Mathematisches Forschungsinstitut Oberwolfach

Report No. 22/2006

Mini-Workshop on Studying Original Sources in Mathematics Education

Organised by
Fulvia Furinghetti (Genova)
Hans Niels Jahnke (Essen)
Jan van Maanen (Groningen)

April 30th – May 6th, 2006

Mathematics Subject Classification (2000): 01, 97.

Introduction by the Organisers

1. General background

In the last thirty years quite some initiatives evolved and much material was developed for using the history of mathematics in the teaching of mathematics at all levels. There is a growing consensus that historical work of pupils and students may contribute to further through:

- providing insights into the development of mathematical concepts;
- developing a deeper understanding of the role of mathematics in our surrounding world and its relation to applications, culture and philosophy; and
- encouraging the perception of the subjective dimensions of mathematics: of aims and intentions in the building of mathematical concepts and algorithms, of alternative methods and of personal and creative aspects.

Among the various possible activities by which historical aspects might be integrated into the teaching of mathematics, the study of an original source is the most demanding and the most time consuming. It requires a detailed and deep understanding of the mathematics in question, of the time when it was written and of the general context of ideas. The aspect of language becomes important in ways which are completely new compared with usual practices of mathematics teaching.

Thus, reading a source is an especially ambitious enterprise, but rewarding and capable of substantially deepening the mathematical understanding.

In principle, the aims and effects which might be pursued through the study of an original source will not be different from those attained by other types of historical activities. However, there are some ideas which are specifically supported by reading mathematical sources.

(1) Studying an original source replaces the usual with something different: it allows student and teacher to see mathematics as an intellectual activity, rather than as just a corpus of knowledge or a set of techniques. For example, Newton's letter to Leibniz of 1676 in which he described how as a young man of 22 years he arrived at the general binomial formula (a cornerstone in his fluxional calculus) is a unique document for a process of mathematical invention progressing by bold generalisations and analogies. Through the reading of the letter, the student more or less feels the presence of the inventor.
(2) Integrating sources in mathematics challenges the learner's perceptions through making the familiar unfamiliar. Coming to grips with a historical text can cause a reorientation of the learner's views and thus deepen his or her mathematical understanding. All too often in teaching, concepts appear as if already existing and they are manipulated with no thought for their construction. Sources remind students that these concepts were invented and that such invention did not happen all by itself. As an example, we might refer to Leibniz' version of the calculus. There are many experiences which show that students are motivated to reflect about the limit approach to calculus when they study Leibniz' way of dealing with infinitely small quantities. Also the teacher may gain insight by concentrating on the unfamiliar. It is often difficult enough to cope with unexpected solutions by students; however, studying sources enables to the teacher and students to keep an open mind.
(3) Integrating the study of sources in mathematics education invites the learner to place the development of mathematics in the scientific and technological context of a particular time and in the history of ideas and societies. One of many examples from antiquity to the present is provided by Heron's textbook (1st century A.D.) on land surveying called *The Dioptra*. Reading parts of it connects the topic of similarity to the context of ancient surveying techniques and shows the astonishingly high achievements of ancient engineers in this and other areas. Such sources may as well provoke students to engage in practical activities (simulations, measurements, theatre), which otherwise would not come to their mind or to the mind of their teacher.
(4) Reading a source is a type of activity which is oriented more to processes of understanding than to final results. The complete meaning of a historical text may remain open, and it occurs quite often that the same text leads to different readings. Of course, this does not entail arbitrariness. The

reader has to give reasons in support of his or her interpretation. As an example we refer to the highly interesting story of negative and/or complex numbers. Reading sources about this topic poses in every case the question whether, and if yes, in which sense these creations were understood as legitimate numbers in different historical times. Doubts that students themselves have from time to time are reflected by the doubts that existed through the ages.

(5) When working with original sources at least three different languages interact in the classroom: the language of the source, the modern terminology of the mathematical topic in question and the everyday language which has evolved in the classroom. This requires of the learner competencies of translation and switching between these languages which are highly desirable from an educational point of view since communication between expert mathematicians and people who want a problem solved mathematically is one of the main problems of applying mathematics.

2. Theoretical and practical orientations

The mini-workshop comprised sessions of different types. Most of the meetings were devoted to traditional presentations of papers. On the other hand, in some sessions the participants discussed the needs and aims of the future development of the field. As a result, research questions were identified which evolved from work in the past and might be helpful in orienting future work. They reflect central issues related to the integration of original sources from the history of mathematics into mathematics education. Each of the questions addresses both the learning of mathematics (by secondary school and university students and by prospective or in-service teachers) and the teaching of mathematics (at the secondary and university levels). In both cases, each of the questions retains its general formulation; however, each is approached differently by the authors according to the target population and their intended educational goals. Thus each question may have more than one answer.

(1) What are the possible epistemological/theoretical basis and frameworks for research and development towards the integration of original sources into the teaching and learning of mathematics?
(2) What are the characteristics of viable models for implementing the integration of original sources in the teaching and learning of mathematics?
(3) What is the actual impact of these models on students' and teachers' learning and understanding of mathematics, and on teachers' teaching practices?
(4) How can historical research and practice inspire, impact, support or supply explanatory frameworks and working tools for research on learning and teaching mathematics?
(5) How can research and practice in mathematics education inspire, support and broaden the research in the history of mathematics in general, and on original sources in particular?

Another issue that came up in the workshop was the problem of upscaling. There is no reason to believe that teaching which is done by an enthusiast with good results can easily and successfully be repeated by the average teacher. It is a welcome development that new materials are being published and that research is being done on projects where "average teachers" are doing the teaching.

In many countries mathematics education standards are in the process of being elaborated. These standards often appear as collections of mathematical problems. The approach of reading sources can be successful in the future only if the community will produce problems in a format adequate to be included among these standard problems. Some contributions during the meeting showed that this is in fact possible.

A related issue is the important role history of mathematics, particularly the reading of sources, might play in the training of teachers for all levels. Studying sources can provide awareness for subtleties in the meaning of mathematical concepts which cannot be afforded otherwise. Thus, the sensitivity of teachers in regard to content-related difficulties of their students might be considerably enhanced. The workshop showed that many activities are in place worldwide which try to take advantage of an approach which includes the study of sources.

3. The Workshop

Most of the contributions during the workshop were related to and inspired by one or more of the research questions outlined in the previous section.

- An important development in recent years is that more empirical research studies on the integration of original sources are being done, many of which include a large number of students. A few of them were presented here (Glaubitz; Clark; Peters; van Maanen, reporting about his student Iris van Gulik-Gulikers).
- Other talks were focusing on theoretical issues based on examples from practice (Arcavi; Bardini, Radford).
- Some papers gave examples from the presenters' own practice with comments on their theoretical background (Barbin; Demattè; Wann-Sheng Horng; van Maanen; Rasfeld; Reich).
- In two sessions the audience was invited to take part in working on historical sources (Pengelley; Jahnke).
- One talk gave an overview on curricula, textbooks and teachers and their roles in making history of mathematics part of mathematics education (Smestad).

The organizers thank the Institute staff for providing a comfortable environment to the participants.

Studying Original Sources in Mathematics Education

Table of Contents

Abstracts

From historical sources to classroom practices

Abraham Arcavi
(joint work with M. Isoda)

History of mathematics can provide many solution approaches (to problems) which are very different from what is common nowadays. Such solution processes may conceal the thinking behind them. Thus, one has to engage in a 'deciphering' exercise in order to understand what was done, what could have been the reasoning behind it and what is the mathematical substrate that makes an unusual method/approach valid and possibly general. Engaging in such an exercise bears some similarities to the process of grasping what lays behind our students' thinking and actions. Even when there are no similarities between the mathematics underlying primary sources and that of our students, experiencing the process of understanding the mathematical approach of a primary historical source can be a sound preparation towards attentive listening to our students. When facing a historical source with a solution approach (foreign to us), we know that the best minds of their time and culture were behind it. Therefore, an historical text cannot be easily dismissed on the basis of the right-wrong dichotomy, as it is commonly the case with students' ideas. An historical source has to be attended to in all its idiosyncrasy, and many times our own understandings cannot be immediately projected onto it, thus one has to delve deep into the text's own nature. Therefore, a first hurdle in the hard task of "decentering" ourselves towards understanding the other's perspective is removed. Repeating these exercises may support (a) the development of the habit of not dismissing any solution; (b) the search of the sense behind an idiosyncratic approach; and (c) the development of tools for understanding (e.g. parsing a text or a reply by a student, posing questions to oneself (or to a peer) around it, paraphrasing in our words and notations, summarizing partial understandings, locating and verbalizing what it is still to be clarified, and contrasting different pieces for coherence). In a sense, this implies some kind of "hermeneutic" (interpretive) practice.

In our study, we used an activity around a source from 'Egyptian Mathematics', a source from Peletier (a French mathematician of the 16th century) and an extract from a solution to an arithmetic problem by a primary school student. These activities were implemented with pre-service and in-service teachers in Japan. We documented and studied the effects of the participants' attempts to make sense of the sources, and the effect it had in their capabilities to better "listen" to a student's idiosyncratic solution.

Our results indicate that the approach is implementable, and that in general, participants profit from it. More specifically, the pre-service teachers in our small sample started to re-think 'teaching practice' by incorporating an important component to it: listening to and understanding the mathematics of "the other" person, and in order to do that they start to ask and question, rather than to evaluate for correctness. Teachers also develop tools to make sense of what the "other" produces (be it a text, or a student), like representations, however sometimes these tools are over-interpretations that impose on the texts or on the students mathematics that can certainly not be there from the beginning.

The full version of the presentation included: a theoretical background relating constructivism to "listening", the challenges and difficulties of listening to students, the design principles for the development of activities around primary sources to support the understanding of the other's perspective, the implementation of such an approach, and the results. The full version of the lecture is based on a paper to appear in a forthcoming special issue of *Educational Studies of Mathematics.*

A multi-week project on mathematical induction and combinatorics for university students, based on Pascal's Traité du Triangle Arithmetique

David J. Pengelley

Blaise Pascal's 1654 Treatise on the Arithmetical Triangle develops a pattern of numbers with simultaneous interpretations as figurates, combinations, and binomial coefficients, and he uses the Triangle to solve the famous "problem of stakes" in an interrupted game of chance. I have been teaching discrete mathematics directly from this rich treatise to beginning university students of mathematics and of secondary teacher preparation. Two fundamental topics are proof by mathematical induction and introduction to combinatorics; in fact Pascal's Treatise provides the first elucidation of the principle of mathematical induction, and then applies it extensively to prove many key properties of the numbers in the Triangle. Mini-workshop participants worked directly with the Treatise, and then analyzed a multi-part project in which students confront, comprehend, and practice mathematical induction and some combinatorics first from Pascal, not from their modern textbook. Participants discussed how students can benefit from original sources as the primary reading material for their curriculum.

Additional material presented for discussion during the mini-workshop and mostly available at math.nmsu.edu/~history, included:

- A paper of mine discussing how I have used Arthur Cayley's original paper on groups (the first) in a course on abstract algebra.
- My first book *Mathematical Expeditions,* of sequences of annotated primary sources for teaching, on five branches of mathematics, used in a beginning university course, and the table of contents of my second book, *Mathematical Masterpieces,* to appear this year, of four sequences in other branches at the upper university level.

- Information on my graduate level course on Using History in Teaching Mathematics.
- A paper on the Bridge Between the Continuous and the Discrete via original sources.
- Various other papers I have written on the pedagogy of teaching with original sources.
- Some historical papers I have written emerging from teaching with original sources, on work of Euclid, Euler, Gauss, Eisenstein.

The problem of points – with students on the tracks of Pascal and Fermat

Peter Rasfeld

The year 1654 is frequently regarded as the year of birth of probability theory. It started off with a correspondence between Pascal and Fermat, in which they tried to solve the problem of dividing the stakes fairly, if a game is prematurely finished. In a project with students (16 years old) it was examined to what extent these letters and other original sources concerning the problem of points can be helpful to develop the nowadays generally accepted solution of Pascal and Fermat. Initially, the following suggestions are supposed: Two players, A and B, have agreed that the winner is the person who has won n points first. The chance for winning a single game is $\frac{1}{2}$ for both and the whole game is stopped after A has obtained a points and B b points ($b < a < n$). Being confronted with this problem for the first time, the students proposed several ratios, e.g. (the explanations are given briefly in brackets)$a : b$ (because A has won a of the $a+b$ points already and B b points), $1 : 0$ (because A has won "in principle"), $1 : 1$ (because none of the players has reached the n agreed points). None of these suggestions were accepted completely by the pupils, and as the problem of points is a rather old and often discussed mathematical problem, it seemed helpful to take a look into the history of mathematics. With the help of some texts (see [3, pp. 11-24]) three propositions were first investigated by the pupils, given by Italian mathematicians before Pascal and Fermat: Luca Pacioli (1445-1515?), $a : b$; Girolamo Cardano (1501-1556), $(1 + 2 + \ldots (n - b)) : (1 + 2 + \ldots (n - a))$; and Nicolo Tartaglia (1499-1557), $(n+a-b) : (n+b-a)$. Pacioli suggested the same ratio as the students themselves, and as this had been declined already, it was not discussed any further. Similarly, the other two propositions were not accepted. It was discussed controversially, whether scores with the same difference of points have to be assessed equally (Tartaglia) or differently (Cardano, Pacioli). This discussion finally led to the idea of dividing the whole stake accordingly to the chances of winning. In a simple example (4 games required to win for both, A has already 2 points and B 1 point) the students obtained a survey of all possible game processes with the help of a tree-diagram and calculated the probabilities by using the path-rules. This procedure was already arduous in this very simple example, with nothing in sight to generalize it and to construct a formula. It seemed reasonable, therefore, to take a look into the history of mathematics once again. With a letter of Pascal to Fermat from the 29th of July, 1654 (see [3, pp. 26-31]) Pascal's algorithmically recursive procedure was investigated and implemented into an Excel-program. But the question of an explicit solution was not answered at all. With a letter of Pascal to Fermat from the 24th of August, 1654 (see [3, pp. 32-38]), in which Pascal describes Fermat's combinatory way (unfortunately Fermat's letter itself concerning this point is lost), the students worked out this procedure. Here it was important that Fermat always extended a game process up to the maximum number of single games. So all game processes became equally probable which allowed a simple counting of the favorable and possible outcomes and saved laborious calculations with the path rules. Finally the counting of the outcomes could be avoid by working out Pascal's method using the arithmetical triangle (see [1, p. 76] and [3, p. 16]). Comparisons with Fermat' procedure gave insight as to why Pascal's method worked and allowed generalizations which led to the searched formula: The proportion of chances (or the proportion, in which the complete stake has to be shared), is $\left[\binom{r}{n-a} + \binom{r}{n-a+1} + \cdots + \binom{r}{r}\right] : \left[\binom{r}{0} + \binom{r}{1} + \cdots + \binom{r}{n-a+1}\right]$, in which r is standing for the maximum number of a game process, given by $r = (n-a)+(n-b)+1$. Retrospectively one can say that the original sources were something like a thread from the first solutions through to the last one. Because the considerations of the mathematicians were represented respectively "only" in the context of examples, the students had to transmit these considerations on other examples and to generalize them. The calculations, usually given in the form of prescriptions, had to be "translated" in terms, equations, or algorithms (for computer programs). Consequently, there was room for students to incorporate their own activities and ideas. It became clear that the problem of points can be solved in quite different and (from our point of view today) more or less acceptable ways. For this to occur, the students had to understand the different explanations and to judge them – including their own positions – critically. The letters of Pascal and Fermat are a beautiful example of how the very same solution of a mathematical problem can be found by using quite different methods. The pupils learned that Pascal found his (first) solution on an algorithmically recursive way, while Fermat did it on a combinatory way. Finally, it was important for the students to take part in the development process leading to the solution of a mathematical problem. Unfortunately, in "normal" school lessons a more or less "ready" mathematics often dominates. Here, the students saw that the solution of problems and the development of new methods can require much time and that wanderings, setbacks, modifications, and new approaches can be contained.

References

[1] A.W.F. Edwards, *Pascal's arithmetical triangle*, John Hopkins Paperback Edition, London (2002).

[2] I. Schneider (Ed.), *Die Entwicklung der Wahrscheinlichkeitstheorie von den Anfängen bis 1933*, Wissenschaftliche Buchgesellschaft, Darmstadt (1988).

[3] M.A. Todhunter (Ed.), *Mathematical theory of probability: ¿From the time of Pascal to that of Laplace*, New York (1965).

Curricula, textbooks and teachers – their roles in making history of mathematics part of mathematics education

Bjørn Smestad

The current situation, generally speaking, seems to be that history of mathematics does not have a strong place in mathematics classrooms. An analysis of the TIMSS 1999 Video Study material shows that out of 638 eighth grade lessons (from seven countries), only 21 included references to the history of mathematics. Most of these had only short bits of information, often consisting of biographical information on mathematicians, whereas information more connected to the actual mathematics was often ignored [4].

In 1997, history of mathematics was included in the curriculum goals for elementary and secondary schools (ages 6-16) in Norway. One of the six general aims of mathematics teaching was "for pupils to develop insight into the history of mathematics and into its role in culture and science" [2]. There were also more specific goals for particular grades. The in-service courses developed for teachers in connection with the broader 1997 reform did not, however, include history of mathematics. A small classroom and interview study [1] suggests that the inclusion of history of mathematics was not considered an important part of the reform.

An analysis [3] of the textbooks that were developed for the 1997 curriculum also revealed problems. An average pupil would see only 36 pages of history of mathematics throughout his ten years of compulsory education. As in the TIMSS 1999 Video Study, a large part of the historical content was biographical information on mathematicians, with little connection to the mathematics concerned. In addition, there were many factual errors in the textbooks. There also seemed to be a lack of consensus on what topics from the history of mathematics would be most meaningful for pupils, as the textbooks showed a bewildering variety in the topics covered. No textbook even mentioned the possibility of studying original sources as a way of working on mathematics.

We may perhaps see the 1997 curriculum as an "experiment" in "upscaling", where several individual teachers' good results were supposed to be repeated in the average classroom. However, the "average teachers" were not given access to the ideas of ones who had succeeded with this approach. This "experiment" now seems to have been abandoned. There is a new curriculum taking effect this year that does not mention the history of mathematics as one of the goals of mathematics teaching.

As a way of finding out more about how this 1997 change in the curriculum may have influenced mathematics teachers, I am currently doing an interview study on Norwegian secondary and high school teachers. This is a phenomenological study where the goal is to gain more knowledge on teachers' attitudes, which may also contribute to the discussion of what can be done to successfully integrate history of mathematics (and original sources) in the average mathematics classroom. An unsurprising finding is that the teachers' background and interest is essential for how history is incorporated. But the study also shows how one single course on the history of mathematics (taking place on one single day) may rekindle an interest that may have been inactive. The teachers interviewed thus far seem to agree that the effect of putting history of mathematics into the curriculum is limited, and that enthusiasm and resources are more important. They do not, however, agree on in what way(s) the students should work on the history. More interviews and much more analysis is still to be done on this study.

What seems clear from all three studies is that history of mathematics is often understood as biography, and that teachers, when including history of mathematics at all, use it simply as a motivation, to humanize the subject. While that is certainly important, work is needed to help teachers see the other positive contributions history of mathematics may make to mathematics teaching. That these positive contributions exist is evident from the other talks of this workshop.

References

[1] Alseth, Bjørnar, Breiteig, Trygve and Brekke, G., *Endringer og utvikling ved R97 som bakgrunn for videre planlegging og justering – matematikkfaget som kasus*, Telemarksforskning-Notodden (2003).

[2] R. Hagness, J. Veiteberg et al. (Ed.), *The Curriculum for the 10-year compulsory school in Norway*, National Centre for Educational Resources, Oslo (1999).

[3] B. Smestad, *Matematikkhistorie i grunnskolens lærebøker*, Høgskolen i Finnmark, Alta (2002). [A shorter version in English: B. Smestad, Historical topics in Norwegian textbooks, in O. B. Bekken & R. Mosvold (Eds.), *Study the masters*, NCM, Göteborg.]

[4] B. Smestad, *History of mathematics in the TIMSS 1999 Video Study*, HPM 2004 Proceedings, Uppsala (2004).

A collection of documents for secondary school students

Adriano Demattè

The book, *Fare matematica con i documenti storici – una raccolta per la scuola secondaria di primo e secondo grado, a-Volume per gli studenti* [1] and *b-Volume per gli insegnanti* [2], is a collection of passages taken from original sources. *"Fare matematica"* ("Doing mathematics") in the title stresses the fact that this book is not only for "reading on mathematics" but rather for operating with problems and exercises. The aim of this publication is to furnish secondary school teachers with a proposal for activities to integrate originals in everyday classroom work. This integration should promote alternative ways of teaching based on working with texts and exercises to reinforce (or sometimes even to introduce) mathematical competencies.

Documents have been chosen to both offer a variety of important authors and to show less famous mathematicians whose works were representative in their lifetime. The book contains 23 Italian documents, 38 European (non-Italian) documents, and 14 documents by non-European authors. Main Italian school topics are present. Sometimes students are asked to conjecture about the causes of certain historical facts and although we know they do not have real expertise, students are able to consider what might have been the antecedents. *Volume b* is a small book for teachers, containing didactical suggestions, solutions to exercises, and a bibliography.

Teachers retain that history of mathematics is supplementary content for students and that, consequently, it requires additional time to include. The use of originals does not require new knowledge or new notions which have to be learnt. More properly, it is doubtful that a student acquires information about mathematics from some historical periods, but that this happens in an implicit way. His/her effort is 'hermeneutically oriented' first of all, or it is 'applications oriented' when s/he engages in solving mathematical problems. In both cases s/he must use (and also reinforce) his *competencies*, i.e., his knowledge and his specific abilities that are applied in new situations. Reading originals sometimes requires a lot of time because it is a complex task that involves considerable linguistic and mathematical abilities. It is not a waste of time because it refers to high-level goals. Interpretation of original texts sometimes is a difficult task and weak students might face considerable obstacles. If the teacher chooses suitable documents and structures appropriate activities, this danger can be avoided.

In my experience, teachers are interested in historical themes, but when they refer to the history of mathematics they do not think about original documents. In order to promote the use of history we would have to provide them with resource material, which could be integrated in curricular activities without requiring more time. The book, *Fare matematica*, tries to satisfy these demands. In my discussion, I posed the problem of teacher training. I consider originals as a starting point to 'persuade' teachers into introducing history in their classrooms.

The question that guided me was, "Can originals give the possibility to tackle the teachers' reasons not to use history?" In the ICME Satellite Meeting of HPM in 2004, Man-Keung Siu provided a "list of unfavourable factors." In relation to the aims of our collection and by means of didactical examples, I discussed the following:

- I have no time for it in class.
- This is not mathematics.
- How can you set questions in a test?
- Students don't like it.
- Students regard it as history and they hate history class.
- Students regard it just as boring as the subject mathematics itself!
- What really happened can be rather tortuous.
- Telling it as it was can confuse rather than to enlighten!
- Students do not have enough general knowledge to appreciate it.
- Progress in mathematics is to make difficult problems routine, so why bother to look back?
- There is a lack of resource material on it.
- There is a lack of teacher training in it!
- I am not a professional historian of mathematics. How can I be sure of the accuracy of the exposition?

- Does it really help to read original texts, which is a very difficult task?
- It is liable to breed cultural shauvinism and parochial nationalism.
- Is there any empirical evidence that students learn better when history of mathematics is made use of in the classroom?

In my opinion, originals can actually provide ways in which to tackle teachers' reasons not to use history. Siu also describes an experiment, "an experimental group used some prepared material with a historical flavour (on the Pythagorean Theorem), while the control group went through the usual sequence of instruction without using those prepared material." The conclusion highlights the positive effect – especially on weak students – of integrating history in mathematics classes. My experience of using parts of the book, *Fare matematica,* substantially confirms Siu's conclusion because in final tests, weak students performed slightly better with respect to previous assessments.

References

[1] A. Demattè (Ed.), *Fare matematica con i documenti storici – una raccolta per la scuola secondaria di primo e secondo grado. Volume per gli studenti*, Editore Provincia Autonoma di Trento – IPRASE del Trentino (in press).

[2] A. Demattè (Ed.), *Fare matematica con i documenti storici – una raccolta per la scuola secondaria di primo e secondo grado. Volume per l'insegnante*, Editore Provincia Autonoma di Trento – IPRASE del Trentino (in press).

[3] F. Furinghetti, *History and mathematics education: a look around the world with particular reference to Italy*, Mediterranean Journal for Research in Mathematics Education **3** (1-2) (2005), pp. 1-20.

[4] H.N. Jahnke, *The use of original sources in the mathematics classroom*, in Fauvel, J. & van Maanen, J. (Eds.), *History in mathematics education: the ICMI study*, Kluwer Academic Publishers, Dordrecht (2000), pp. 291-328.

[5] M.K. Siu, *No, I don't use history of mathematics in my class: Why?*, in Furinghetti, F., Kaijser, S. & Vretblad, A. (Eds.), *History and Pedagogy of Mathematics, Fourth summer university proceedings*, Uppsala (Sweden) (2005).

The historical and epistemological pillars of the "indeterminate"

Caroline Bardini

This presentation is the introductory part of the joint talk, "Unknowns, variables and parameters," presented by Dr. Luis Radford and Dr. Caroline Bardini. It is devoted to discussing the historical evolution of the representation of the "given" in a mathematical text and to highlight the epistemology that underpins it.

Symbolical mathematical texts prior to Viète had the following peculiarity: the given data of a problem was necessarily numbers. Indeed, whenever a problem involved known and unknown objects, the "known" (or, in other words, the given data) had to be a "common known", shared by all. Whereas in many texts prior to Viète the unknown was represented by the means of a symbol (the necessity of such representation was easy to conceive precisely because the information was not known), what was given and known from all was exclusively designated by a numerical representation.

Hence, at that stage of the 16th century, if on the one hand there was, in the geometrical framework, objects (geometrical figures) considered as "generic" (despite the singularity of each drawing), there was not, on the other hand, a representation system to convey the idea of "any" numbers within the symbolic writing framework.

Thus, as long as the data given was not specified, or, in other words, as long as the given of a mathematical text was not *fully given,* because of the universal characteristic that geometrical representations convey, Geometry was the only tool mathematicians could use to rigorously unfold their proofs. And, despite the awareness of Arabic algebra, Geometry has remained, for many mathematicians after Euclid (Luca Pacioli, Tartaglia, Clavius, etc.) the Golden Path to the truth.

The major contribution of Viète was hence the introduction, for the very first time, of a symbolic representation, other than numerical, for the "given" of (and in) a mathematical text. With such convention, the given acquired, now also within the written symbolical framework, the attribute of *being general* ("quelconque" in French).

The fact that the given data was no longer explicit in its symbolic representation would automatically have a major consequence in what concerns the interpretation of a mathematical text -and this is the core of our presentation. In front of such new symbolic system, from now on the reader would have to consider the "given" as being, despite an apparent contradiction, *at the same time* **arbitrary**.

In fact, Viète's contribution shook the nature itself of the concept of the "known." By establishing such symbolic system, the "given" was thus, in the written symbolic framework, split into two categories: the "explicitly given" (which value was known by all), and its symmetric: the non-specific "indeterminate."

A duality that was prior to Viète and exclusive to Geometry was thus now also present in the symbolic writing. By conceiving to designate not only the unknown but also *the given quantities* of a problem by the means of letters, an arbitrary but fixed object, until then only possibly represented through Geometry, could now appear within the symbolic representation. The dialectic of the "particular yet general" or, in other words, of the "specific yet unspecified" that underpins the concept of "indeterminate" was, thanks to Viète, also shared by what we now retrospectively call Algebra. However, despite the richness of this system, which imported all the advantages of the proofs procedures specific to the numerical domain at the same time it let the universal feature of the solutions, the epistemological complexity of "arbitrarily given" that the literal representation conveys remained rather problematic. Grasping the concept of variable is, as Russell acknowledges, far from being evident and the analysis of some of our classroom practice shows the depth of this issue.

References

[1] C. Bardini, *Le rapport au symbolisme algébrique. Une approche didactique et épistémologique*, Thèse de doctorat, Paris (2003).

[2] Hopital, *Analyse des infiniment petits pour l'intelligence des courbes*, Paris, Montalant (1696).

[3] B. Russel, *The principles of mathematics*, London (1903).
[4] M. Serfati, *La dialectique de l'indéterminé, de Viète à Frege et Russel*, in M. Serfati (Ed.), *La recherche de la vérité,* pp. 145-174, Paris, ACL Editions (1999).
[5] R. Witmer, *The analytic art: Nine studies in algebra, geometry, and trigonometry from the Opus restitutae mathematicae analyseos, seu, Algebrâ novâ by François Viète*, in M. Serfati (Ed.), *La recherche de la vérité,* Ohio, Kent State University Press (1983).

Acknowledgement

This project was funded by the European Union (Marie-Curie Outgoing International Fellowship - Université Paris7/University of Melbourne).

Variables, unknowns, and parameters or the insinuation of mathematical generality

Luis Radford

Variables, unknowns, and parameters are different ways to deal with the general in mathematics. They allow one to move along different layers of mathematical generality. From the viewpoint of the historical conceptual development of mathematics, using signs to designate them has been carried out through a lengthy process. In Pythagoras' time, proving mathematical properties concerning numbers, for instance, was often done through the use of pebbles. Euclid's theory of numbers, as developed in the *Elements*, in contrast, uses other signs (such as lines and letters) although the traces of the pebble-based technique is still visible in Euclid's proving style ([1], [3]). Certainly, the recourse to lines and letters in the Greek mathematical tradition led to new forms of conveying generality that remained beyond the Pythagorean use of pebbles. However, this generality did not reach its full potential before the emergence of alphanumeric algebraic symbolism in the 16th century. For example, in the text known as *Anaphorikos* (see [2]), Hypsikles, who is considered the first mathematician to have dealt with polygonal numbers in a deductive manner, states Proposition 1 as follows:

> If any number of terms is considered such that <starting from the greatest> every two successive ones have the same difference, {*the terms*} being even in number, then, the difference between {*the sum of*} half the number of terms [starting from the greatest], from {*the sum of*} the remaining ones, is equal to the multiple of the common difference by the square of half the number of terms.

Even if the numbers are not specified (they are *any* numbers, represented by segments), what Hypsikles proves is only good for *six* numbers however. Generality remains implicit or insinuated: if the number of terms were to be 8 or 10 or any other *particular* even number, the proving process could be carried out in a similar manner. But the proof cannot be provided *in general*, that is to say, as valid *stricto sensu* for *any* even number. Diophantus, in his treatise on polygonal numbers, faced the same difficulty (see [4]). There is a difference though. In his proof, Hypiskles talks about *six* numbers; Diophantus considers four numbers but does not mention the word "four." He talks about the amount of numbers (in general), even if the proof exhibits four numbers (Diophantus says: "Soient $AB, B\Gamma, B\Delta, BE$ des nombres en quantité quelconque ..." (op. cit, p. 280)). The lack of signs to name and objectify the arbitrary given and yet indeterminate amount of numbers that these propositions talk about confines the proof process to an implicit or insinuated layer of mathematical generality that was only to be expanded in the 16th century by Vieta.

The difficulties that past mathematicians encountered in dealing with the general are somewhat similar (not identical) to the ones contemporary students face. In fact, as my classroom research shows, often, the students encounter difficulties in distinguishing, e.g. variables and parameters and using them in an appropriate manner. One of the research questions that I am currently investigating is the nature of the cognitive difficulties related to the concept of parameter in high school students. The results suggest that numerically-centered concepts of variables conveyed by current curricula around the world may result in an obstacle for a suitable understanding of parameters. Recourse to original historical sources may prove to be enlightening to find new ways to work out solutions to this problem. The work of Euclid, Diophantus, Hypsikles and others insinuates that non-numerical, relational conceptual aspects of variables may help students to make sense of variables and parameters. An open problem is to design appropriate classroom activities that can provide students with rich mathematical experiences, allowing them to move along different layers of generality and to acquire a deep conceptual understanding about variables, unknowns and parameters.

References

[1] W. Lefèvre, *Rechensteine und Sprache*, Klett-Cotta, Stuttgart (1981).
[2] K. Manitius, *Des Hypsikles schrift anaphorikos nach überlieferung und inhalt kritisch behandelt,* Lehmannsche Buchdruckerei, Dresden (1888).
[3] L. Radford, *On culture and mind. A post-Vygotskian semiotic perspective, with an example from Greek mathematical thought*, in M. Anderson, A. Sáenz-Ludlow, S. Zellweger, & V. Cifarelli (Eds.), *Educational perspectives on mathematics as semiosis: From thinking to interpreting to knowing,* (pp. 49-79). Legas Publishing, Ottawa (2003).
[4] P. Ver Ecke, *Diopante d'Alexandrie. Les six livres arithmétiques et le livre des nombres polygones,* Desclée de Brower, Liège (1926).

The different readings of original sources : An experience in pre-service teaching

Evelyne Barbin

How does one read an original mathematical text? At first glance, there are two opposite ways. The first one is a mathematical way which consists of interpreting the text in modern terms with our knowledge of mathematics. The second one is an historical way which replaces the text in an historical context and interprets it in this context. How is this done with students? The first way is not satisfactory because it is a way to bannish an historical approach; the second one is difficult.

So, I would like to show there are other ways to read and to work with original sources and that it is interesting to combine them.

Here, I present my experience on teaching historical relations between figures and numbers, given in a multi-disciplinary license (third year) course for future teachers in primary schools. This course is neither a course on the history of mathematics nor a mathematics course. The purpose of the course is to introduce mathematical ideas, notions, and procedures by the way of history. The course is composed of twelve lectures of one hour each and twenty-four practical seances of one hour and a half each. It includes Babylonian and Egyptian mathematics, Greek and Arabic mathematics, and algebra from the Renaissance to Descartes.

A large portion of the course is based on reading original sources, with approximately sixty texts studied. There are different kinds of reading original sources proposed to the students, and I distinguish eight types of reading and working with original texts. The types are: to interpret in an historical context; to translate; to compare translations of an original text; to compare original texts; to write in the same way as an original text; to pass from one way to another; to interpret a text with modern language; and to compare two interpretations. For instance, we compare two translations (in French) which give the first problem of the Babylonian tablet BM 13901 because it is a good way to show that translations are linked to interpretations. The first translation by Thureau Dangin uses words like "to add", "to subtract", and "to multiply", and leads to an algebraic interpretation of Babylonian tablets. The second translation, by Jens Hoyrup, uses words which lead to a geometrical interpretation.

To present the different types of reading we give as an example the teaching of "Pythagorean numbers and the problem of irrationality." The purpose of such an example is to study contexts and proofs of irrationality in Greek mathematics. Five texts are given to students. The first one is an extract of Arithmetical Introduction by Nicomaque of Gerase on triangular numbers and square numbers. Students are asked to interpret it in historical context in order to obtain pentagonal numbers in the same way (form, sides, generation). The second text is an excerpt of Commentaries on the First Book of Euclid of Proclus of Lycie on two methods to calculate Pythagorean triples. Here, students have to interpret it in a modern language. The next text is an extract of Exposition of Mathematical Knowledges by Theon of Smyrne on side numbers and diagonal numbers. Students have to interpret it in historical context with geometrical figures and have also to justify it in a modern way. The fourth text is from Euclid's Elements, Book X, proposition CXVII, on the incommensurability of diagonal and side of a square. Using this text, students must interpret using modern words. The last text is an excerpt of Theetete by Plato, about Theodore of Cyrene and problem of incommensurability. Students are asked to interpret this text in the historical context to obtain geometrical construction and proof of irrationality in way of Euclid.

Difficulties in reading an original source in its historical context come from the lake of historical knowledges, but not only. They also come from the necessity of a "dépaysement": to make strange what is too familiar to us. Such is the case with the famous Geometry of Descartes. So, it is interesting before reading such a text to explain to students that many contemporaries of Descartes did not understand his text. Many historical aspects must be given to appreciate the novelty of the "deconstruction" of geometrical figures in simple lines and the arithmetization of geometry by Descartes. For this purpose, it is interesting to compare proofs by Greek mathematicians to proofs by Cartesian mathematicians, and to read extracts of the Rules for the direction of mind by Descartes. A "dépaysement" exists to show that Descartes proposed a new practice for the geometer. This practice included not contemplating figures but calculating with numbers and letters and to show that Descartes gave a new conception of deduction, which was not a deduction of propositions by means of logical operations but a deduction of lines by means of arithmetical operations.

Some examples and comments on integrating original mathematical texts in mathematics education

Constantinos Tzanakis

Original sources in mathematics education may be used (a) in the classroom via excerpts and worksheets based on them; or (b) by the teacher only, to deepen his/her understanding of a subject and enhance his/her awareness of mathematical results and activities.

Several relevant points are illustrated by means of three examples:

(A) *Ancient Greek mathematical texts in the teaching of Euclidean Geometry in high school: A cross-curricular approach*
Excerpts from Euclid's *Elements* and Proclus' *Commentary,* have been used in the classroom concerning: (a) different proofs of the equality of the angles in an isosceles triangle as they appear in Euclid, Proclus and Pappus; (b) the construction of the bisector of an angle; (c) the triangle inequality for the sides of a triangle; and (d) the sum of the angles of a triangle. Students' reactions and the follow-up discussions were presented. Other possible excerpts from ancient texts, especially concerning the concept of *diorismos,* as well as additional comments on the differences among the various proofs both from a mathematical and a more general epistemological and cultural point of view were also discussed. This activity is placed in the current educational milieu in Greece.

(B) *On appreciating the subtleties inherent in an "elementary" concept: the case of the concept of (instantaneous) speed*
Excerpts from original texts can be used in the classroom and/or by the teacher to help reveal the conceptual subtleties inherent to this (seemingly) elementary concept, in particular the following two issues:

(a) Elementary school pupils and to some extent, junior high school students, encounter difficulties in understanding the concept of speed (even for uniform rectilinear motion) as a new magnitude emerging from the ratio of two **different** physical magnitudes. Although this difficulty is at least in part didactically-driven, there is an epistemological issue which has to do with the fact that ratios and products of magnitudes is the only way to create new ones, but conceiving them in this way is not straight forward, neither was this so in the past. Excerpts from ancient Greek texts, e.g. from Archimedes' *On Spirals*, and Autolycos' *On the moving sphere*, clearly suggests that for the Greeks such ratios were meaningless – hence the absence of the concept of speed from ancient Greek thought (uniform motion is defined using only an adverb, not a noun!). This is an issue that can be discussed further in the context of teacher education programs.

(b) High school students and sometimes university students cannot easily grasp the concept of instantaneous speed. Once again, this is in part didactically-driven, but it is definitely a concept whose founding is obviously related with epistemologically crucial concepts (i.e., limit, procedures involving an infinite number of steps). Excerpts from Newton's *Principia* reveal two things that: (i) It is not easy to give a satisfactory **verbal** definition, without a deeper understanding of the limit concept, as it is currently done in Greek schools; and (ii) instantaneous speed is a generic concept for Newton, in order to introduce the derivative concept and make it plausible. It is an example of what could be "Physical Mathematics" (a term coined from Pólya), which should not be ignored in teaching. Both (i) & (ii), together with Aristotle's account in his *Physics* and of Zeno's *paradox of the arrow* could be used in teacher education programs to increase their awareness of the subtleties inherent in seemingly elementary concepts.

(C) *Looking for a generalization of complex numbers – Hamilton's invention of quaternions*

From Hamilton's very detailed *Preface* to his *Lectures on Quaternions* one can see the deeper reasons for searching for a generalization of complex numbers, as well as the reasons for Hamilton's failure for several years. This is directly related to the mid-19th century problem of specifying the conditions under which abstract algebraic operations become legitimate, the limitations put by the fact that the concept of function was not yet so central in mathematics and the importance of geometrical and physical considerations.

Excerpts from this long *Preface*, reveal:

(i) Hamilton's original motivation – his conception of algebra as "the science of time" and how far away this "problematique" is from today's mathematics.

(ii) The specific mathematical question that motivated his research: since complex numbers can describe in a well-defined way plane rotations (as it is seen e.g. in excerpts from Wessel's works), is it possible to find some generalized complex numbers such that a similar description is possible for space rotations?

(iii) That the question in (ii) is one of the examples from mid-19th century efforts to generalize algebra beyond the usual number sets (e.g. as it appears in Peacock's *Principle of Permanence* etc).

Additionally, it is possible to discuss a posteriori the reasons for Hamilton's long-time failure to conceive quaternions. It seems that this has to do with the fact that multiplication (of any two quantities) was conceived exclusively as an operation; the possibility to conceive the one number as an operator acting on a set of numbers (i.e. function) was perhaps not widely common idea at that time. Hamilton's procedure could be reconstructed in a modern context, to introduce students to quaternions and relate it to other subjects (geometry, mechanics of rigid bodies, quantum mechanics) to undergraduates in a more or less elementary way.

The historical roots of vector calculus: J.W. Gibbs (1839-1903)

Karin Reich

Though several years ago vector calculus became a standard discipline within mathematics in high schools, mathematics education has hardly taken notice of this fact. Thus, there are good reasons for presenting an original source of vector calculus which has had influence and is still understandable today. The selection of a reasonable text is quite difficult, because the history of vector calculus is more or less chaotic.

There are two primary roots and various developments of vector calculus. The first root goes back to Hamilton's quaternions which were discovered in 1843. The second root is Hermann Grassmann's "Extension theory" which was first published in 1844, with a second totally reworked edition presented in 1862. Both authors, William Hamilton (1805 - 1865) and Hermann Grassmann (1809 - 1877) are unreasonable to use with students because they are by far too difficult to understand for undergraduate students. Hamilton's quaternions possessed one real and three imaginary components: i, j, and k. It was the American physicist Josiah Willard Gibbs who transformed Hamilton's quaternions into vectors with three real components i, j, k. For Gibbs this meant:

$$i.i = j.j = k.k = +1.$$

These three-dimensional vectors were easy to use and could be applied without problems in physics and technology.

Gibbs belonged to the very first mathematicians who received a Ph.D. in the United States (Yale, 1858). In 1881 and 1884 his "Elements of Vector Analysis" were first published privately. There were subsequent editions by Gibbs' student Edwin Bidwell Wilson (1879 - 1964). Wilson graduated in 1899 from Harvard and afterwards he attended Yale where he earned his PhD in 1901. In 1903, the year of Gibbs' death, Wilson published a revised version of Gibbs' lectures. A second edition followed in 1907 and a third in 1913.

Gibbs' presentation is indeed compatible with modern vector calculus introduced in high schools, though the terminology and symbols are different. In his first chapter "Concerning the algebra of vectors" Gibbs gave the following very simple definition: "If anything has magnitude and direction, its magnitude and direction together constitute what is called a vector," with vectors being written with small Greek letters: α, β, and so forth. Further definitions concerned scalars, equal vectors, negative vectors, multiplication of vectors by scalars and unit vectors. Then followed the main operations addition, subtraction, and the two kinds of multiplication, called the direct and skew product (modern terminology: dot and cross product or interior and exterior product):

$$\begin{aligned}\alpha &= xi + yj + zk, \qquad \beta = x'i + y'j + z'k;\\ \alpha \cdot \beta &= xx' + yy' + zz',\\ \alpha \times \beta &= (yz' - zy')i + (zx' - xz')j + (xy' - yx')k.\end{aligned}$$

The result of the skew product was a (one-dimensional) vector, but for Grassmann, the result of an exterior multiplication had been a plane of two dimensions.

Gibbs dedicated the further chapters to differential and integral calculus of vectors, linear vector functions and transcendental functions of dyadics.

Historically, especially for the physicists who followed Gibbs, his vector calculus became an important tool within electrodynamics (Oliver Heaviside, August Föppl, Arnold Sommerfeld). Mathematicians, however, rejected vector calculus as a whole, no matter what kinds of definitions were used. The crucial question was, is vector calculus only a new notation or does vector calculus need to be recognized as more, i.e. as a mathematical progress? Poincaré's answer was typical for a physicist: "Dans les sciences mathématiques une bonne notation a la même importance philosophique qu'une bonne classification dans les Sciences naturelles". In 1905 and 1915 relativity theory was published. This was a change as far as the development of vector calculus was concerned. Tensor calculus became the main tool and this kind of tensor calculus had its roots in invariant theory, i.e. in algebra.

Perceiving history of mathematics: How reading original sources improves students motivation on mathematics

Katja Peters

The Herzog August Library in Wolfenbüttel, one of the most important scientific libraries in Germany, focuses on collecting books printed or written during the 17th century. During the 1980s a teacher of the Wolfenbüttel secondary school developed in cooperation with the library the so-called "Schülerseminare". This is a special course offering for students taking part in an advanced course in German literature, biology, or history during the last two years of secondary school. A course may apply for being invited by the library for three days studying original sources.

On the first day of the course students are welcomed by the staff of the institute. First, they get to know the topic of the seminar. Particular old sources are shown, suggesting an idea of what will be featured over the next three days. During the next two hours they are introduced to the worthiness and the careful use of old sources, the electronic catalogue of books, the reference library, and using microfiches. Afterwards, students decided which topic they wanted to work on. This time was clearly limited to the first day – otherwise the risk of ending up in failure is great. On the second and third day the students work on their chosen topic and before they go home in the evenings some of the groups present the results of their work.

Since the library did not offer a special program on studying mathematics, I compiled a list of sources which could be used during the seminar. Reading mathematical sources is not very easy for young students. Either they understand the mathematical context but not the language – because most of the books are written in Latin – or they understand the language but not really the mathematics. Most of the books in German were written after 1750 and if the pupils are not trained on reading sources it is very difficult to understand the thoughts of 18th century mathematicians.

To keep the first experience with the history of mathematics as easy as possible I pre-selected the sources according to the following criteria: special language German; written, printed or of importance during the 17th century; mathematical level (not too difficult); and the condition of the book. Before starting the seminar, the staff of the library chose the books which could be offered to the students. The students then decided on their topics, either on their own or in groups. Their list offered a wide variety of interests. For example, it included surveying statements on Chinese and Indian mathematics or the number π, intensive studies of sources about extracting roots, calculation "on the line" by Adam Ries, and the use of the astrolabe and early calculating machines.

During the next two days the students worked intensively. None of them were late in the mornings and several used the free time in the evenings to continue studying in the library. They lost their inhibition for working in a scientific library and learned to appreciate their own abilities of independent study. Most of them completed work on their chosen topic. They described their main difficulties with reading original sources as understanding the unfamiliar presentation of mathematics in the past and the different use or lack of technical terms. These problems could be addressed by asking the teachers or other pupils. It was amazing to watch groups of students discussing mathematical topics in the cafeteria or during lunch time. Approximately four months later we used the developed material to create an exhibition for parents and other students at school. Most of the participants did not have any problems in remembering the contents despite the long intervening period. Each student emphasized the improvement of his or her attitude to mathematical studies. The students indicated that they would have no problems with concentrated work over a longer period of time and rated their performance as fine.

References

[1] K. Peters, *Mathematikgeschichte hautnah erfahren – oder: Wie Quellenlektüre für Mathematik begeistern kann*, in W. Hein & P. Ullrich (Eds.), *Mathematik im Fluss der Zeit*, Tagung zur Geschichte der Mathematik in Attendorn/Neu-Listernohl (28.5. bis 1.6.2004), *Algorismus* **44** (2004), pp. 534-548 .

Using original sources in teaching: One teacher's experience with personal study and curricular inclusion

Kathleen M. Clark

In 2004 - 2005, I conducted a study which investigated five secondary mathematics teachers' efforts to study and use the history of logarithms in their teaching. A professional development experience, constructed to reflect the features of effective professional development identified by Garet, Porter, Desimone, Birman, and Yoon (2001) and Smith [4], was designed to engage teachers in the study of the historical development of logarithms (taken primarily from 16th and 17th century European developments). Modifications of activities found in the *Exponentials and Logarithms* module [1], as well as various print and electronic resources, were used during the professional development.

Two primary research questions guided the study. First, the study addressed how teachers with different background knowledge and experiences responded to the professional development. Second, the study investigated how teachers' background variables and experience with the professional development influenced the teachers' personal mathematical knowledge and instructional practice. Exploratory case study methodology was used to describe the experiences of five participants; four teaching in a public high school and one teaching in a private day school. Data sources used in the case study included teacher background, attitudes, and content knowledge instruments; participant observation during all professional development sessions and classroom instruction (during a unit on logarithms); and semi-structured interviews.

Studying teachers' experiences with the history of mathematics and the subsequent translation of those experiences into their teaching was of particular interest to me. Barbin [2] indicated that in order to investigate the impact of using history in the mathematics classroom, two kinds of materials must be studied. First, we should collect experiences of teachers who use history, including their aims, steps, problems that arise in teaching, and the advantages and disadvantages they report. Second, we should collect questionnaires and conduct interviews of teachers and students about their study of mathematics (p. 90). Within the collection of experiences of teachers who use history of mathematics also exists the particular experiences related to teachers' use of historical sources. Although several international examples exist which describe how teachers are successful with "bringing historical texts into the classroom" [3, p. 155], examples of such work at the secondary level in the United States remain difficult to find.

Consequently, for this presentation I highlighted the participants' experiences with an interpretation of Napier's two particle argument (presented by me during one of the professional development sessions at each of two school sites) and their engagement with an online translation of Napier's *Mirifici logarithmorum canonis descriptio* [6]. Of the five participants, only one, Mandy Wilson (a pseudonym), chose to incorporate Napier's *Descriptio* into the unit she planned which introduced her students to the historical development of logarithms. In my presentation, I described Mandy's professional development and personal learning experiences which contributed to one classroom experience focused on Napier's *Descriptio*.

Mandy planned to use the *Descriptio* in two different ways. First, she intended to provide students with an exposure to an original source in order to highlight aspects of the language and the mathematical content for later use when studying the historical development of logarithms. Mandy was also interested in alerting students to various details associated with original conceptions of mathematical developments. The second purpose was derived from Mandy's desire to share the two particle argument with students. To this end, she believed that students' prior experience with the language and content of the *Descriptio* would provide a foundation for following and appreciating the argument. Mandy chose to assign her students to read the *Descriptio* (Wright's 1616 translation). Although she guided students through only the first four pages (of 29 total in the translation), Mandy expected her students to complete the reading prior to her presentation of the two particle argument – an argument in which Napier used a kinematic model to define *logarithm*.

Student reactions to their first experience with a (translated) original source were illuminating. When asked about her first exposure to studying logarithms in another course, Lynn (one of Mandy's students) replied, "I didn't like logarithms because we really didn't talk about what they were." After grappling with the *Descriptio* and the two particle argument, Lynn shared that she was quite frustrated with the language, vocabulary, and mathematical concepts that she encountered when reading the document. By the end of their study of the historical development of logarithms, however, Mandy observed that Lynn and another student relented on the reluctance they initially exhibited and that they benefited from "having to do some thinking about the topic; it's not just a calculator approach to the world" (M. Wilson, interview, 4/20/05).

Mandy was committed to providing students with opportunities with which to "help students understand the human element of mathematics" and to allow them to "see mathematics as an important element of the historical development of man's thought process" (M. Wilson, Attitudes Survey, 11/04/04). Mandy's view of education, which was parallel to her school's educational philosophy, enabled her to provide an experience with original sources that secondary students do not typically have. Further research aimed at investigating the mathematical and pedagogical understandings that both pre-service and in-service teachers hold after engaging in the study of original mathematical texts may be the key to creating the the potential for more students to have such experiences.

References

[1] D.L. Anderson, R. Berg, A.W. Sebrell & D.W. Smith, *Exponentials and logarithms*, in V. Katz & K. D. Michalowicz (Eds.), *Historical modules for the teaching and learning of secondary mathematics*, Mathematical Association of America, Washington D.C. (2004).

[2] E. Barbin, *Integrating history: research perspectives*, in J. Fauvel & J. van Maanen (Eds.), *History in mathematics education: The ICMI study* (pp. 63 - 90), Kluwer, Boston (2000).

[3] M.G. Bartolini Bussi & A. Sierpinska, *Historical formation and student understanding in mathematics*, in J. Fauvel & J. van Maanen (Eds.), *History in mathematics education: The ICMI study* (pp. 143 - 170), Kluwer, Boston (2000).

[4] M.S. Garet, A.C. Porter, L. Desimone, B.F. Birman & K.S. Yoon, *What makes professional development effective? Results from a national sample of teachers*, American Educational Research Journal **38**(4) (2001), pp. 915-945.

[5] M.S. Smith, *Practiced-based professional development for teachers of mathematics*, National Council of Teachers of Mathematics, Reston, VA (2001).

[6] E. Wright, *John Napier: A description of the admirable table of logarithms*, (1616, transcribed from an original translation). Retrieved November 1, 2004 from http://www.johnnapier.com/table_of_logarithms_001.htm.

Reading al-Khwarizmi's treatise on quadratic equations with 9th-graders: An empirical study

Michael R. Glaubitz

The integration of historico-mathematical elements into the traditional mathematics syllabus is assumed by many to provide valuable enhancements to the mathematical and scientific literacy of learners. Yet others, among them in-service teachers, question the usefulness of this approach in view of the thematic pressure and lack of time in the typical classroom situation. As a matter of fact, rather little is known empirically about the effectiveness or possible drawbacks of historico-mathematical teaching in class. In order to research this question, I conducted a comparative study, in which more than 250 typical German 9th-graders, in ten classes, participated. The students learned about a core subject matter from the syllabus (quadratic equations) *with* or *without* paying substantial regards to the historical origin and development of today's thinking and solving schemes related to solving quadratic equations.

I devised two parallel teaching units: one historical, including the reading of original sources and the other quite conventional and assembled from various standard textbooks. The material was collected and organized in two specially designed workbooks for the students and the lessons were given to them by their respective, regular mathematics teachers.

All students received an identical and quite conventional introduction to quadratic equations and learned to solve them by completing the square and by using the formula (3 lessons). Seven of the participating classes then read and studied the historical material, which included – amongst other elements – excerpts from al-Khwarizmi's famous 'al-jabr' ($\sim$ 820 A.D.) [1], while three control classes pursued the conventional treatment of quadratic equations and concerned themselves with standard exercises and applications (6 lessons respectively). Both groups then underwent an identical achievement test and a second test six to eight weeks later in order to find out if one group performed better than the other. In addition, I have collected the workbooks and data from two student questionnaires (made anonymous), in which they could comment upon the teaching unit and their general beliefs on mathematics and school. Also, most of the lessons were videotaped and transcribed.

The sample of participating students and teachers covered a variety of different learning and teaching styles and a broad spectrum of beliefs about mathematics, school, and education. However, from the questionnaire responses we can conclude that mathematics is generally seen as a rather popular subject among the participating students. While the statement "I like school" reaches a mediocre average value of 2.62 on a 1 to 4-scale, with 1 representing "not at all" to 4 representing "very much", the statement "I like maths" averages 3.08. This value is further exceeded by the 3.23 average corresponding to the statement "I like to learn about maths in ancient times." Thus, the substantial historical enrichments in this teaching unit were clearly appreciated by most students. In addition, many of them felt that the methodical focus in the lessons changed. Routine activities like 'calculating', 'working with formulas', or 'proving' became less important in their eyes, while the main stress was put on hermeneutical and communicative activities like 'reading mathematical texts', 'discussing with others' or 'varying the modes of representation.' As a consequence, some of the students' beliefs were questioned. For example, mathematics no longer represented a subject in which the main concern is (or should be) calculating, doing many problems, and learning things (i.e. formulas, theorems) by heart. Many students said that the importance of understanding contexts and reasoning became more apparent to them, whereas they would not believe that the contents of the historical unit were of any use for later classes or for their professional career.

As mentioned before, both the experimental and the control classes underwent an identical achievement test directly after the respective lessons. The experimental classes achieved an average mark of 2.89 on the test, on a scale from 1 meaning 'excellent' to 6 meaning 'unsatisfactory.' By comparison, the control classes averaged 3.30. Prior to the study, the corresponding achievement test averages were 3.16 (experimental classes) and 3.29 (control classes). It was found that the experimental classes performed 0.27 better than before, whereas the control classes' achievement was essentially unchanged (-0.01). Six weeks later, the experimental classes averaged 3.04 (which is still 0.12 better than their pre-study value), whereas the control classes dropped to 3.53, which is 0.24 worse than their average score prior to the study and 0.49 below the experimental classes' average in the second test. The overall drop in the second test score is certainly due to short-term memory effects.

Still much work has to be done in order to find out more about the complex interrelations of the aforementioned data. Nevertheless, it is obvious that the historical teaching (including original sources) did not damage the students' skills. To the contrary, it may have had the opposite effect. Furthermore, the historical material seems to have improved the quality of the teaching by introducing hitherto unknown or underestimated hermeneutical and communicative elements and making students think about the subject and their relations to it.

References

[1] Muhammad ibn Musa al-Khwarizmi, *The algebra of Mohammed ben Musa, edited and translated by Frederic Rosen*, reprint of the London 1831 edition, Olms, Hildesheim (1986).

Teaching Heron's formula in context

Wann-Sheng Horng

In my talk, I presented episodes devoted to seminars (including graduate student class, in-service training program) and high school teaching practices. Based on their reflections and feedbacks, I tried to use Heron's formula to demonstrate a theme, that is, how teachers first learn the primary sources about Heron's formula and their Chinese versions and then teach the formula in due context can enhance their pedagogical content knowledge.

Among the episodes, four different Chinese versions of the proof, which are basically due to introduction by Italian Jesuit Jacques Rho in 1634, had been discussed in my seminars on social history of mathematics and on reading of ancient mathematical texts respectively. In this connection, I asked my graduate students, most of them high school teacher attending my weekly seminar, to design a teaching project on the topic. Besides this activity, I was also invited to give a lecture (on a similar topic) to senior high school mathematics teachers who attended one in-service teaching seminar for three hours. Feedback was requested in order to understand whether they were aware of the related methodological sensitivity.

During my on-going seminars on the subject, Mr. Jun-hong Su, one of my doctoral students, gave a lecture to his colleagues who are teaching gifted classes in Taipei City High School, in which he referred to proofs by Mei Wending (1633 - 1721) and Li Shanlan (1811 - 1882). In addition, he also brought to his two gifted classes three versions of the proof, namely one that was covered in the textbook and the other two by Heron and Mei Wending. A questionnaire on how the students reflected on the teaching activities was collected. In addition, the teaching of two more classes were conducted in order to serve as reference.

Mr. Su's teaching experiment of the formula drew upon a significant issue, namely how the Chinese versions of proof for the formula evolved over time. This might also well explain why his fellow students were so enthusiastic about the connection between law of cosines and the proof of the formula in the history of Chinese mathematics after the 16th century. Therefore, Mr. Su was invited to serve as the guest editor of the special issue (April 2006, available in Chinese) of the *HPM Tongxun* in order to put together all of the materials developed by my team. It should be noted that Ms. Hui-Yu Su, the editor of the *HPM Tongxun*, proposes Heron's formula as a mediator to vertically integrate from elementary to senior high school level the topics concerning the area formula for a triangle. Meanwhile, Mr. Jun-hong Su has also concluded our odyssey on Heron.

In preparing for a full text, I will go on to discuss the following issue, namely how does the introduction of ancient mathematical texts intervene teachers' teaching practice as they prepare to teach Heron's formula in context? Since most participants of my seminars are in-service teachers, they have a dual role to play in this connection, both as a historian and a teacher. Is the status responsible for their awareness of integrating history and pedagogy of mathematics (HPM) in their teaching practice? If the answer is yes, how and to what extent can we assist in this effort? And how can we have an optional way to achieve the goal? As a conclusion, I will try to explain these data in terms of Hans Niels Jahnke's hermeneutic two-fold circle and its modified version, hermeneutic tetrahedron.

Students working on their own ideas: Bernoulli's lectures on the differential calculus in grade 11

Hans Niels Jahnke

During his first stay in Paris in 1691 - 1692 Johann Bernoulli (1667-1748) imparted the basic ideas and techniques of the new Leibnizian differential calculus to the Marquis de l'Hospital, a then prominent member of the French Academy of the Sciences. On this occasion Bernoulli wrote a Latin manuscript of about 40 pages on the differential calculus which served as model for the first published textbook on this topic, the Analyse des infiniment petits pour l'intelligence de lignes courbes (1696) [1], written by de l'Hospital. Bernoulli's manuscript was lost for more than two hundred years until a copy of it was detected in the library of the university of Basel in 1922 [2]. A German translation of the manuscript appeared in 1924 [3]. Sections of Bernoulli's manuscript were read with students of an advanced mathematical course ("Leistungskurs") in grade 11 at a Gymnasium near Bielefeld, Germany (see [4] for details). The students had already been introduced to the fundamentals of the differential calculus and they knew the concepts of limit and derivative and how to apply them in order to determine tangents, extrema values and points of inflection. In Bernoulli's manuscript they found a conceptual framework completely different from their own, comprising the notions of differentials and of "infinitely small quantities." In Postulate 1 Bernoulli stated explicitly that "a quantity which is decreased or increased by an infinitely smaller quantity is neither decreased nor increased" and in Postulate 2 it is said that "every curved line consists of infinitely many segments which are infinitely small." Thus, there exists a considerable diachronic tension between Leibniz' and Bernoulli's original notions of the calculus and the modern limit-based approach. When reading Bernoulli's manuscript the students had to think themselves into the ideas of mathematicians living at a different time and in a different culture and using a different conceptual framework. One can characterize this cognitive constellation by the notion of "alienation" ("Verfremdung", Bertold Brecht), a term well-known in the theory of literature. The students' ideas of the calculus are alienated by reading the source in order that this very process leads them to a better and more reflexive understanding. The teaching sequence began with a unit on Bernoulli's biography, his postulates of the differential calculus and the rules for forming differentials derived from the postulates. It continued with the construction of a tangent to the parabola and the determination of extrema values; both topics well known to the students in the framework of limits and derivatives. In an additional unit two letters exchanged between Leibniz and Johann Bernoulli about infinitely small quantities were read, and at the urging of the students Bernoulli's methods for calculating points of inflection were studied.

The students showed themselves to be very inventive in interpreting Bernoulli's postulates on infinitely small quantities. In a sense, the Leibniz-Bernoulli conceptual framework proved to be near to the intuitive thinking of many of them, and, thus, they started to think more consciously about the concepts of limit and derivative they had previously learned. Also, the students accepted that in the modern approach a tangent is defined as the limiting object of a sequence of secants; whereas for Leibniz and Bernoulli a tangent was simply the rectilinear continuation of an infinitely small segment.

An expressive experience was to see that some mathematical ideas of young Johann Bernoulli were rather imperfect and later on eliminated from calculus. For example, Bernoulli considered the behavior of the point of intersection of the abscissa and the tangent in a neighborhood of the point of inflection as a method to determine points of inflection. Though this method works in most cases they realized that there are many difficult exceptions.

Reading Bernoulli's manuscript proved a viable means for obtaining a better understanding of the foundations of the differential calculus.

References

[1] G. M. d. l'Hospital, *Analyse des infiniment petits pour l'intelligence de lignes courbes*, Impriméric Royale, Paris (1696).

[2] J. Bernoulli, *Lectiones de calculo differentialium*, Manuskript 1691/92, *Verhandlungen der Naturforschenden Gesellschaft in Basel* **38** (1923), pp. 1 - 23.

[3] P. Schafheitlin, *Die Differentialrechnung von Johann Bernoulli aus dem Jahre 1691/92. Nach der in der Basler Universitätsbibliothek befindlichen Handschrift übersetzt, mit einem Vorwort und Anmerkungen versehen*, Akademische Verlagsgesellschaft, Leipzig (1924).

[4] H.N. Jahnke, *Historische Reflexion im Unterricht. Das erste Lehrbuch der Differentialrechnung (Bernoulli 1692) in einer elften Klasse*, mathematica didactica **18**, (2) (1995), pp. 30 - 58.

Original Sources: some educational practice

Jan van Maanen

In the discussion about the integration of original sources in mathematics education several questions come up, such as:

- what do we understand by "original" (or "primary", as it is also called)?
- what arguments do we have to convince teachers and educational politicians that it will 'pay off' to work with original sources in mathematics lessons?
- what experimental evidence do we have to base these arguments upon?
- what are the requirements for the educational setting and does one setting work better than the other?

In an attempt to answer these questions I shall discuss an activity with 'original sources', which was conducted at a grammar school near my university, in the north of the Netherlands. It is a one-day project for small groups (ages 16 and 17). The day of the project was well-chosen, since most of the other classes were not at school, because of working weeks and examinations, so there was ample working space for the pupils, also in the computer rooms. The day, for the first time in May 2002 and repeated in subsequent years until 2005, starts with an introduction about the aim and procedures of the project. The aim is that the pupils solve a mathematical problem, set in a historical document, and they then write a report about it in which the historical context is sketched as well. A special feature of the problems is that the pupils receive them in a photocopied version, but the books from which the copies were made are available during the day, and the pupils may consult them. Three problems come from a geometry text printed c. 1610 (by Sybrandt Hansz. Cardinael), one from algebra textbooks (Stampioen 1639; De Graaf 1672), and one from an arithmetic manuscript written around 1620. These five sources are all in Dutch, but since the school also teaches Latin a sixth problem is proposed from Euler's *Introductio in analysin infinitorum* (1748).

Generally, the teams, which consisted of three or four pupils and which had from 9:30 until 16:00 in order to complete their task, divided the work. After they understood the problem, which generally was not at all easy because of the linguistic aspects, some worked on the solution, whereas others in the library and via the internet did research about the historical background. This was the period in which the teachers, one being their own mathematics teacher and the other -the author of this report- an expert in history of mathematics, received many questions. Euler's problem, for example, requires students to be able to calculate by hand a sharp rational approximation to given square root. This does not exist in the standard curriculum, but in this case the pupils needed to know the method and they were quite eager to learn it.

One of the interesting aspects of the project was that the difference with the methods and notation already known to the pupils provoked them to think about the value of their own methods. One longer quote may illustrate this. It is given by four pupils who calculated the height of a 13-14-15 triangle with the help of a geometrical transformation, whereas their own method for solving the problem was to apply the cosine-rule:

> Now that we had the chance to compare several solutions of the geometrical problem by Sybrandt Hansz. Cardinael we have come to the conclusion that we find our present methods more practical than the method of calculation used by Sybrandt Hansz. Cardinael. This may be because we did not yet know the methods he exploited, but it is certain that our way is easier than his, since we only have to use one formula, whereas Sybrandt Hansz. Cardinael needed much more reasoning. Furthermore, we did not have to construct and make drawings in the triangle, whereas Sybrandt Hansz. Cardinael had to make many drawings, e.g. squares.
>
> Naturally, our method is in fact only well applicable with the help of modern calculators. Sybrandt Hansz. Cardinael should have used many complicated tables for this, or it must have cost him much time and calculating effort. Therefore it is probable that our method was not the easiest in his time, and that his method was more practical for him since it did not require additional tools.
>
> [pupils evaluating the project; written conclusion 8 May 2003]

Such judgements are common in the reports. Pupils point at the greater efficiency of our present methods and notation, but often they also realize that the old methods have valuable sides too.

Part of the material, e.g. two problems by Cardinael, is also available through the internet (`http://www.math.rug.nl/didactiek/BSP/cardinael/cardinael.html` and `.../cardinael2.html` ; 11 May 2006).

Some conclusions seem justified: the quality and appeal of the source are important, really old material which nevertheless can be read by the pupils without edition or translation works well. The setting is important, especially the length of the time slot and the availability of an expert worked well. But much more research needs to be done in order to answer the four questions above more fully. Especially relevant are the questions whether such a project could be scaled up, and how pupils would react if the project would be given to them by a teacher who has no affinity for history.

Reporters: Kathleen M. Clark, Michael R. Glaubitz

Participants

Prof. Dr. Abraham Arcavi
Department of Science Teaching
Weizmann Institute of Science
76100 Rehovot
ISRAEL

Prof. Dr. Evelyne Barbin
Departement de Mathematiques
Universite de Nantes
B.P. 92208
F-44322 Nantes Cedex 3

Caroline Bardini
Department of Science &
Mathematics Education
The University of Melbourne
Parkville 3010
Victoria
AUSTRALIA

Dr. Kathy Clark
Mid-Atlantic Center for Mathematics
Teaching and Learning
University of Maryland
2226 H.R.W.Benjamin Building
College Park, MD 20742-1175
USA

Adriano Dematte
via Madonnina 14
I-38050 Povo di Trento

OStR. Michael Glaubitz
Fachbereich Mathematik
Universität Duisburg-Essen
Campus Essen
Universitätsstr. 3
45117 Essen

Prof. Dr. Wann-Sheng Horng
Department of Mathematics
National Taiwan Normal University
88 sec. 4, Ting Chou Road
Taipei
TAIWAN

Prof. Dr. Hans Niels Jahnke
Fachbereich Mathematik
Universität Duisburg-Essen
Campus Essen
Universitätsstr. 3
45117 Essen

Prof. Dr. Jan A. van Maanen
Freudenthal Instituut
Universiteit Utrecht
Post Bus 9432
NL-3506 GK Utrecht

Prof. Dr. David Pengelley
Mathematical Sciences
New Mexico State Unversity
Las Cruces, NM 88003-0001
USA

StR'in Katja Peters
Arndtstr. 28
52064 Aachen

Prof. Dr. Luis Radford
Ecole des Sciences de l'Education
Universite Laurentienne
Sudbury, Ont. P3E 2C6
CANADA

Dr. Peter Rasfeld
Fakultät für Mathematik
Universität Bielefeld
Universitätsstr. 25
33615 Bielefeld

Prof. Dr. Karin Reich
Institut für Geschichte der Naturwissenschaften, Mathematik und Technik der Universität Hamburg
Bundesstr. 55
20146 Hamburg

Bjorn Smestad
Faculty of Education
Oslo University College
4 St. Olavs Plass
N-0130 Oslo

Prof. Dr. Costas Tzanakis
Department of Education
University of Crete
74100 Rethymnon Crete
GREECE