* are the irreducible constituents of $R^{\mathbf{G}}_{\mathbf{T}_w}(\mathrm{Id})$. The $\mathbf{G}^F$-conjugacy classes of maximal tori are parameterized by the F-classes of the Weyl group W, and for $\chi \in \mathrm{Irr}(W)$ we define the *almost characters* $R_\chi := \frac{1}{|W|}\sum_{w\in W}\chi(w)R^{\mathbf{G}}_{\mathbf{T}_w}(\mathrm{Id})$ [to simplify we assume $(\mathbf{G}, F)$ split]. The *fake degree* of χ is the degree of R_χ.

Lusztig's Fourier matrix S is the decomposition matrix $R_\chi/$ unipotent characters. We complete it to a square matrix by using that the R_χ are some of Lusztig's *character sheaves*, which give a basis of the class functions on $\mathbf{G}^F$. We add the other unipotent character sheaves. The blocks of S are *Lusztig's families of unipotent characters.*

For an F-stable Levi subgroup $\mathbf{L}$ of an F-stable parabolic subgroup $\mathbf{P}$ we have Harish-Chandra induction $R^{\mathbf{G}}_{\mathbf{L}}$. An irreducible character is *cuspidal* if it does not occur in a proper Harish-Chandra induced. If λ is a cuspidal unipotent character of $\mathbf{L}^F$, then the group $W_{\mathbf{G}}(\mathbf{L},\lambda) := N_{\mathbf{G}^F}(\mathbf{L},\lambda)/\mathbf{L}^F$ is a Coxeter group and $\mathrm{End}_{\mathbf{G}^F}(R^{\mathbf{G}}_{\mathbf{L}}(\lambda))$ is a Hecke algebra for $W_{\mathbf{G}}(\mathbf{L},\lambda)$. Thus unipotent characters are parameterized by triples $(\mathbf{L},\lambda,\chi)$ where $\chi \in \mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L},\lambda))$.

We have $R^{\mathbf{G}}_{\mathbf{T}_w}(g) = \sum_i (-1)^i \,\mathrm{Trace}(g \mid H^i_c(\mathbf{X}_w, \mathbb{Q}_\ell))$. Each $H^i_c(\mathbf{X}_w, \mathbb{Q}_\ell))$ is a $\mathbf{G}^F \times {<}F{>}$-module. For a unipotent character ρ, the eigenvalues of F on the ρ-isotypic part of $H^i_c(\mathbf{X}_w, \mathbb{Q}_\ell))$ are a power of $q^{1/2}$ times a root of unity λ_ρ, and λ_ρ is independent of i or w. λ_ρ is the *eigenvalue of Frobenius* attached to ρ.

Reflection data. By the isogeny theorem, $(X(\mathbf{T}), Y(\mathbf{T}), \Phi, \Phi^\vee, \phi, q)$ describes $(\mathbf{G}, F)$, where ϕ is an element of finite order of $\mathrm{GL}(X(\mathbf{T}))$ such that $F = q\phi$ [when $\mathbf{G}$ is split we have $\phi = \mathrm{Id}$]. We claim that all the data above (unipotent characters, degrees, Fourier matrix, eigenvalues of Frobenius) can be obtained as a combinatorial game starting from just:

- $W \subset \mathrm{GL}(V)$, where $V = X(\mathbf{T}) \otimes \mathbb{C}$.

- $\phi \in \mathrm{GL}(V)$ of finite order, normalizing W.
- q.

This is the *reflection data* for $\mathbf{G}$.

Hecke algebras for complex reflection groups. Let W be an arbitrary irreducible complex reflection group. We assume W *well generated*; thus it has $\dim V$ generators. The braid group is

$$B := \Pi_1((V - \{\text{reflecting hyperplanes for } W\})/W, x);$$

it can be generated by the same number of *braid reflections*, and has a presentation by *braid relations* of the form $w_1 = w_2$ where w_1, w_2 are positive words of the same length in the braid reflections (Bessis). If $\mathbf{s}$ is a braid reflection, its image in W is a *distinguished* reflection, i.e. its non-trivial eigenvalue is of the form $e^{2i\pi/e}$. One gets a presentation of W by adding the relations $\mathbf{s}^e = 1$. We have explicit Coxeter-like diagrams for the presentations (Broué-Malle-Rouquier, Bessis-M.).

For a reductive group, the Hecke algebra of $\mathbf{G}$ is the quotient of $\mathbb{C}[q]B$ by $(\mathbf{s} - q)(\mathbf{s} + 1) = 0$. For a complex reflection group, the *Spetsial Hecke algebra* $\mathcal{H}$ of W is the quotient of $\mathbb{C}[q]B$ by $(\mathbf{s} - q)(\mathbf{s} - \zeta_e)(\mathbf{s} - \zeta_e^2)\dots(\mathbf{s} - \zeta_e^{e-1}) = 0$ for a reflection of order e.

This algebra $\mathcal{H}$ should be symmetric with "canonical" symmetrizing trace. we should have the "Tits deformation theorem": for a specialization which does not map q to not a root of unity, $\mathcal{H} \mapsto \mathbb{C}[W]$; τ specializes to the canonical trace on $\mathbb{C}W$. Write $\tau = \sum_{\chi \in \mathrm{Irr}(W)} \chi/S_\chi$; the S_χ are the *Schur elements*; they are Laurent polynomials in some root of q ($\mathcal{H}$ splits over an extension of the form $\mathbb{C}[q^{1/a}]$).

Spetsial groups. The *generic degree* of χ is $D_\chi := S_{\mathrm{Id}}/S_\chi$. We call W Spetsial if D_χ are polynomials in q. The Spetsial groups are $G(e,1,r), G(e,e,r)$ and

group	4	5	6	7	8	9	10	11	12	13	14	15	16
dimension	2	2	2	2	2	2	2	2	2	2	2	2	2
Spetsial	*		*		*						*		

group	17	18	19	20	21	22	23	24	25	26	27
dimension	2	2	2	2	2	2	3	3	3	3	3
Spetsial							H_3	*	*	*	*

group	28	29	30	31	32	33	34	35	36	37
dimension	4	4	4	4	4	5	6	6	7	8
Spetsial	F_4	*	H_4		*	*	*	E_6	E_7	E_8

They are all well-generated.

Spetses. We call *Spetses* a list of: unipotent degrees, parameters for relative cylotomic Hecke algebras, eigenvalues of Frobenius, Fourier matrices which satisfy (many) axioms.

Lusztig knew already a solution for Coxeter groups which are not Weyl groups (except the Fourier matrix for H_4 which was determined by Malle in 1994). Malle gave a solution for imprimitive Spetsial complex reflection groups in 1995, and also proposed (unpublished) data for most primitive Spetsial groups. But it was not clear if the data above were uniquely defined. We can now show there is a unique solution for all primitive complex reflection groups.

We start from the principal series where we know (by definition) the parameters. We find the intersection with the "series $R_{T_w}^{\mathbf{G}}(\mathrm{Id})$" where $\mathbf{w}$ is a dth-root of $\boldsymbol{\pi}$ by the test $\mathrm{Dim}(\rho)(\zeta_d) \neq 0$. We manage to recover the parameters of the Hecke algebra "$\mathrm{End}_{\mathbf{G}^F} R_{T_w}^{\mathbf{G}}(\mathrm{Id})$" from this. We know when we have found all degrees by

$$\sum_{\rho \text{ unipotent}} \mathrm{Dim}(\rho)\overline{\mathrm{Dim}(\rho)} = \sum_{\chi \in \mathrm{Irr}(W)} (\text{Fake degree}(\chi))^2.$$

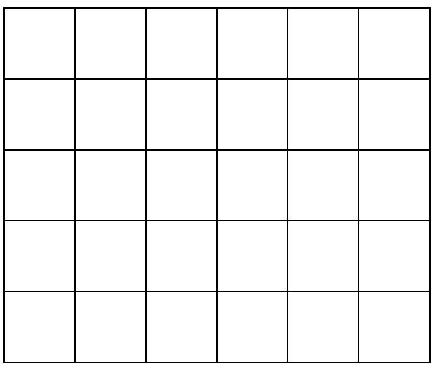
This equality holds restricted to each Lusztig family. Each family intersects the principal series, and Lusztig has described the intersection, defining thus *families of characters of the Hecke algebra*. Rouquier has observed that they are blocks of the Hecke algebra over the ring $\mathbb{Z}[\{\zeta_{e_s}\}_{s \,\in\, \{\text{reflections of } W\}}, q, q^{-1}, \{(q^n-1)^{-1}\}_{n>1}]$. This makes sense for arbitrary complex reflection groups and the *Rouquier blocks* have been computed by Broué-Kim and Malle-Rouquier.

In the talk, we went on describing axioms on the Fourier matrices and families allowing us to construct Spetses for Spetsial groups.

Rank Polynomials

Gordon James
(joint work with Marco Brandt, Richard Dipper and Sinéad Lyle)

Consider a rectangular a by b array of boxes with a less than or equal to b; for example



where $a = 5$ and $b = 6$. Think of this as a street map in an American city, and choose a shortest route from the northwest corner to the southeast corner. It is a well known puzzle to determine the number of such shortest routes, and the answer is easily seen to be the binomial coefficient $\binom{a+b}{a}$.

Next, label each corner on our street plan by an ordered pair (i,j), where this refers to the intersection of the ith horizontal line from the top with the jth vertical line from the left. How many shortest routes have the property that all corners we visit have j greater than or equal to i? Once we recognise that we are counting the number of standard (b,a)-tableaux, we see that the answer is $\binom{a+b}{a} - \binom{a+b}{a-1}$.

We shall now present q-analogues of the last two results. Given a shortest route p let $N(p)$ denote the number of boxes below p. Then the sum over shortest routes p of $q^{N(p)}$ is $\begin{bmatrix} a+b \\ a \end{bmatrix}$, the q-binomial coefficient which counts the number of a-dimensional subspaces of an $(a+b)$-dimensional space over the field of q elements.

Note that $q^{N(p)}$ is the number of ways of filling the boxes below p with elements from the field of q elements. Suppose that X is a filling of the boxes below p with elements of the field of q elements. We say that X is *good* if, for each corner (i,j) through which p passes, the matrix whose bottom left hand corner is $(a+1,1)$ and whose top right hand corner is (i,j) has rank at most $j-i$. We define the rank polynomial $r(p)$ of p to be the number of ways of filling the boxes below p with elements of the field of q elements such that the filling is good. Note that $r(p)=0$ unless all corners on p have j greater than or equal to i.

The polynomials $r(p)$ are difficult to calculate - they are truly rank polynomials. With great difficulty we have been able to prove the following result.

Theorem 1. *The sum over shortest routes* p *of* $r(p)$ *is equal to* $\begin{bmatrix} a+b \\ a \end{bmatrix} - \begin{bmatrix} a+b \\ a-1 \end{bmatrix}$.

The motivation for introducing rank polynomials lies in the theory of $F\,\mathrm{GL}_n(q)$-modules where F is a field of characteristic coprime to q. Assume that $a+b=n$, with a at most b. Let $M^{(b,a)}$ denote the vector space consisting of all F-linear combinations of a-dimensional subspaces of the n-dimensional space on which $\mathrm{GL}_n(q)$ acts. Then $M^{(b,a)}$ is an $F\,\mathrm{GL}_n(q)$-module of dimension $\begin{bmatrix} a+b \\ a \end{bmatrix}$.

Now, given a subspace U of $M^{(b,a)}$ it is easy to associate with each shortest route p a subspace $U(p)$, such that the dimension of U equals the sum over p of the dimension of $U(p)$.

Taking U to be the Specht module $S^{(b,a)}$ and p to be a shortest route, we may state our main theorem.

Theorem 2. *We have that the dimension of* $S^{(b,a)}(p)$ *is equal to the rank polynomial* $r(p)$.

The proof consists first of finding $r(p)$ linearly independent elements in $S^{(b,a)}(p)$, and then using Theorem 1 and the fact that the dimension of $S^{(b,a)}$ is known to be equal to $\begin{bmatrix} a+b \\ a \end{bmatrix} - \begin{bmatrix} a+b \\ a-1 \end{bmatrix}$ to complete the proof.

Theorem 2 represents a major step towards answering a question first raised in [4] and on which partial progress had been made in [1, 3, 5].

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The group of endotrivial modules for the symmetric group

Nadia Mazza
(joint work with Jon Carlson, Daniel Nakano)

Endotrivial modules are known to play an important role in the modular representation theory of finite groups. For one thing, they are the building blocks of the endo-permutation modules. For another, they are a part of the Picard group of self equivalences of the stable category of kG-modules in the case that G is a finite group and k is a field of characteristic p. A few years ago, Jon Carlson and Jacques Thévenaz completed a program to classify the endotrivial kG-modules over p-groups [4]. Building on this result, Bouc [1] has competed a similar program to classify the endo-permutation modules over p-groups.

More recently, we (i.e. Jon Carlson, Nadia Mazza and Daniel Nakano) started a project to classify the endotrival modules over families of finite simple groups and related groups (cf. [3]). We have found the structure of the group of endotrivial modules and the actual generators for the group, in almost all cases, for the finite groups of Lie type in the defining characteristic. In addition, we also established the classification of endotrivial modules in case the considered group has a normal Sylow p-subgroup. Both results build on the classification for p-groups and on results in [2]. In the case of the finite groups of Lie type, as expected, the group of endotrivial modules is cyclic, generated by the class of the syzygy module $\Omega(k)$, except in the cases that the group has Lie rank 1, or that the Lie rank is 2 and the field of definition of the group is small. Let us recall that $\Omega(k)$ is endotrivial for any group G, and moreover, its class generates a direct summand of $T(G)$, which has infinite order unless G has normal p-rank 1 (i.e. if G has a Sylow p-subgroup cyclic or quaternion).

The natural next case, which we handle in the present collaboration, is the group of endotrivial modules for the symmetric (and alternating) groups. In spite of the advanced nature of the representation theory of symmetric groups, the situation has turned out to be considerably more difficult than we expected. To this point, we have only partial results, which we describe after a brief recall of the needed notions.

We consider the symmetric group G on n letters, p a prime number and k an algebraically closed field of characteristic p. All kG-modules are finitely generated left kG-modules. A kG-module M is endotrivial provided that its endomorphism algebra $\mathrm{End}_k M$ is isomomorphic, as kG-module, to the direct sum $k \oplus$ (proj) of the trivial module and a projective module. Equivalently, M is endotrivial if $\mathrm{End}_k M \cong k$ in the stable module category $\mathbf{stmod}(kG)$. The tensor product (over k, with diagonal G-action) of endotrivial kG-modules induces a structure of abelian group on the set $T(G)$ of isomorphism classes of endotrivial modules in $\mathbf{stmod}(kG)$. We call $T(G)$ the *g*roup of endotrivial kG-modules. It is known that $T(G)$ is finitely generated and its torsion free rank equals the number $c_2(G)$ of G-conjugacy classes of maximal elementary abelian p-subgroups of G of p-rank 2, or $c_2(G)+1$ if G has p-rank at least 3. Let us point out that, however the torsion free rank can (relatively easily) be computed, if it is at least 2, then we have no method that would allow us to determine a complete set of generators for $T(G)$ unless $p=2$ and the 2-Sylow subgroups are dihedral.

The general strategy that we adopt to handle the problem can be outlined as follows. First, we determine the group of endotrivial modules for the normalizer N of a Sylow p-subgroup in G. For this, we use the results on the group of endotrivial modules in the case of a normal Sylow p-subgroup. Thereafter, in order to go from $T(N)$ to $T(G)$, we apply, on one hand, general facts of group theory (such as the Green correspondence and the Mackey formula), and, on the other hand, we rely on the well-known theory of the representation of the symmetric group (cf. [6]).

For the symmetric group G, in the case that $p=2$, the problem is reasonably straightforward because the Sylow 2-subgroups are self normalizing except in a couple of well known cases. If p is odd, then we could only answer the question in the case that the Sylow p-subgroups of G are abelian. A surprise in the odd characteristic case for the symmetric groups is that there are several Young modules which are endotrivial.

We strongly suspect that for $p>2$ and $n \geq p^2$, the group $T(G)$ has no torsion, beyond that coming from the sign representation. In addition to this obstacle, the torsion free rank is two when $p^2 \leq n < p^2+p$, in which case, we have no method to find generators for the torsion free part of the group.

Our main result is the following.

Theorem. Let $G,\ p,\ k$ and $T(G)$ as defined above.

(1) If $p=2$, then

$$T(G) \cong \begin{cases} 0 & \text{if } n \leq 3 \\ \mathbb{Z}^2 & \text{if } n = 4,\ 5 \\ \mathbb{Z} & \text{if } n \geq 6 \end{cases}$$

More precisely, if $n=4$, the Specht module $S^{(3,1)}$ corresponding to the partition $(3,1)$ of 4 is endotrivial, and $T(G)$ is generated by its class and the class of $\Omega(k)$. If $n=5$, then the kG-Green correspondents of the modules for $n=4$ are endotrivial and their classes generate $T(G)$.

(2) If $p \geq 3$, then

$$T(G) \cong \begin{cases} 0 & \text{if } n < p \\ \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n = p,\ p+1 \\ \mathbb{Z}/(p-1)\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } p+2 \leq n < 2p \\ \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } 2p \leq n < 3p \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } 3p \leq n < p^2 \end{cases}$$

In particular, if $2p \leq n < 3p$, then the Young modules labeled by the partitions $(n-p, p)$ are endotrivial and their class is an element of order 2 in $T(G)$.

We end this abstract with a remark on the results of our joint work that were not presented in the talk. Namely, we worked and proved similar results for the alternating groups, for the same range of n's.

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On Blocks of finite reductive groups

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(joint work with Paul Fong & Bhama Srinivasan)

[relying on work of M.Broué–G.Malle–J.Michel, M.Cabanes–M.Enguehard, and of course G.Lusztig]

1. Notation and motivations

For G a finite group, $\mathrm{Irr}(G)$ denotes the set of irreducible characters of G over $\bar{\mathbb{Q}}$. For $\chi \in \mathrm{Irr}(G)$, its Schur element is

$$s_\chi := |G|/\chi(1)\,.$$

For ℓ a prime, and $\delta \in \mathbb{N}$, set

$$\mathrm{Irr}(G)_\delta := \{\chi \mid (s_\chi)_\ell = \ell^\delta\}$$

and for $r \in (\mathbb{Z}/\ell\mathbb{Z})^\times$, set

$$\mathrm{Irr}(G)_{\delta,r} := \{\chi \mid (\ell^{-\delta} s_\chi) \equiv \pm r \bmod \ell\}\,.$$

1.1. **Global–local conjectures.** Let G be a finite group, let ℓ be a prime, let S be an ℓ–subgroup of G. We set $|S|_\ell = \ell^a$.

1.1.1. *"Direct counting" conjectures.* They directly relate "global" datum to "local" datum. Note that they are not stated here in full generality.

Conjecture 1 (Alperin–McKay)**.**

$$|\operatorname{Irr}(G)_a| = |\operatorname{Irr}(N_G(S))_a|\,.$$

Conjecture 2 (Isaacs–Navarro)**.**

$$|\operatorname{Irr}(G)_{a,r}| = |\operatorname{Irr}(N_G(S))_{a,r}|\,.$$

1.1.2. *Alternating sums conjectures.* (Knörr–Robinson, Dade, Uno)

The MAKRODINU (**M**ckay**A**lperin**K**nörr**RO**binson**D**ade**I**saacs**N**avarro**U**no) conjecture may be stated as follows.

Conjecture 3 (MAKRODINU)**.** *Assume $O_\ell(G) = 1$. Given an ℓ–block B of G, we have*

$$\sum_{C,b} (-1)^{|C|}\, |\operatorname{Irr}(N_G(C), b)_{\delta,r}| = 0$$

where C runs over chains of ℓ–subgroups modulo G–conjugation and b runs over ℓ–blocks of $N_G(C)$ such that $b^G = B$.

1.2. **General motivation.** Assume B is a block of B with abelian defect group D. Let b be the corresponding block of $N_G(D)$. Then

Conjecture 4 (ADC)**.** *There is an isotypie*

$$\mathbb{Z}\operatorname{Irr}(G, B) \xrightarrow{\sim} \mathbb{Z}\operatorname{Irr}(N_G(D), b)$$

Since an isotypie exchanges $\operatorname{Irr}(G, B)_{\delta,r}$ and $\operatorname{Irr}(N_G(D), b)_{\delta,r}$, we see that

Proposition 5. *In the abelian defect group case, conjecture (ADC) implies conjecture (MAKRODINU).*

We are looking for bijections which extend (ADC) to the nonabelian defect case and would imply (MAKRODINU). This is what is achieved for finite reductive groups (at least with some "good" restrictions, and for unipotent blocks).

2. Prerequisites on finite reductive groups

2.1. **Notation.**

- $\mathbf{G}$ is a connected reductive algebraic group over $\bar{\mathbb{F}}_q$, endowed with a Frobenius–like endomorphism F. The pair $(\mathbf{G}, F)$, or by abuse of language the group $G := \mathbf{G}(q) := \mathbf{G}^F$, is a *finite reductive group*.
- $\mathbb{G}$ is the *type* of G, consisting of the root datum of $\mathbf{G}$ endowed with the automorphism defined by F. There is a polynomial in $\mathbb{Z}[x]$

$$|\mathbb{G}|(x) = x^N \prod_d \Phi_d(x)^{a(d)} \quad \text{such that } |\mathbb{G}|(q) = |G|\,.$$

- The set UnIrr(G) of unipotent characters of G is naturally parametrized by a set UnIrr($\mathbb{G}$). For $\rho \in \mathrm{UnIrr}(\mathbb{G})$ there is $\mathrm{Deg}_\rho(x) \in \mathbb{Q}[x]$ such that $\mathrm{Deg}_\rho(q) = \chi_\rho(1)$. The (polynomial) Schur element of ρ is

$$s_\rho(x) := |\mathbb{G}|(x)/\,\mathrm{Deg}_\rho(x) \in \mathbb{Z}[x]\,.$$

2.2. **Generic Sylow theory.** A Φ_d group is a finite reductive group whose polynomial order is a power of $\Phi_d(x)$ (hence it is a torus). The centralisers of Φ_d–subgroups are called *d-split subgroups*.

Theorem 6. *(1) Maximal Φ_d–subgroups of G have (polynomial) order $\Phi_d(x)^{a(d)}$ and they are all conjugate by G.*

(2) If $(\mathbf{S}, F)$ is a Sylow Φ_d-subgroup, the automiser $N_G(\mathbf{S})/C_G(\mathbf{S})$ is a (complex) reflection group in its natural representation on $\mathbb{C} \otimes X(\mathbf{S})$.

2.3. **Jordan decomposition.** By work of Lusztig and al., there is a partition ("*Jordan decomposition into rational series*")

$$\mathrm{Irr}(G) = \dot{\bigcup} \mathrm{Irr}(G,(s))$$

where s runs over a complete set of representatives of the $\mathbf{G}^*(q)$–conjugacy classes of semisimple elements of $\mathbf{G}^*(q)$ ($\mathbf{G}^*$ the dual group of $\mathbf{G}$).

We shall use Jordan decomposition of irreducible characters in a particular case.

Definition 7 (*cf.* [1] or [2])**.** *The prime ℓ is said to be excellent for $\mathbb{G}$ if the following conditions are satisfied :*

(VG1) *ℓ is good for the root datum of $\mathbb{G}$,*

(VG2) *ℓ does not divide the order of the automorphism of the root datum defined by the Frobenius endomorphism F,*

(VG3) *ℓ does not divide $|(Z\mathbf{G}/Z^0\mathbf{G})^F| \times |(Z\mathbf{G}^*/Z^0\mathbf{G}^*)^{F^*}|$.*

A semi-simple element is said to be excellent if its order is only divisible by excellent primes.

Assume that s is an excellent element of $\mathbf{G}^*(q)$. Then the connected component $C_{\mathbf{G}^*}(s)^0$ of its centralizer in $\mathbf{G}^*$ is a Levi subgroup of $\mathbf{G}^*$, and there exists a pair $(\mathbf{G}(s), \hat{s})$, where $\mathbf{G}(s)$ is a Levi subgroup of $\mathbf{G}$ and $\hat{s}$ is a linear character of $\mathbf{G}(s)^F$ in general position, such that

- $(\mathbf{G}(s), \hat{s})$ is in duality with $(C_{\mathbf{G}^*}(s)^0, s)$,
- $\mathrm{Irr}(G,(s)) = \{\epsilon_\mathbf{G}\epsilon_{\mathbf{G}(s)} R^G_{G(s)}(\hat{s}\rho) \mid (\rho \in \mathrm{UnIrr}(G(s)))\}$

3. Sketch of results

3.1. (d,s)–**Harish–Chandra theory.** Choose d, and (s) such that s is excellent. Define the relation $(M_1, \mu_1) \leq (M_2, \mu_2)$ on pairs (M d-split , $\mu \in \mathrm{Irr}(M,(s))$, by condition $M_1 \subseteq M_2$ and $< R^{M_2}_{M_1} \mu_1\,,\, \mu_2 > \neq 0\,.$

We denote by $\mathrm{Irr}(G,(M,\mu))$ the set of irreducible constituents of $\mathrm{Irr}\, R^G_M(\mu)$.

Theorem 8 (Generalized Harish-Chandra)**.** (1) *The relation $\leq$ is an order relation.*

(2) *Given $\gamma \in \mathrm{Irr}(G,(s))$, minimal pairs smaller than (G,γ) are all conjugate under G. They are called $(d,(s))$-cuspidal.*

(3) *We have a partition*

$$\mathrm{Irr}(G,(s)) = \dot{\bigcup} \mathrm{Irr}(G,(M,\mu))$$

where (M,μ) runs over the set (up to G-conjugacy) of d-cuspidal pairs such that $\mu \in \mathrm{Irr}(M,(t))$ with t conjugate to s under $\mathbf{G}^*(q)$.

3.2. **Main theorems.** By Jordan decomposition of characters, the next theorem is an immediate application of results of [1] about unipotent characters.

Theorem 9. *Let s be excellent semisimple element of G. Let (M,μ) be $(d,(s))$-cuspidal pair. Then the group $W_G(M,\mu) := N_G(M,\mu)/M$ is a (complex) reflection group.*

Assume now that H is a finite reductive group, and that G is a d–split Levi subgroup in H. Let $\tilde{G}$ be a group such that

$$G \subseteq \tilde{G} \subseteq N_H(G)\,,$$

and let s be a excellent semisimple element of G.

If (M,μ) is a $(d,(s))$ cuspidal pair in G, we denote by $\mathrm{Irr}(\tilde{G},(M,\mu))$ the set of characters of $\tilde{G}$ whose restriction down to $\tilde{G}$ is a linear combination of characters in $\mathrm{Irr}(G,(M,\mu))$, and we denote by $\mathrm{Irr}(N_{\tilde{G}}(M,\mu),\mu)$ the set of irreducible characters of $N_{\tilde{G}}(M,\mu)$ which live above μ.

Theorem 10 (Totally proved only for classical groups)**.** *Let $\tilde{G}$, s and (M,μ) be as above. There is an isometry looking like a "partial perfect isometry"*

$$R = R^{\tilde{G}}_{(M,\mu)} \,:\, \mathbb{Z}\,\mathrm{Irr}(N_{\tilde{G}}(M,\mu),\mu) \xrightarrow{\sim} \mathbb{Z}\,\mathrm{Irr}(\tilde{G},(M,\mu))$$

with the following properties.

Assume ℓ is excellent, $\ell \nmid q$, and that d is the order of q modulo ℓ.

(a) $R^{\tilde{G}}_{(M,\mu)}$ *is equivariant under the action of* $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$.

(b) $s_{R\chi}/s_\chi \equiv 1 \bmod \ell$ *for all* $\chi \in \mathrm{Irr}(N_{\tilde{G}}(M,\mu),\mu)$.

In particular, the map R induces bijections

$$\mathrm{Irr}(N_{\tilde{G}}(M,\mu),\mu)_{\delta,r} \xrightarrow{\sim} \mathrm{Irr}(\tilde{G},(M,\mu))_{\delta,r}\,.$$

Comments

- Relies on [1], plus some miraculous ingredients, like :
  - For s an ℓ-element, $|G|_p/|G(s)|_p \equiv \pm 1 \bmod \ell$.
  - If ℓ is excellent, whenever $\rho \in \mathrm{UnIrr}(\mathbb{G})$ and $d \in \mathbb{N}$ are such that $\Phi_{d\ell}(x)$ divides $s_\rho(x)$ then $\Phi_d(x)$ divides $s_\rho(x)$.
- It is in the spirit of global–local methods, since a d-split Levi subgroup M is the centralizer of an ℓ-subgroup.

3.3. **Compatibility of various partitions.** We combine and refine partitions :

• *Jordan decomposition :*

$$\mathrm{Irr}(G) = \dot{\bigcup} \mathrm{Irr}(G,(s)) \quad (s \text{ semi–simple element of } G^* \text{ modulo conjugation})$$

• *Generalized* $(d,(s))$*–Harish–Chandra theory (for excellent* s*) :*

$$\mathrm{Irr}(G,(s)) = \dot{\bigcup} \mathrm{Irr}(G,(M,\mu)) \quad ((M,\mu) \text{ a } (d,(s))\text{–cuspidal pair up to conjugacy})$$

• *Block decomposition :* For $\mathrm{Irr}(G,B,(s)) := \mathrm{Irr}(G,B) \cap \mathrm{Irr}(G,(s))$, we have

$$\mathrm{Irr}(G) = \dot{\bigcup} \mathrm{Irr}(G,B) \quad \text{and} \quad \mathrm{Irr}(G,B) = \dot{\bigcup} \mathrm{Irr}(G,B,(s)) .$$

Theorem 11 ([2])**.** *Let* ℓ *be excellent for* G*, and let* B *be a unipotent* ℓ*–block of* G*. There exists a* $(d,(s))$*-cuspidal pair* (M,μ) *such that*

$$\mathrm{Irr}(G,B,(s)) = \mathrm{Irr}(G,(M,\mu)) .$$

Comments.

• This is in the spirit of the first work by Fong–Srinivasan and Schewe, who described blocks in connection with Deligne-Lusztig induction.

• Note that the group $W_G(M,\mu)$ is the automizer of an abelian ℓ-subgroup, namely the Sylow ℓ–subgroup of the center of M : $W_G(M,\mu) = N_G(M,\mu)/M = N_G(Z_\ell M, b_\mu)/C_G(Z_\ell M)$.

Thus the preceding result may be seen as a (small) step towards a "generalization" of (ADC) :

$$\begin{array}{ccc}
\mathrm{Irr}(G,B) & = & \dot{\bigcup}\, \mathrm{Irr}(G,B,(s)) \\
 & & \| \\
 & & \mathrm{Irr}(G,(M,\mu)) \\
 & \text{fake-isotypie} \rightarrow & \updownarrow \\
 & & \mathrm{Irr}\, W_G(M,\mu)
\end{array}$$

4. Application to MAKRODINU

Theorem 12. *Conjecture (MAKRODINU) holds under the following hypothesis :*

(1) $G = \mathbf{G}(q)$ *is a finite reductive group,*
(2) ℓ *is a prime such that* $\ell \nmid q$ *and which is excellent for* $\mathbb{G}$*,*
(3) B *is a unipotent* ℓ*–block of* G*.*

Comments on the proof.

One may replace (following Jianbei An) chains of elementary abelian ℓ-subgroups of G by chains of "generic elementary abelian" subgroups, namely by chains of d-split subgroups. In other words, MAKRODINU is equivalent to

$$\sum_{C,b}(-1)^{|C|}\,|\,\mathrm{Irr}(N_G(C),b)_{\delta,r}| = 0$$

where C runs over the set of chains (up to conjugation) of d-split Levi subgroups of G.

To prove the above equality it is enough to prove the refined equality

$$\sum_{C,b}(-1)^{|C|}\,|\,\mathrm{Irr}(N_G(C),b,(s))_{\delta,r}| = 0\,,$$

whenever s is an ℓ–element of $\mathbf{G}^*(q)$. This is an immediate consequence of the main theorem above.

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