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Mini-Workshop: Zeta Functions, Index and Twisted K-Theory; Interactions with Physics

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Abstract. The meeting provided an opportunity for number theorists, physicists, topologists and analysts to discuss recent developments and interactions in the area of multiple zeta functions, twisted K-theory, Hopf algebras, and global analysis.

Mathematics Subject Classification (2000): 11F03, 11M06, 16W30, 81T15, 81T18, 81T75, 55N15, 58J30, 58J32, 58J42.

Introduction by the Organisers

This mini-workshop brought together number theorists, analysts, geometers and mathematical physicists to discuss current issues at the common boundary of mathematics and physics. Topics covered included the number theoretic and algebraic structures underlying renormalization, twisted K-theory and higher algebraic structures, modular forms, and arithmetic and spectral zeta functions. A particular theme was around developing interconnections between arithmetic (multiple) zeta functions, spectral zeta functions associated with elliptic operators (and related spectral invariants such as spectral flow) and current issues in physics such as renormalization and mirror symmetry. Multiple zeta functions appear in index theory and K-theory via their relation to anomalies, in number theory in their relation to polylogarithms, in renormalization questions in perturbative quantum field theory and Hopf algebras, in duality issues and in twisted K-theory for index theorems for projective families of elliptic operators, thereby providing a rich set of overlapping topics with common analytical issues.

This meeting was organized around one hour talks, four each day, with plenty of time between talks for informal discussion and a 45 minute talk in the afternoon for students; three graduate students were among the 16 participants. Some participants lectured for two hours in order to have time to introduce the audience to the subject before entering the technical details.

The organizers and participants would like to thank the *Mathematisches Forschungsinstitut Oberwolfach* for providing a pleasant and stimulating enviroment for this meeting.

Workshop: Mini-Workshop: Zeta Functions, Index and Twisted K-Theory; Interactions with Physics

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Abstracts

What a geometric series can do in Quantum Field Theory

Dirk Kreimer
(joint work with Karen Yeats)

In my two talks I discussed the structure of nonperturbative quantum field theory from the viewpoint of Dyson–Schwinger equations and the Hopf algebra of perturbation theory. They covered three recent papers [1, 4, 5].

1. Dyson–Schwinger equations

First we considered Dyson–Schwinger equations for one-particle irreducible renormalized Green functions. The talks focussed on the question how to treat the non-linearity of such Dyson–Schwinger equations systematically in the context of a renormalizable quantum field theory. Such a theory provides a finite set $\mathcal{R}$ of amplitudes which need renormalization. These amplitudes are in one-to-one correspondence with the monomials in the underlying Lagrangian L,

$$L = \sum_{r\in\mathcal{R}} \varphi(r). \tag{1.1}$$

For a given superficially divergent amplitude $r \in \mathcal{R}$ we let Γ^r be the sum

$$\Gamma^r = \mathbb{I} + \sum_{\Gamma} \alpha^{|\Gamma|} \frac{\Gamma}{\mathrm{sym}(\Gamma)} \tag{1.2}$$

over all 1PI graphs Γ contributing to that amplitude, where $0 < \alpha \ll 1$ is a loop-counting small parameter.
Following [4] we have

$$\Gamma^r = \mathbb{I} + B_+(\Gamma^r, Q(\{\Gamma^i\})). \tag{1.3}$$

The Hochschild one-cocycle

$$B_+^r(\Gamma^r, Q) = \sum_{k\geq 1} \alpha^k B_+^{k;r}(\Gamma^r Q^k) \tag{1.4}$$

is a sum of one-cocycles $B_+^{k,r}$ and Q is a monomial in the Γ^r. The uniqueness of Q implies the Slavnov–Taylor identities for the renormalized couplings [4], and in this way internal symmetries of a Lagrangian field theory are captured by Hochschild cohomology.
We set

$$\Gamma^r = \mathbb{I} + \sum_j c_j^r \alpha^j \tag{1.5}$$

and those c_j^r are the linear generators of a sub-Hopf algebra [4]:

Theorem 1. There exists maps $B_+^{k;r}$, polynomials $P_{k,j}^r$ in those linear generators and integers s_r such that

$$\Gamma^r = \mathbb{I} + \sum_k \alpha^k B_+^{k;r}\left(\Gamma^r Q^k\right), \tag{1.6}$$

$$\Delta B_+^{k;r} = B_+^{k;r} \otimes \mathbb{I} + \left(\mathrm{id} \otimes B_+^{k;r}\right)\Delta, \tag{1.7}$$

$$Q = \alpha \prod_{r\in\mathcal{R}} [\Gamma^r(\alpha)]^{s_r}, \tag{1.8}$$

$$\Delta c_k^r = \sum_{j=0}^{k} P_{k,j}^r \otimes c_{k-j}^r, \tag{1.9}$$

which make the system $\{c_k^r\}$ into a sub Hopf algebra $H(\Delta, m, S, \varepsilon)$ of the Feynman graph Hopf algebra.

We refer the reader to [4] for details.
Feynman rules φ are then defined in accordance with the Hochschild cohomology so that the iterative structure of subintegrals corresponds to iterated applications of Hochschild one-cocycles.
In perturbation theory the renormalized Feynman rules φ_R allow to write

$$G_R^r(\alpha, L) = \varphi_R(\Gamma^r) = 1 + \sum_k \alpha^k \varphi_R(c_k^r)(L). \tag{1.10}$$

We can expand in a different manner

$$G_R^r(\alpha, L) = 1 + \sum_k \gamma_k^r(\alpha) L^k, \tag{1.11}$$

and the renormalization group dictates relations between the γ_k^r. Here, we restrict ourselves to Green functions which depend on a single kinematical variable $L = \ln q^2/\mu^2$, which is legitimate if one studies short-distance singularities.
Straightforward algebra delivers the desired formula for the expansion in L:

$$\gamma_k^r(\alpha) = \frac{1}{k}\left[\gamma_1^r(\alpha)\gamma_{k-1}^r(\alpha) + \sum_j s^j \gamma_1^j(\alpha)\alpha\partial_\alpha \gamma_{k-1}^r(\alpha)\right]. \tag{1.12}$$

Next, we introduced the Mellin transforms $F(\rho)$ for each Hopf primitive $\varphi(B_+(\mathbb{I}))$. Then, the Dyson–Schwinger equation determines the remaining unknown $\gamma_1(\alpha)$,

$$\gamma \cdot L = \alpha(1 + \gamma \cdot \partial_{-\rho})^{-1}[e^{-L\rho} - 1]F(\rho)|_{\rho=0}, \tag{1.13}$$

where we evaluate the rhs at $\rho = 0$ after taking derivatives.
This equation determines the Taylor coefficients $\gamma_{1,j}$ through the Taylor coefficients of the Mellin transform.
We hence have resolved the computation of a problem in non-perturbative physics, an infinite resummation of graphs, to the determination of Mellin transforms of Hopf primitives. This summarizes a research program which I have been pursuing for a couple of years, and emphasizes the need to understand better the Mellin

transforms of those primitives. The Taylor coefficients of these Mellin transforms are interesting periods, starting from the residue assigned to the underlying graph, which is a particular important period of motivic origin [2].
In the talk, we then looked at the results of [3] as an example and rederived the resummation $\widetilde{\Sigma}(a,p)$ of one-loop fermion propagators in Yukawa theory, whose Mellin transform is a geometric series. Particularly intriguing is the possibility to find connections to ζ functions as described in [5]. In the talk we discussed the appearance of single Riemann $\zeta(2l+1)$, $l \geq 1$, for coupled Dyson–Schwinger equations following [5].

2. A functional equation

Inspection of the solution in [3] shows

$$\widetilde{\Sigma}(a,p) = -\frac{\sqrt{a/(2\pi)}}{\exp(p^2)\mathrm{erfc}(p)} \times \widetilde{\Sigma}\left(\frac{(\exp(p^2)\mathrm{erfc}(p))^4}{a/(2\pi)^2}, p\right), \tag{2.1}$$

where a is now the loop counting parameter and p is another variable such that with $z = e^{2L}$, $p = \frac{d}{dz}\sqrt{\frac{2}{a}}\left(z - z\widetilde{\Sigma}\left(\mu^2\sqrt{z}\right)\right)$. Note that on the lhs of (2.1) we have a weak coupling expansion for a, on the rhs we have a strong coupling expansion, hence an expansion in $1/a$.
With $T = a/(2\pi)$, $u = (\exp(p^2)\mathrm{erfc}(p))^{-4}$, and $Z(T,u) = \widetilde{\Sigma}(a,p)$ we get a functional equation reminiscent of a functional equation for a ζ-function in two variables for the function field case for the non-perturbative renormalized Green function

$$Z(T,u) = -u^{\frac{5}{4}-1}T^{2(\frac{5}{4}-1)}Z\left(\frac{1}{Tu}, u\right). \tag{2.2}$$

The propagator coupling duality of this Green function can now be expressed with $u = \exp(s+t)$, $T = \exp(-t)$, and $\zeta(s,t) = \exp(\frac{t-s}{8})Z(T,u)$ as

$$\zeta(s,t) = -\zeta(t,s). \tag{2.3}$$

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Locality and Dyson Schwinger equations from Hochschild cohomology of renormalization Hopf algebras

Christoph Bergbauer
(joint work with Dirk Kreimer)

In quantum field theory, Feynman graphs and the corresponding analytic expressions – Feynman integrals – constitute the building blocks of perturbative expansions. In realistic quantum field theories, Feynman integrals typically diverge when their graphs contain cycles. The process of assigning a sensible value to these divergent integrals is called renormalization. In simple cases it suffices, for example, to subtract off the first terms of the Taylor series with respect to the external momentum. In more general cases, that is when already subgraphs contain cycles and thus already subintegrals diverge, the renormalization process is described by the Bogoliubov recursion. It was Kreimer's discovery that the solution of the Bogoliubov recursion is essentially given by the antipode map of a Hopf algebra $(\mathcal{H}, m, \Delta, 1, \varepsilon)$ on rooted trees [4] where the rooted trees keep track of nested and disjoint subdivergences. Similarly, Hopf algebras based directly on Feynman graphs can be considered [2]. As an algebra, $\mathcal{H}$ is free commutative, and the coproduct Δ cuts rooted trees into pieces. We describe the first Hochschild coalgebra cohomology of $\mathcal{H}$ [3] and show how the fact that certain linear endomorphisms B_+ satisfy the 1-cocycle condition

$$\Delta B_+ = (id \otimes B_+)\Delta + B_+ \otimes 1 \tag{0.1}$$

translates into physics: A proof that the renormalization procedure provides a finite result using local counterterms is easily afforded by (0.1). This is illustrated using a two-loop example in dimensional regularization with minimal subtraction and on-shell scheme, respectively.

On the nonperturbative side, the same Hochschild 1-cocycles provide (combinatorial) Dyson-Schwinger equations, for example

$$X = 1 + \alpha B_+(X^2) + \alpha^2 B_+(X^3)$$

with solution $X \in \mathcal{H}[[\alpha]]$. An important consequence of B_+ satisfying the cocycle condition is that if X is decomposed as $X = \sum X_n \alpha^n$, the X_n generate a Hopf subalgebra of $\mathcal{H}$. Indeed,

$$\Delta X_n = \sum_{i=0}^{n} \left(\sum_{l_1+\ldots+l_{i+1}=n-i} X_{l_1} \ldots X_{l_{i+1}} \right) \otimes X_i, \tag{0.2}$$

which holds for various other Dyson-Schwinger equations as well [1]. See Kreimer's talk for physical implications.

The talk is based on the recent review paper [1] which also contains a proof of (0.2).

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Invariants of Graph C^*-Algebras

Alan Carey

(joint work with Adam Rennie, John Phillips)

I discussed some recent (unpublished) work on invariants of graph C^*-algebras constructed via extensions of the semifinite local index theorem in noncommutative geometry (see [3, 4, 5]. The motivation for this study stems from three sources. The first is a desire to construct invariants for algebras which are purely infinite and for which K-theory provides little information. Our invariants depend not only on the algebra but also on an automorphic circle action. The key idea is to use states or weights on the algebra that are KMS with respect to the circle action. The second motivation arises from comments of Matilde Marcolli on directed graphs that are associated to Mumford curves. The latter are algebraic curves over finite field extensions of the p-adic completion of the rationals (for p a fixed prime). The idea here is that there may be a way to get information about Mumford curves using invariants from the associated graph algebras. The third motivation arises from the possibility of using the graph algebra presentation of $SU_q(2)$ to obtain new information about this algebra. (Contributions by David Pask, Ryzsard Nest, Jens Kaad, Matilde Marcolli and Kester Tong are gratefully acknowledged.)

Let $E = (E^0, E^1)$ be a directed graph, where E^0 consists of the vertices and E^1 the edges with $r, s : E^1 \to E^0$. A *Cuntz-Krieger E-family* in a C^*-algebra B consists of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ satisfying the *Cuntz-Krieger relations*

$$S_e^* S_e = p_{r(e)} \text{ for } e \in E^1 \text{ and } p_v = \sum_{\{e:s(e)=v\}} S_e S_e^* \text{ whenever } v \text{ is not a sink.}$$

We let $A(E)$ be the corresponding universal C^*-algebra generated by the operators satisfying these relations. We can consider circle actions on such algebras by mapping some or all of the operators S_e to multiples by a phase zS_e where $|z| = 1$. This does not change the defining relations and can be seen to define an automorphic circle action on the algebra. (Note that higher dimensional tori also act automorphically.)

We will consider the fixed point algebra F under such a circle action and take the projection $\Phi_0 : A(E) \to F$. We assume there exists a faithful trace τ on F and we extend it to a state on $A(E)$ by composition with Φ_0, ie $\tau \circ \Phi_0$. We can form the GNS Hilbert space $L^2(A(E), \tau)$. This is a bimodule for $A(E)$ acting on the left and F acting on the right. We introduce the rank one operators $\Theta^R_{x,y} z = x\Phi_0(y^*.z)$ for $x, y, z \in A(E)$. We define $\mathcal{N}$ to be the von Neumann algebra generated by the rank one operators. Then one can prove there exists a trace $\tilde{\tau}$ on $\mathcal{N}$ such that $\tilde{\tau}(\Theta^R_{x,y}) = \tau \circ \Phi_0(y^*x)$.

In the case of the Cuntz algebra O_n for example we can apply this procedure to the canonical gauge action and obtain the standard type $III_{1/n}$ representation corresponding to the KMS state $\tau \circ \Phi_0$ with τ in this case being the unique faithful trace on the fixed point algebra. There is a modular operator Δ on $L^2(O_n, \tau \circ \Phi)$ such that the gauge action σ_t on O_n is given by $a \to \Delta^{it} a \Delta^{-it}$. (In the talk I gave a number of examples but in this summary I will stick to O_n) We want to extract from these constructions some invariants of the pair F, O_n. There is a dense subalgebra O_c of O_n such that the map σ defined by $\sigma(a) = \Delta a \Delta^{-1}$. is a non $*$-automorphism (polynomials in the generators will suffice for this purpose). We say that a unitary u in a matrix algebra $\tilde{O}_c$ over O_c satisfies the **modular condition** with respect to the non$*$ automorphism σ given by conjugation with $\Delta \otimes Id$ (where Id is the identity matrix) if both the operators

$$u\sigma(u^*), \quad u^*\sigma(u)$$

are in a matrix algebra $\tilde{F}$ over the algebra F.

Let $\Delta = e^{\log nD}$ where D is an unbounded densely defined operator. We would like to construct a semifinite spectral triple using D but D does not satisfy the summability condition that is

$$\tilde{\tau}((1 + D^2)^{-1/2-r})$$

is not trace class for all $r > 0$ So we define a new functional on $\mathcal{N}$ by

$$\tau_\Delta(T) := \tilde{\tau}(\Delta T).$$

Then as $\tilde{\tau}$ is a faithful semifinite normal trace on $\mathcal{N}$, and Δ is a positive invertible operator affiliated to $\mathcal{N}$ we may show that τ_Δ is a faithful semifinite normal weight on $\mathcal{N}$. If we restrict τ_Δ to the fixed point algebra of $\mathcal{N}$, say $\mathcal{M}$ under the conjugation action of Δ^{it} then we get a faithful semifinite normal trace. Similar remarks apply when we work with matrix algebras over these algebras. Henceforth we will drop the $\otimes Id$ from the notation for simplicity.

We now introduce (a special case of) the analytic spectral flow formula of [1, 2]. This formula starts with a semifinite spectral triple $(\mathcal{A}, \mathfrak{H}, \mathcal{D})$ and computes the spectral flow from $\mathcal{D}$ to $u\mathcal{D}u^*$, where $u \in \mathcal{A}$ is unitary with $[\mathcal{D}, u]$ bounded, in the case where $(\mathcal{A}, \mathfrak{H}, \mathcal{D})$ is of dimension $p \geq 1$. Thus for any $n > p$ we have by Theorem 9.3 of [2]:

$$sf_{\tau_\Delta}(\mathcal{D}, u\mathcal{D}u^*) = \frac{1}{C_{n/2}} \int_0^1 \tau(u[\mathcal{D}, u^*](1 + (\mathcal{D} + tu[\mathcal{D}, u^*])^2)^{-n/2}) dt,$$

with $C_{n/2} = \int_{-\infty}^{\infty}(1+x^2)^{-n/2}dx$.
Theorem[7] The pairing (denoted $< >$) between the K-class of u and the K-homology class represented by the spectral triple is given by

$$< [u], (\mathcal{A}, \mathfrak{H}, \mathcal{D}) >= \lim_{r\to 0^+} r\tau(u[\mathcal{D}, u^*](1+\mathcal{D}^2)^{-1/2-r}).$$

In particular, the limit on the right exists.
Definition Let u_t be a continuous path of modular unitaries such that $u_t\sigma(u_t^*)$ and $u_t^*\sigma(u_t)$ are also continuous paths in F. Then we say that u_t is a modular homotopy, and say that u_0 and u_1 are modular homotopic. There is a binary operation on modular homotopy classes which makes these into a semigroup.
Definition. Let $K_1(A, \sigma)$ be the abelian semigroup with one generator $[u]$ for each unitary $u \in M_l(A)$ satisfying the modular condition and with the following relations:

1) $[1] = 0$,
2) $[u] + [v] = [u \oplus v]$,
3) If $u_t,\ t \in [0,1]$ is a continuous paths of unitaries in $M_l(A)$ satisfying the modular condition then $[u_0] = [u_1]$.

Example. For $S_\mu \in O_{nc}$ we write $P_\mu = S_\mu S_\mu^*$. Then for each μ, ν we have a unitary

$$u_{\mu,\nu} = \begin{pmatrix} 1-P_\mu & S_\mu S_\nu^* \\ S_\nu S_\mu^* & 1-P_\nu \end{pmatrix}.$$

It is simple to check that this a self-adjoint unitary satisfying the modular condition.

This construction generalises to other partial isometries with range and source projections in F. One of the main results we obtain, from which one can compute particular numerical values for the spectral flow corresponding to unitaries in the above example is the following theorem.

Theorem For any modular unitary of the form u_v with $v \in \tilde{O}_c$ a partial isometry with range and source projections in $\tilde{F}$ and $v\sigma(v^*),\ v^*\sigma(v) \in \tilde{F}$, and any Dixmier trace φ_ω we have

$$sf_\varphi(\mathcal{D}, u_v\mathcal{D}u_v^*) = \lim_{r\to 0} r\varphi(u_v[\mathcal{D}, u_v^*](1+\mathcal{D}^2)^{-1/2-r}) = \frac{1}{2}\varphi_\omega(u_v[\mathcal{D}, u_v^*](1+\mathcal{D}^2)^{-1/2}).$$

The functional

$$\tilde{O}_c \otimes \tilde{O}_c \ni a_0 \otimes a_1 \to \lim_{r\to 0} r\varphi(a_0[\mathcal{D}, a_1](1+\mathcal{D}^2)^{-1/2-r})$$

is a twisted b, B-cocycle. Moreover, the spectral flow depends only on the modular homotopy class of u_v.

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T-duality in string theory via noncommutative geometry

Varghese Mathai

(joint work with P. Bouwknegt, J. Evslin, K. Hannabuss, J. Rosenberg)

We begin by recalling T-duality in Type II string theory with trivial background H-flux on spacetime that is compactified on a torus, $M \times \mathbb{T}^n$. Then the T-dual spacetime is topologically the same spacetime $M \times \hat{\mathbb{T}}^n$, where $\hat{\mathbb{T}}^n$ denotes the dual torus, and T-duality is realized via the smooth analog of the Fourier-Mukai transform, which we now recall. Consider the correspondence

(0.1)
$$\begin{array}{ccccc} & & M \times \mathbb{T}^n \times \hat{\mathbb{T}}^n & & \\ & \swarrow^{p} & & \searrow^{\hat{p}} & \\ M \times \mathbb{T}^n & & & & M \times \hat{\mathbb{T}}^n \end{array}$$

Recall that there is a canonical line bundle $\mathcal{P}$ called the Poincaré line bundle, over the torus $\mathbb{T}^n \times \widehat{\mathbb{T}}^n$, which is defined as follows. Consider the free action of $\mathbb{Z}^n$ on $\mathbb{R}^n \times \widehat{\mathbb{T}}^n \times \mathbb{C}$ given by,

$$\begin{aligned} \mathbb{Z}^n \times (\mathbb{R}^n \times \widehat{\mathbb{T}}^n \times \mathbb{C}) &\to \mathbb{R}^n \times \widehat{\mathbb{T}}^n \times \mathbb{C} \\ (n, (r, \rho, z)) &\to (r + n, \rho, \rho(n) z) \end{aligned}$$

The Poincaré line bundle is defined as the quotient, $\mathcal{P} = (\mathbb{R}^n \times \widehat{\mathbb{T}}^n \times \mathbb{C})/\mathbb{Z}^n$. We denote its pullback to the correspondence space $M \times \mathbb{T}^n \times \hat{\mathbb{T}}^n$ also by $\mathcal{P}$. In the late 1990s, it was argued by Minasian-Moore, Horava and Moore-Witten that for a spacetime X, in Type IIA string theory, the RR fields are classified by $K^0(X)$ and the charges classified by $K^1(X)$ whereas in Type IIB string theory, the RR fields are classified by $K^1(X)$ and the charges classified by $K^0(X)$.

In this case, T-dualizing on $\mathbb{T}^n$, the *Buscher rules* for the Ramond-Ramond (RR) fields are concisely encoded by the Fourier-Mukai transform,

$$T_! = \hat{p}_! \left(\mathcal{P} \otimes p^!(\ \cdot\ ) \right).$$

which induces a T-duality isomorphism of K-theories,

$$T_! : K^\bullet(M \times \mathbb{T}^n) \xrightarrow{\cong} K^{\bullet+n}(M \times \hat{\mathbb{T}}^n). \tag{0.2}$$

That is, T-duality in the absence of a background field, gives an equivalence betweenType IIA string theory and Type IIB string theory.

This formulation of T-duality continues to work nicely for principal circle bundles with background H-flux, [8, 6]. However, it is problematic for general for higher rank principal torus bundles with background H-flux. To overcome this, we reformulate the Fourier-Mukai theory in terms of noncommutative geometry. We begin by recalling some fundamental facts about C^*-algebras.

Let A be a C^*-algebra with an action α of a locally compact group H. i.e. there is a homomorphism $\alpha : H \to \mathrm{Aut}(A)$ such that $h \mapsto \alpha_h(a)$ is norm continuous $\forall a \in A$. Consider the space $C_c(H, A)$ of all compactly supported A-valued continuous functions on H. It is a $*$-algebra as follows: for $f_j \in C_c(H, A)$ and $g, h \in H$ and the product is given by α-twisted convolution,

$$f_1 * f_2(h) = \int_H f_1(g)\alpha_g(f_2(g^{-1}h))dg$$

and $*$-operator

$$f^*(g) = \Delta(g)^{-1}\alpha_g(f(g^{-1})^*)$$

where $\Delta : H \to \mathbb{R}^+$ is the modular function relating left and right Haar measure on H.

The **crossed product** C^*-algebra $A \rtimes_\alpha H$ is the completion of $C_c(H, A)$ in a universal norm.

(1) **P. Green's theorem (version 1)**: Let A be a C^*-algebra with an action α of a locally compact group H, and suppose that H is a normal closed subgroup of another locally compact group G. Then one can form the induced C^*-algebra

$$\begin{aligned} B &= \mathrm{Ind}_H^G(A, \alpha) \\ &= \{f : G \to A : f(t+g) = \alpha(g)(f(t)),\ t \in G, g \in H\} \end{aligned}$$

which has an action β of G given by translation. Then the C^*-algebras, $A \rtimes_\alpha H$ and $B \rtimes_\beta G$ are Morita equivalent.

P. Green's theorem (version 2): Let G be a locally compact group, and H, K be closed subgroups of G. Then the C^*-algebras $C(G/K) \rtimes H$ and $C(H\backslash G) \rtimes K$ are Morita equivalent.

(2) **Connes Thom isomorphism**: Let A be a C^*-algebra with an action α of $G = \mathbb{R}^d$. Then there is a natural isomorphism,

$$K_j(A) \cong K_{j+d}(A \rtimes_\alpha \mathbb{R}^d).$$

The striking aspect of the result is that the K-theory of the crossed product algebra $A \rtimes_\alpha \mathbb{R}^d$ is independent of the action α.

(3) **Takai duality**: Let A be a C^*-algebra with an action α of $G = \mathbb{R}^d$. Then the crossed product C^*-algebra $A \rtimes_\alpha \mathbb{R}^d$ has a natural action $\hat{\alpha}$ of the Pontrjagin dual group $\hat{\mathbb{R}}^d$ given by

$$(\hat{\alpha}_{g'}(f))(g) = \langle g, g'\rangle f(g)$$

for all $f \in C_c(\mathbb{R}^d, A)$, $g \in \mathbb{R}^d$, $g' \in \hat{\mathbb{R}}^d$. Then Takai duality asserts that the C^*-algebras A and $A \rtimes_\alpha \mathbb{R}^d \rtimes_{\hat\alpha} \hat{\mathbb{R}}^d$ are Morita equivalent.

We are now in a position to rephrase T-duality (i.e the Fourier-Mukai transform) in terms of noncommutative geometry. We first make the following observations.

(1) $C(M \times \mathbb{R}^n/\mathbb{Z}^n) \rtimes \mathbb{R}^n$ and $C(M \times \mathbb{R}^n \backslash \mathbb{R}^n) \rtimes \mathbb{Z}^n$ are Morita equivalent, by P. Green's theorem (version 2). Simplifying, we see that $C(M \times \mathbb{R}^n \backslash \mathbb{R}^n) \rtimes \mathbb{Z}^n = C(M) \otimes C^*(\mathbb{Z}^n)$, is isomorphic to $C(M) \otimes C(\widehat{\mathbb{T}}^n) = C(M \times \widehat{\mathbb{T}}^n)$ by the Fourier transform. Moreover, the radius R of the torus transforms as $R \leftrightarrow 1/R$.

(2) $K_j(C(M \times \mathbb{R}^n/\mathbb{Z}^n) \rtimes \mathbb{R}^n)$ and $K_{j+n}(C(M \times \mathbb{R}^n/\mathbb{Z}^n))$ are isomorphic by Connes Thom isomorphism theorem.

(3) $C(M \times \mathbb{R}^n/\mathbb{Z}^n) \rtimes \mathbb{R}^n \rtimes \widehat{\mathbb{R}}^n$ and $C(M \times \mathbb{R}^n/\mathbb{Z}^n)$ are Morita equivalent, by Takai duality.

Therefore the Fourier-Mukai transform is equivalent to taking the crossed product with $\mathbb{R}^n$ of the algebra of continuous functions on spacetime, $C(M \times \mathbb{R}^n/\mathbb{Z}^n)$.

We are now in a position to further abstract T-duality (i.e. the Fourier-Mukai transform) for general C^*-algebras as follows.

Let A belong to some subcategory $\mathfrak{C}$ of C^*-algebras, and $A \to \mathsf{T}(A)$ be a covariant functor on $\mathfrak{C}$ satisfying the following two properties:

(1) $(A, \mathsf{T}(A))$ are K-equivalent. (we allow for shifts in degree).

(2) $(A, \mathsf{T}(\mathsf{T}(A)))$ are Morita equivalent.

Then we call $\mathsf{T}(A)$ an *abstract T-dual* of A.

Example. Let A be a G-C^*-algebra, where $G = \mathbb{R}^n$. Set $\mathsf{T}(A) = A \rtimes G$. Then it is easily deduced that $\mathsf{T}(A)$ is an abstract T-dual of A.

This reformulation of T-duality (i.e the Fourier-Mukai transform) enables us in [5], to give a complete characterization of T-duality for principal 2-torus-bundles with H-flux. As noticed in [7] for instance, principal torus bundles with H-flux do not necessarily have a T-dual which is a torus bundle. A big puzzle has been to explain these mysterious "missing T-duals." It turns out that every principal 2-torus-bundle with H-flux does indeed have a T-dual, but in the missing cases (which we characterize), the T-dual is non-classical and is a continuous field of stabilized noncommutative tori, or in other words, a bundle of Kronecker foliated tori. This suggests an unexpected link between classical string theories and noncommutative string theories, obtained by compactifying on noncommutative tori.

In general, for higher rank principal torus bundles [4, 1], it appears to be necessary to even further abstract the notion of T-duality (i.e. the Fourier-Mukai transform) to C^*-algebras internal to a category with non-trivial associator, as done in [3, 2].

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Quasi-symmetric functions, multiple zeta values, and rooted trees

Michael E. Hoffman

My first talk was about the algebra of multiple zeta values, and the second about Hopf algebras of rooted trees. A thread that connects the two is the Hopf algebra QSym of quasi-symmetric functions. First defined by Gessel [4], QSym consists of those formal power series $f \in \mathbf{Q}[[t_1, t_2, \dots]]$ (each t_i having degree one), such that f has bounded degree, and the coefficient in f of

$$t_{i_1}^{p_1} t_{i_2}^{p_2} \cdots t_{i_k}^{p_k}$$

equals the coefficient in f of $t_1^{p_1} t_2^{p_2} \cdots t_k^{p_k}$ whenever $i_1 < i_2 < \cdots < i_k$. As a vector space, QSym is generated by the monomial quasi-symmetric functions

$$M_{p_1 p_2 \cdots p_k} = \sum_{i_1 < i_2 < \cdots < i_k} t_{i_1}^{p_1} t_{i_2}^{p_2} \cdots t_{i_k}^{p_k}.$$

The algebra Sym of symmetric functions is a proper subalgebra of QSym: for example, M_{11} and $M_{12} + M_{21}$ are symmetric, but M_{12} is not.

As an algebra, QSym is generated by those monomial symmetric functions corresponding to Lyndon words in the positive integers [11, 6]. The subalgebra of $\mathrm{QSym}^0 \subset \mathrm{QSym}$ generated by all Lyndon words other than M_1 has the vector space basis consisting of all monomial symmetric functions $M_{p_1 p_2 \cdots p_k}$ with $p_k > 1$ (together with $M_\emptyset = 1$). There is a homorphism $\mathrm{QSym}^0 \to \mathbf{R}$ given by sending each t_i to $\frac{1}{i}$; that is, the monomial quasi-symmetric function $M_{p_1 \cdots p_k}$ is sent to the multiple zeta value

$$\zeta(p_k, p_{k-1}, \dots, p_1) = \sum_{i_1 > i_2 > \cdots > i_k \geq 1} \frac{1}{i_1^{p_k} i_2^{p_{k-1}} \cdots i_k^{p_1}}. \tag{0.1}$$

In particular, the subalgebra $\mathrm{Sym}^0 = \mathrm{Sym} \cap \mathrm{QSym}^0$ (which is the subalgebra of Sym generated by the power-sum symmetric functions M_i with $i > 1$) has a homomorphic image in the reals generated by the values $\zeta(i)$ of the Riemann zeta function at integers $i > 1$. It is also convenient to think of $M_{p_1\cdots p_k}$ as the monomial $x^{p_k-1}y\cdots x^{p_1-1}y$ in the noncommutative polynomial ring $\mathbf{Q}\langle x,y\rangle$ (with QSym^0 corresponding to $\mathfrak{H}^0 := \mathbf{Q}1 + x\mathbf{Q}\langle x,y\rangle y$), so that the quantity (0.1) is the image under a homomorphism $\zeta : \mathfrak{H}^0 \to \mathbf{R}$ of $x^{p_k-1}y\cdots x^{p_1-1}y$. In fact, it appears that all identities of multiple zeta values follow from the interaction between the algebra structure of QSym and a second algebra structure on $\mathfrak{H}^0$ coming from the shuffle product in $\mathbf{Q}\langle x,y\rangle$; see, e.g., [7, 9].

To give QSym the structure of a graded connected Hopf algebra, one defines a coproduct Δ by

$$\Delta(M_{p_1\cdots p_k}) = \sum_{j=0}^{k} M_{p_1\cdots p_i} \otimes M_{p_{i+1}\cdots p_k}.$$

This coproduct makes the power-sum symmetric functions M_i primitive, and the elementary symmetric functions $M_{1\dots 1}$ divided powers. Using this Hopf algebra structure, one can define an action of QSym on $\mathbf{Q}\langle x,y\rangle$ that makes $\mathbf{Q}\langle x,y\rangle$ a QSym-module algebra (see [7] for details). In terms of this action one can state a result of Y. Ohno [12] as follows: for any word w of $\mathfrak{H}^0$ and nonnegative integer i,

$$\zeta(h_i \cdot w) = \zeta(h_i \cdot \tau(w)).$$

(Here $\cdot$ denotes the action, h_i is the complete symmetric function of degree i, and τ is the antiautomorphism of $\mathbf{Q}\langle x,y\rangle$ that exchanges x and y.)

My second talk concerned the relationship between QSym and some Hopf algebras of trees (or more precisely forests) defined by Kreimer [10] and Foissy [2]. Kreimer's commutative Hopf algebra $\mathcal{H}_K$, which has as its algebra generators rooted trees, is the graded dual of the noncommutative Hopf algebra $\mathcal{T}$ of rooted trees defined by Grossman and Larson [5]. Foissy's noncommutative Hopf algebra $\mathcal{H}_F$, which is generated by planar rooted trees, is self-dual.

Now Sym is a self-dual Hopf algebra. The larger Hopf algebra QSym is commutative but not cocommutative, and so cannot be self-dual: its graded dual is the Hopf algebra NSym of noncommutative symmetric functions in the sense of Gelfand *et al.* [3]. As an algebra, NSym is the noncommutative polynomial algebra $\mathbf{Q}\langle e_1, e_2, \dots\rangle$, with e_i in degree i, and the e_i are divided powers. There is an abelianization homomorphism NSym $\to$ Sym sending e_i to the elementary symmetric function of degree i.

The Hopf algebra structure on $\mathcal{H}_K$ is such that the "ladder" trees ℓ_i (where ℓ_i is the unbranched tree with i vertices) are divided powers: so the map φ : Sym $\to \mathcal{H}_K$ sending the ith elementary symmetric function to ℓ_i is a Hopf algebra homomorphism. In fact, there is a commutative diagram of Hopf algebras

$$\begin{array}{ccc} \mathrm{NSym} & \xrightarrow{\Phi} & \mathcal{H}_F \\ \downarrow & & \downarrow \\ \mathrm{Sym} & \xrightarrow{\varphi} & \mathcal{H}_K \end{array} \tag{0.2}$$

where Φ sends e_i to the unbranched planar rooted tree having i vertices. (The map $\mathcal{H}_F \to \mathcal{H}_K$ sends each planar rooted tree to the corresponding rooted tree, and forgets order in products.) The commutative diagram (0.2) dualizes to give

$$\begin{array}{ccc} \mathrm{QSym} & \xleftarrow{\Phi^*} & \mathcal{H}_F \\ \uparrow & & \uparrow \\ \mathrm{Sym} & \xleftarrow{\varphi^*} & \mathcal{T} \end{array} \tag{0.3}$$

and the diagram (0.3) makes it easy to establish some interesting properties of the elements of the Hopf algebras involved. For example, if for a rooted tree t we let $|t|$ be the number of non-root vertices of t and $\mathrm{Symm}(t)$ the symmetry group of t, then

$$\kappa_n = \sum_{|t|=n} \frac{t}{|\,\mathrm{Symm}(t)|} \in \mathcal{T}$$

can be seen to form a set of divided powers in $\mathcal{T}$, and $\varphi^*(\kappa_n) = h_n$, the complete symmetric function of degree n. In fact, $\varepsilon_n := (-1)^n S(\kappa_n)$, where S is the antipode in $\mathcal{T}$, is an element that maps under φ^* to the nth elementary symmetric function: further, $n!\varepsilon_n$ is exactly the rooted tree in which n vertices are directly connected to the root.

If we define an operator $\mathfrak{N} : \mathcal{T} \to \mathcal{T}$ by $\mathfrak{N}(t) = \ell_2 \circ t$, where $\circ$ is the Grossman-Larson product, then we can define coefficients $n(t; t')$ by

$$\mathfrak{N}^k(t) = \sum_{|t'|=|t|+k} n(t; t')t'.$$

If $\ell_1 = \bullet$ is the tree consisting of just the root vertex, then $n(\bullet; t)$ is nonzero for every rooted tree t: in the terminology of Brouder [1], $n(\bullet; t)$ is the "tree multiplicity" of t. For a forest $t_1 \cdots t_k$ of rooted trees, let $B_+(t_1 \cdots t_k)$ be the rooted tree obtained by attaching the root of each t_i to a new root vertex. Then using diagram (0.3) it is easy to see that

$$n(\bullet; B_+(\ell_{n_1}\ell_{n_2} \cdots \ell_{n_k})) = \frac{1}{m_1!m_2!\cdots} \binom{n_1 + \cdots + n_k}{n_1\ n_2\ \cdots\ n_k},$$

where m_i is the number of the n_j equal to i. Cf. equation (1) of [1].

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Some Aspects of CQFT in $D = 4$ Dimensions

Ansgar Schneider

In quantum field theory (QFT) the concept of global conformal invariance (GCI) has far reaching consequences. E.g. it implies Huygens principle and rationality of the Wightman distributions (the n-point-functions) [NT]. Furthermore the 2-point-function of a field is fixed and characterized by the spin $(j_1, j_2), j_k \in \frac{1}{2}\mathbb{N}$ and the scaling dimension $\delta \in \mathbb{N}$ of the field. The condition of Wightman positivity leads to so-called unitary bounds for δ, namely $\delta \geq 1 + j_1 + j_2$,if $j_1 j_2 = 0$, and $\delta \geq 2 + j_1 + j_2$,if $j_1 j_2 \neq 0$. Thus a scalar field ($j_1 = j_2 = 0$) must have at least scaling dimension $\delta \geq 1$, a vector field ($j_1 = j_2 = \frac{1}{2}$) at least $\delta \geq 3$.

All in all the condition of GCI is rather strong and gives a priori much structure to the theory. One may hope that it is one of the most promising approaches in constructive QFT.

Due to [NST] a scalar field satisfying GCI and having scaling dimension $\delta = 4$ is of particular interest. There is a natural candidate for such a field, namely the scalar Wick square $\Phi := \sum_{\mu=0}^{3} :B_\mu B^\mu:$ of the free spin 1 vector field $B = (B_\mu)_{\mu=0,1,2,3}$ of dimension 2, which itself violates the axiom of Wightman positivity. I.e. $\{B(f) \mid f \in \mathcal{S}(\mathbb{R}^4, \mathbb{C}^4)\}$ are operators on a (non degenerate) indefinite(!) inner product ("Fock") space $\big(D = \bigoplus_n \mathcal{S}(\mathbb{R}^{4n}, (\mathbb{C}^4)^{\otimes n})^{\mathrm{sym}}, \langle\_\,,\_\rangle_D\big)$. Keep in mind that one of the major difficulties in constructive QFT is to satisfy the axiom of Wightman positivity, which is caused by its non-linear nature. E.g. the Wightman distributions

$$W_n : \mathcal{S}(\mathbb{R}^{4n}) \ni h_1 \otimes \cdots \otimes h_n \mapsto \langle \Omega, \Phi(h_1) \dots \Phi(h_n)\Omega \rangle_D ,$$
$$\Omega := \text{vacuum} := (1, 0, 0, \dots),$$

of our real scalar field Φ have to fulfil

$$\sum_{n,m=0}^{N} W_{n+m}(\tilde{\varphi}_n \otimes \varphi_m) \geq 0,$$

for all $N \in \mathbb{N}$ and all test functions $\varphi_k \in \mathcal{S}(\mathbb{R}^{4k})$. Here we used the notation $\tilde{\varphi}_k(x_1, \dots, x_k) := \bar{\varphi}_k(x_k, \dots, x_1)$.

By construction Φ satisfies all the linear Wightman axioms and only the non-linear axiom of Wightman positivity is unclear.

In fact Wightman positivity is satisfied at the level of the 2-point-function, since the Wick square has scaling dimension $\delta = 4$ and the unitary bound for a scalar field is $\delta \geq 1$, furthermore, and that is remarkable, Wightman positivity is satisfied also at the level of the 4-point-function [NRT]. So one might conjecture that Wightman positivity is satisfied in general.

Unfortunately some careful analysis [A] on the generated algebra of the scalar Wick square shows that Wightman positivity fails at the level of the 12-point-function, i.e.

$$\sum_{n,m=0}^{6} W_{n+m}(\tilde{\varphi}_n \otimes \varphi_m) \geq 0$$

is **not** satisfied for all test functions $\varphi_k \in \mathcal{S}(\mathbb{R}^{4k})$.

More precisely one can show that one can approximate all bilocal field operators $B(f)B(g), f, g \in \mathcal{S}(\mathbb{R}^{4k}, \mathbb{C}^4)$, by sixth order elements of $\mathcal{A}(\Phi(h)|h \in \mathcal{S}(\mathbb{R}^4))$, the operator algebra generated by Φ. As easily seen, the bilocal operators violate Wightman positivity, hence the Wick square does.

Since one may define Φ as a certain limit of a sum of bilocal operators $B(f)(g)$, the above result can be formulated as an equality of algebras

$$\lim \mathcal{A}(\Phi(h)|h \in \mathcal{S}(\mathbb{R}^4)) = \lim \mathcal{A}(B(f)B(g))|f, g \in \mathcal{S}(\mathbb{R}^4, \mathbb{C}^4)),$$

and we know that elements of the r.h.s violate Wightman positivity.

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Sheaf theory for smooth and topological stacks and twisted cohomology

U. Bunke
(joint work with Th. Schick and M. Spitzweck)

1. Twisted cohomology - de Rham models

We consider a smooth manifold X with a closed three from λ. Then we can form the two-periodic λ-twisted de Rham complex

$$\cdots \stackrel{d_\lambda}{\to} \Omega^{even}(X) \stackrel{d_\lambda}{\to} \Omega^{odd}(X) \stackrel{d_\lambda}{\to} \Omega^{even}(X) \stackrel{d_\lambda}{\to} \dots \ , \tag{1.1}$$

where $d_\lambda := d_{dR} + \lambda$.

Definition 1.1. The cohomology of the complex (1.1) is called the λ-twisted two-periodic cohomology of X and will be denoted by $H^*(X;\lambda)$.

The important motivation for this definition is that $H^*(X;\lambda)$ can serve as a target of the Chern character from twisted K-theory. This distinguishes the choice of a closed three from among the possibility to choose closed froms of arbitrary odd degree.

A natural choice of the twist for twisted K-theory $K^*(X;P)$ is a PU-principal bundle $P \to X$, where PU is the projective unitary group of a separable infinite-dimensional Hilbert spaces. The definition of twisted cohomology above has the following draw-backs.

(1) $H^*(X;\lambda)$ does not depend in a functorial way on P. In fact, it depends on the choice of the closed form λ.
(2) The image of the Chern character is a lattice. In order to study integrality questions one should be able to define an integral version of twisted cohomology. This goes beyond the de Rham model.
(3) The use of the de Rham model is tied to the smooth case. A generalization to orbifolds is possible. But twisted cohomology should be defined in the topological category.

2. Non-periodic twisted de Rham cohomology

Let $\Omega(X)[[z]]$ denote the space of formal power-series of smooth differential forms on X, where z is a formal variable of degree $\deg(z) = 2$. On this space we define the differential $d^z_\lambda := d_{dR} + \lambda\frac{d}{dz}$. The complex $(\Omega(X)[[z]], d^z_\lambda)$ admits an action of $T := \frac{d}{dz}$ of degree -2. We form the $\mathbb{N}^{op}$-indexed system

$$\mathcal{S}et : 0 \leftarrow \Omega(X)[[z]] \stackrel{T}{\leftarrow} \Omega(X)[[z]][2] \stackrel{T}{\leftarrow} \Omega(X)[[z]][4] \stackrel{T}{\leftarrow} \Omega(X)[[z]][6] \dots$$

of complexes of abelian groups. We introduce the shifts so that the connecting maps have degree zero.

Lemma 2.1. *The limit* $\lim \mathcal{S}et$ *in the category of complexes of abelian groups* $C(\mathtt{Ab})$ *is isomorphic to the complex (1.1).*

Localizing this construction to open subsets of X we obtain a complex of sheaves $U \to (\Omega(U)[[z]], d^z_{\lambda_{|U}})$ which we will denote by $\Omega_X[[z]]_\lambda$.

3. Gerbes and twisted sheaves

In order to construct $\Omega_X[[z]]_\lambda$ we first associate to $P \to X$ the smooth gerbe of its U-reductions $f : G \to X$. To each smooth stack X we associate a Grothendieck site $\mathbf{X}^\infty$ of manifolds over X (with submersive structure maps). The covering families of an object $M \to X$ are the coverings of M by families of open subsets. If $f : X \to Y$ is a map of smooth stacks, then we define an adjoint pair of functors $f^* : \mathtt{Sh}\mathbf{Y}^\infty \Leftrightarrow \mathtt{Sh}\mathbf{X}^\infty : f_*$. The functor f_* is left-exact and thus has a right-derived descendent $Rf_* : D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathbf{X}^\infty) \to D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathbf{Y}^\infty)$.

Let $\mathbf{R}_G \in \mathtt{Sh}_{\mathtt{Ab}}\mathbf{G}^\infty$ denote the constant sheaf on $\mathbf{G}^\infty$ with value $\mathbf{R}$.

Theorem 2. There is a non-canonical isomorphism

$$Rf_*(\mathbf{R}_G) \cong \Omega_X[[z]]_\lambda \text{ in } D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathbf{X}^\infty).$$

It is now easy to generalize to the topological category and arbitrary coefficients.

4. T-duality and periodization diagrams

Recall that automorphisms of a gerbe over a stack X are classified by $H^2(X; \mathbb{Z})$. We consider the diagram

$$\begin{array}{ccc} \mathtt{pr}_X^* G & \xrightarrow{u} & \mathtt{pr}_X^* G \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & G & \\ & \downarrow{\scriptstyle f} & \\ & X & \end{array},$$

where $\mathtt{pr}_X : X \times T^2 \to X$ is the projection, and the automorphism u of gerbes over $X \times T^2$ is classified $1 \times \mathtt{or}_{T^2}$. We set $m := p \circ u$. We call the diagram

$$\mathtt{pr}_X^* G \overset{p}{\underset{m}{\rightrightarrows}} G \xrightarrow{f} X \tag{4.1}$$

the periodization diagram.

5. Integration and periodization

Consider topological stacks X and Y and a map $f : X \to Y$. Since $Rf_* : D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathbf{X}) \to D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathbf{Y})$ admits a left-adjoint f^* we have a unit transformation $\alpha_f : \mathtt{id} \to Rf_* \circ f^*$. Assume in addition that $f : X \to Y$ is a locally trivial bundle of oriented closed manifolds of dimension $n \in \mathbb{N}$.

Lemma 5.1. *There is a natural transformation*

$$\int_f : Rf_* \circ f^* \to \mathtt{id} \ ,$$

the integration map.

Note that the map $p : \mathtt{pr}_X^* G \to G$ in (4.1) is a T^2-principal bundle.

Definition 5.2. The periodization map associated to $G \to X$ is the transformation

$$T_G : Rf_* \circ f^* \stackrel{\alpha_m}{\to} Rf_* \circ Rm_* \circ m^* \circ f^* \stackrel{f\circ m = f \circ p}{\to} Rf_* \circ Rp_* \circ p^* \circ f^* \stackrel{\int_p}{\to} Rf_* \circ f^* \ .$$

For any sheaf $\mathcal{F} \in D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathbf{X})$ we consider the diagram

$\mathcal{S}et_G(\mathcal{F})$:

$$0 \leftarrow Rf_* \circ f^*(\mathcal{F}) \stackrel{T_G}{\leftarrow} Rf_* \circ f^*(\mathcal{F})[2] \stackrel{T_G}{\leftarrow} Rf_* \circ f^*(\mathcal{F})[4] \stackrel{T_G}{\leftarrow} Rf_* \circ f^*(\mathcal{F})[6] \stackrel{T_G}{\leftarrow} \dots$$

in the unbounded derived category $D(\mathtt{Sh}_{\mathtt{Ab}}\mathbf{X})$.

Definition 5.3. The periodic G-twisted sheaf associated to $\mathcal{F}$ is defined by

$$P_G(\mathcal{F}) := \mathtt{holim}\mathcal{S}_G(\mathcal{F}) \ .$$

Note that the homotopy limit of an $\mathbb{N}^{op}$-indexed diagram in a triangulated category as $D(\mathtt{Sh}_{\mathtt{Ab}}\mathbf{X})$ is well-defined up to non-canonical isomorphism.

Theorem 3. There exists a refinement of the constructions above providing a functor

$$P_G : D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathbf{X}) \to D(\mathtt{Sh}_{\mathtt{Ab}}\mathbf{X})$$

which depends functorially on G.

Definition 5.4. We define the periodic G-twisted cohomology of X with coefficients in $\mathcal{F} \in D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathbf{X})$ by

$$H^*_{per}(X, \mathcal{F}) := H^*(X; P_G(\mathcal{F})) \ .$$

6. Examples

As an illustration we calculate the two-periodic cohomology $H^*(X; P_G(R_X))$ in the case where X is a point and $\mathcal{F} = R_X$ for certain abelian groups R.

Proposition 6.1.

R	H^{ev}	H^{odd}
$\mathbf{Q}$	$\mathbf{Q}$	0
$\mathbb{Z}/n$	0	0
$\mathbf{Q}/$	$\mathbb{A}_f^{\mathbf{Q}}$	0
	0	$\mathbb{A}_f^{\mathbf{Q}}/\mathbf{Q}$

,

where $\mathbb{A}_f^{\mathbf{Q}}$ *denotes the finite adeles of* $\mathbf{Q}$.

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A formula for the basic zeta coefficient for pseudodifferential boundary operators

Gerd Grubb

In the introductory part of the talk we have recalled some of the modern generalizations of trace functionals, to operators that are not necessarily trace-class. For a classical pseudodifferential operator (ψdo) A of order σ on a closed manifold X there is, on one hand, the noncommutative residue $\operatorname{res} A$ introduced by Wodzicki [14] and Guillemin [8] ca. 1984; a functional depending, in local coordinates, only on a finite number of homogeneous symbol terms (we call such functionals *symbolic*). It vanishes on operators of noninteger or low order and is expressible by an integral over X of a form $\operatorname{res}_x(A)\,dx$, where $\operatorname{res}_x(A)$ is in local coordinates the integral of the fiber trace tr of the symbol term $a_{-n}(x,\xi)$ for $|\xi|=1$. On the other hand there is the canonical trace $\operatorname{TR} A$ introduced by Kontsevich and Vishik [9] ca. 1995, defined only for a subset of the operators and *global* (depending on the full structure). It extends the usual trace and is expressible by an integral over X of a form $\operatorname{TR}_x A\,dx$, where $\operatorname{TR}_x A$ is a finite-part integral of $\operatorname{tr} a(x,\xi)$ in ξ in the sense of Hadamard (see Lesch [10] and Grubb [5] for this characterization). The expression $\operatorname{TR}_x A$ is defined pointwise in local coordinates, but is not in general invariant under coordinate changes.

When P_1 is an auxiliary m'th order elliptic operator on X with no eigenvalues on $\overline{\mathbb{R}}_-$, one can define the generalized zeta function $\zeta(A,P_1,s)=\operatorname{Tr}(AP_1^{-s})$ for large $\operatorname{Re} s$ and extend it meromorphically to the complex plane, with simple poles at $(\sigma+n-j)/m,\ j=0,1,2,\dots$. The basic zeta coefficient $C_0(A,P_1)$ is then defined as the regular value of $\zeta(A,P_1,s)$ at $s=0$, i.e., the coefficient of s^0 in the Laurent expansion at $s=0$. In a number of cases where $\operatorname{res} A$ vanishes, $C_0(A,P_1)$ equals $\operatorname{TR} A$.

It is known from works of Okikiolu [12], Kontsevich and Vishik [9], Melrose and Nistor [11], that the *trace defects* $C_0(A,P_1)-C_0(A,P_2)$ and $C_0([A,A'],P_1)$ are symbolic, expressible in terms of residues of classical ψdo's involving $\log P_1$ and $\log P_2$. Paycha and Scott [13] have recently shown a formula for $C_0(A,P_1)$ itself:

$$C_0(A,P_1)=\int_X\big(\operatorname{TR}_x A-\tfrac{1}{m}\operatorname{res}_{x,0}(A\log P_1)\big)\,dx, \tag{0.1}$$

where the terms in the integrand are defined in local coordinates, $\mathrm{res}_{x,0}(A \log P_1)$ being defined from the term in the symbol of $A \log P_1$ that is homogeneous of degree $-n$ (without a log-factor).

The main purpose of the talk is to report on how this formula can be extended to operators on manifolds with boundary. From here on X denotes a compact n-dimensional C^∞ manifold with boundary $\partial X = X'$. We can assume that $X \subset \widetilde{X}$ for a smooth n-dimensional manifold $\widetilde{X}$ without boundary, so that ∂X is its boundary there. $\widetilde{X}$ is provided with a vector bundle $\widetilde{E}$, and $E = \widetilde{E}|_X$. We consider an operator $B = P_+ + G$ on X of order $\sigma \in \mathbb{Z}$ acting in E, where $P_+ = r^+ P e^+$ is the truncation to X of a classical ψdo P defined on $\widetilde{X}$ and satisfying the transmission condition at X', and G is a singular Green operator (s.g.o.) of class 0, as defined by Boutet de Monvel [1]. (Here r^+ restricts from $\widetilde{X}$ to X, and e^+ extends by 0 from X to $\widetilde{X}$.) Taking the trace with respect to the normal variable x_n near the boundary induces a classical ψdo $\mathrm{tr}_n G$ on X' from G.

For such operators, Fedosov, Golse, Leichtnam and Schrohe [2] defined a non-commutative residue (symbolical), and Grubb and Schrohe [7] introduced a canonical trace (global) in particular cases; the latter is an integral of finite-part integrals $\mathrm{TR}_x P$ and $\mathrm{TR}_{x'}(\mathrm{tr}_n G)$ that can always be defined pointwise in local coordinates.

With P_1 denoting an auxiliary second-order elliptic differential operator on $\widetilde{X}$ having no eigenvalues on $\overline{\mathbb{R}}_-$, we can define the zeta function $\zeta(B, P_{1,+}, s) = \mathrm{Tr}(B(P_1^{-s})_+)$ for large $\mathrm{Re}\, s$ and show that it extends meromorphically across 0, having a simple pole there; then $C_0(B, P_{1,+})$ denotes the regular value at 0.

Trace defect formulas for $C_0(B, P_{1,+}) - C_0(B, P_{2,+})$ and $C_0([B, B'], P_{1,+})$ were worked out in [7] and [4].

One of the difficulties with generalizing (0.1) is that little is known about logarithms of operators in the Boutet de Monvel calculus. One can easily define $(\log P_1)_+$, but its behavior under compositions with operators in the Boutet de Monvel calculus needs also to be studied, and so do derived operators such as $G^+(\log P_1) = r^+(\log P_1)e^- J$ and $G^-(\log P_1) = J r^-(\log P_1)e^+$, where J stands for reflection in the boundary. The latter are generalized s.g.o.s with a certain singular behavior at the boundary.

The formula we have found contains extra terms stemming from the truncation at the boundary. Namely, $C_0(B, P_{1,+})$ is a finite sum of locally defined pieces of the form:

$$
\begin{aligned}
&\int_X [\mathrm{TR}_x P - \tfrac{1}{2} \mathrm{res}_{x,0}(P \log P_1)]\, dx + \tfrac{1}{2} \int_{X'} \mathrm{res}_{x'} \mathrm{tr}_n(G^+(P) G^-(\log P_1))\, dx' \\
&\qquad + \int_{X'} [\mathrm{TR}_{x'} \mathrm{tr}_n G - \tfrac{1}{2} \mathrm{res}_{x',0} \mathrm{tr}_n(G(\log P_1)_+)]\, dx'.
\end{aligned}
\tag{0.2}
$$

The strategy of the proof is to combine the knowledge of the trace defect formulas from [4] with an exact calculation in one particular case, where P_1 is taken as simple as possible. The symbol estimates are based on the parameter-dependent symbol calculus of [3]. For the calculation of the contribution from G we moreover need to invoke Laguerre expansions, treating the diagonal part of the symbol of G in a different way than the off-diagonal part. Details are written up in [6].

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Relative pairing in cyclic cohomology and divisor flows

Matthias Lesch

(joint work with Henri Moscovici, Markus Pflaum)

Cyclic cohomology of associative algebras, viewed as a noncommutative analogue of de Rham cohomology, provides via its pairing with K-theory a natural extension of the Chern-Weil construction of characteristic classes to the general framework of noncommutative geometry [1]. In this capacity, cyclic cohomology has been extensively and successfully exploited to produce geometric invariants for K-theory classes (see Connes [2] for an impressive array of such applications, that include proving the Novikov conjecture for all word-hyperbolic groups [3]).

In this work [5] we present a new application of this method to the construction of geometric invariants in the relative setting, which takes full advantage of the excision property not only in topological K-theory but also in (periodic) cyclic cohomology (cf. Wodzicki [8], Cuntz–Quillen [4]).

After reviewing the relative Chern character and the relative pairing in the general framework of cyclic (co)homology, and briefly illustrating it in the familiar context of de Rham (co)homology on manifolds with boundary, we recast in this light the divisor flow for suspended pseudodifferential operators introduced by Melrose [7], as well as its multiparametric versions defined in Lesch–Pflaum [6]. More precisely, we show that the divisor flow for parametric pseudodifferential operators on a closed manifold can be expressed as the pairing between a relative cyclic class determined by the regularization à la Melrose of the operator trace together with its symbolic 'boundary' and the relative K_1-group of the pair consisting of parametric pseudodifferential operators together with their symbol algebra. This representation gives a clear and conceptual explanation to all the essential features of the divisor flow – its homotopy nature, additivity and integrality. In addition, it provides a cohomological formula for the spectral flow along a smooth path of self-adjoint elliptic first order differential operators, between any two invertible such operators on a closed manifold.

In the sequel we give some more details. For those notions which are mentioned but not explained here see [5]. Consider two unital Fréchet algebras $\mathcal{A}, \mathcal{B}$ and

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{A} \stackrel{\sigma}{\longrightarrow} \mathcal{B} \longrightarrow 0$$

an exact sequence of Fréchet algebras and unital homomorphisms such that $\mathcal{A}$ (and hence $\mathcal{B}$) is a good Fréchet algebra.

It is well–known that the Chern character in noncommutative geometry is a natural transformation

$$\mathrm{ch}_\bullet : K_\bullet(\mathcal{A}) \longrightarrow HC_\bullet(\mathcal{A})$$

from K–theory to cyclic homology. Since excision holds in many cases for K–theory and cyclic (co)homology there seemed to be no need to develop the corresponding relative theories. We show in various examples that this point of view should be questioned.

Therefore, we first identify the *relative* objects in the various theories. Let $E\ell\ell_\infty(\mathcal{A}) = \sigma^{-1}(\mathrm{GL}_\infty(\mathcal{B})), \mathrm{GL}_\infty(\mathcal{B}) = \lim\limits_{N\to\infty} \mathrm{GL}_N(\mathcal{B})$, and denote by

$$\pi_1(E\ell\ell_\infty(\mathcal{B}), \mathrm{GL}_\infty(\mathcal{A}); a_0)$$

the set of homotopy classes of paths $(a_s)_{0\leq s\leq 1} \subset E\ell\ell_\infty(\mathcal{B})$ with $a_s|_{s=0} = a_0 \in \mathrm{GL}_\infty(\mathcal{A})$ and $a_1 \in \mathrm{GL}_\infty(\mathcal{A})$. It is clear that for different $a_0, \tilde{a}_0$ there is a canonical bijection between the corresponding homotopy sets. Hence it suffices to consider $a_0 = I$.

Theorem.

(1) $\pi_1\big(E\ell\ell_\infty(\mathcal{A}), \mathrm{GL}_\infty(\mathcal{A}); I\big)$ *is canonically isomorphic to the relative K-theory group* $K_1(\mathcal{A},\mathcal{B}) = K_1(J)$.

(2) *The relative cyclic homology* $HC_\bullet(\mathcal{A},\mathcal{B})$ *is the homology of the first quadrant double complex* $\big(\mathrm{Tot}_\bullet^\oplus \mathcal{B}C_{\bullet,\bullet}(\mathcal{A},\mathcal{B}), \widetilde{b}+\widetilde{B}\big)$, *where* $\mathcal{B}C_{p,q}(\mathcal{A},\mathcal{B}) = \mathcal{B}C_{p,q}(\mathcal{A}) \oplus \mathcal{B}C_{p,q+1}(\mathcal{B})$, *while*

$$\widetilde{b} = \begin{pmatrix} b & 0 \\ -\sigma_* & -b \end{pmatrix}, \quad \textit{and } \widetilde{B} = \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}.$$

(3) *The Chern character of a relative K-theory class represented by* $(a_s)_{0\le s\le 1}$ *is given by a pair representing a relative cyclic homology class as follows:*

$$\mathrm{ch}_\bullet\left((a_s)_{0\le s\le 1}\right) = \left(\mathrm{ch}_\bullet(a_1) - \mathrm{ch}_\bullet(a_0), \int_0^1 \not{c}\mathrm{h}\left(\sigma_*(a_s), \sigma_*(\dot{a}_s)\right) ds\right).$$

Here, $\mathrm{ch}_\bullet$ *is the ordinary odd Chern character and* $\not{c}\mathrm{h}$ *is the transgressed Chern character.*

Connes' concept of a cycle over an algebra has a natural extension to the relative case:

Definition. *A relative cycle of degree* k *over* $(\mathcal{A},\mathcal{B})$ *consists of the following data:*

(1) *differential graded unital algebras* (Ω, d) *and* $(\partial\Omega, d)$ *over* $\mathcal{A}$ *resp.* $\mathcal{B}$ *together with a surjective unital homomorphism* $r : \Omega \to \partial\Omega$ *of degree* 0,

(2) *unital homomorphisms* $\varrho_{\mathcal{A}} : \mathcal{A} \to \Omega^0$ *and* $\varrho_{\mathcal{B}} : \mathcal{B} \to \partial\Omega^0$ *such that* $r \circ \varrho_{\mathcal{A}} = \varrho_{\mathcal{B}} \circ \sigma$,

(3) *a graded trace* $\int$ *on* Ω *of degree* k *such that*

$$\int d\omega = 0\,, \quad \textit{whenever} \quad r(\omega) = 0.$$

The graded trace $\int$ induces a unique closed graded trace $\int'$ on $\partial\Omega$ of degree $k-1$, such that Stokes' formula

$$\int d\omega = \int' r\omega\,, \qquad \text{for } \omega \in \Omega$$

is satisfied.

The boundary $(\partial\Omega, d, \int')$ is just a cycle over the algebra $\mathcal{B}$. For $(\Omega, d, \int)$, this is in general not the case, unless the trace $\int$ is closed.

We next define the *character* of a relative cycle C. Define $(\varphi_k, \psi_{k-1}) \in C^k(\mathcal{A}) \oplus C^{k-1}(\mathcal{B})$ as follows:

$$\varphi_k(a_0,\dots,a_k) := \frac{1}{k!}\int \varrho(a_0)d\varrho(a_1)\dots d\varrho(a_k),$$
$$\psi_{k-1}(b_0,\dots,b_{k-1}) := \frac{1}{(k-1)!}\int' \varrho(b_0)d\varrho(b_1)\dots d\varrho(b_{k-1}).$$

Then (φ_k, ψ_{k-1}) is a relative cyclic cocycle in $\mathrm{Tot}_\oplus^k \mathcal{B}C^{\bullet,\bullet}(\mathcal{A},\mathcal{B})$.

In [5] it is shown that a de Rham cohomology class on a manifold with boundary naturally gives rise to a cycle with boundary. As a less trivial example in loc. cit. we study in detail the algebra of parametric pseudodifferential operators on a closed manifold. We show that the divisor flow [7] can be expressed expressed as the pairing between a relative cyclic class determined by the regularization à la Melrose of the operator trace together with its symbolic 'boundary' and the relative K_1-group of the pair consisting of parametric pseudodifferential operators together with their symbol algebra.

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Arithmetic Mirror Symmetry

Noriko Yui

Definition:A smooth projective variety X of dimension 3 over $\mathbf{C}$ is called a *Calabi–Yau* threefold if (1) $H^i(X, \mathcal{O}_X) = 0$ for each $i = 1, 2$, and (2) the canonical bundle $\mathcal{K}_X$ is trivial.

Introduce the Hodge numbers $h^{i,j}(X) := \dim_{\mathbf{C}} H^j(X, \Omega^i_X)$ for $0 \leq i, j \leq 3$. Then (1) $\Leftrightarrow$ $h^{1,0}(X) = h^{2,0}(X) = 0$ and (2) implies that $h^{3,0}(X) = p_g(X) = 1$. The Hodge numbers are concocted to the Hodge diamond:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{1,2} & & 1 \\
 & 0 & & h^{2,2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

The Serre duality asserts that $h^{1,1} = h^{2,2}$ and the complex conjugation gives rise to the equality $h^{1,2} = h^{2,1}$. The Betti numbers are given by

$$B_0 = B_6 = 1,\ B_1 = B_5 = 0, B_2 = h^{1,1} = h^{2,2} = B_4,\ B_3 = 2(1 + h^{2,1})$$

and the Euler characteristic is $\chi = 2(h^{1,1} - h^{2,1})$.

Topological Mirror Symmetry Conjecture:*Given a family of Calabi–Yau threefolds $\mathcal{M}$, there is a mirror family of Calabi–Yau threefolds $\mathcal{W}$ in the sense that the mirror map interchanges the Hodge numbers:*

$$h^{1,1}(\mathcal{W}) = h^{2,1}(\mathcal{M}),\ h^{2,1}(\mathcal{W}) = h^{1,1}(\mathcal{M})$$

so that the Euler characteristics change the sign: $\chi(\mathcal{W}) = -\chi(\mathcal{M})$. Furthermore, there is a linear isomorphism

$$\Phi : H^{1,1}(\mathcal{W}) \to H^{2,1}(\mathcal{M})$$

which identifies the Type IIA and the Type IIB prepotentials.

In this talk, I will introduce the so-called "Arithmetic Mirror Symmetry" for Calabi–Yau threefolds defined over $\mathbb{Q}$. Our aim is to interpret the mirror symmetry phenomenon by means of arithmetic invariants of the Calabi–Yau threefolds, e.g., zeta-functions, Galois representations and L-series.

Suppose that X is a Calabi–Yau threefold defined over $\mathbb{Q}$. Then X always has an integral model. Let p be a "good" prime. We define the congruence zeta-function $\zeta_p(X,T)$ of $X \pmod p$ by encoding the numbers of $\mathbf{F}_q$–rational points on $X \pmod p$ for $q = p^a$, $a \in \mathbf{N}$. We have

$$\zeta_p(X,T) = \prod_{i=0}^{6} P_p^i(T)^{(-1)^{i+1}}$$

where $P_p^i(T) := \det(1 - \mathrm{Frob}_p^* T \mid H_{et}^i(\bar{X}, \mathbf{Q}_\ell)) \in 1 + T\mathbf{Z}[T]$ with degree B_i. The i-th L-series of X is

$$L_i(X,s) := L(H_{et}^i(\bar{X}, \mathbf{Q}_\ell), s) = \prod_{p:\text{good}} \frac{1}{P_p^i(p^{-s})},$$

and globally, the L-series of X is $L(X,s) := \prod_{i=0}^{6} L_i(X,s)^{(-1)^i}$.

Construction of Calabi–Yau orbifolds: We construct Calabi–Yau threefolds using orbifolding construction in weighted projective 4-spaces. The starting point is the Fermat hypersurface of degree m in $\mathbf{P}^4$:

$$X_0^m + X_1^m + X_2^m + X_3^m + X_4^m = 0.$$

Let $Q = (q_0, q_1, q_2, q_3, q_4) \in \mathbf{N}^5$. We say that $< m, Q >$ is an *admissible pair* if

(1) $\gcd(q_0, \cdots, \widehat{q_j}, \cdots, q_4) = 1$ for every j, $0 \le j \le 4$ ($\hat{q}_j$ means deletion of q_j),

(2) $q_i | m$ for every i, $0 \le i \le 4$ and

(3) $m = q_0 + q_1 + q_2 + q_3 + q_4$ (the Calabi–Yau condition).

Let $\mu_Q = \prod_{i=0}^{5} \mu_{q_i}$ where $\mu_{q_i} = \mathrm{Spec}(\mathbf{Q}[X]/(X^{q_i} - 1)$ is a finite group (scheme) of order q_i for each i. Then μ_Q acts on V by componentwise multiplication. Define the quotient $Y := V/\mu_Q$. Then the Calabi–Yau condition (3) guarantees that Y is a singular Calabi–Yau threefold. Y has only cyclic quotient singularities, which are defined over $\mathbf{Q}$. These singularities are well understood. The exists a crepant resolution, X, of Y, i.e.,a resolution preserving the triviality of the canonical sheaf of Y. X is a smooth Calabi–Yau threefold defined over $\mathbf{Q}$. There are altogether 147 admissible pairs with $< 5, (1, 1, 1, 1, 1) >$ the smallest and $< 1806, (1, 42, 258, 602, 903) >$ the largest.

Fermat motives: One problem here is that the Betti numbers B_3 could be rather large, implying the associated Galois representations could have very high dimension. To go around this difficulty, we introduce the concept of "Fermat motives", which are defined explicitly by means of projectors. The cohomological realizations of Fermat motives enable us to decompose any Weil cohomology groups of X into the product of motivic cohomology groups and the cohomology group associated to the singular locus. In particular, the polynomial $P_p^3(T)$ factors into the product of motivic characteristic polynomials (of degree at most $\varphi(m)$ where φ is the Euler function), and the polynomials associated to the singular locus. Passing to Fermat motives, we are able to compute the congruence zeta-functions and L-series for our 147 Calabi–Yau orbifolds.

Mirror Calabi–Yau threefolds: We construct mirror partners of our Calabi–Yau threefolds by the Greene–Plesser method, namely, first deforming them and then taking quotients by discrete groups of symmetries. This process switches the Hodge numbers and we obtain mirror partners satisfying Topological Mirror Symmetry Conjecture. We can compute the congruence zeta-functions and L-series of these mirror Calabi–Yau threefolds. One consequence of the topological mirror symmetry can be observed arithmetically as in the following theorem.

Theorem: *The characteristic polynomial of the motive M_Q corresponding to the weight $Q = (q_0, q_1, q_2, q_3, q_4)$ divides both $P_p^3(\mathcal{M}, T)$ and $P_p^3(\mathcal{W}, T)$ in $\mathbb{Z}[T]$. In particular, the motive M_Q remain invariant under the mirror map.*

"Quantum" zeta-function and "quantum L-series: String theorists wish to have a relation something like T-duality for zeta-functions and L-series of mirror pairs of Calabi–Yau threefolds. However, the classical zeta-functions and L-series are NOT the right kind of objects. This leads us to the search of a "quantum" zeta-function and L-series. We offer two candidates for such a quantum object. We argue at local level, i.e., for "quantum" zeta-functions. For a mirror pair $(\mathcal{M}, \mathcal{W})$, we define $\zeta_p(\mathcal{M}, \mathcal{W}, p^{-s}) := P_p^3(\mathcal{M}, p^{-s}) \cdot P_p^3(\mathcal{W}, p^{-s})^{-1}$. Then $\zeta_p(\mathcal{M}, \mathcal{W}, s)$ satisfies the required "T-duality", but results in throwing away the most essential factor $P_p^3(M_Q, p^{-s})$ corresponding to the weight motive M_Q. The other candidate is the slope zeta-function due to D. Wan. Note that the reciprocal roots of $P_p^i(\mathcal{M}, T)$ are all algebraic integers. Factor $\zeta_p(\mathcal{M}, T)$ over $\overline{\mathbf{Z}}_p$, and write $\zeta_p(\mathcal{M}, T) = \prod_i (1 - \alpha_i T)^{\pm 1}$ where α_i are algebraic integers in $\bar{\mathbf{Z}}_p$. Then the (normalized) slopes $\mathrm{ord}_q(\alpha_i)$ are rational numbers in the interval $[0, 3]$. The *slope* zeta-function is defined by $S_p(\mathcal{M}, u, T) := \prod_i (1 - u^{\mathrm{ord}_q(\alpha_i)} T)^{\pm 1}$.

Proposition: *Let $(\mathcal{M}, \mathcal{W})$ be a mirror pair of Calabi–Yau orbifolds over $\mathbf{Q}$ constructed above. Let p be a good prime. Suppose that both $\mathcal{M} \otimes \mathbf{F}_p$ and $\mathcal{W} \otimes \mathbf{F}_p$ are ordinary, that is, the Newton polygon coincides with the Hodge polygon. Then*

$$S_p(\mathcal{W}, u, T) = \frac{1}{S_p(\mathcal{M}, u, T)}.$$

The ordinarity condition holds for primes $p \equiv 1 \bmod m$.

However, neither candidates would fit the bill, and our search for the right "quantum" zeta-functions goes on.

Remark: Several people have asked the question of how our Arithmetic Mirror Symmetry is related to the Mirror Symmetry story in String Theory. The idea of mirror symmetry first appeared as a physics prediction that two different physical theories (Type IIA and Type IIB) in space-time dimension 10, would give rise, after compactification, isomorphic physical theories in space-time dimension 4, except that the role of A-model and B-model correlation functions (prepotentials) are reversed. In particular, this would imply that there is a mirror pair of Calabi–Yau threefolds $\mathcal{M}$ and $\mathcal{W}$ and a linear isomorphism (mirror map) relating the A-model prepotential (which count the number of rational curves on $\mathcal{M}$) and the B-model prepotential (which can be computed by the period integrals on $\mathcal{W}$). This theory has presented a spectacular advance in Enumerative Geometry, e.g., counting the number of rational curves (or higher genus curves), the Gromov-Witten invariants, the BPS states, on Calabi–Yau threefolds. However, this theory misses completely the dimension zero objects, namely, points on Calabi–Yau threefolds. Arithmetic Mirror Symmetry tries to remedy this situation.

The book *Mirror Symmetry V* (AMS/IP Stud. in Adv. Math. Series (2006), to appear) contains a number of articles addressing "Arithmetic Mirror Symmetry".

A splitting formula for the $su(N)$ spectral flow of the odd signature operator coupled to a path of $SU(N)$ connections

Benjamin Himpel

Introduction. A splitting formula is a tool for computing spectral flow of a path of self-adjoint elliptic differential operators, which is roughly the net number of eigenvalues that change sign. We consider a 3-manifold $M = X \cup_T Y$ split along a torus, a trivialized $SU(N)$ bundle over M, and the odd signature operator D_{A_t} coupled to a path $A_t \in \mathcal{A} = \Omega^1(M) \otimes su(N)$ of $SU(N)$ connections, $t \in [0, 1]$. The odd signature operator twisted by an $SU(N)$-connection A is given by

$$\begin{aligned} D_A : \Omega^{0+1}(M) \otimes su(N) &\to \Omega^{0+1}(M) \otimes su(N) \\ (\alpha, \beta) &\mapsto (d_A^*\beta, d_A\alpha + *d_A\beta), \end{aligned}$$

where $d_A\omega := d\omega + [A, \omega]$ is the twisted exterior derivative on the 0- and 1-forms $\Omega^{0+1}(M) \otimes su(N) := \Omega^0(M) \otimes su(N) \oplus \Omega^1(M) \otimes su(N)$, $[\cdot, \cdot]$ is the wedge product on forms combined with the Lie bracket on the coefficients, and d_A^* is its dual.

Because the space of all connections is contractible, the spectral flow depends only on the endpoints. We let $\mathrm{SF}(A_0, A_1) := \mathrm{SF}(D_{A_t})$. Henceforth we assume, that A_0 and A_1 are flat, that is, $d_A^2 = 0$, because by $\dim(\mathrm{Ker} D_{A_\varepsilon}) = \dim(H^{0+1}(M, d_{A_\varepsilon}))$ the spectral flow turns out to be a topological invariant. We are interested in finding a formula of the form

$$\mathrm{SF}(A_0, A_1) = \mathrm{SF}(D_{A_t}|_X; \mathcal{P}_t^+) + \mathrm{SF}(D_{A_t}|_Y; \mathcal{P}_t^-) + C(A_0) - C(A_1),$$

where $\mathcal{P}_t^{\pm}$ are paths of certain Atiyah-Patodi-Singer boundary conditions and C is a correction term, which only depends on A_ε, $\varepsilon = 0, 1$.

The usual situation to keep in mind, in which a splitting formula can be applied to compute spectral flow, is when $A_t|_X$ is flat and Y is a simple manifold, e.g. a solid torus. In such a situation $\mathrm{SF}(D_{A_t}|_X; \mathcal{P}_t^+)$ may be computed by topological means, while $\mathrm{SF}(D_{A_t}|_Y; \mathcal{P}_t^-)$ may be computed once and for all by a different method. See for example [8]. In this talk I will also present a splitting formula, which can be useful whenever the representation varieties of X and Y are connected. Note that this is the case whenever X and Y are complements of torus knots.

Motivation. The connection between topology and Chern-Simons gauge theory is given by the identification of $SU(N)$ representations and flat $SU(N)$ connections up to conjugation and the action of the gauge group $\mathcal{G}$ respectively. The flat $SU(N)$ connections are precisely the critical points of the Chern-Simons function $cs(A) = \frac{1}{8\pi^2}\int_M \mathrm{tr}(A \wedge dA + A \wedge A \wedge A)$ on the space of $SU(N)$ connections, and the Chern Simons function factors through $\mathcal{A}/\mathcal{G}$ as an $\mathbf{R}/$ valued function.

Taubes [11] laid the groundwork for new topological invariants based on gauge theory by showing that the $SU(2)$ Casson invariant for integral homology 3-spheres [1] has a gauge theoritical interpretation as the Euler characteristic on $\mathcal{A}/\mathcal{G}$ in the spirit of the Poincaré-Hopf theorem. See also [9] for a new proof of this result. Taubes realized that the Hessian of the Chern-Simons function and the odd signature operator coupled to the same path of $SU(2)$ connections have the same spectral flow. Floer extended this idea around the same time to instanton Floer homology, which has the $SU(2)$ Casson invariant as its Euler characteristic, by viewing the critical points of the Chern-Simons function as a /8 graded Morse complex.

Generalizations of the $SU(2)$ Casson invariant to $SU(N)$ [2, 4, 5] via gauge theory followed, as well as integer graded instanton Floer homology [6], providing finer topological invariants of homology 3-spheres. In order to compute any of these invariants, one needs to compute the spectral flow of the odd signature operator coupled to a path of $SU(N)$ connections.

Witten [12] introduced new 3-manifold invariants for each $k \in \mathbb{Z}$ by the integral $Z_k(M) = \int_{\mathcal{A}/\mathcal{G}} e^{2\pi k i cs(A)}$, which is as beautifully intuitive as it is mathematically non-rigorous. However, this invariant can be defined rigorously by the axioms of topological quantum field theory, as well as interpreted as an asymptotic expansion by the method of stationary phase. The conjectured correspondence between these two approaches is known as the asymptotic expansion conjecture [7]. Among other things, one must compute spectral flow to verify the asymptotic expansion conjecture.

Some Details. We may assume that M is cylindrical in a neighborhood of T, because the spectral flow does not depend on the metric. Given a connection A on M, we may assume after gauge transformation, that A is in cylindrical form and flat near T, with $A|_T = a_{u,v} := -i\mathrm{diag}(u)\, dl - i\mathrm{diag}(v)\, dm \in \mathcal{A}_T$,

$u, v \in \Lambda := \{u \in \mathbf{R}^N \mid \sum_{j=1}^N u_j = 0\}$. The restriction of $\Omega^{0+1}(M; su(N))$ to T is isomorphic to $\Omega^{0+1+2}(T; su(N))$. Therefore, boundary conditions are certain Lagrangian subspaces of $L^2(\Omega^{0+1+2}(T; su(N)))$. Our Atiyah-Patodi-Singer boundary conditions for D_A are of the form $\mathcal{P}^+_{u,v,\theta} := \mathcal{P}^+_{a_{u,v}} \oplus L_{u,v,\theta}$, continuously parametrized by $\Lambda \oplus \Lambda \oplus (S^1)^{N-1}$ with $A|_T = a_{u,v}$, where $\mathcal{P}^+_a$ is the L^2-span of the positive eigenspace of de Rham operator twisted by a

$$\begin{aligned} S_a\colon \Omega^{0+1+2}(T; su(N)) &\rightarrow \Omega^{0+1+2}(T; su(N)) \\ (\alpha, \beta, \gamma) &\mapsto (*d_a\beta, - * d_a\alpha - d_a * \gamma, d_a * \beta). \end{aligned}$$

and $L_{u,v,\theta}$ are finite Lagrangian subspaces of $\mathrm{Ker} S_{a_{u,v}} \cong H^{0+1+2}(T, d_{a_{u,v}})$. The definition of these boundary conditions are technical, but explicit, making spectral flow computations along a path of flat connections possible. The correction terms $C(A_\varepsilon)$, $\varepsilon = 0, 1$, are certain Maslov triple indices of Lagrangian subspaces of $\mathrm{Ker}(S_{a_\varepsilon})$ associated to A_ε as defined in [8, 10].

An Application. There has been a recent conjecture by [3] relating the $SU(3)$ Casson invariant of unions of certain torus knots $M = T_{2,p} \cup_T T_{2,p'}$ to the $SU(2)$ Casson knot invariants of its pieces: $\lambda_{SU(3)}(M) = 4\lambda_{SU(2)}(T_{2,p})\lambda_{SU(2)}(T_{2,p'})$. For the $SU(3)$ Casson invariant one needs to compute $SU(3)$ spectral flow from the trivial connection to other flat connections. In order to apply the splitting formula in this situation we will refine it in the following way.

It is a simple consequence of the splitting formula that the spectral flow of the odd signature operator coupled to a loop of $SU(N)$ connections on a manifold with boundary only depends on its restriction to the boundary. Let $\tilde{\rho}_t$ be an arbitrary loop in $\Lambda^2 \times (S^1)^{N-1}$. Let A_t be a loop of $SU(N)$ connections on the solid torus restricting to a_{ρ_t} on the boundary, such that ρ_t lifts to $\tilde{\rho}_t$. Let $\mathrm{SF}(\tilde{\rho}_t) := \mathrm{SF}(D_{A_t}|_S; \mathcal{P}^+_{\tilde{\rho}_t})$. This can be computed just like in [8] for $SU(2)$.

Consider two flat connections A_0 and A_1 on $M = X \cup_T Y$. Let B_t and B'_t be paths of $SU(N)$ connections on X and Y respectively with $A_\varepsilon|_X = B_\varepsilon$ and $A_\varepsilon|_Y = B'_\varepsilon$, $\varepsilon = 0, 1$, with $\tilde{\rho}$ and $\tilde{\rho}'$ the corresponding paths in $\Lambda^2 \times (S^1)^{N-1}$. Then

$$\mathrm{SF}(A_0, A_1) = \mathrm{SF}(D_{B_t}|_X; \mathcal{P}^+_{\tilde{\varrho}_t}) + \mathrm{SF}(D_{B'_t}|_Y; \mathcal{P}^-_{\tilde{\varrho}'_t}) + \mathrm{SF}(\tilde{\rho}_{1-t} * \tilde{\rho}'_t) + C(A_0) - C(A_1).$$

Since the representation variety of the complement of a torus knot is path connected, we can find for two flat connections A_0 and A_1 paths of flat connections B_t and B'_t on $T_{2,p}$ and $T_{2,p'}$ as above. Now we can compute the $SU(3)$ spectral flow and check the conjecture.

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Chen integrals of symbols and renormalised multiple zeta values

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(joint work with Dominique Manchon)

The operator P defined on functions on integers $f : \mathbb{N} \to \mathbb{C}$ by

$$P(f)(n) = \sum_{n>m>0} f(m)$$

is a Rota-Baxter operator of weight $\theta = -1$:

$$P(f)\,P(g) = P\,(f\,P(g)) + P\,(g\,P(f)) + P(fg).$$

When applied to $f(n) = n^{-s_1}$, $g(n) = n^{-s_2}$, these relations lead to "stuffle" or "second shuffle relations" for multiple zeta functions [1]. For double zeta functions, they read:

$$\zeta(s_1)\,\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2)$$

where $\zeta(s) = \sum_{n>0} n^{-s}$ for $s > 1$ and $\zeta(s_1, s_2) = \sum_{n_1>n_2} n_1^{-s_1} n_2^{-s_2}$ for $s_1 > 1$ and $s_2 \geq 1$.

Correspondingly, starting from $f \in L^1(\mathbb{R}^+, \mathbb{C})$, one can build the map $\tilde{P}_r(f) : \mathbb{R}^+ \to \mathbb{C}$ defined for $r > 0$ by:

$$\tilde{P}_r(f)(y) = \int_{y \geq x \geq r} f(x)\,dx.$$

Then the classical Rota-Baxter relation (of weight zero)

$$\tilde{P}_r(f)\,\tilde{P}_r(g) = \tilde{P}_r(f\,\tilde{P}_r(g)) + \tilde{P}_r(g\,\tilde{P}_r(f))$$

[1] We refer the reader to e.g. [H], [Z], [W], [ENR], [CEMP], [Mi], [Zu] among a long list of articles on algebraic relations obeyed by multiple zeta functions

is an integration by parts in disguise. When applied to $f(x) = x^{-s_1}\,\chi(x)$, $g(x) = x^{-s_2}\,\chi(x)$, with χ a smooth cut-off function around the origin which is 1 outside the interval $]-r, r[$ for some positive real number r, this leads to shuffle relations for continuous analogs of multiple zeta functions. These generalise the following relation:

$$\tilde{\zeta}_r(s_1)\,\tilde{\zeta}_r(s_2) = \tilde{\zeta}_r(s_1, s_2) + \tilde{\zeta}_r(s_2, s_1),$$

where $\tilde{\zeta}_r(s) = \int_{x\geq r} x^{-s}\,dx$ for $s > 1$ and $\tilde{\zeta}_r(s_1, s_2) = \int_{x_1 > x_2 \geq r} x_1^{-s_1} x_2^{-s_2}\,dx_1\,dx_2$ for $s_1 > 1$ and $s_2 \geq 1$.

We use a renormalisation procedure à la Connes and Kreimer [CK] to obtain an extension of multiple zeta functions at all integer arguments (in particular at nonpositive arguments) which obey the *stuffle* (or *second shuffle*) relations. The requirement that the extended multiple zeta functions should also obey stuffle relations naturally leads to requiring that certain maps define characters on some Hopf algebra equipped with the stuffle product (the coproduct being the deconcatenation). The extension of multiple zeta functions at integer arguments $s_i \geq 1$ (which respects the stuffle product) carried out by number theorists can be understood in those terms.[2] [3]

It was shown in [MP1] that the Euler-Zagier-Hoffmann multiple zeta function

$$\zeta(s_1, \cdots, s_k) := \sum_{0 < n_k < \cdots < n_1} \frac{1}{n_1^{s_1}} \cdots \frac{1}{n_k^{s_k}}$$

relates via an Euler-MacLaurin formula to a Chen integral of symbols on $\mathbb{R}$ (assuming temporarily that the sums and integrals converge):

$$\begin{aligned} \tilde{\zeta}_r(s_1, \cdots, s_k) &:= \int_{[r,+\infty[}^{Chen} f_1(s_1) \otimes \cdots \otimes f_k(s_k) \\ &= \int_{r \leq x_k \leq \cdots \leq x_1} d\,x_1 \cdots d\,x_k\, f_1(s_1)(x_1) \cdots f_k(s_k)(x_k), \end{aligned}$$

where $f_i(x) = \chi(x)\,|x|^{-s_i}$.

The f_i's can be interpreted as classical symbols on $\mathbb{R}$. Using the well-known extension of the ordinary Lebesgue integral to cut-off integrals on (log-)polyhomogeneous symbols, one can define cut-off Chen integrals for tensor products $\sigma_1 \otimes \cdots \otimes \sigma_k$ of classical symbols:

$$-\!\!\!\!\!\!\int_{|\xi| \geq r}^{Chen} \sigma_1 \otimes \cdots \otimes \sigma_k \quad = \quad -\!\!\!\!\!\!\int_{r \leq |\xi_k| < \cdots < |\xi_1|} \sigma_1(\xi_1) \cdots \sigma_k(\xi_k).$$

[2] In [GZ], the authors have a similar approach using a Birkhoff factorisation to renormalise multiple zeta functions. Our construction relates to theirs via a Mellin transform.

[3] There is another extension of multiple zeta functions at integer arguments $s_i \geq 1$, which is compatible with the *shuffle* relations. But compatibility of an extension to integer arguments $s_i \geq 1$ with both sets of relations is impossible (see e.g. [ENR], [W]).

¿From there, using the Euler-Mac-Laurin formula, one can then extend ordinary Chen sums to cut-off Chen sums:

$$\sum^{Chen} \sigma_1 \otimes \cdots \otimes \sigma_k \quad = \sum_{0<|n_k|<\cdots<|n_1|} \sigma_1(n_1)\cdots\sigma_k(n_k).$$

A holomorphic regularisation $\mathcal{R} : \sigma \mapsto \sigma(z)$ on classical symbols (e.g. dimensional regularisation) leads to meromorphic maps

$$\begin{aligned} \Phi_r^{\mathcal{R}}(\sigma_1 \otimes \cdots \otimes \sigma_k) : z &\mapsto \;-\!\!\!\!\!\!\int_{|\xi|\geq r}^{Chen} \sigma_1(z) \otimes \cdots \otimes \sigma_k(z), \\ \Psi^{\mathcal{R}}(\sigma \otimes \cdots \otimes \sigma_k) : z &\mapsto \sum^{Chen} \sigma_1(z) \otimes \cdots \otimes \sigma_k(z) \end{aligned}$$

with poles of order $\leq k+1$.
The meromorphicity property for cut-off Chen integrals follows from that of cut-off integrals of holomorphic families of log-polyhomogeneous symbols. The meromorphicity property of cut-off Chen sums of holomorphic symbols, which we derive in dimension 1, then follows from the Euler-MacLaurin formula.
One thereby obtains two characters $\Phi_r^{\mathcal{R}}$, resp. $\Psi^{\mathcal{R}}$ with values in meromorphic functions defined on Hopf algebras, the tensor algebra of classical symbols with constant coefficients on $\mathbb{R}$ equipped with the appropriate shuffle, resp. stuffle product. The **Birkhoff factorisation** of these characters then provides renormalised values at $z = 0$

$$\varphi_r^{\mathcal{R}}(\sigma_1 \otimes \cdots \otimes \sigma_k) = -\!\!\!\!\!\!\int_{|\xi|\geq r}^{Chen,\mathcal{R}} \sigma_1 \otimes \cdots \otimes \sigma_k, \quad \psi^{\mathcal{R}}(\sigma \otimes \cdots \otimes \sigma_k) = \sum^{Chen,\mathcal{R}} \sigma_1 \otimes \cdots \otimes \sigma_k$$

which obey the same shuffle, resp. stuffle relations. Applying this to $\sigma_i(\xi) = |\xi|^{-s_i}\chi(\xi)$ and a regularisation $\sigma \mapsto \sigma(z)(\xi) = \sigma(\xi)\,|\xi|^{\frac{z}{1+\mu z}}$ for some real number μ leads to renormalised "multiple zeta values" $\tilde{\zeta}_r^{\mu}(s_1, \cdots, s_k)$, resp. $\zeta^{\mu}(s_1, \cdots, s_k)$ which obey the expected shuffle, resp. stuffle relations.
In particular, the latter gives renormalised multiple zeta values (depending on the parameter μ) at all integers (positive or not), which verify the stuffle relations. Confirming known results, we show that the renormalised multiple zeta values *at integers* $s_i \geq 1$ is completely determined by the finite part $\mathrm{fp}_{z=0}\zeta(1 + \frac{z}{1+\mu z})$.

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